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We give an elementary construction of any real closed field in terms of Nash function fields. We also give a characterization of any Archimedean field in terms of fields of Nash functions.

Introduction

In the study of Hilbert’s 17th problem, orderings of a real field \( k \) are of importance (see [Alonso 1986; Alonso et al. 1984; Artin 1927; Artin and Schreier 1927a; 1927b; Bochnak and Efroymson 1980; Bröker 1982; Dubois 1981; Guangxing 2005; Marshall 2003; Prestel and Delzell 2001; Schwartz 1980]). By the Artin–Schreier theorem [Artin 1927; Artin and Schreier 1927a; 1927b], the study of such orderings amounts to considering real closures of \( k \). The aim of this article is to construct a universal model of an arbitrary real closed field. To this end, we construct, in terms of Nash functions, all real closures of the rational function fields \( \mathbb{Q}(\Lambda_T) \), where \( \Lambda_T = (\Lambda_t : t \in T) \) and \( T \neq \emptyset \) is a system of any number of variables. This suffices to achieve our purpose, because any real closed field \( R \) is order-preserving isomorphic to a real closure of some field \( \mathbb{Q}(\Lambda_T) \) (Corollary 5.5). If \( T = \emptyset \), then \( \mathbb{Q}(\Lambda_T) = \mathbb{Q} \), and the above is obvious. We assume the Kuratowski–Zorn lemma, so the set \( T \) can be well-ordered, provided \( T \neq \emptyset \).

L. Bröker [1982] proved in his ultrafilter theorem that there exists a one-to-one correspondence between the family of ultrafilters and the family of orderings in \( \mathbb{Q}(\Lambda_T) \), or equivalently with the real closures of \( \mathbb{Q}(\Lambda_T) \). We prove that there exists a one-to-one correspondence between the family of orderings in \( \mathbb{Q}(\Lambda_T) \) and the family of plain filters (Theorem 5.2, Proposition 2.4, and Corollary 2.5). By a plain filter we mean a filter \( \Omega \) of subsets of \( \mathbb{R}^T \) with these properties:

1. Any \( U \in \Omega \) is a nonempty open connected semialgebraic set.
2. For any algebraic set \( V \subseteq \mathbb{R}^T \), where \( V = P^{-1}(0) \) and \( P \in \mathbb{Q}[\Lambda_T] \), some connected component of \( \mathbb{R}^T \setminus V \) belongs to \( \Omega \).

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(3) For any $U_1, U_2 \in \Omega$, there exists $U_3 \in \Omega$ such that $U_3 \subset U_1 \cap U_2$.

The correspondence between orderings and plain filters is as follows: For any ordering $\succ$ of $\mathbb{Q}(\Lambda_T)$, there exists a unique plain filter $\Omega$ such that $f > 0$ if and only if $f > 0$ on some $U \in \Omega$, where $\succ$ is the usual ordering on $\mathbb{R}$. Conversely, any plain filter $\Omega$ determines a unique ordering $\succ$ of $\mathbb{Q}(\Lambda_T)$ in this way.

The main result of this article is Theorem 5.2, where we give a construction of any real closure of $\mathbb{Q}(\Lambda_T)$ in terms of Nash functions. The main idea and motivation for the above considerations was a geometric construction of the algebraic closure of $\mathbb{C}(\Lambda_1, \ldots, \Lambda_m)$ [Spodzieja 1996]. More precisely, for any plain filter $\Omega$ of open connected semialgebraic sets and any $U \in \Omega$, the ring $\mathcal{N}(U)$ of $\mathbb{Q}$-Nash functions (see Section 1) on $U$ is a domain. In $\bigcup_{U \in \Omega} \mathcal{N}(U)$, we introduce an equivalence relation $\sim$: $(f_1 : U_1 \to \mathbb{R}) \sim (f_2 : U_2 \to \mathbb{R})$ if and only if $f_1|_{U_1} = f_2|_{U_2}$ for some $U_3 \in \Omega$. The set $\mathcal{N}_\Omega$ of equivalence classes of $\sim$ with the usual operations of addition and multiplication is a field, which is a real closure of $\mathbb{Q}(\Lambda_T)$ (see Theorem 5.2, and compare [Spodzieja 1996, Theorem 2.4 and Corollary 2.5]). One can view $\mathcal{N}_\Omega$ as the inverse limit of the étale topology $\bigcup_{U \in \Omega} \mathcal{N}(U)$ of $\mathbb{R}^T$ [Grothendieck 1967].

In Section 3, we prove that an ordering $\succ$ of $\mathbb{Q}(\Lambda_T)$ is Archimedean if and only if the set $\bigcap_{U \in \Omega} U$ is nonempty for the plain filter $\Omega$ determining $\succ$; and if that is the case, this set has exactly one point (Theorem 3.1). In Section 4, we give some examples of non-Archimedean orderings corresponding to the one in [Spodzieja 1996].

1. Preliminaries

Let $\mathbb{K}$ be the field $\mathbb{Q}$ of rational, $\mathbb{R}$ of real, or $\mathbb{C}$ of complex numbers. Let $T$ be a nonempty set. We denote by $\Lambda_T = (\Lambda_t : t \in T)$ a system of independent variables $\Lambda_t$, by $\mathbb{K}[\Lambda_T]$ the ring of polynomials in $\Lambda_T$ over $\mathbb{K}$, and by $\mathbb{K}(\Lambda_T)$ the quotient field of $\mathbb{K}[\Lambda_T]$. Note that for any $P \in \mathbb{K}(\Lambda_T)$, we have $P \in \mathbb{K}(\Lambda_{t_1}, \ldots, \Lambda_{t_m})$ for some finite number of indices $t_1, \ldots, t_m \in T$.

We denote by $\mathbb{K}^T$ the set of all functions $T \to \mathbb{K}$ equipped with the unique topology for which all projections $\mathbb{K}^T \ni x \mapsto x(t) \in \mathbb{K}$, $t \in T$ are continuous.

Let $\mathbb{L}$ be a subfield of $\mathbb{K}$. A subset of $\mathbb{K}^T$ is called $\mathbb{L}$-algebraic, or simply algebraic if $\mathbb{L} = \mathbb{K}$, when it is defined by a finite system of equations $P = 0$, where $P \in \mathbb{L}[\Lambda_T]$. Any $\mathbb{L}$-algebraic set in $\mathbb{K}^T$ is of the form $\{x \in \mathbb{K}^T : (x(t_1), \ldots, x(t_m)) \in V\}$, where $m \in \mathbb{N}, t_1, \ldots, t_m \in T$, and $V \subset \mathbb{K}^m$ is an $\mathbb{L}$-algebraic subset of $\mathbb{K}^m$.

