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Given a genus-g Heegaard splitting of a 3-manifold, the Goeritz group is defined to be the group of isotopy classes of orientation-preserving homeomorphisms of the manifold that preserve the splitting. In this work, we show that the Goeritz groups of genus-2 Heegaard splittings for lens spaces $L(p, 1)$ are finitely presented, and give explicit presentations of them.

1. Introduction

It is well known that every closed orientable 3-manifold M can be decomposed into two handlebodies of the same genus. This is what we call a Heegaard splitting of the manifold, and the genus of the handlebodies is called the genus of the splitting. Given a genus-g Heegaard splitting of M, the *Goeritz group* of the splitting, which we will denote by $\mathcal{G}_g$, is the group of isotopy classes of orientation-preserving homeomorphisms of M that preserve each of the handlebodies of the splitting setwise. In particular, this group is interesting when the manifold is the 3-sphere or a lens space since it is well known from [Waldhausen 1968; Bonahon 1983; Bonahon and Otal 1983] that they have unique Heegaard splittings for each genus up to isotopy. In this case, each Goeritz group depends only on the genus of the splitting, and so we can define the *genus-g Goeritz group* $\mathcal{G}_g$ of each of those manifolds without mentioning a specific Heegaard splitting. For the 3-sphere, it was shown in [Goeritz 1933; Scharlemann 2004] that $\mathcal{G}_2$ is finitely generated, and subsequently in [Akbas 2008; Cho 2008] that $\mathcal{G}_2$ is finitely presented and its finite presentation was introduced. Further, in [Koda 2011], a natural generalization of a Goeritz group is studied, namely, the group of isotopy classes of orientation-preserving homeomorphisms of the 3-sphere preserving an embedded genus-two handlebody which is possibly knotted.

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In this work, we show that the Goeritz group $\mathcal{G}_2$ of each of the lens spaces $L(p, 1)$ is finitely presented. In the main theorem, Theorem 5.4, their explicit presentations are given. For the genus-2 Goeritz groups of the other lens spaces, and for the higher genus Goeritz groups of the 3-sphere and lens spaces, it is conjectured that they are all finitely presented, but it is still known to be an open problem.

We generalize the method developed in [Cho 2008]. We find a tree on which $\mathcal{G}_2$ for $L(p, 1)$ acts such that the quotient of the tree by the action of $\mathcal{G}_2$ is a single edge, and then apply the well known theory of groups acting on trees due to Bass and Serre (see [Serre 1980]). Such a tree will be found in the barycentric subdivision of the disk complex for one of the handlebodies of the splitting. For arbitrary lens spaces $L(p, q)$, finding such trees, if they exist, is a much more complicated problem than $L(p, 1)$, which will be fully discussed in [Cho and Koda 2012].

Throughout the paper, we simply denote by $\mathcal{G}$ the genus-2 Goeritz group $\mathcal{G}_2$ of a lens space. We use the standard notation $L(p, q)$ with $p \geq 2$ for a lens space with its basic properties found in standard textbooks. For an example, we refer to [Rolfsen 1976]. For a genus-1 Heegaard splitting of $L(p, 1)$, any oriented meridian circle of a solid torus of the splitting is identified with a $(p, 1)$-curve (or a $(p, p-1)$-curve) on the boundary of the other solid torus after a suitable choice of oriented longitude and meridian of the other solid torus is made. The triple $(V, W; \Sigma)$ will denote a genus-2 Heegaard splitting of a lens space $L = L(p, q)$. That is, $L = V \cup W$ and $V \cap W = \partial V = \partial W = \Sigma$, where V and W are handlebodies of genus two.

The disks D and E in a handlebody are always assumed to be properly embedded, and their intersection is transverse and minimal up to isotopy. In particular, if D intersects E, then $D \cap E$ is a collection of pairwise disjoint arcs that are properly embedded in both D and E. Finally, $\mathrm{Nbd}(X)$ will denote a regular neighborhood of X, and $\mathrm{cl}(X)$ the closure of X for a subspace X of a polyhedral space where the ambient space will always be clear from the context.

