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MARIO VELÁSQUEZ, EDWARD BECERRA AND HERMES MARTINEZ

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TWISTED K-THEORY FOR THE ORBIFOLD $[*/G]$

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The second author dedicates this paper to Heset María

The main result of this paper establishes an explicit ring isomorphism between the twisted orbifold K-theory ${}^{\omega}K_{\mathrm{orb}}([*/G])$ and $R(D^{\omega}(G))$ for any element $\omega \in Z^3(G; S^1)$. We also study the relation between the twisted orbifold K-theories ${}^{\alpha}K_{\mathrm{orb}}(\mathscr{X})$ and ${}^{\alpha'}K_{\mathrm{orb}}(\mathscr{Y})$ of the orbifolds $\mathscr{X} = [*/G]$ and $\mathscr{Y} = [*/G']$, where G and G' are different finite groups, and $\alpha \in Z^3(G; S^1)$ and $\alpha' \in Z^3(G'; S^1)$ are different twistings. We prove that if G' is an extraspecial group with prime number p as an index and order p^n (for some fixed $n \in \mathbb{N}$), under a suitable hypothesis over the twisting α' we can obtain a twisting α on the group $(\mathbb{Z}_p)^n$ such that there exists an isomorphism between the twisted K-theories ${}^{\alpha'}K_{\mathrm{orb}}([*/G'])$ and ${}^{\alpha}K_{\mathrm{orb}}([*/(\mathbb{Z}_p)^n])$.

1. Introduction

The twisted K-theory is a successful example of the increasing flow of physical ideas into mathematics. Brought from the physical setup, the twisted orbifold K-theory has been, for the last twenty-five years, a fruitful field of ideas and development in K-theory and algebraic topology. It emerged from two sources: the consideration of the D-brane charge on a smooth manifold by Witten [1998], and the concept of discrete torsion on an orbifold by Vafa [2001]. Although for any element $\alpha \in H^3(\mathscr{X}; \mathbb{Z})$ one can associate the twisted K-theory ${}^{\alpha}K(\mathscr{X})$, its structure is simpler if the element α lies in the image of the pullback associated to the map $X \to *$. In such a case, we call this element a *discrete torsion* since we can see it as an element in the cohomology $H^3(G; \mathbb{Z})$.

On the other hand, an orbifold is a type of generalization of a smooth manifold. It is a topological space locally modeled as a quotient of a manifold by an action of a finite group. When X represents an orbifold, the twisted K-theory is more

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interesting because it is naturally related to equivariant theories if we specialize in the orbifold $\mathcal{X} = [X/G]$, where X is a smooth manifold and G is a compact Lie group acting almost freely on X. In the case of orbifolds, we have another important advantage to work with; it is the cohomological counterpart given by the Chen–Ruan cohomology on orbifolds $H^*_{CR}(\mathcal{X}; \mathbb{C})$ related to K-theory by the Chern character. The Chen–Ruan cohomology of orbifolds has an interesting nontrivial internal product which makes it an algebra. This product can be presented in the setup of the K-theory to obtain a stringy product on the K-theory of the orbifold $K_{\text{orb}}(\mathcal{X})$ (see [Becerra and Uribe 2009; Adem and Ruan 2003]). If the orbifold considered has the form $[X/G]$, then the twisted orbifold K-theory can be related to the equivariant K-theory of the spaces of fixed points by the G-action on X.

On the other hand, the tensor product defines a product

$$^{\alpha}K_{\text{orb}}(\mathcal{X}) \otimes {}^{\beta}K_{\text{orb}}(\mathcal{X}) \to {}^{\alpha+\beta}K_{\text{orb}}(\mathcal{X})$$

for any pair of elements α and β in $H^3(\mathcal{X}; \mathbb{Z})$. In fact, one can obtain a stringy product for the twisted K-theory on orbifolds by using the stringy product defined on each space of fixed points to define an explicit stringy product in each ${}^{\alpha}K(\mathcal{X})$ for any $\alpha \in H^3(\mathcal{X}; \mathbb{Z})$. Nevertheless, the crucial information to define the stringy product on the twisted K-theory of orbifolds does not lie in $H^3(\mathcal{X}; \mathbb{Z})$; instead it lies in $H^4(\mathcal{X}; \mathbb{Z})$. Given an element ϕ in $H^4(\mathcal{X}; \mathbb{Z})$, it defines an element $\theta(\phi)$ in $H^3(\wedge X; \mathbb{Z})$, where $\wedge\mathcal{X}$ is the inertia orbifold associated to $\mathcal{X}$. Hence, we can define a stringy product over the twisted K-theory orbifold ${}^{\theta(\phi)}K_{\text{orb}}(\wedge\mathcal{X})$ by using a suitable structure of the inertia orbifold $\wedge X$. One such product structure is based on the map $\theta(\phi)$ called the *inverse transgression map*, which is considered to be the inverse of the classical transgression map.

For this paper, the stringy product in $K_{\text{orb}}(\mathcal{X})$ has a trivial expression as we will restrict our observations to the case in which $\mathcal{X} = [\{*\}/G]$, where G is a finite group.

The main result in this paper is to present an explicit relation between the twisted Drinfeld double $D^{\omega}(G)$ and the twisted orbifold K-theory ${}^{\omega}K_{\text{orb}}([*/G])$ for an element ω of discrete torsion (see Section 4). This allows us to relate the twisted orbifold K-theories ${}^{\omega}K_{\text{orb}}([*/G])$ and ${}^{\omega'}K_{\text{orb}}([*/G'])$ for two orbifolds $[*/G]$ and $[*/G']$ with the twistings $\omega \in H^4(G; \mathbb{Z})$ and $\omega' \in H^4(G'; \mathbb{Z})$. To obtain such a relation, we modify the stringy product defined in [Adem et al. 2007] by an element in $R_{\alpha_g}(C(g) \cap C(h))$.

2. Pushforward map in the twisted representation ring

In this section, we introduce the pushforward map. Although this map can be defined for almost complex manifolds, we will focus only on the case of homogeneous spaces G/H. To define this map, let us recall the Thom isomorphism theorem in equivariant K-theory.

Fact [Segal 1968, Proposition 3.2]. *Let X be a compact G-manifold and $p : E \to X$ be a complex G-vector bundle over X. There exists an isomorphism*

$$\phi : K_G^*(X) \to K_G^*(E, E \setminus E_0), \quad \phi([F]) := p^*(F) \otimes \lambda_{-1}(E),$$

where E_0 is the zero section and the class $\lambda_{-1}(E)$ is the Thom class associated to $[E]$.

Remark 2.1. We need to recall how to define the *normal bundle*. If M and N are G-manifolds (that means a manifold with a smooth G-action) and $f : M \to N$ is a G-embedding, we can define a (real) vector bundle τ such that $df(TM) \oplus \tau \cong TN$ (for details in this construction, consult [tom Dieck 1987]). If the map f is not a G-embedding, we can consider $f : M \to N \times D^j$ (D^j is the unitary disk in $\mathbb{R}^j$ with the trivial G-action) for sufficiently large j and by Corollary 1.10 [Wasserman 1969] we can approximate f by an immersion g_f, then we define the normal bundle for f as the normal bundle of g_f.