If $\mathbb{L}$ is a subfield of $\mathbb{R}$, then we assume that $\mathbb{L}$ is an ordered field with order induced from $\mathbb{R}$.

Let $\mathbb{L}$ be a subfield of $\mathbb{R}$. A subset of $\mathbb{R}^T$ is called $\mathbb{L}$-semialgebraic when it is defined by a finite alternative of finite systems of inequalities $P > 0$ or $P \geq 0$, where $P \in \mathbb{L}[\Lambda_T]$. Analogously to the above, any $\mathbb{L}$-semialgebraic set in $\mathbb{R}^T$ is of the form
\{x \in \mathbb{R}^T : (x(t_1), \ldots, x(t_m)) \in X\}, \text{ where } m \in \mathbb{N}, t_1, \ldots, t_m \in T, \text{ and } X \subset \mathbb{R}^m \text{ is an } \mathbb{L}\text{-semialgebraic subset of } \mathbb{R}^m. \text{ A set is called open basic } \mathbb{L}\text{-semialgebraic if it has the form } \{x \in \mathbb{R}^T : g_i(x) > 0, \ i = 1, \ldots, n\}, \text{ for some } n \in \mathbb{N} \text{ and } g_i \in \mathbb{L}[\Lambda_T], \ i = 1, \ldots, n.

We now list some basic properties of algebraic and semialgebraic sets in infinite-dimensional real vector spaces, which follow easily from their analogues in finite-dimensional spaces [Benedetti and Risler 1990; Bochnak et al. 1987; Bochnak and Efroymson 1980; Efroymson 1974; 1976; 1981; Mostowski 1976; Prestel and Delzell 2001; Tancredi and Tognoli 2006; Tworzewski 1990].

**Proposition 1.1.** Let \( \mathbb{L} \) be a subfield of \( \mathbb{R} \) (or \( \mathbb{K} \) in (a)).

(a) The family of \( \mathbb{L}\text{-algebraic sets in } \mathbb{K}^T \) is closed with respect to union and intersection of a finite number of sets.

(b) The family of \( \mathbb{L}\text{-semialgebraic sets in } \mathbb{R}^T \) is closed with respect to complement, union, and intersection of a finite number of sets.

(c) (Tarski–Seidenberg) Let \( \pi_{t_1, \ldots, t_m} : \mathbb{R}^T \ni x \mapsto (x(t_1), \ldots, x(t_m)) \in \mathbb{R}^m, \text{ where } t_1, \ldots, t_m \in T. \text{ If } X \subset \mathbb{R}^T, Y \subset \mathbb{R}^m \text{ are } \mathbb{L}\text{-semialgebraic sets, then } \pi_{t_1, \ldots, t_m}(X) \text{ and } \pi_{t_1, \ldots, t_m}^{-1}(Y) \text{ are } \mathbb{L}\text{-semialgebraic sets, too.}

(d) For any \( \mathbb{L}\text{-semialgebraic set } X \subset \mathbb{R}^T, \text{ the interior } \text{Int} \ X, \text{ closure } \overline{X}, \text{ and the boundary } \partial X \text{ are } \mathbb{L}\text{-semialgebraic sets.}

Let \( \mathbb{L} \) be a subfield of \( \mathbb{R} \). A function \( f : U \to \mathbb{R}, \text{ where } U \subset \mathbb{R}^T \) is an open \( \mathbb{L}\text{-semialgebraic set, is called an } \mathbb{L}\text{-Nash function if } f \text{ is analytic and there exists a nonzero polynomial } P \in \mathbb{L}[\Lambda_T, Z] \text{ such that } P(\lambda, f(\lambda)) = 0 \text{ for } \lambda \in U. \text{ In fact, } f \text{ depends on a finite number of variables, so the analyticity of } f \text{ is clear. The ring of } \mathbb{L}\text{-Nash functions in } U \text{ is denoted by } \mathcal{N}^L(U).

The next result follows via R. Thom’s lemma (see for instance [Bochnak et al. 1987, Proposition 2.5.4 and the arguments of Theorems 2.3.6 and 2.4.4]) from the fact that any \( \mathbb{L}\text{-semialgebraic set in a finite-dimensional vector space over } \mathbb{R} \text{ is the disjoint union of a finite number of } \mathbb{L}\text{-semialgebraic sets which are homeomorphic to Cartesian products of intervals.}

**Proposition 1.2.** Let \( \mathbb{L} \) be a subfield of \( \mathbb{R} \). Any connected component of an \( \mathbb{L}\text{-semialgebraic subset of } \mathbb{R}^T \) is \( \mathbb{L}\text{-semialgebraic.}

A function \( f : U \to \mathbb{C}, \text{ where } U \subset \mathbb{C}^T \) is an open set, is called a \( \mathbb{C}\text{-Nash function if } f \text{ is holomorphic and there exists a nonzero polynomial } P \in \mathbb{C}[\Lambda_T, Z] \text{ such that } P(\lambda, f(\lambda)) = 0 \text{ for } \lambda \in U. \text{ The ring of } \mathbb{C}\text{-Nash functions in } U \text{ is denoted by } \mathcal{N}^C(U).

For the basic properties of Nash functions and semialgebraic sets in finite-dimensional vector spaces, see, for instance, [Benedetti and Risler 1990; Bochnak et al. 1987; Bochnak and Efroymson 1980; Efroymson 1974; 1976; 1981; Mostowski
1976; Nash 1952; Tancredi and Tognoli 2006; Tworzewski 1990]. From these
properties, we immediately obtain:

**Proposition 1.3.** Let \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{K} = \mathbb{C} \), let \( \mathbb{L} \) be a subfield of \( \mathbb{K} \), and let \( U \subset \mathbb{K}^T \)
be an open connected set. Then \( \mathcal{N}^\mathbb{K}(U) \) is a domain, provided \( U \) is semialgebraic
when \( \mathbb{K} = \mathbb{R} \). In particular \( \mathcal{N}^\mathbb{Q}(U) \) is a domain.

### 2. Orderings in \( \mathbb{Q}(\Lambda_T) \)

Let \( T \) be a nonempty set. A family \( \Omega \) of subsets of \( \mathbb{R}^T \) will be called a c-filter
(connected sets filter) if it satisfies these conditions:

(i) Any \( U \in \Omega \) is a nonempty open connected \( \mathbb{Q} \)-semialgebraic set.

(ii) For any \( \mathbb{Q} \)-algebraic set \( V \subset \mathbb{R}^T \), there exists \( U \in \Omega \) such that \( V \cap U = \emptyset \).