2. Primitive elements of the free group of rank two

The fundamental group of the genus-2 handlebody is the free group $\mathbb{Z} * \mathbb{Z}$ of rank two. We call an element of $\mathbb{Z} * \mathbb{Z}$ *primitive* if it is a member of a generating pair of $\mathbb{Z} * \mathbb{Z}$. Primitive elements of $\mathbb{Z} * \mathbb{Z}$ have been well understood. For an example we refer [Osborne and Zieschang 1981] to the reader. A key property of the primitive elements of the free group of rank two is the following, which is a direct consequence of Corollary 3.3 in [Osborne and Zieschang 1981]:

Proposition 2.1. *Fix a generating pair $\{x, y\}$ of $\mathbb{Z} * \mathbb{Z}$, and let w be a primitive element of $\mathbb{Z} * \mathbb{Z}$. Then for some $\epsilon \in \{1, -1\}$ and some $n \in \mathbb{Z}$, some cyclically reduced form of w is a product of terms of the form $x^{\epsilon} y^{n}$ or $x^{\epsilon} y^{n+1}$, or else a product of terms of the form $y^{\epsilon} x^{n}$ or $y^{\epsilon} x^{n+1}$.*

From the proposition, the cyclically reduced forms of a primitive element are very restrictive. For example, if w is a primitive element of $\mathbb{Z}*\mathbb{Z}$, then no cyclically reduced form of w in terms of x and y can contain x and x^{-1} (and y and y^{-1}) simultaneously.

A simple closed curve in the boundary of a genus-2 handlebody W represents an element of $\pi_1(W) = \mathbb{Z}*\mathbb{Z}$. We call a pair of essential disks in W a *complete meridian system* for W if the union of the two disks cuts up W into a 3-ball. Given a complete meridian system $\{F, G\}$, assign symbols x and y to circles ∂F and ∂G respectively. Suppose that an oriented simple closed curve l on ∂W meets $\partial F \cup \partial G$ transversely and minimally. Then l determines a word in terms of x and y which can be read off from the intersections of l with ∂F and ∂G (after a choice of orientations of ∂F and ∂G), and hence l represents an element of the free group $\pi_1(W) = \langle x, y\rangle$.

In this set up, the following is a simple criterion for the primitiveness of the elements represented by such a simple closed curve:

Lemma 2.2. *With a suitable choice of orientations of ∂F and ∂G, if a word determined by the simple closed curve l contains one of the subwords yxy^{-1} or $xyxy^n$ for $n \geq 3$, then any element in $\pi_1(W)$ represented by l cannot be a primitive element.*

Proof. Let Σ' be the 4-holed sphere cut up from ∂W along $\partial F \cup \partial G$, and denote by f_+ and f_- (respectively g_+ and g_-) the boundary circles of Σ' that came from ∂F (respectively ∂G).

Suppose first that a word represented by l contains a subword of the form yxy^{-1}. Then we may assume that there are two arcs l_+ and l_- of $l \cap \Sigma'$ such that l_+ connects f_+ and g_+, and l_- connects f_+ and g_- as in Figure 1, left. Since $|l \cap f_+| = |l \cap f_-|$ and $|l \cap g_+| = |l \cap g_-|$, we must have two other arcs m_+ and m_- of $l \cap \Sigma'$ such that m_+ connects f_- and g_+, and m_- connects f_- and g_-. We see then that there exists no arc component of $l \cap \Sigma'$ that meets only one of f_+, f_-, g_+ or g_-. That is, any word determined by l contains neither $x^{\pm 1}x^{\mp 1}$ nor $y^{\pm 1}y^{\mp 1}$, and so each word is cyclically reduced, but a word determined by l already contains both y and y^{-1}, and so l cannot represent a primitive element of $\pi_1(W)$ by Proposition 2.1.

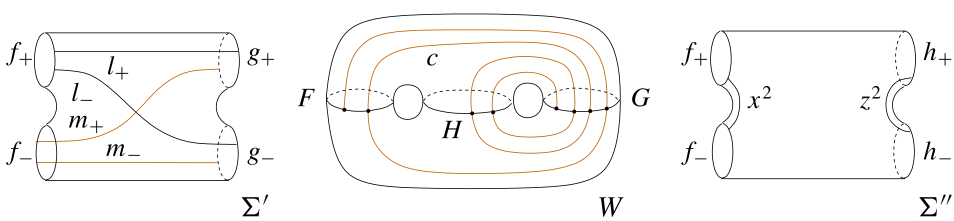

Figure 1. The 4-holed spheres Σ' and Σ''.