Now, we proceed to define the pushforward map $f_* : K_G^*(X) \to K_G^*(Y)$ for a differentiable map $f : X \to Y$ between almost complex G-manifolds by letting τ be the normal bundle associated to the map $f : X \to Y$. We define the pushforward, which will be denoted by f_*, as the composition

$$K_G^*(X) \xrightarrow{\phi} K_G^*(\tau, \tau \setminus \tau_0) \xrightarrow{j} K_G^*(Y \times D^j, (Y \times D^j) \setminus g_f(X)) \xrightarrow{i_\sharp} K_G^*(Y \times D^j) \cong K_G^*(Y),$$

where ϕ is the Thom isomorphism. The map j is given by excision, the map $i_\sharp$ is the pullback map induced by the inclusion, and the last isomorphism is induced by the natural inclusion. The pushforward map can be defined also in the twisted case (see [Carey and Wang 2008]). Consider the following diagram of inclusions:

$$\begin{tikzcd} G & K \arrow[l] \\ H \arrow[u] & H \cap K \arrow[l] \arrow[u] \end{tikzcd}$$

from which we get a diagram of surjections:

$$\begin{tikzcd} G/G & G/K \arrow[l, "i_1"'] \\ G/H \arrow[u, "j_1"] & G/(H \cap K) \arrow[l, "i_2"'] \arrow[u, "j_2"'] \end{tikzcd}$$

Using this diagram we obtain the map

$$j_{2*} \circ i_2^* : K_G^*(G/H) \to K_G^*(G/K), \quad [E] \mapsto [\lambda_{-1}(\tau_{j_2}) \otimes i_2^*(p^*(E))],$$

where τ_{j_2} is the normal bundle of j_2, and the map

$$i_1^* \circ j_{1*} : K_G^*(G/H) \to K_G^*(G/K), \quad [E] \mapsto [i_1^*(\lambda_{-1}(\tau_{j_1})) \otimes i_1^*(p^*(E))].$$

Afterwards, we compare the two maps and we conclude that the *obstruction bundle* is $\lambda_{-1}(i_1^*(\tau_{j_1})/\tau_{j_2})$. This means that

$$i_1^* \circ j_{1*}([E]) = j_{2*} \circ i_2^*([E]) \otimes \lambda_{-1}(i_1^*(\tau_{j_1})/\tau_{j_2}). \tag{2-1}$$

We consider the particular case of the groups $H = C_G(x)$ and $K = C_G(y)$, where x and y are elements in the group G and $C_G(x)$ and $C_G(y)$ denote their centralizers in G. Then by (2-1) we get an obstruction bundle which is denoted as $\gamma_{x,y}$.

3. Twisted orbifold K-theory for the orbifold $[*/G]$

The goal of this section is to consider a K-theory structure on an orbifold structure defined by the trivial action of a finite group over the space $\{*\}$. This is a particular case of a more general kind of spaces that are obtained by almost free actions of a compact Lie group over compact manifolds. These spaces naturally have an orbifold structure that sets a basis for all developments in this paper. When the manifold is one point and the group is finite, all the hypotheses in the already defined theory hold.

Let us consider the inertia orbifold $\wedge[*/G]$ for a finite group G. We define the orbifold K-theory for the orbifold $[*/G]$ as the module

$$K_{\mathrm{orb}}([*/G]) := K(\wedge[*/G]) \cong \bigoplus_{(g)} K([*/C_G(g)]) \cong \bigoplus_{(g)} K_{C_G(g)}(*),$$

where (g) denotes the class of conjugation of the element $g \in G$ and $C_G(g)$ denotes the centralizer of the element $g \in G$. In this case, the orbifold K-theory introduced in [Adem et al. 2007] turns out to be simply $K_G(*)$, which is additively isomorphic to the group $\bigoplus_{(g)} R(C_G(g))$ (see [Adem and Ruan 2003]), where $R(C_G(g))$ denotes the Grothendieck ring associated to the semigroup of isomorphism classes of linear representations of the group $C_G(g)$, and the sum is taken over conjugacy classes. The product structure in $K_G(*)$ is defined as follows: consider the maps

$$\begin{aligned}
e_1 &: C_G(g) \cap C_G(h) \times C_G(g) \cap C_G(h) \to C_G(g), & e_1(a,b) &= a,\\
e_2 &: C_G(g) \cap C_G(h) \times C_G(g) \cap C_G(h) \to C_G(h), & e_2(a,b) &= b,\\
e_{12} &: C_G(g) \cap C_G(h) \times C_G(g) \cap C_G(h) \to C_G(gh), & e_{12}(a,b) &= ab.
\end{aligned}$$

Note that for any element $\tau \in G$, the map $\phi_\tau : G \to G$ defined by $\phi_\tau(g) = \tau g \tau^{-1}$ implies that $\phi_\tau \circ e_i = e_i \circ (\phi_\tau, \phi_\tau)$. Thus, the maps e_i are ϕ_τ-equivariant for any element $\tau \in C_G(g)$. Given E in $R(C_G(g))$ and F in $R(C_G(h))$, we define the product

$$E \star F := e_{12*}(e_1^*(E) \otimes e_2^*(F) \otimes \gamma_{g,h}) \in R(C_G(gh)). \tag{3-1}$$

Since the action is trivial, it follows from Theorem 2.2 in [Segal 1968] that $R(C_G(g)) = K_{C_G(g)}(*)$. Thus, the product can be seen as

$$\star : K_{C_G(g)}(*) \times K_{C_G(h)}(*) \to K_{C_G(gh)}(*)$$

in the setup of equivariant K-theory. We note that the product defined in (3-1) is analogous to the stringy and twisted stringy product defined in [Becerra and Uribe 2009] in the case in which G is an abelian group. Let α be a cocycle in $Z^3(G; S^1)$, i.e., $\alpha : G \times G \times G \to S^1$ satisfies $\alpha(a,b,c)\alpha(a,bc,d)\alpha(d,c,d) = \alpha(ab,c,d)\alpha(a,b,cd)$ for all $a,b,c,d \in G$. We proceed to define the twisted orbifold K-theory ${}^{\alpha}K_{\mathrm{orb}}([*/G])$.

For the global quotient $[X/G]$ and the element $\alpha \in Z^3(G; S^1)$, the twisted orbifold K-theory is defined as the sum

$${}^{\alpha}K_{\mathrm{orb}}([X/G]) := \bigoplus_{g \in C} {}^{\alpha_g}K_{C_G(g)}(X^g), \tag{3-2}$$

where C is a set of representatives of the conjugacy classes in G and α_g is the inverse transgression map (see below for details). In particular, if G is an abelian group, the set C is the group G. For every group H and $\beta \in Z^2(H; S^1)$ we take its associated group H_β given by the central extension

$$1 \to S^1 \to H_\beta \to H \to 1.$$

Recall that the group H_β is the set $S^1 \times H$, with the group operation defined by

$$(s_1, h_1) * (s_2, h_2) := (s_1 s_2 \beta(h_1, h_2), h_1 h_2).$$

The twisted equivariant K-theory ${}^{\beta}K_H(X)$ is defined as the class of H_β-equivariant vector bundles such that the action of the center S^1 restricts to multiplication on the fibers. In the case of the space $X = \{*\}$, the twisted equivariant K-theory ${}^{\beta}K_H(*)$ coincides with $R_\beta(H)$, the Grothendieck ring of classes of projective representations for the group H (see [Karpilovsky 1993] for a precise definition of $R_\beta(H)$).