(iii) For any \( U_1, U_2 \in \Omega \), there exists \( U_3 \in \Omega \) such that \( U_3 \subset U_1 \cap U_2 \).

**Proposition 2.1.** Let \( \Omega \) be a c-filter of subsets of \( \mathbb{R}^T \). The set \( \partial \Omega := \bigcap_{U \in \Omega} \overline{U} \) has
at most one point. Moreover, whenever \( T \) is a finite set, \( \partial \Omega \neq \emptyset \) if and only if there
exists a bounded set \( U \in \Omega \).

**Proof.** If \( x_1, x_2 \in \partial \Omega \) with \( x_1 \neq x_2 \), then for some polynomial \( f \in \mathbb{Q}[\Lambda_T] \), we have
\( f(x_1) < 0 < f(x_2) \). Hence, for some \( W \in \Omega \) such that \( W \cap f^{-1}(0) = \emptyset \), we have
both \( f(x) < 0 \) and \( f(x) > 0 \) for some \( x \in W \). This contradiction gives the first part
of the assertion.

Now let \( T = \{t_1, \ldots, t_m\} \). Suppose that \( \partial \Omega \neq \emptyset \) and each \( W \in \Omega \) is an unbounded
set. Take \( x_0 \in \partial \Omega \), and let \( f = (\Lambda_T) = \Lambda_{t_1}^2 + \cdots + \Lambda_{t_m}^2 - r \), where \( r \in \mathbb{Q} \) and
\( r > x_0^2(t_1) + \cdots + x_0^2(t_m) \). Then \( f^{-1}(0) \cap W = \emptyset \) for some \( W \in \Omega \). Since \( W \) is a
connected unbounded set, \( x_0 \) is not an accumulation point of \( W \). This contradicts
the choice of \( x_0 \). Now assume that some \( W \in \Omega \) is bounded. Then it is easy to
see that there exists a sequence of nonempty compact sets \( C_1 \supset C_2 \supset \cdots \) with
diameters decreasing to 0 and such that \( U \cap C_n \neq \emptyset \) for all \( U \in \Omega \) and \( n \in \mathbb{N} \). Then
there exists \( x \in \bigcap_{n \in \mathbb{N}} C_n \) belonging to \( \partial \Omega \).

Let us fix a c-filter \( \Omega \) and define a relation \( >_\Omega \) in \( \mathbb{Q}(\Lambda_T) \) by

\[
\begin{align*}
0 \rightarrow \Omega 0 & \iff \text{there exists } U \in \Omega \text{ such that } f(x) > 0 \text{ for all } x \in U, \\
0 \rightarrow \Omega g & \iff f - g >_\Omega 0.
\end{align*}
\]

Let \( \Omega \) be a family of subsets of \( \mathbb{R}^T \). If an ordering \( > \) of \( \mathbb{Q}(\Lambda_T) \) satisfies \( f > 0 \)
if and only if \( f > 0 \) on some \( U \in \Omega \), we say that \( \Omega \) determines the ordering \( > \).

**Lemma 2.2.** The relation \( >_\Omega \) is an ordering in \( \mathbb{Q}(\Lambda_T) \), or in other words, a total
ordering satisfying

\[
f >_\Omega g \Rightarrow f + h >_\Omega g + h \quad \text{and} \quad f >_\Omega 0, \quad g >_\Omega 0 \Rightarrow fg >_\Omega 0.
\]
Proof. The relation $\succ_{\Omega}$ is well-defined. Indeed, if $f \in \mathcal{Q}(\Lambda_T)$ and $f \neq 0$, then the union of the sets of zeros and poles of $f$ is contained in some $\mathcal{Q}$-algebraic set $V \subseteq \mathbb{R}^m$. Hence, by (i) and (ii), for some $U \in \Omega$, the values $f(x)$ have a fixed sign for all $x \in U$. Moreover, if for some $U_1, U_2 \in \Omega$ we have $f(x) > 0$ for $x \in U_1$ and $f(x) \leq 0$ for $x \in U_2$, then $0 < f(x) \leq 0$ for $x \in U_1 \cap U_2$, and $U_1 \cap U_2 \neq \emptyset$ by (iii). This is impossible. It is easy to see that the remaining conditions are also satisfied. □

Proposition 2.3. Let $\Omega_1, \Omega_2$ be $c$-filters. If the orderings $\succ_{\Omega_1}$ and $\succ_{\Omega_2}$ are equal, then $\Omega = \{U \cup W : U \in \Omega_1, W \in \Omega_2\}$ is a $c$-filter determining the ordering $\succ_{\Omega_1}$.

Proof. Since $\Omega_1$ and $\Omega_2$ are $c$-filters, it suffices to prove that $U \cap W \neq \emptyset$ for all $U \in \Omega_1$ and $W \in \Omega_2$. Suppose $U \cap W = \emptyset$ for some $U \in \Omega_1$ and $W \in \Omega_2$. Let $U = U_1 \cup \cdots \cup U_k \cup V$ be a decomposition of $U$ into disjoint basic open semialgebraic sets $U_1, \ldots, U_k$ and a semialgebraic set $V$ included in an algebraic set. By (i) and (ii), there exists $U' \in \Omega_1$ such that $U' \subset U_i$ for some $i \in \{1, \ldots, k\}$. Since $U_i = \{x \in \mathbb{R}^T : f_j(x) > 0, j = 1, \ldots, n\}$ for some $f_1, \ldots, f_n \in \mathcal{Q}[\Lambda_T]$, by the assumption we have $f_1, \ldots, f_n \succ_{\Omega_1} 0$, and so there exists $W_1 \in \Omega_2$ such that $f_j(x) > 0$ for all $x \in W_1$ and $j = 1, \ldots, n$. By (iii), there exists $W_2 \in \Omega_2$ such that $W_2 \subset W \cap W_1$ and $f_j(x) > 0$ for all $j = 1, \ldots, n$ and $x \in W_2$. Thus $W_2 \subset U$, which contradicts the assumption. □

Now let $\succ$ be an ordering in $\mathcal{Q}(\Lambda_T)$, and let

$$\mathcal{U}_\succ = \left\{ \bigcap_{i=1}^n f_i^{-1}((0, +\infty)) \subseteq \mathbb{R}^T : f_i \in \mathcal{Q}(\Lambda_T), f_i \succ 0 \text{ for } i = 1, \ldots, n, n \in \mathbb{N} \right\},$$

where we regard $f \in \mathcal{Q}(\Lambda_T)$ as a function $f : \mathbb{R}^T \to \mathbb{R}$. By the definition of $\mathcal{U}_\succ$ and the Tarski transfer principle (see [Tarski 1948; Seidenberg 1954]), we find that $\emptyset \not\in \mathcal{U}_\succ$. Moreover, the relation $\succ$ is defined by

$$f > 0 \iff \text{there exists } U \in \mathcal{U}_\succ \text{ such that } f(x) > 0 \text{ for all } x \in U.$$

The sets of the family $\mathcal{U}_\succ$ may be disconnected, so $\mathcal{U}_\succ$ is not a $c$-filter. We will prove that the ordering $\succ$ is defined by some $c$-filter.