Next, suppose that a word represented by l contains a subword of the form $xyxy^n$ for $n \geq 3$. We may assume there is an arc c of $l \cap \Sigma'$ connecting f_+ and g_+ in Σ'. Consider the circle which is the frontier of a regular neighborhood of $f_+ \cup c \cup g_+$ in Σ'. This circle bounds a disk H in W, and $\{F, H\}$ forms a complete meridian system of W. Assigning symbols x and z to ∂F and ∂H respectively, the circle l represents an element of $\pi_1(W) = \langle x, z \rangle$ (see Figure 1, middle).

Let Σ'' be the 4-holed sphere cut up from ∂W along $\partial F \cup \partial H$, and denote by f_+ and f_- (respectively h_+ and h_-) the boundary circles of Σ'' that came from ∂F (respectively ∂H). There are two arcs of $l \cap \Sigma''$ such that one connects f_+ and f_-, and the other one connects h_+ and h_-. We may assume that these two arcs represent subwords of the form x^2 and z^2 (see Figure 1, right). Thus there exists no arc component of $l \cap \Sigma''$ that meets only one of f_+, f_-, h_+ and h_-. That is, each word represented by l is cyclically reduced. But a word determined by l already contains both x^2 and z^2, and so l cannot represent a primitive element of $\pi_1(W)$ by Proposition 2.1 again. □

3. Primitive disks in a handlebody

Recall that $(V, W; \Sigma)$ denotes a genus-two Heegaard splitting of a lens space $L = L(p, q)$ with $p \geq 2$. We call an essential disk E in V *primitive* if there exists an essential disk E' in W such that ∂E intersects $\partial E'$ transversely in a single point. Such a disk E' is called a *dual disk* of E. Note that E' is also primitive in W with a dual disk E, and $W \cup \mathrm{Nbd}(E)$ and $V \cup \mathrm{Nbd}(E')$ are both solid tori. Primitive disks are necessarily nonseparating. We call a pair of disjoint, nonisotopic primitive disks in V a *primitive pair* in V. Similarly, a triple of pairwise disjoint, nonisotopic, primitive disks (if it exists) is a *primitive triple*.

A nonseparating disk E_0 properly embedded in V is called *semiprimitive* if there is a primitive disk E' in W such that $\partial E'$ is disjoint from ∂E_0. With a suitable choice of oriented meridian and longitude circles on the boundary of the solid torus obtained by cutting up W along E', the oriented boundary circle ∂E_0 can be considered a $(p, 1)$-curve on the boundary of the solid torus, if $q = 1$.

Any simple closed curve on the boundary of W represents an element of $\pi_1(W)$, which is the free group of rank two. We can interpret primitive disks algebraically as follows, which is a direct consequence of [Gordon 1987]:

Lemma 3.1. *Let D be a nonseparating disk in V. Then D is primitive if and only if ∂D represents a primitive element of $\pi_1(W)$.*

Note that no disk can be both primitive and semiprimitive since the boundary circle of a semiprimitive disk in V represents the p-th power of a primitive element of $\pi_1(W)$.

Let D and E be essential disks in V, and suppose that D intersects E transversely and minimally. Let $C \subset D$ be a disk cut up from D by an outermost arc β of $D \cap E$ in D such that $C \cap E = \beta$. We call such a C an *outermost subdisk* of D cut up by $D \cap E$. The arc β cuts E into two disks, say G and H. Then we have two essential disks E_1 and E_2 in V which are isotopic to disks $G \cup C$ and $H \cup C$ respectively. We call E_1 and E_2 the *disks from surgery* on E along the outermost subdisk C of D cut up by $D \cap E$. Observe that E_1 and E_2 each have fewer arcs of intersection with D than E had, since at least the arc β no longer counts.