Returning to the case of the orbifold $[*/G]$ for a finite group G, the twisted orbifold K-theory defined in (3-2) takes the form

$${}^{\alpha}K_{\mathrm{orb}}([*/G]) := \bigoplus_{g \in C} {}^{\alpha_g}K_{C_G(g)}(*) \cong \bigoplus_{g \in C} R_{\alpha_g}(C_G(g)).$$

Inverse transgression map. We review the inverse transgression map for finite groups to describe the multiplicative structure in the module ${}^{\alpha}K_{\text{orb}}([*/G])$. Throughout this subsection we follow the development presented in Section 3.2 in [Becerra and Uribe 2009]. Let us recall the definition of the inverse transgression map for a global quotient $[M/G]$. For $g \in G$, consider the action of $C_G(g) \times \mathbb{Z}$ on $M^g = \{x \in M \mid gx = x\}$ given by $(h, m) \cdot x := hg^m x$ and the homomorphism

$$\psi_g : C_G(g) \times \mathbb{Z} \to G, \quad (h, m) \mapsto hg^m.$$

Thus, the inclusion $i_g : M^g \to M$ becomes a ψ_g equivariant map and induces a homomorphism

$$i_g^* : H_G^*(M; \mathbb{Z}) \to H^*_{C_G(g)\times\mathbb{Z}}(M^g; \mathbb{Z}).$$

From the isomorphisms

$$\begin{aligned} H^*_{C_G(g)\times\mathbb{Z}}(M^g; \mathbb{Z}) &\cong H^*(M^g \times_{C_G(g)} \times EC_G(g) \times B\mathbb{Z}; \mathbb{Z}) \\ &\cong H^*_{C_G(g)}(M^g; \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(S^1; \mathbb{Z}), \end{aligned}$$

we have, for each k,

$$i_g^* : H_G^k(M; \mathbb{Z}) \to H^k_{C_G(g)}(M^g; \mathbb{Z}) \oplus H^{k-1}_{C_G(g)}(M^g; \mathbb{Z}).$$

Hence, we define the inverse transgression map as the map induced by projecting on the second factor

$$\tau_g : H_G^k(M; \mathbb{Z}) \to H^{k-1}_{C_G(g)}(M^g; \mathbb{Z}).$$

In the particular case that $[M/G] = [*/G]$ the definition above turns into:

Definition 3.1. For any element $\alpha \in Z^3(G; S^1)$, the inverse transgression map is defined as the map

$$\tau_g : H_G^k(*; \mathbb{Z}) \to H^{k-1}_{C_G(g)}(*; \mathbb{Z})$$

induced by τ_g on each k.

Product in the twisted case. Take $\alpha \in Z^3(G; \mathbb{Z})$. Let us consider the orbifold $[*/G]$ where G is a finite group. Now, consider the module

$$(3\text{-}3) \qquad {}^{\alpha}K_{\text{orb}}([*/G]) := \bigoplus_{g\in C} R_{\alpha_g}(C_G(g)),$$

where $\alpha_g \in H^2(C(g); \mathbb{Z})$ denotes the inverse transgression map of α. The goal of this subsection is to define an associative product for this module; specifically, we show that it's possible to endow the module ${}^{\alpha}K_{\text{orb}}([*/G])$ with a ring structure. For simplicity, we denote $C_G(g)$ as $C(g)$. Consider the inclusion maps of groups

$$i_g : C(g) \cap C(h) \to C(g), i_h : C(g) \cap C(h) \to C(h)$$

and

$$i_{gh} : C(g) \cap C(h) \to C(gh)$$

for $g, h \in G$. These maps induce the restriction maps

$$\begin{aligned} i_g^* &: H^2(C(g); S^1) \to H^2(C(g) \cap C(h); S^1), \\ i_h^* &: H^2(C(h); S^1) \to H^2(C(g) \cap C(h); S^1), \end{aligned}$$

and the morphism i_{gh} induces a map

$$(i_{gh})_* : H^2(C(g) \cap C(h); S^1) \to H^2(C(gh); S^1),$$

which is the induction morphism in group cohomology (see for example [tom Dieck 1987]).

Given $E \in R_{\alpha_g}(C(g))$, we consider it as a $C(g)_{\alpha_g}$-module that restricts to multiplication on the fibers over S^1. Therefore, we get the following commutative diagram for the inclusion i_g and the identity map s on S^1:

$$\begin{tikzcd} 1 \arrow[r] & S^1 \arrow[r] & C(g)_{\alpha_g} \arrow[r] & C(g) \arrow[r] & 1 \\ 1 \arrow[r] & S^1 \arrow[r] \arrow[u, "s"] & (C(g) \cap C(h))_{i_g^*(\alpha_g)} \arrow[r] \arrow[u, "(s, i_g)"] & C(g) \cap C(h) \arrow[r] \arrow[u, "i_g"] & 1 \end{tikzcd} \tag{3-4}$$

This implies that any $C(g)_{\alpha_g}$-module restricts to a $(C(g) \cap C(h))_{i_g^*(\alpha_g)}$-module, denoted by $i_g^*(E)$. In particular, for any $(E, F) \in R_{\alpha_g}(C(g)) \times R_{\alpha_h}(C(h))$, we get the map

$$\begin{gathered} R_{\alpha_g}(C(g)) \times R_{\alpha_h}(C(h)) \to R_{i_g^*(\alpha_g)}(C(g) \cap C(h)) \times R_{i_h^*(\alpha_h)}(C(g) \cap C(h)), \\ (E, F) \mapsto (i_g^*(E), i_h^*(F)). \end{gathered}$$

Now, from the central extensions

$$\begin{aligned} 1 \to S^1 \to (C(g) \cap C(h))_{i_g^*(\alpha_g)} \to C(g) \cap C(h) \to 1, \\ 1 \to S^1 \to (C(g) \cap C(h))_{i_h^*(\alpha_h)} \to C(g) \cap C(h) \to 1 \end{aligned}$$

induced by $i_g^*(\alpha)$ and $i_h^*(\alpha) \in H^2(C(g) \cap C(h); S^1))$, we get

$$\begin{multline} 1 \to S^1 \times S^1 \to (C(g) \cap C(h))_{i_g^*(\alpha_g)} \times (C(g) \cap C(h))_{i_h^*(\alpha_h)} \\ \to C(g) \cap C(h) \times C(g) \cap C(h) \to 1 \end{multline}$$

For $E \in R_{i_g^*(\alpha_g)}(C(g))$ and $F \in R_{i_g^*(\alpha_g)}(C(g))$, the tensor product $E \otimes F$ is naturally a $(C(g) \cap C(h))_{i_g^*(\alpha_g)} \times (C(g) \cap C(h))_{i_h^*(\alpha_h)}$-module that restricts to multiplication on the fibers by elements of S^1. By considering the action restricted to the diagonal

$$\Delta(C(g)\cap C(h)) \subset C(g)\cap C(h)\times C(g)\cap C(h),$$

we get the central extension

$$1 \to S^1 \to (C(g)\cap C(h))_{i_g^*(\alpha_g)i_h^*(\alpha_h)} \to \Delta(C(g)\cap C(h)) \to 1$$

corresponding to the element

$$i_g^*(\alpha_g)i_h^*(\alpha_h) \in H^2(C(g)\cap C(h); S^1).$$

Thus, we have

$$R_{i_g^*(\alpha_g)}(C(g)\cap C(h)) \times R_{i_h^*(\alpha_h)}(C(g)\cap C(h)) \to R_{i_g^*(\alpha_g)i_h^*(\alpha_h)}(C(g)\cap C(h)),$$
$$(E, F) \mapsto E\otimes F.$$

Now, since $i_g^*(\alpha_g)i_h^*(\alpha_h) = i_{gh}^*(\alpha_{gh})$ in $H^2(C(g)\cap C(h); S^1)$ since the cocycles are cohomologous (see [Adem et al. 2007, Proposition 4.3]), it follows that

$$R_{i_g^*(\alpha_g)i_h^*(\alpha_h)}(C(g)\cap C(h)) \cong R_{i_{gh}^*(\alpha_{gh})}(C(g)\cap C(h)).$$

Therefore, the induction map can be defined as

$$R_{i_{gh}^*(\alpha_{gh})}(C(g)\cap C(h)) \to R_{(i_{gh})_* i_{gh}^*(\alpha_{gh})}(C(gh)), \quad A \mapsto \mathrm{Ind}_{C(g)\cap C(h)}^{C(gh)}(A).$$

Thus, a product on the module (3-3) can be obtained from the previously described morphisms to get

$$R_{\alpha_g}(C(g)) \times R_{\alpha_h}(C(h)) \to R_{\alpha_{gh}}(C(gh)) \tag{3-5}$$

defined by

$$(E, F) \mapsto E \star_\alpha F := \mathrm{Ind}_{C(g)\cap C(h)}^{C(gh)}(i_g^*(E)\otimes i_h^*(F)\otimes \gamma_{g,h}),$$

where $\gamma_{g,h}$ is defined as the excess bundle as in Section 2.