Proposition 2.4. There exists a unique $c$-filter $\Omega$ with the following properties:

(a) For any $f \in \mathcal{Q}(\Lambda_T)$, we have $f \succ_{\Omega} 0$ if and only if $f > 0$.

(b) For any $U \in \Omega$, there exists a $\mathcal{Q}$-algebraic set $V \subset \mathbb{R}^T$ such that $U$ is a connected component of $\mathbb{R}^T \setminus V$.

(c) For any $\mathcal{Q}$-algebraic set $V \subset \mathbb{R}^T$, some connected component of $\mathbb{R}^T \setminus V$ belongs to $\Omega$.

Proof. Let $\mathcal{F}$ be the family of all connected components of sets $U \in \mathcal{U}_\succ$. 
Claim 1. Every $U \in \mathcal{U}_\succ$ has a connected component $U_0$ such that $U_0 \cap W \neq \emptyset$ for any $W \in \mathcal{U}_\succ$.

Let $U \in \mathcal{U}_\succ$ and let $U = U_1 \cup \cdots \cup U_n$ be the decomposition into connected components. Assume to the contrary that there exist $W_1, \ldots, W_n \in \mathcal{U}_\succ$ such that $U_i \cap W_i = \emptyset$ for $i = 1, \ldots, n$. Then $U \cap W_1 \cap \cdots \cap W_n = \emptyset$, which is impossible. This gives Claim 1.

Claim 2. Each $U \in \mathcal{U}_\succ$ has exactly one connected component $S_U$ that intersects every $W \in \mathcal{U}_\succ$.

Let $U \in \mathcal{U}_\succ$, and let $U_1, \ldots, U_p$ be the connected components of $U$. Then

$$(1) \quad U = \bigcap_{i=1}^{s} \{ x \in \mathbb{R}^T : g_l(x) > 0 \}$$

for some nonzero polynomials $g_l \in \mathbb{Q}[^T \Lambda_T]$, with $g_l > 0$ for $l = 1, \ldots, s$, and

$$U_i = [f_i^{-1}(0) \cap U_i] \cup \bigcup_{j=1}^{n} \bigcap_{k=1}^{m} \{ x \in \mathbb{R}^T : f_i, j, k(x) > 0 \}, \quad i = 1, \ldots, p,$$

for some nonzero polynomials $f_i, f_i, j, k \in \mathbb{Q}[^T \Lambda_T]$. Denote by $\epsilon_i, j, k$ the sign of $f_i, j, k$ in the ordering $\succ$. Then $\epsilon_i, j, k \neq 0$ and $\epsilon_i, j, k f_i, j, k > 0$ for any $i, j, k$. Observe that for some $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, n\}$, we have $f_i, j, k > 0$ for $k = 1, \ldots, m$. Indeed, in the opposite case,

$$\emptyset = \bigcap_{l=1}^{s} \bigcap_{i=1}^{p} \bigcap_{j=1}^{n} \bigcap_{k=1}^{m} \{ x \in \mathbb{R}^T : g_l(x) > 0, \epsilon_i, j, k f_i, j, k(x) > 0 \} \in \mathcal{U}_\succ,$$

which is impossible. So, for some $i_0 \in \{1, \ldots, p\}$ and $j_0 \in \{1, \ldots, n\}$,

$$U' = \bigcap_{k=1}^{m} \{ x \in \mathbb{R}^T : f_{i_0}, j_0, k(x) > 0 \} \in \mathcal{U}_\succ,$$

and $U' \cap U_j = \emptyset$ for $j \neq j_0$. Hence, by Claim 1, $S_U = U_{j_0}$ is the unique connected component of $U$ satisfying Claim 2.

Claim 3. The family $\Omega = \{ S_U : U \in \mathcal{U}_\succ \}$ is a c-filter.

Since for every $\mathbb{Q}$-algebraic set $V \subseteq \mathbb{R}^T$ there exists $U \in \mathcal{U}_\succ$ such that $U \cap V = \emptyset$, we have $S_U \cap V = \emptyset$. Hence, it suffices to prove that for any $S_{U_1}, S_{U_2} \in \Omega$, there exists $S_{U_3} \in \Omega$ contained in $S_{U_1} \cap S_{U_2}$. Indeed, by the argument of Claim 2, there exist $W_1, W_2 \in \mathcal{U}_\succ$ such that $W_1 \subset S_{U_1}$ and $W_2 \subset S_{U_2}$. Hence, $S_{W_1 \cap W_2} \subset W_1 \cap W_2 \subset S_{U_1} \cap S_{U_2}$ and $S_{W_1 \cap W_2} \in \Omega$.

Claim 4. The c-filter $\Omega$ defined in Claim 3 satisfies the assertion of Proposition 2.4.
Part (a) is obvious.

Let $U \in \mathcal{U}_\prec$ be of the form (1), $f = g_1 \ldots g_s$, and $V = f^{-1}(0)$. Then, by the
definition of $S_U$, we see that $S_U$ is a connected component of $\mathbb{R}^T \setminus V$. This gives (b).

Let $V = f^{-1}(0)$ be a $\mathbb{Q}$-algebraic subset of $\mathbb{R}^T$. Then $U = \{x \in \mathbb{R}^T : f^2(x) > 0\} = \mathbb{R}^T \setminus V \in \mathcal{U}_\succ$ and $S_U \in \Omega$ is a connected component of $\mathbb{R}^T \setminus V$. This gives (c) and completes the proof. □

We call the $\mathcal{C}$-filter defined in Proposition 2.4 the \textit{plain filter} for the ordering $\succ$ and denote it by $\succ$.

From Proposition 2.4, we immediately obtain:

**Corollary 2.5.** The mapping $\succ \mapsto \succ$ is a one-to-one correspondence between the
set of orderings of $\mathbb{Q}(\Lambda_T)$ and the set of plain filters.

**Remark 2.6.** From the ultrafilter theorem [Bröker 1982], we see that for any
ultrafilter $\mathcal{F}$ of subsets of $\mathbb{R}^T$, there exists a plain filter $\Omega \subset \mathcal{F}$.