Since E and D are assumed to intersect minimally, E_1 and E_2 are isotopic to neither E nor D. In particular, if both D and E are nonseparating, then the resulting disks E_1 and E_2 are both nonseparating and they are not isotopic to each other. Further, E_1 and E_2 are meridian disks of the solid torus V cut up by E, and the boundary circles ∂E_1 and ∂E_2 are not isotopic to each other in the two holed torus ∂V cut up by ∂E.

Theorem 3.2. *Let $(V, W; \Sigma)$ be the genus-two Heegaard splitting of the lens space $L = L(p, 1)$ with $p \geq 2$. Let D and E be primitive disks in V which intersect each other transversely and minimally. Then one of the two disks from surgery on E along an outermost subdisk of D cut up by $D \cap E$ is primitive. Furthermore, it has a common dual disk with E.*

Proof. We will prove the theorem only for $p \geq 5$. The cases of $p \in \{2, 3, 4\}$ will be similar but simpler.

Let C be an outermost subdisk of D cut up by $D \cap E$. The choice of a dual disk E' of E determines a unique semiprimitive disk E_0 in V, namely, the meridian disk E_0 of V disjoint from $E \cup E'$. Among all the dual disks of E, choose one, denoted by E' again, so that the semiprimitive E_0 determined by E' intersects C minimally. Further, there is a unique semiprimitive disk E'_0 in W disjoint from $E \cup E'$. We give symbols x and y to oriented $\partial E'$ and $\partial E'_0$ respectively to have $\pi_1(W) = \langle x, y \rangle$. For convenience, we simply identify the boundary circles $\partial E'$ and $\partial E'_0$ with the assigned symbols x and y respectively. Notice that the circle y is disjoint from ∂E and intersects ∂E_0 in p points in the same direction, and x is disjoint from ∂E_0 and intersects ∂E in a single point. Thus we may assume that ∂E_0 and ∂E determine the words y^p and x respectively.

Let Σ_0 be the 4-holed sphere ∂V cut up by $\partial E \cup \partial E_0$. We regard Σ_0 as a 2-holed annulus where the two boundary circles came from ∂E_0 and the two holes came from ∂E. Then $y \cap \Sigma_0$ is the union of p spanning arcs which cut Σ_0 into p rectangles, and x is a single arc connecting two holes which are contained in a single rectangle. See Figure 2, left.

Suppose first that C is disjoint from E_0. Note that one of the disks from surgery on E along C is E_0, which is semiprimitive. The arc $C \cap \Sigma_0$ is the frontier of

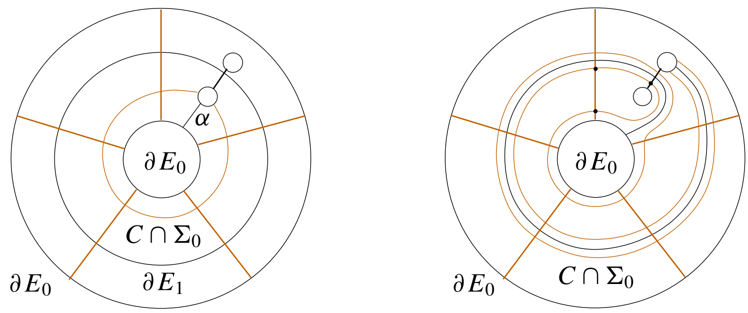

Figure 2. The 2-holed annulus Σ_0 in $L(5, 1)$.

a regular neighborhood of the union of one boundary circle of Σ_0 and an arc α connecting the boundary circle to a hole. Observe that the arc α is disjoint from $y \cap \Sigma_0$, otherwise a word of ∂D must contain yxy^{-1} (after changing orientation if necessary) which contradicts that D is primitive, by Lemma 2.2. See Figure 2, right. Consequently, if we denote by E_1 the disk from surgery that is not E_0, then ∂E_1 intersects $\partial E'$ in a single point. That is, the resulting disk E_1 is primitive with the common dual disk E' of E. See Figure 2, left.

From now on, we assume that C intersects E_0. Let C_0 be an outermost subdisk of C cut up by $C \cap E_0$. The arc $C_0 \cap \Sigma_0$ is the frontier of a regular neighborhood of one hole of Σ_0 and an arc, say α_0, connecting the hole to a boundary circle of Σ_0. By the same reasoning as in the case of α, the arc α_0 is disjoint from $y \cap \Sigma_0$. Thus one of the disks from surgery on E_0 along C_0 is E, and the other one, denoted by E_1 again, is primitive since ∂E_1 intersects $\partial E'$ in a single point as in the previous case. Note that $|C \cap E_1| < |C \cap E_0|$ from the surgery construction. See Σ_0 in Figure 3.