Definition 3.2. By using the restriction notation, we define the twisted stringy product in the module ${}^{\alpha}K_{\mathrm{orb}}([*/G])$ as the map

$$R_{\alpha_g}(C(g)) \times R_{\alpha_h}(C(h)) \to R_{\alpha_g\alpha_h}(C(gh))$$
$$(E, F) \mapsto I_{C(g)\cap C(h)}^{C(gh)}\big(\mathrm{Res}_{C(g)\cap C(h)}^{C(g)}(E) \otimes \mathrm{Res}_{C(g)\cap C(h)}^{C(h)}(F)\otimes \gamma_{g,h}\big),$$

where $\mathrm{Res}_{C(g)\cap C(h)}^{C(g)}$ denotes the restriction of α-twisted representations of $C(g)$ to $i_g(\alpha)$-representations of $C(g)\cap C(h)$ (and likewise for h).

4. Twisted orbifold K-theory and the algebra $D^\omega(G)$

The goal of this section is to give an introduction of the *twisted Drinfeld double* $D^\omega(G)$ and to show an explicit relation with the twisted orbifold K-theory. Our main reference is [Witherspoon 1996]. Let us recall the definition and the main properties of the twisted Drinfeld double to clarify the nature of this structure and its representations. From a different point of view, we can also obtain some properties of the stringy product defined on the sections above, using the properties of the representations of the twisted Drinfeld double. Namely, the Grothendieck ring of these representations is isomorphic to the twisted orbifold K-theory with the structure induced by the stringy product, which will be proven on page 481. Because of the associativity of the tensor product of the $D^\omega(G)$-modules, this yields a proof of the associativity of the stringy product defined above; see Corollary 4.3.

Let G be a finite group and k an algebraically closed field. Let ω be an element in $Z^3(G, k^*)$, that is, a function $\omega : G \times G \times G \to k^*$ such that

$$\omega(a, b, c)\,\omega(a, bc, d)\,\omega(d, c, d) = \omega(ab, c, d)\,\omega(a, b, cd)$$

for all $a, b, c, d \in G$. We define the quasitriangular quasi-Hopf algebra $D^\omega(G)$ as the vector space $(kG)^* \otimes (kG)$, where $(kG)^*$ denotes the dual of the algebra kG (see [Drinfel'd 1987]) and the algebra structure in $D^\omega(G)$ is given as follows: consider the canonical basis $\{\delta_g \otimes \bar{x}\}_{g,x\in G}$ of $D^\omega(G)$, where δ_g is the function such that $\delta_g(h) = 1$ if $h = g$ and 0 otherwise. We denote $\delta_g \otimes \bar{x}$ by $\delta_g\bar{x}$. Now, we define the product of elements in the basis by

$$(\delta_g\bar{x})(\delta_h\bar{y}) = \omega_g(x, y)\,\delta_g\delta_{xhx^{-1}}\overline{xy}, \tag{4-1}$$

where ω_g is the image of ω via the inverse transgression map of the element $g \in G$ as in Definition 3.1. The multiplicative identity for this product is the element $1_{D^\omega(G)} = \bigoplus_{g\in G} \delta_g\bar{1}$. Now, we use the notation δ_g for the element $\delta_g\bar{1}$. The coproduct $\Delta : D^\omega(G) \to D^\omega(G) \otimes D^\omega(G)$ in the algebra $D^\omega(G)$ is defined by the map

$$\Delta(\delta_g\bar{x}) = \bigoplus_{h\in G} \gamma_x(h, h^{-1}g)(\delta_h\bar{x}) \otimes (\delta_{h^{-1}g}\bar{x}), \tag{4-2}$$

where

$$\gamma_x(h, l) = \frac{\omega(h, l, x)\,\omega(x, x^{-1}hx, x^{-1}lx)}{\omega(h, x, x^{-1}lx)}.$$

The algebra $D^\omega(G)$ endowed with these operations is usually called the twisted Drinfeld double.

Representations of $D^\omega(G)$. Let U, V be modules over the algebra $D^\omega(G)$. Consider the tensor product $U \otimes V$ as a $D^\omega(G)$-module endowed with the action from $D^\omega(G)$, induced by the coproduct Δ. Note that the field k can be considered as a trivial $D^\omega(G)$-module, which is the multiplicative identity for the tensor product of $D^\omega(G)$-modules. In particular, for $k = \mathbb{C}$, we define the ring of representations $R(D^\omega(G))$ of $D^\omega(G)$ as the $\mathbb{C}$-algebra generated by the set of isomorphism classes of $D^\omega(G)$-modules with the direct sum of modules as the sum operation and the tensor as the product operation. We define the ideal $R_0(D^\omega(G))$ generated by all combinations $[U] - [U'] - [U'']$ (brackets denoting the isomorphism class) where $0 \to U' \to U \to U'' \to 0$ is a short exact sequence of $D^\omega(G)$-modules. Now, we define the Grothendieck ring $R(D^\omega(G))$ as the quotient between $\operatorname{Rep}(D^\omega(G))$ and the ideal $R_0(D^\omega(G))$.

The algebra $D^\omega(G)$ is quasitriangular with

$$A = \bigoplus_{g,h\in G} \delta_g \bar{1} \otimes \delta_h \bar{g} \quad \text{and} \quad A^{-1} = \bigoplus_{g,h\in G} \omega_{ghg^{-1}}(g, g^{-1})^{-1} \delta_g \bar{1} \otimes \delta_h \overline{g^{-1}}.$$

Thus, $A\Delta(a)A^{-1} = \sigma(\Delta(a))$ for all $a \in D^\omega(G)$, where σ is the automorphism that exchanges the images in the coproduct. Therefore, if U and V are $D^\omega(G)$-modules, this equation implies that $U \otimes V$ and $V \otimes U$ are isomorphic as $D^\omega(G)$-modules; that is, the algebra $R(D^\omega(G))$ is commutative. Now, assume that $\beta : G \times G \to \mathbb{C}^*$ is a cochain with coboundary

$$\delta\beta(a, b, c) = \beta(b, c)\beta(a, bc)\beta(ab, c)^{-1}\beta(a, b)^{-1}.$$

Then, the algebra $D^{\omega\delta\beta}(G)$ is isomorphic to $D^\omega(G)$ given through the map

$$\upsilon(\delta_g \bar{x}) = \frac{\beta(g, x)}{\beta(x, xgx^{-1})} \delta_g \bar{x}.$$

In particular, we get the isomorphism

$$\upsilon^* : R(D^{\omega\delta\beta}(G)) \overset{\cong}{\to} R(D^\omega(G)).$$

Next, we consider the following theorem (compare [Willerton 2008, Theorem 19]):

Theorem 4.1. *The ring $R(D^\omega(G))$ is additively isomorphic to the ring*

$$\bigoplus_{(g)\subset G} R_{\omega_g}(C(g)),$$

where (g) denotes the conjugacy class of $g \in G$.