**Remark 2.7.** It is easy to observe that the statements of this section hold with $\mathbb{Q}$ replaced by $\mathbb{R}$.

### 3. Archimedean orderings in $\mathbb{Q}(\Lambda_T)$

Let $\succ$ be an ordering of $\mathbb{Q}(\Lambda_T)$. Then one can assume that $T$ is linearly ordered by

$$t_1 \succ t_2 \iff \Lambda_{t_1} \succ \Lambda_{t_2}.$$ 

If $f \succ g$, then we also write $g \prec f$.

**Theorem 3.1.** The following conditions are equivalent:

(a) The field $(\mathbb{Q}(\Lambda_T), \succ)$ is Archimedean.

(b) There exists $x_\succ \in \partial \Omega_\succ$ such that the set of coordinates of $x_\succ$ is algebraically
independent over $\mathbb{Q}$.

(c) There exists $x_\succ \in \partial \Omega_\succ$ such that $f > 0$ if and only if $f(x_\succ) > 0$.

(d) There exists $x_\succ \in \partial \Omega_\succ$ such that $x_\succ \in U$ for any $U \in \Omega_\succ$.

**Proof.** Assume (a). Then for any $t_1, \ldots, t_n \in T$ with $t_1 < \cdots < t_n$, and for the
projection $\pi_{t_1, \ldots, t_n} : \mathbb{R}^T \mapsto (x(t_1), \ldots, x(t_n)) \in \mathbb{R}^n$, the family

$$\Omega_{t_1, \ldots, t_n} = \{\pi_{t_1, \ldots, t_n}(U) : U \in \Omega\}$$

determines an Archimedean order in $\mathbb{Q}(\Lambda_{t_1}, \ldots, \Lambda_{t_n})$. Thus for some $W \in \Omega_{t_1, \ldots, t_n}$,
the function $f = \Lambda_{t_1}^2 + \cdots + \Lambda_{t_n}^2$ is bounded on $W$. So the set $W$ is bounded.
Hence, by Proposition 2.1, there exists $(x_1, \ldots, x_n) \in \partial \Omega_{t_1, \ldots, t_n}$. Since the
projections $\pi_{t_1, \ldots, t_n}$ are open, it is easy to observe that, for $t_{k_1}, \ldots, t_{k_j} \in \{t_1, \ldots, t_n\}$
with $t_{k_1} < \cdots < t_{k_j}$, we have $(x_{k_1}, \ldots, x_{k_j}) \in \partial \Omega_{t_{k_1}, \ldots, t_{k_j}}$. Consequently, there
exists \( x \in \mathbb{R}^T \) such that for any \( t_1, \ldots, t_n \in T \) with \( t_1 < \cdots < t_n \), we have \( \pi_{t_1, \ldots, t_n}(x) \in \partial \Omega_{t_1, \ldots, t_n} \). Summing up, \( x \in \partial \Omega \). The set of coordinates of \( x \) is algebraically independent over \( \mathbb{Q} \): otherwise, \( f(x) = 0 \) for some nonzero polynomial \( f \in \mathbb{Q}[\Lambda_T] \), and so \( f \) is infinitesimal. This contradicts (a) and gives (b).

Assume (b). Then any nonzero \( f \in \mathbb{Q}(\Lambda_T) \) with \( f > 0 \) is defined at \( x_\prec \). Moreover, \( f(x_\prec) \neq 0 \), so \( f(x_\prec) > 0 \). Conversely, assume that \( f(x_\succ) > 0 \). Then obviously for some connected component \( U \) of \( f^{-1}(0, +\infty) \), we have \( U \in \Omega_\succ \) and \( f(x) > 0 \) for \( x \in U \). Summing up, we obtain (c).

The implication (c) \( \Rightarrow \) (d) is trivial.

Now assume (d). Then we immediately obtain (b), and hence, no \( f \in \mathbb{Q}(\Lambda_T) \) is infinitesimal, and the field \( (\mathbb{Q}(\Lambda_T), \succ) \) is Archimedean. This gives (a) and completes the proof. \( \square \)

**Remark 3.2.** The assertion of Theorem 3.1 also holds for every c-filter determining \( \succ \) in place of the plain filter \( \succ \).

Theorem 3.1 implies:

**Corollary 3.3.** Let \( T \) be a finite set. Then the set of Archimedean orderings of \( \mathbb{Q}(\Lambda_T) \) is a dense subset of the space of orderings in \( \mathbb{Q}(\Lambda_T) \) in the path topology (see, for instance, [Marshall 2008]) of the real spectrum \( \text{Sper}(\mathbb{Q}[\Lambda_T]) \).

4. Examples of non-Archimedean orderings

Let \( m \) be a fixed positive integer and \( \Lambda \) a system of \( m \) variables \( \Lambda_1, \ldots, \Lambda_m \).

Take any \( P \in \mathbb{R}[\Lambda] \). Let \( \Gamma_P \subset \mathbb{R}^m \) be a set defined by

\[
\Gamma_P = \{ (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m : P(\lambda_1, \ldots, \lambda_m, \lambda_m + \gamma) = 0 \text{ for some } \gamma \in [0, \infty) \}. \]

We define a polynomial \( \omega(P) \in \mathbb{R}[\Lambda_1, \ldots, \Lambda_{m-1}] \) (or a number \( \omega(P) \in \mathbb{R} \) if \( m = 1 \)) by \( \omega(P) = 0 \) for \( P = 0 \), and \( \omega(P) = P_0 \) for \( P \neq 0 \), where

\[
P = P_0 \Lambda_m^d + P_1 \Lambda_m^{d-1} + \cdots + P_d
\]

and \( P_i \in \mathbb{R}[\Lambda_1, \ldots, \Lambda_{m-1}] \) (or \( P_i \in \mathbb{R} \) if \( m = 1 \)) for \( i = 0, \ldots, d \) and \( P_0 \neq 0 \).

Let us define sets \( W_P \subset \mathbb{R}^m \), for \( P \in \mathbb{R}[\Lambda] \). The definition will be inductive with respect to the number of variables \( \Lambda_1, \ldots, \Lambda_m \). For \( P \in \mathbb{R}[\Lambda] \), we put

\[
W_P = \begin{cases} 
\mathbb{R} \setminus \Gamma_P \subset \mathbb{R} & \text{if } m = 1, \\
(\mathbb{R}^m \setminus \Gamma_P) \cap (W_{\omega(P)} \times \mathbb{R}) \subset \mathbb{R}^m & \text{if } m > 1.
\end{cases}
\]

By the Tarski–Seidenberg theorem — see Proposition 1.1(c) — the sets \( W_P \) are semialgebraic for all \( P \in \mathbb{R}[\Lambda] \).