Let Σ_1 be the 4-holed sphere ∂V cut up by $\partial E \cup \partial E_1$. We regard Σ_1 as a 2-holed annulus, like Σ_0, where the two boundary circles came from ∂E_1 and the two holes came from ∂E. Then $y \cap \Sigma_1$ is the union of p spanning arcs which cut Σ_1 into p rectangles as in the case of Σ_0, but the two holes, which came from ∂E, are now contained in different consecutive rectangles, and $x \cap \Sigma_1$ is the union of two arcs each joining a hole and a boundary circle of Σ_1 as in Figure 3. If the original subdisk C is disjoint from E_1, then we are done since E_1 is the desired primitive disk resulting from the surgery.

Suppose that C also intersects E_1, and let C_1 be an outermost subdisk of C cut up by $C \cap E_1$. Then $C_1 \cap \Sigma_1$ is the frontier of a regular neighborhood of the union of one hole of Σ_1 and an arc, say α_1, connecting the hole to a boundary circle. The arc α_1 is also disjoint from $y \cap \Sigma_1$ by the same reasoning as for α_0. Thus if we denote by E_2 the disk from surgery on E_1 along C_1 that is not E, then ∂E_2 represents a word $xyxy^{p-1}$. See Σ_1 in Figure 3.

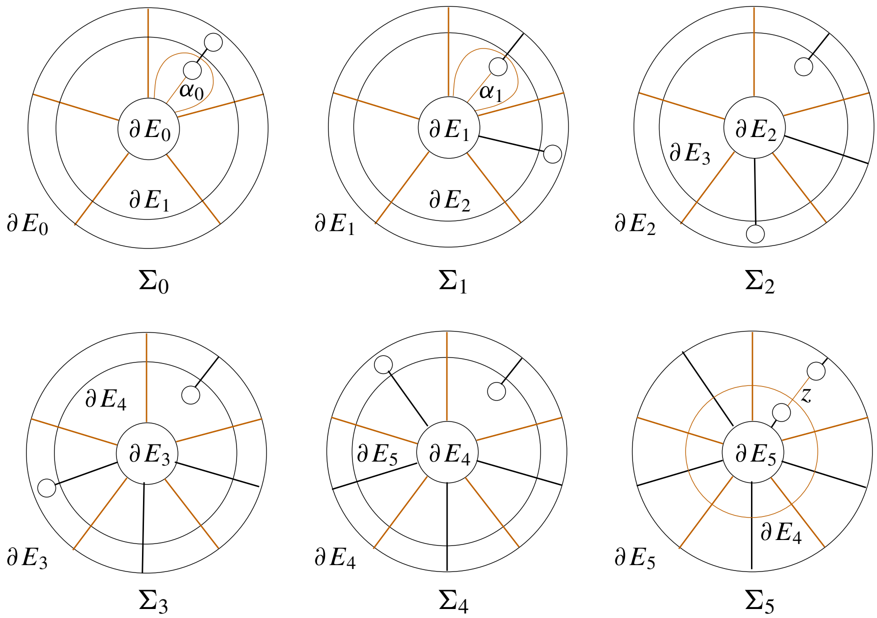

Figure 3. The sequence of 2-holed annuli from the consecutive surgeries for $L(5, 1)$.