Proof. For all $x \in G$, we take the subspaces

$$S^\omega(x) := \bigoplus_{g\in C(x)} \mathbb{C}\delta_x \bar{g} \quad \text{and} \quad D^\omega(x) := \bigoplus_{g\in G} \mathbb{C}\delta_x \bar{g}$$

of $D^\omega(G)$. Then $S^\omega(x)$ is a subalgebra of $D^\omega(G)$ with identity element $\delta_x \bar{1}$ such that, from the product defined in $D^\omega(G)$, it follows that $S^\omega(x) \cong R_{\omega_x} C(x)$ where

$R_{\omega_x}C(x)$ is defined in [Karpilovsky 1993]. Given $(g) \subset G$, consider

$$D^{\omega}((g)) := \bigoplus_{h \in (g)} D^{\omega}(h).$$

Note that $D^{\omega}(G) \cong \bigoplus_{(g)\subset G} D^{\omega}((g))$ (additively). For an element h in a fixed conjugacy class (g), take a $S^{\omega}(h)$-module (i.e., a $R_{\omega_h}C(h)$-module) U, and define the map

$$U \mapsto U \otimes_{S^{\omega}(h)} D^{\omega}(h),$$

whose image is a $D^{\omega}((g))$-module if we take the action of $D^{\omega}((g))$ on it as right multiplication in the second factor. On the other hand, for a $D^{\omega}((g))$-module V, we define the map

$$V \mapsto V\delta_h\bar{1},$$

whose image is a $R_{\omega_h}C(h)$-module. Thus, there is an equivalence between $R_{\omega_h}C(h)$-modules and $D^{\omega}((g))$-modules. Therefore, from [Karpilovsky 1993, Theorem I.3.2], we have $R_{\omega_h}(C(h)) \cong R(D^{\omega}((g)))$ for any $h \in (g)$, and the theorem follows. □

From [Dijkgraaf et al. 1991], we get that it is possible to explicitly describe the morphism using the *induction DPR* which is defined on each $R_{\alpha}(C(g))$ for $g \in G$. Namely, let (ρ, V) be a twisted representation of the group $C(g)$ and define the representation $\psi((\rho, V)) := (\pi_{\rho}, A)$ of $D^{\omega}(G)$ as given by

$$A := \operatorname{Ind}_{C(g)}^{G}(V), \quad \pi_{\rho} := \pi_{\rho}(\delta_k\bar{x})x_j \otimes v = \delta_k\delta_{x_s g x_s^{-1}} \frac{\omega_k(x, x_j)}{\omega_k(x_s, r)} x_s \otimes \rho(r)v, \tag{4-3}$$

where x_j is a representative of a class in $G/C(g)$, $r \in C(g)$ and the element x_s is a representative of a class in $G/C(g)$, such that $xx_j = x_s r$.

Relation between $R(D^{\omega}(G))$ and the twisted K-theory of the orbifold $[*/G]$. Let us consider an element $\omega \in Z^3(G; S^1)$. By (3-3) the twisted orbifold K-theory of the orbifold $[*/G]$ is the ring

$${}^{\omega}K_{\mathrm{orb}}([*/G]) = \bigoplus_{(g)\subset G} {}^{\omega_g}K_{C(g)}(*) \cong \bigoplus_{(g)\subset G} R_{\omega_g}(C(g)).$$

By Theorem 4.1, there exists an additive isomorphism between $R(D^{\omega}(G))$ and the twisted orbifold K-theory ${}^{\omega}K_{\mathrm{orb}}([*/G])$. We will show that if we endow this ring with the twisted product $\star_{\alpha}$, then the additive isomorphism is in fact a ring isomorphism. The DPR induction is defined as $(I_{C(g)}^{G}(E), \rho_{\pi})$ where (E, π) is an element in $R_{\omega_g}(C(g))$. Let us consider two elements E and F in $R_{\omega_g}(C(g))$ and $R_{\omega_h}(C(h))$ respectively. The tensor product of the DPR-induction of these elements can be related to the twisted product $\star$ via the Frobenius reciprocity as follows:

$$\begin{aligned} I^G_{C(g)}(E)\otimes I^G_{C(h)}(F) &\cong I^G_{C(g)}\big(E\otimes R^G_{C(g)}(I^G_{C(h)}(F))\big)\\ &\cong I^G_{C(g)}\big(E\otimes I^G_{C(g)}(R^{C(h)}_{C(g)\cap C(h)}(F)\otimes\gamma_{g,h})\big)\\ &\cong I^G_{C(g)}\big(I^{C(g)}_{C(g)\cap C(h)}(R^{C(g)}_{C(g)\cap C(h)}(E)\otimes R^{C(h)}_{C(g)\cap C(h)}(F)\otimes\gamma_{g,h})\big)\\ &\cong I^G_{C(g)\cap C(h)}\big(R^{C(g)}_{C(g)\cap C(h)}(E)\otimes R^{C(h)}_{C(g)\cap C(h)}(F)\otimes\gamma_{g,h}\big)\\ &\cong I^G_{C(gh)}\big(I^{C(gh)}_{C(g)\cap C(h)}(R^{C(g)}_{C(g)\cap C(h)}(E)\otimes R^{C(h)}_{C(g)\cap C(h)}(F)\otimes\gamma_{g,h})\big)\\ &\cong I^G_{C(gh)}(E\star_\omega F). \end{aligned}$$

Proposition 4.2. *There exists a ring isomorphism*

$$({}^{\omega}K_{\mathrm{orb}}([*/G]),\star_\omega)\cong(R(D^\omega(G)),\otimes).$$

Proof. DPR induction defines a morphism $\phi:\bigoplus_{(g)\subset G}R_\omega(C(g))\to R(D^\omega(G))$. Moreover, we proved above that for $E\in R_{\omega_g}(C(g))$ and $F\in R_{\omega_h}(C(h))$, we have

$$\phi(E)\otimes\phi(F)=\phi(E\star_\omega F);$$

that is, it is a ring homomorphism. By Theorem 4.1, the result follows. □

Corollary 4.3. *The stringy product $\star_\omega$ is associative.*

5. Twisted K-theory for an extraspecial p-group

The goal of this section is to establish a relation between the twisted orbifold K-theories for the orbifolds $[*/H]$ and $[*/G]$, where H is an extraspecial group with exponent p, order p^{2n+1} and $G=(\mathbb{Z}_p)^{2n+1}$. For an odd prime number p, a p-group H is called *extraspecial* if its center $Z(H)$ is a cyclic group of order p, that is $Z(H)\cong\mathbb{Z}_p$, and $H/Z(H)$ is an elementary abelian group. Any extraspecial p-group has order p^{2n+1} for some $n\in\mathbb{N}$. On the other hand, for any n there exist two extraspecial groups of order p^{2n+1} such that a group has exponent p and the other group has exponent p^2. The motivation for these kinds of relations comes from works such as [Goff et al. 2007], where these relations are studied for $p=2$, and to some extent results due to A. Duman [2009]. However, there exists a deeper interest to study these kinds of relations by establishing correspondences with the twisted Drinfeld algebras. In particular, the following result is of utmost importance for obtaining the results of this section:

Theorem 5.1 [Naidu and Nikshych 2008, Corollary 4.20]. *Let H be a finite group $\omega'\in Z^3(H;S^1)$ such that*

- *H contains an abelian normal subgroup K,*
- *$\omega'|_{K\times K\times K}$ is trivial in cohomology (in $H^3(K;S^1)$),*

- *there exists an H-invariant 2-cochain μ over H such that $\delta(\mu)|_{K\times K\times K} = \omega'|_{K\times K\times K}$.*

Then, there exists a group G and an element $\omega \in Z^2(G; S^1)$ such that $R(D^{\omega}(G)) \cong R(D^{\omega'}(H))$.

From the relation established in the previous section between the twisted Drinfeld's algebras and the twisted orbifold K-theory, we get the following corollary under the same assumptions as in the last theorem.