Analogously to Theorem 1.1 of [Spodzieja 1996], we prove the following proposition, which gives an example of c-filter.
Proposition 4.1. The family $\mathcal{W} = \{ W_P : P \in \mathbb{R}[\Lambda] \}$ satisfies these conditions:

- $R_0$. $W_P \subset \{ \lambda \in \mathbb{R}^m : P(\lambda) \neq 0 \}$.
- $R_1$. $W_P \cap W_Q = W_{P Q}$.
- $R_2$. For $P \neq 0$, $W_P$ is an unbounded subset of $\mathbb{R}^m$.
- $R_3$. For $P \neq 0$, $W_P$ is an open, connected and simply connected set.

Moreover, one can demand that

- $R_4$. $W_P = \mathbb{R}^m$ for $P = \text{const}$, $P \neq 0$.

In particular, $\mathcal{W}$ contains the c-filter

$$
\Omega = \{ W_P : P \in \mathbb{Q}[\Lambda] \}.
$$

Lemma 4.2. Let $1 \leq i_1 < \cdots < i_m \leq n$, and let $P \in \mathbb{R}[\Lambda_{i_1}, \ldots, \Lambda_{i_m}]$. Let $Q \in \mathbb{R}[\Lambda_1, \ldots, \Lambda_n]$ be a polynomial of the form

$$
Q(x_1, \ldots, x_n) = P(x_{i_1}, \ldots, x_{i_m}), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.
$$

Then $W_P \subset \mathbb{R}^m$, $W_Q \subset \mathbb{R}^n$, and

$$
W_Q \subset \{ (x_1, \ldots, x_n) \in \mathbb{R}^n : (x_{i_1}, \ldots, x_{i_m}) \in W_P \}.
$$

Proof. For $P = 0$ or $n = m$, the assertion is trivial. Assume that $P \neq 0$ and $n > m$.

Consider the case $n = m + 1$. Then there exists $1 \leq j \leq n$ such that

$$(\Lambda_{i_1}, \ldots, \Lambda_{i_m}) = (\Lambda_1, \ldots, \Lambda_{n-j}, \Lambda_{n-j+2}, \ldots, \Lambda_n),$$

under the obvious convention for $j = 1$ and $j = n$. Denote the $i$-th iteration of $\omega$ by $\omega^j$, where $\omega^0(P) = P$. Then, for $(x_1, \ldots, x_{n-i}) \in \mathbb{R}^{n-i}$,

$$
\omega^j(Q)(x_1, \ldots, x_{n-i}) = \begin{cases} 
\omega^j(P)(x_1, \ldots, x_{n-j}, x_{n-j+2}, \ldots, x_{n-i}) & \text{if } 0 \leq i \leq j - 2, \\
\omega^j(P)(x_1, \ldots, x_{n-j}) & \text{if } i = j - 1, \\
\omega^{j-1}(P)(x_1, \ldots, x_{n-i}) & \text{if } j \leq i \leq n.
\end{cases}
$$

Hence,

$$
\Gamma_{\omega^j(Q)} = \{ (x_1, \ldots, x_{n-i}) \in \mathbb{R}^{n-i} : (x_1, \ldots, x_{n-j}, x_{n-j+2}, \ldots, x_{n-i}) \in \Gamma_{\omega^j(P)} \}
$$

for $0 \leq i \leq j - 2$, and

$$
\Gamma_{\omega^{j-1}(Q)} = \{ (x_1, \ldots, x_{n-j+1}) \in \mathbb{R}^{n-j+1} : (x_1, \ldots, x_{n-j}) \in \Gamma_{\omega^{j-1}(P)} \}
$$

and $\Gamma_{\omega^j(Q)} = \Gamma_{\omega^{j-1}(P)}$ for $j \leq i \leq n$. In particular, $W_{\omega^j(Q)} = W_{\omega^{j-1}(P)}$ for $j \leq i \leq n$. 
Summing up, by (3),
\[
W_Q = \bigcap_{i=0}^{n} \left[ (\mathbb{R}^{n-i} \setminus \Gamma_{\omega^i}(Q)) \times \mathbb{R}^i \right]
\]
\[
= \bigcap_{i=0}^{j-2} \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : (x_1, \ldots, x_{n-j}, x_{n-j+2}, \ldots, x_{n-i}) \in \mathbb{R}^{n-i-1} \setminus \Gamma_{\omega^i}(P) \right\}
\]
\[
\cap \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : (x_1, \ldots, x_{n-j}) \in \mathbb{R}^{n-j} \setminus \Gamma_{\omega^j}(P) \right\} \cap \left[ W_{\omega^j}(Q) \times \mathbb{R}^j \right]
\]
\[
\subset \bigcap_{i=0}^{j-2} \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : (x_1, \ldots, x_{n-j}, x_{n-j+2}, \ldots, x_{n-i}) \in \mathbb{R}^{n-i-1} \setminus \Gamma_{\omega^i}(P) \right\}
\]
\[
\cap \left[ W_{\omega^j}(Q) \times \mathbb{R}^j \right]
\]
\[
= \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : (x_{i_1}, \ldots, x_{i_m}) \in W_P \right\}.
\]
This gives the assertion for \( n = m + 1 \). Hence, by an easy induction with respect to \( n - m \), we obtain the assertion. \( \Box \)

Let \( T \) be a linearly ordered set and let \( > \) be the ordering of \( T \).
For any \( t_1, \ldots, t_m \in T \), \( t_1 < \cdots < t_m \), we define a projection map
\[
\pi_{t_1, \ldots, t_m} : \mathbb{R}^T \ni x \mapsto (x(t_1), \ldots, x(t_m)) \in \mathbb{R}^m.
\]
Define a family \( \Omega \) of semialgebraic subsets \( U \) of \( \mathbb{R}^T \) by
\[
U = (\pi_{t_1, \ldots, t_m})^{-1}(W_P),
\]
where \( m \in \mathbb{N}, t_1, \ldots, t_m \in T, t_1 < \cdots < t_m \), and \( P \in \mathbb{Q}[\Lambda_{t_1}, \ldots, \Lambda_{t_m}] \setminus \{0\} \).