We continue such a construction repeatedly whenever C also intersects the next disk. For each $1 \leq j \leq p-1$, if C intersects E_j, then we obtain the disk E_{j+1} from surgery on E_j along an outermost subdisk C_j cut up by $C \cap E_j$. We see that $|C \cap E_{j+1}| < |C \cap E_j|$ from the surgery construction. In the 2-holed annulus Σ_j, the arc $C_j \cap \Sigma_j$ is the frontier of a regular neighborhood of the union of a hole of Σ_j and an arc α_j connecting the hole to a boundary circle. The arc α_j is disjoint from $y \cap \Sigma_j$, and so ∂E_{j+1} represents a word of the form $(xy)^j xy^{p-j}$. In particular, notice that the disk E_p is semiprimitive and E_{p-1} is primitive, since there is a primitive disk E'' in W disjoint from ∂E_p that intersects ∂E_{p-1} in a single point. Such an E'' is not hard to find. In the final 2-holed annulus Σ_5 in Figure 3, the arc z is the boundary circle of E'' in Σ_p. Note that z is disjoint from $x \cup y$, and so it does bound a disk E'' in the 3-ball W cut up by $E' \cup E'_0$. Also, z intersects ∂E_{p-1} in a single point and is disjoint from ∂E_p.

We remark that each of the arcs α_j, $j \in \{0, 1, \ldots, p-1\}$, is disjoint from the circle y due to the fact that D is primitive. There are infinitely many arcs α_0 that are not isotopic to each other in Σ_0, but each arc α_j in Σ_j with $j \geq 1$ is unique up to isotopy. Therefore, once E_1 is determined, we have the unique sequence of disks $E_2, E_3, \ldots, E_p$ only under the condition that each α_j is disjoint from y.

Claim. *For each $j \in \{2, 3, \ldots, p-1\}$, the subdisk C intersects E_j.*

Proof of claim. Suppose not, and let E_j be the first disk disjoint from C. First, suppose that $j \in \{2, 3, \ldots, p-3\}$. Then C is disjoint from E_j and intersects E_{j-1}, and so the arc $\partial C \cap \Sigma_j$ gives a subword of ∂D of the form $(yx)^j y^{p-j}$ which implies that D is not primitive by Lemma 2.2 again, which is a contradiction. Next, suppose that $j = p-2$. That is, C is disjoint from E_{p-2} and intersects E_{p-3}. Then one of the resulting disks from surgery on E along C is E_{p-2}, and the other one is exactly E_{p-1}, which is a disk in the sequence of disks in the previous construction. The subdisk C is disjoint from $E_{p-2} \cup E_{p-1}$, and consequently, C necessarily intersects the semiprimitive disk E_p in the previous construction in a single arc. That is, $|C \cap E_p| = 1$. But from the consecutive surgery constructions for $j \in \{2, 3, \ldots, p-3\}$, we have $1 \leq |C \cap E_{p-3}| < |C \cap E_0|$, which contradicts the minimality of $|C \cap E_0|$. Similarly, if $j = p-1$, then we have the same contradiction on the minimality, since C is disjoint from E_p in this case. This proves the claim.

By the claim, we can do surgery on E_{p-1} along C_{p-1} and one resulting disk from surgery is E_p, the semiprimitive disk. But $|C \cap E_{j+1}| < |C \cap E_j|$ for each $j \in \{1, 2, \ldots, p-1\}$, and consequently $|C \cap E_p| < |C \cap E_0|$, which contradicts the minimality of $|C \cap E_0|$ again.

Therefore the primitive disk E_1 is a disk from surgery on E along C, and E' is also a dual disk of E_1, and so we complete the proof. We note that the other disk from surgery is either E_0 or E_2 depending on whether C is disjoint from E_0 or not. □

Theorem 3.3. *Let $(V, W; \Sigma)$ be the genus-two Heegaard splitting of the lens space $L = L(p, 1)$ with $p \geq 2$. Then, for every primitive pair $\{D, E\}$ of V, D and E have a common dual disk. In particular, the two disks of each primitive pair have a unique common dual disk if $p \geq 3$, and have exactly two common dual disks if $p = 2$ which form a primitive pair in W.*

Proof. The proof of the existence of a common dual disk goes almost in the same way as that of Theorem 3.2, by taking the primitive disk D disjoint from E instead of the outermost subdisk C in Theorem 3.2. That is, when we choose a dual disk E' of E so that $|\partial D \cap \partial E_0|$ is minimal where E_0 is the unique semiprimitive disk in V disjoint from $\mathrm{Nbd}(E \cup E')$, the primitive disk D must be E_1, having the common dual disk E' of E.