Corollary 5.2. *There exists a ring isomorphism*

$$^{\omega}K_{\mathrm{orb}}([*/G]) \cong {}^{\omega'}K_{\mathrm{orb}}([*/H]).$$

Now, we follow with a nice application of this result.

Proposition 5.3. *Let H be an extraspecial group with order p^{2n+1} and exponent p. Then*

$$K_{\mathrm{orb}}([*/H]) \cong {}^{\omega}K_{\mathrm{orb}}([*/(\mathbb{Z}_p)^{2n+1}])$$

for some nontrivial twisting ω.

Proof. Let H be an extraspecial group. From definition we may assume $K = Z(H) \cong \mathbb{Z}_p$. Now, suppose there exists $\mu \in C^2(H; S^1)$ such that $\delta(\mu)|_{K\times K\times K} = \omega'|_{K\times K\times K}$, which is H-invariant; that is, if we take the action of H on the 2-cochains $C^2(H; S^1)$ defined by ${}^{y}\mu := \mu(yx_1y^{-1}, yx_2y^{-1})$, then ${}^{y}\mu = \mu$ in $C^2(H; S^1)$ for all $y \in H$. Now, since $K = Z(H)$, it follows that ${}^{y}\mu|_K = \mu|_K$ for all $y \in H$. Thus, for all $y \in H$ there exists a 1-chain η_y on H such that $\delta\eta_y = {}^{y}\mu/\mu = 1$. Since K is abelian, we can define the map

$$\nu : H/K \times H/K \to C^1(H; S^1), \quad (y_1, y_1) \mapsto \frac{{}^{y_2}\eta_{y_1}\eta_{y_2}}{\eta_{y_1y_2}}.$$

Lemma 5.4 [Naidu 2007, Lemma 4.2, Corollary 4.3]. *The function ν defines an element in $H^2(H/K; \hat{K})$).*

However, this element represents a short exact sequence

$$1 \to \hat{K} \to \hat{K} \times_{\nu} H/K \to H/K \to 1,$$

where the product in $\hat{K} \times_{\nu} H/K$ is defined by the formula

$$(\rho_1, x_1)(\rho_2, x_2) := (\nu(x_1, x_2)\rho_1\rho_2, x_1x_2).$$

Now, the element $\omega \in Z^3(G; S^1)$, with $G := \hat{K} \times_{\nu} H/K$, is defined for all (ρ_1, x_1), (ρ_2, x_2), (ρ_3, x_3) in $\hat{K} \times_{\nu} H/K$ by the formula

$$\omega\big((\rho_1, x_1)(\rho_2, x_2)(\rho_3, x_3)\big) := \big(\nu(x_1, x_2)(u(x_3))\big)(1)\rho_1(k_{x_2,x_3}),$$

where $u : H/K \to H$ is a function such that, when composed with the projection

$p : H \to H/K$, yields $p(u(x)) = x$, and $k_{x_2,x_3} \in H$ is an element that satisfies $u(x_1)u(x_2) = k_{x_1,x_2}u(p(u(x_1)u(x_2)))$.

Clearly, when ω' is the trivial 3-cocycle, we can choose μ to be trivial and so ν is also trivial. By definition of an extraspecial group, H/K is an elementary abelian group and if ν is trivial, it follows easily that $\hat{K} \times_\nu H/K \cong (\mathbb{Z}_p)^{2n+1}$. It remains to show that ω is nontrivial in $H^3(G; S^1)$. Take $H = \{h_1, \dots, h_{p^{2n+1}}\}$, $K = Z(H) = \{z_1, \dots, z_p\}$ and $\hat{K} = \{\rho_1, \dots, \rho_p\}$, with ρ_i nontrivial for $i \neq 1$. Denote the quotient group $H/K = \{x_1K, \dots, x_{p^{2n}}K\}$, with $x_1 = 1_{H/K}$. Now, we define the function $u : H/K \to H$ such that $u(x_iK) = x_i(z_i^{-1})$ for $x_iK \in H/K$, $z_j \in K$. Consider the element $((\rho, x_iK), (\rho, x_iK), (\rho, x_iK))$ with $\rho \in \hat{K}$ fixed and nontrivial. Since ν is trivial, the element ω is reduced to $\rho(k_{x_iK,x_iK}) = \rho(z_i) \neq 1$, which implies that ω is nontrivial. □

Twisted orbifold K-theory for the orbifold $[*/(\mathbb{Z}_p)^n]$**.** With the above result, to calculate the orbifold K-theory structure for $[*/H]$, with H an extraspecial p-group, we only have to calculate the twisted orbifold K-theory for $[*/(\mathbb{Z}_p)^n]$ and a twist element in $H^3((\mathbb{Z}_p)^n; S^1)$, following the constructions presented in Section 3. Because all those constructions are based on the inverse transgression map, we proceed to give an explicit way of calculating it. Later we present an example with a particular twist element, having no trivial inverse transgression map.

*. Inverse transgression map for the group $(\mathbb{Z}_p)^n$ Let us consider the following commutative diagram given by two natural short exact sequences:

(5-1)
$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\times p"] \arrow[d] & \mathbb{Z} \arrow[r, "\pi"] \arrow[d] & \mathbb{Z}_p \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{Z}_p \arrow[r, "\times p"] & \mathbb{Z}_{p^2} \arrow[r, "\tau"] & \mathbb{Z}_p \arrow[r] & 0 \end{tikzcd}$$

where π and τ are the natural projections, and likewise for the downarrow maps. These two exact sequences in the diagram above induce long exact sequences

$$\cdots \longrightarrow H^{k-1}(BG; \mathbb{Z}_p) \xrightarrow{\partial} H^k(BG; \mathbb{Z}) \xrightarrow{(\times p)_*} H^k(BG; \mathbb{Z}) \xrightarrow{(\pi)_*} H^k(BG; \mathbb{Z}_p) \xrightarrow{\partial} H^{k+1}(BG; \mathbb{Z}) \longrightarrow \cdots, \tag{5-2}$$

$$\cdots \longrightarrow H^{k-1}(BG; \mathbb{Z}_p) \xrightarrow{\beta} H^k(BG; \mathbb{Z}_p) \xrightarrow{(\times p)_*} H^k(BG; \mathbb{Z}_{p^2}) \xrightarrow{(\tau)_*} H^k(BG; \mathbb{Z}_p) \xrightarrow{\beta} H^{k+1}(BG; \mathbb{Z}_p) \longrightarrow \cdots. \tag{5-3}$$

Remark 5.5. The connection morphism β of the long exact sequence (5-3) is known as the Bockstein map. It induces a map $\beta : H^*(BG; \mathbb{Z}_p) \to H^*(BG; \mathbb{Z}_p)$

which has the multiplicative property

$$\beta(xy) = \beta(x)y + (-1)^{\deg(x)} x\beta(y).$$

Since G is a p-group, $H^k(BG; -)$ is also a p-group and this implies that the morphism $(\times p)_*$ in the long exact sequences (5-2) and (5-3) is the zero map. Thus, π_* and τ_* are injective maps and $H^k(BG;\mathbb{Z}) \cong H^k(BG;\mathbb{Z}_{p^2})$. On the other hand, by the exactness of the sequence (5-3), we have $H^k(BG;\mathbb{Z}_{p^2}) \cong \operatorname{Ker}\big(\beta : H^k(BG;\mathbb{Z}_p) \to H^{k+1}(BG;\mathbb{Z}_p)\big)$ and then

$$H^k(BG;\mathbb{Z}) \cong \operatorname{Ker}\big(\beta : H^k(BG;\mathbb{Z}_p) \to H^{k+1}(BG;\mathbb{Z}_p)\big).$$