**Proposition 4.3.** The family \( \Omega \) is a c-filter.

**Proof.** By Proposition 4.1 (condition \( R_2 \)), any \( U \in \Omega \) is a nonempty set.

Let \( V \subseteq \mathbb{R}^T \) be a \( \mathbb{Q} \)-algebraic set, and let \( P \in \mathbb{Q}[\Lambda_T] \setminus \{0\} \) be such that \( V = \{ x \in \mathbb{R}^T : P(x) = 0 \} \). Then \( P \in \mathbb{Q}[\Lambda_{t_1}, \ldots, \Lambda_{t_m}] \setminus \{0\} \) for some \( t_1, \ldots, t_m \in T \), \( t_1 < \cdots < t_m \), and \( U = (\pi_{t_1, \ldots, t_m})^{-1}(W_P) \). Applying Proposition 4.1 (condition \( R_0 \)), we obtain that \( U \) satisfies (i).

Let \( U_1, U_2 \in \Omega \). Let \( t_1, \ldots, t_m, u_1, \ldots, u_n \in T \) satisfy \( t_1 < \cdots < t_m \) and \( u_1 < \cdots < u_n \), and assume moreover that for some \( P \in \mathbb{Q}[\Lambda_{t_1}, \ldots, \Lambda_{t_m}] \) and \( Q \in \mathbb{Q}[\Lambda_{u_1}, \ldots, \Lambda_{u_n}] \) we have \( U_1 = (\pi_{t_1, \ldots, t_m})^{-1}(W_P) \) and \( U_2 = (\pi_{u_1, \ldots, u_n})^{-1}(W_Q) \). Let \( v_1, \ldots, v_s \in T \), \( v_1 < \cdots < v_s \), be such that \( \{t_1, \ldots, t_m\} \cup \{u_1, \ldots, u_n\} \subset \{v_1, \ldots, v_s\} \), and let \( \overline{P}, \overline{Q} \in \mathbb{R}[\Lambda_{v_1}, \ldots, \Lambda_{v_s}] \) be polynomials of the form (4) determined by \( P \) and \( Q \), respectively. Then, by Proposition 4.1 (condition \( R_1 \)) and Lemma 4.2,
\[
(\pi_{v_1, \ldots, v_s})^{-1}(W_{\overline{P} \overline{Q}}) = (\pi_{v_1, \ldots, v_s})^{-1}(W_{\overline{P}}) \cap (\pi_{v_1, \ldots, v_s})^{-1}(W_{\overline{Q}}) \subset U_1 \cap U_2.
\]
This gives (ii).
Take any \( U \in \Omega \). There exist \( t_1, \ldots, t_m \in T \) and \( P \in \mathbb{R}[\Lambda_{t_1}, \ldots, \Lambda_{t_m}] \setminus \{0\} \) such that \( t_1 < \cdots < t_m \) and \( U = (\pi_{t_1, \ldots, t_m})^{-1}(W_P) \). By Proposition 4.1 (condition \( R_3 \)), \( U \) satisfies (iii). This completes the proof. \( \square \)

From the definition of the family \( \Omega \), we immediately obtain:

**Corollary 4.4.** For any \( t_1, t_2 \in T \), we have \( t_1 > t_2 \) if and only if \( \Lambda_{t_1} > \Omega \Lambda_{t_2} \).

Let \( \mathcal{Q} \in \mathbb{Q}[\Lambda_T] \setminus \{0\} \) and let \( \Omega_\mathcal{Q} \) be a family of semialgebraic subsets \( U \) of \( \mathbb{R}^T \) defined by

\[
(6) \quad U = (\pi_{t_1, \ldots, t_m})^{-1}(W_P \cap W_\mathcal{Q}),
\]

where \( m \in \mathbb{N}, t_1, \ldots, t_m \in T, t_1 < \cdots < t_m \), and \( P, \mathcal{Q} \in \mathbb{Q}[\Lambda_{t_1}, \ldots, \Lambda_{t_m}] \setminus \{0\} \). By Proposition 4.3, we have:

**Corollary 4.5.** \( \Omega_\mathcal{Q} \) is a c-filter.

Let \( X \subset \mathbb{R}^T \) be an open semialgebraic set and let \( \hat{x} \in X \) be a point with rational coordinates. There exist \( t_1, \ldots, t_k \in T, t_1 < \cdots < t_k \), and an open semialgebraic set \( Y \subset \mathbb{R}^k \) such that \( X = \{x \in \mathbb{R}^T : (x(t_1), \ldots, x(t_k)) \in Y\} \). Hence, there exists \( r > 0 \) such that

\[
B := \{x \in \mathbb{R}^T : \max_{i=1,\ldots,k} |x(t_i) - \hat{x}(t_i)| < r\} \subset X.
\]

Let

\[
P_0 = \Lambda_{t_1} \cdots \Lambda_{t_k} (\Lambda_{t_1}^2 + \cdots + \Lambda_{t_k}^2 - 1/r^2),
\]

let \( U_0 = (\pi_{t_1, \ldots, t_k})^{-1}(W_{P_0}) \), and let \( F : U_0 \to \mathbb{R}^T \) be a mapping defined by

\[
F(x)(t) = \begin{cases} \hat{x}(t) + 1/x(t) & \text{for } x \in U_0, t \in \{t_1, \ldots, t_k\}, \\ x(t) & \text{for } x \in U_0, t \in T \setminus \{t_1, \ldots, t_k\}. \end{cases}
\]

**Proposition 4.6.** \( \{F(U) : U \in \Omega_{P_0}\} \) is a c-filter subset of \( X \). In particular, for any open semialgebraic set \( Y \subset \mathbb{R}^T \), there exists c-filter subsets of \( Y \).

**Proof.** By Lemma 4.2, any set \( U \in \Omega_{P_0} \) is a subset of \( U_0 \). Moreover, \( F \) is an open semialgebraic mapping, so \( F(U) \) is semialgebraic for \( U \in \Omega_{P_0} \). Hence, \( \{F(U) : U \in \Omega_{P_0}\} \) satisfies conditions (i)–(iii). \( \square \)

From Proposition 4.6 and Theorem 3.1, we have that:

**Corollary 4.7.** The set of c-filters defined in Proposition 4.6 is a dense subset of the space of orderings in \( \mathbb{Q}(\Lambda_T) \) in the path topology of the real spectrum \( \text{Sper}(\mathbb{Q}[\Lambda_T]) \). Moreover, any ordering determined by such a c-filter is not Archimedean.

**Remark 4.8.** It is easy to see that the results of this section hold if we replace \( \mathbb{Q} \) by \( \mathbb{R} \).
5. Fields of Nash functions

Let $T$ be a nonempty set. We denote by $N(X)$ the domain of $\mathbb{Q}$-Nash functions on an open connected semialgebraic set $X \subset \mathbb{R}^T$.