Relation to the inverse transgression map. By definition, the inverse transgression map τ_g is a map defined between the groups $H^k(BG;\mathbb{Z})$ and $H^{k-1}(BC_G(g);\mathbb{Z})$. Since $G = (\mathbb{Z}_p)^n$ is an abelian group, the inverse transgression map can be factorized as

$$\begin{aligned}\tilde{\tau}_g : \operatorname{Ker}\big(\beta : H^k(BG;\mathbb{Z}_p) \to H^{k+1}(BG;\mathbb{Z}_p)\big)& \\ \to \operatorname{Ker}\big(\beta &: H^{k-1}(BG;\mathbb{Z}_p) \to H^k(BG;\mathbb{Z}_p)\big).\end{aligned}$$

Consider the cohomology ring $H^*(BG;\mathbb{Z}_p) \cong \mathbb{F}_p[x_1,\dots,x_n] \otimes \Lambda[y_1,\dots,y_n]$ with $|x_i| = 2$ and $|y_i| = 1$ for $i = 1,\dots,n$. By the calculations above we need to find a polynomial $p(x_1,\dots,x_n,y_1,\dots,y_n) \in \mathbb{F}_p[x_1,\dots,x_n] \otimes \Lambda[y_1,\dots,y_n]$ of degree k, such that $\beta(p) = 0$ and $\tilde{\tau}_g(p) \neq 0$ for some $g \in G$.

To obtain the desired polynomial, first we do the calculation of the inverse transgression map. Take an element $g = (a_1,\dots,a_n) \in G$ and consider the map

$$G \times \mathbb{Z} \to G \times \langle g \rangle \to G$$

defined by

$$(h,m) \mapsto (h, g^m) \mapsto hg^m.$$

At the level of cohomology we get

$$\begin{array}{llll} \text{(5-4)} & H^*(BG;\mathbb{F}_p) \to H^*(BG \times B(\mathbb{Z}_p);\mathbb{F}_p) & \to & H^*(BG \times B\mathbb{Z};\mathbb{F}_p), \\ & \qquad x_i \mapsto x_i + a_i w & \mapsto & x_i, \\ & \qquad y_i \mapsto y_i + a_i z & \mapsto & y_i + a_i z, \end{array}$$

where

$$\begin{aligned} H^*(BG;\mathbb{F}_p) &= \mathbb{F}_p[x_1,\dots,x_n] \otimes \Lambda[y_1,\dots,y_n], \\ H^*(BG \times B(\mathbb{Z}_p);\mathbb{F}_p) &= \mathbb{F}_p[x_1,\dots,x_n,w] \otimes \Lambda[y_1,\dots,y_n,z], \\ H^*(BG \times B\mathbb{Z};\mathbb{F}_p) &= \mathbb{F}_p[x_1,\dots,x_n] \otimes \Lambda[y_1,\dots,y_n,z]. \end{aligned}$$

Now, for the products $x_i y_j, x_i x_j, y_i y_j \in H^*(BG;\mathbb{F}_p)$ we can obtain the calculation of the inverse transgression maps. For the first product $x_i y_j$ we get

$$(x_i y_j) \mapsto (x_i + a_i w)(y_j + a_i z) = x_i y_j + x_i a_j z + a_i w y_j + a_i a_j w z \tag{5-5}$$

in $H^*(BG \times B(\mathbb{Z}_p); \mathbb{F}_p)$ and $(x_i y_j) \mapsto x_i y_j + x_i a_j z$ in $H^*(BG \times B\mathbb{Z}; \mathbb{F}_p)$, from Definition 3.1 it follows that $\tilde{\tau}_g(x_i y_j) = x_i a_j$. For the second product $x_i x_j$ we get

$$(x_i x_j) \mapsto (x_i + a_i w)(x_j + a_i w) = x_i x_j + x_i a_j w + a_i w x_j$$

in $H^*(BG \times B(\mathbb{Z}_p); \mathbb{F}_p)$ and $(x_i x_j) \mapsto x_i x_j$ in $H^*(BG \times B\mathbb{Z}; \mathbb{F}_p)$; hence

$$\tilde{\tau}_g(x_i x_j) = 0. \tag{5-6}$$

Finally, for the product $y_i y_j$ we get

$$(y_i y_j) \mapsto (y_i + a_i z)(y_j + a_i z) = y_i y_j + y_i a_j z + a_i z y_j \tag{5-7}$$

in $H^*(BG \times B(\mathbb{Z}_p); \mathbb{F}_p)$ and $(y_i y_j) \mapsto y_i y_j + (a_j y_i - a_i y_j) z$ in $H^*(BG \times B\mathbb{Z}; \mathbb{F}_p)$; therefore $\tilde{\tau}_g(y_i y_j) = (a_j y_i - a_i y_j)$.

Since we are interested in calculating the inverse transgression map for elements $\alpha \in H^4(G; \mathbb{Z})$, we consider only polynomials of degree 4 in

$$H^*(BG; \mathbb{F}_p) = \mathbb{F}_p[x_1, \dots, x_n] \otimes \Lambda[y_1, \dots, y_n].$$

Now, we present some examples of the inverse transgression map. It is easiest to consider the cases $n = 2$ and $n = 3$. In the first case the inverse transgression map is a trivial map. In the latter the inverse transgression map is more interesting.

Example 5.6. $\underline{n = 2}$. For $p \neq 2$ we have $H^*(BG; \mathbb{F}_p) = \mathbb{F}_p[x_1, x_2] \otimes \Lambda[y_1, y_2]$ with $|y_i| = 1$ and $|\beta y_i| = |x_i| = 2$. Thus, we can just consider linear combinations of the polynomials $p_1(x_1, x_2, y_1, y_2) := x_1 x_2$, $p_2(x_1, x_2, y_1, y_2) := x_1 y_1 y_2$ and $p_3(x_1, x_2, y_1, y_2) := x_2 y_1 y_2$. For p_1 the calculations leading up to (5-6) show that $\tilde{\tau}_g(p_1) = 0$. Thus we need to find a $(\mathbb{Z}_p)$-linear combination p of the polynomials p_2 and p_3 such that $\beta(p) = 0$. But we have

$$\beta(p_3) = x_2(\beta(y_1) y_2 - y_1 \beta(y_2)) = x_2(x_1 y_2 - y_1 x_2),$$
$$\beta(p_2) = x_1(\beta(y_1) y_2 - y_1 \beta(y_2)) = x_1(x_1 y_2 - y_1 x_2).$$

Therefore, there does not exists such a $(\mathbb{Z}_p)$-linear combination.

$\underline{n = 3}$. By analyzing the degree of the polynomials, we obtain the element

$$p(x_1, x_2, x_3, y_1, y_2, y_3) = x_1 y_2 y_3 - x_2 y_1 y_3 + x_3 y_1 y_2, \tag{5-8}$$

which satisfies the condition $\beta(p) = 0$. To check this, we use the property of β

noted in Remark 5.5:

$$\begin{aligned}\beta(p) &= \beta(x_1y_2y_3)-\beta(x_2y_1y_3)+\beta(x_3y_1y_2)\\ &= \beta(x_1)y_2y_3+x_1\beta(y_2y_3)-\beta(x_2)y_1y_3-x_2\beta(y_1y_3)+\beta(x_3)y_1y_2+x_3\beta(y_1y_2)\\ &= x_1\beta(y_2)y_3-x_1y_2\beta(y_3)-x_2\beta(y_1)y_3+x_2y_1\beta(y_3)+x_3\beta(y_1)y_2-x_3y_1\beta(y_2)\\ &= x_1x_2y_3-x_1y_2x_3-x_2x_1y_3+x_2y_1x_3+x_3x_1y_2-x_3y_1x_2\\ &= 0.\end{aligned}$$

The inverse transgression map for an element $g=(a_1,a_2,a_3)\in(\mathbb{Z}_p)^3$ evaluated in the polynomial p gives

$$\begin{aligned}\tau_g(p) &= \tau_g(x_1y_2y_3)-\tau_g(x_2y_1y_3)+\tau_g(x_3y_1y_2)\\ &= x_1(a_3y_2-a_2y_3)-x_2(a_3y_1-a_1y_3)+x_3(a_2y_1-a_1y_2)\\ &= a_1(x_2y_3-x_3y_2)+a_2(x_3y_1-x_1y_3)+a_3(x_1y_2-x_2y_1).\end{aligned} \tag{5-9}$$

Lemma 5.7. *Let $g=(a_1,a_2,a_3)$ and $h=(b_1,b_2,b_3)$ be elements in $G=(\mathbb{Z}_p)^3$. The double inverse transgression map of p is equal to*

$$\tau_h\tau_g(p)=[(a_1,a_2,a_3)\times(b_1,b_2,b_3)]\cdot(x_1,x_2,x_3). \tag{5-10}$$

Remark 5.8. With $n=3$, this example shows that for $n\geq 3$ the inverse transgression map is nontrivial. We can always consider the element $p(x_1,\dots,x_n,y_1,\dots,y_n)=x_iy_jy_k-x_jy_iy_k+x_ky_iy_j$ as being in $H^4((Z_p)^n;\mathbb{Z})$. By calculations similar to those leading to (5-9), we can prove that $\beta(p)=0$ while $\tau_g(p)\neq 0$ for $g\in(\mathbb{Z}_p)^n$.

By using the inverse transgression map for the group $(\mathbb{Z}_p)^n$ presented above and by the decomposition formula presented in Theorem 3.6 in [Becerra and Uribe 2009], we can calculate the explicit structure of the twisted orbifold K-theory for the orbifold $[*/(\mathbb{Z}_p)^3]$ and the twist element α as the element in $H^3((\mathbb{Z}_p)^3;S^1)$ associated to the polynomial defined in (5-8) via the isomorphism $H^3((\mathbb{Z}_p)^3;S^1)\cong H^4((\mathbb{Z}_p)^3;\mathbb{Z})$. Note that in this case

$${}^{\alpha}K_{\mathrm{orb}}([*/(\mathbb{Z}_p)^3])=\bigoplus_{g\in(\mathbb{Z}_p)^3}{}^{\alpha_g}K_{(\mathbb{Z}_p)^3}(*)\cong\bigoplus_{g\in(\mathbb{Z}_p)^3}R_{\alpha_g}((\mathbb{Z}_p)^3).$$

Now, for each $g\in\mathbb{Z}_p$, the decomposition formula implies

$$R_{\alpha_g}((\mathbb{Z}_p)^3)\otimes\mathbb{Q}\cong\prod_{g,h\in(\mathbb{Z}_p)^3}\left(\mathbb{Q}(\zeta_p)_{h,\alpha_g}\right)^{(\mathbb{Z}_p)^3},$$

where ζ_p is a p-root of the unity. Note that the action of $(\mathbb{Z}_p)^3$ on $\mathbb{Q}(\zeta_p)_{h,\alpha_g}$ is to multiply by the double inverse transgression map evaluated on $k\in(\mathbb{Z}_p)^3$, $\tau_h(\alpha_g)(k)$.

By Lemma 5.7, we get

$$\big(\mathbb{Q}(\zeta_p)_{h,\alpha_g}\big)^{(\mathbb{Z}_p)^3} = \begin{cases} \mathbb{Q}(\zeta_p) & \text{if } g = \lambda h,\ \lambda \in \mathbb{Z}_p, \\ 0 & \text{else.} \end{cases} \tag{5-11}$$

So, for $h \neq 0$, we have

$$R_{\alpha_g}((\mathbb{Z}_p)^3) \otimes \mathbb{Q} = \prod_{\lambda \in \mathbb{Z}_p} \mathbb{Q}(\zeta_p),$$

while for $g = 0$, we get

$$R_{\alpha_1}((\mathbb{Z}_p)^3) \otimes \mathbb{Q} = \prod_{\lambda \in (\mathbb{Z}_p)^3} \mathbb{Q}(\zeta_p).$$

Then, the twisted orbifold K-theory module for the orbifold $[*/(\mathbb{Z}_p)^3]$ turns out to be

$${}^{\alpha}K_{\mathrm{orb}}([*/(\mathbb{Z}_p)^3]) \otimes \mathbb{Q} = \prod_{\lambda \in \mathbb{Z}_p} \mathbb{Q}(\zeta_p) \ \oplus \prod_{\lambda \in (\mathbb{Z}_p)^3} \mathbb{Q}(\zeta_p)$$

and the product structure is defined via the product of the elements in $\mathbb{Q}(\zeta_p)$.

6. Final remarks

With the result presented in Section 4 about the Grothendieck ring associated to the semigroup of representations of the twisted Drinfeld double $D^{\omega}(G)$ and the twisted orbifold K-theory, we found a nice relation between two structures coming from different sources. As we already said, the orbifold $[*/G]$ is a particular case of a more general kind of orbifolds obtained by the almost free action of a compact Lie group G on a compact manifold M. With a little more structure, the stringy product introduced in Section 3 can be extended to a stringy product on the module ${}^{\alpha}K_{\mathrm{orb}}([M/G])$ (in the same way as in [Becerra and Uribe 2009]), where $[M/G]$ denotes the orbifold structure obtained by the almost free action (see [Adem and Ruan 2003] for the details of this structure). Therefore, under suitable hypotheses we can think about the twisted orbifold K-theory ${}^{\omega}K_{\mathrm{orb}}([M/G])$ as a more general object which coincides with the Grothendieck ring $R(D^{\omega}(G))$ if G is a finite group and $M = \{*\}$. Nevertheless, we shall explore the interpretation and consequences of this more general object. Next, we focus our attention on the results obtained in Section 5, where we establish an explicit relation between the twisted orbifold K-theories of the orbifolds $[*/H]$ and $[*/(\mathbb{Z}_p)^n]$, where H is a particular extraspecial p-group. In the same spirit, we look for some general relation between the twisted orbifold K-theories ${}^{\alpha}K_{\mathrm{orb}}([M/G])$ and ${}^{\beta}K_{\mathrm{orb}}([M/K])$ of the orbifolds $[M/G]$ and $[M/K]$, for suitable twistings $\alpha \in H^3(G; S^1)$ and $\beta \in H^3(H; S^1)$, and appropriate actions of the finite groups G and K on a compact manifold M. In the same way, we hope that some analogous results may be obtained if G and K are compact Lie groups acting almost freely on a compact manifold M. By our preliminary observations, in order to obtain such results, some hypothesis on the almost free actions of the compact Lie groups G and K must be added.

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MARIO VELÁSQUEZ
UNIVERSIDAD DE LOS ANDES
DEPARTAMENTO DE MATEMÁTICAS
CARRERA 1 NO 18A–12 EDIFICIO H
BOGOTÁ
COLOMBIA
ma.velasquez109@uniandes.edu.co

EDWARD BECERRA
DEPARTMENT OF MATHEMATICS
UNIVERSIDAD NACIONAL DE COLOMBIA
CARRERA 30 CALLE 45, CIUDAD UNIVERSITARIA
BOGOTÁ
COLOMBIA
esbecerrar@unal.edu.co

HERMES MARTINEZ
ESCUELA DE MATEMÁTICAS
UNIVERSIDAD SERGIO ARBOLEDA
CALLE 74 NO. 14–14
BOGOTÁ
COLOMBIA
hermes.martinez@ima.usergioarboleda.edu.co

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