Let $>$ be an ordering in $\mathbb{Q}(\Lambda_T)$ and let $\Omega_>$ be the plain filter of subsets of $\mathbb{R}^T$ determining $>$. Let us introduce in $\bigcup_{U \in \Omega_>} N(U)$ a relation $\sim_>$ by

$$(f_1: U_1 \to \mathbb{R}) \sim_>(f_2: U_2 \to \mathbb{R}) \iff \exists U \in \Omega_> (U \subset U_1 \cap U_2 \text{ and } f_1|_U = f_2|_U).$$

From Proposition 2.4, we immediately see that $\sim_>$ is an equivalence relation. The equivalence class of $\sim_>$ determined by $f: U \to \mathbb{R}$ is denoted by $[f]_>$, and the set of all such classes by $N_>$. The set $N_>$ is linearly ordered by

$$[f]_> > 0 \iff \exists U \in \Omega_> (f \in N(U) \text{ and } f(x) > 0 \text{ for } x \in U).$$

**Proposition 5.1.** The set $N_>$, together with the usual operations

$$[f_1]_> + [f_2]_> = [f_1|_U + f_2|_U]_>, \quad [f_1]_> \cdot [f_2]_> = [f_1|_U f_2|_U]_>,$$

where $f_1 \in N(U_1), f_2 \in N(U_2)$, and $U \in \Omega_>, U \subset U_1 \cap U_2$, is a real field.

**Proof.** Since the ring $N(U)$ is a domain for any $U \in \Omega_>$, so is $N_>$. We prove that any nonzero $f \in N_>$ has an inverse in $N_>$. Indeed, there exists $U \in \Omega_>$ such that $f \in N(U)$. Since $f \neq 0$, the set $f^{-1}(0)$ is contained in some proper $\mathbb{Q}$-algebraic subset of $\mathbb{R}^T$. Then, by the definition of c-filter, one can assume that $f(\lambda) \neq 0$ for any $\lambda \in U$. Thus $1/f \in N(U)$, so $f$ has an inverse in $N_>$. Summing up, $N_>$ is a field. Since $-1 \in N(U)$ is not a sum of squares in $N(U)$, it follows that $-1 \in N_>$ is not a sum of squares in $N_>$. □

**Theorem 5.2.** The field $N_>$ is a real closure of the field $(\mathbb{Q}(\Lambda_T), >)$.

**Proof.** Take any irreducible polynomial $P \in N_>[Z]$ of odd degree $d$ with respect to $Z$. Then there exists $U \in \Omega_>$ such that $P \in N(U)[Z]$. Let $t_1, \ldots, t_m \in T$, and let $\tilde{U} \subset \mathbb{R}^m$ be an open connected semialgebraic set such that $U = \{x \in \mathbb{R}^T : (x(t_1), \ldots, x(t_m)) \in \tilde{U}\}$. By using the Hermite method (for $\tilde{U}$) we deduce that there exists a decomposition $U = U_1 \cup \cdots \cup U_k \cup V$ of $U$ into disjoint open basic $\mathbb{Q}$-semialgebraic sets $U_1, \ldots, U_k$ and a semialgebraic set $V$ included in an algebraic set such that $P(x, Z)$ has the same number of zeroes for all $x \in U_i$ and each of these zeroes is single. By (i) and (ii) in the definition of a c-filter, there exists $U' \in \Omega_>$ such that $U' \subset U_i$ for some $i \in \{1, \ldots, k\}$. Then there exists $k \in \mathbb{N}, k > 0$ such that $P(x, Z)$ has exactly $k$ zeroes for $x \in U'$, and so there exist functions $\xi_1, \ldots, \xi_k: U' \to \mathbb{R}$ with $\xi_1(x) < \cdots < \xi_k(x)$ such that $P(x, \xi_i(x)) = 0$ for $x \in U', i = 1, \ldots, k$. As $\xi_i(x)$ are single zeroes of $P(x, Z)$, by the Implicit Function Theorem, $\xi_i$ is a Nash function for $i = 1, \ldots, k$. As $N_>$ is a real field...
(Proposition 5.1), \( \mathcal{N}_\succ \) is a real closed field. Since \( \mathcal{N}_\succ \) is an algebraic extension of \( \mathbb{Q}(\Lambda_T) \), by the Artin–Schreier Theorem, it is a real closure of \( (\mathbb{Q}(\Lambda_T), \succ) \). \( \square \)

**Remark 5.3.** The above results of this section also hold for an arbitrary \( c \)-filter determining \( \succ \) in place of the plain filter \( \Omega_\succ \). The results also hold if we put \( \mathbb{R} \) in place of \( \mathbb{Q} \).

From Theorems 3.1 and 5.2, we recover the familiar result that any Archimedean field can be embedded in \( \mathbb{R} \).

**Corollary 5.4.** Let \( \Omega_\succ \) be a plain filter of subsets of \( \mathbb{R}^T \) determining an Archimedean ordering \( \succ \) of \( \mathbb{Q}(\Lambda_T) \), and let \( x_\succ \in \bigcap_{U \in \Omega_\succ} U \). Then the mapping

\[
\mathcal{N}_\succ \ni f \mapsto f(x_\succ) \in \mathbb{R}
\]

is an order-preserving monomorphism.

From Theorem 5.2, we immediately obtain:

**Corollary 5.5.** Let \( R \) be a real closed field with ordering \( \succ \), and let \( T \) be the transcendence basis of \( R \) over \( \mathbb{Q} \) whose existence is guaranteed by the Kuratowski–Zorn lemma. Assume that \( T \neq \emptyset \) and let \( \Lambda_T = (\Lambda_t : t \in T) \) be a system of independent variables. Then the field \( R \) is order-preserving isomorphic to a real closure of the rational functions field \( \mathbb{Q}(\Lambda_T) \), i.e., to some field \( \mathcal{N}_\succ \).

**Remark 5.6.** Let \( \mathbb{K} \) be an algebraically closed field of characteristic zero. Then \( \mathbb{K} = R[i] \), where \( i^2 = -1 \), for some real closed field \( R \). Let \( T \subset R \) be the transcendence basis of \( \mathbb{K} \) over \( \mathbb{Q} \). Assume that \( T \neq \emptyset \). Then \( \mathbb{K} \) is isomorphic to an algebraic closure of \( \mathbb{Q}(\Lambda_T) \). By Theorem 1.1 of [Spodzieja 1996], one can introduce a filter \( \Omega_\subset \) of open, connected, and simply connected semialgebraic subsets \( U \) of \( \mathbb{C}^T \) satisfying conditions (i), (ii), and (iii). Then, analogously to [Spodzieja 1996], one can introduce a geometric construction of the algebraic closure of \( \mathbb{Q}(\Lambda_T) \) in terms of complex Nash functions.

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