Now, let E' be a common dual disk of D and E. Let E_0 and E'_0 be the unique meridian disks of V and W respectively that are disjoint from $\mathrm{Nbd}(E \cup E')$ (see Figure 4, left). Cut the surface Σ along $\partial E' \cup \partial E'_0$ to obtain the 4-holed sphere Σ'. Then $\partial E \cap \Sigma'$ is a single arc in Σ' connecting the two holes coming from $\partial E'$, and $\partial D \cap \Sigma'$ consists of $p-1$ parallel arcs connecting the two holes coming from $\partial E'_0$ and two arcs connecting the holes coming from $\partial E'$ to $\partial E'_0$ on opposite sides, as in Figure 4.

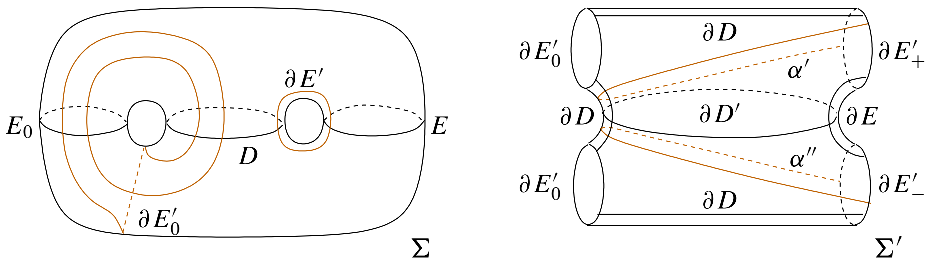

Figure 4. The surfaces Σ and Σ' for $L(2, 1)$.

Let D' be a common dual disk of D and E which is not isotopic to E'. Then an outermost subdisk C' of D' cut up by $D' \cap (E' \cup E_0')$ would intersect ∂D if C' is incident to E'. Denote by $\partial E_+'$ and $\partial E_-'$ the two holes of Σ' which came from $\partial E'$. We may assume that the endpoints of the arc $\alpha' = C' \cap \Sigma'$ meet $\partial E_+'$. Since $|\partial D' \cap \partial E_+'| = |\partial D' \cap \partial E_-'|$, we must have one more arc component α'' of $\partial D' \cap \Sigma'$ other than $C' \cap \Sigma'$ whose endpoints meet $\partial E_-'$ (see Figure 4, right). The arc α'' also intersects ∂D, and so $\partial D'$ intersects ∂D in more than one point, which contradicts that D' is a dual disk of D. Similarly, if C' is incident to E_0', then D' cannot be a dual disk of E. Thus we see that D' is disjoint from $E' \cup E_0'$.

If $p \geq 3$, there is no possibility of such a disk D' which is disjoint from $E' \cup E_0'$ and is not isotopic to E', and so E' is the unique common dual disk. If $p = 2$, there is a unique circle in Σ' which is not boundary parallel and which intersects ∂E and ∂D exactly once (see the circle $\partial D'$ in Figure 4, right). So we have exactly two common dual disks D' and E' and in this case they are disjoint from each other. □

Given a primitive disk D in V, there are infinitely many (nonisotopic) primitive disks each of which forms a primitive pair together with D. But any primitive pair can be contained in at most one primitive triple, proved as follows:

Theorem 3.4. *Let $(V, W; \Sigma)$ be the genus-two Heegaard splitting of the lens space $L = L(p, 1)$ with $p \geq 2$. Then there is a primitive triple of V if and only if $p = 3$. In this case, every primitive pair is contained in a unique primitive triple.*

Proof. Let $\{E, E_1\}$ be a primitive pair of V. Choose a common dual disk E' of E and E_1 given by Theorem 3.3. There are unique semiprimitive disks E_0 in V and E_0' in W disjoint from $\mathrm{Nbd}(E \cup E')$. Let Σ_1 be the 4-holed sphere ∂V cut up by $\partial E \cup \partial E_1$, and as in Figure 3 again, consider Σ_1 as a 2-holed annulus with two boundary circles coming from ∂E_1 and two holes from ∂E. We give symbols x and y to $\partial E'$ and $\partial E_0'$ respectively as in the proof of Theorem 3.2.

The boundary of any primitive disk E_2 in V disjoint from E and E_1, if it exists, lies in Σ_1, and it is the frontier of a regular neighborhood of the union of a boundary circle, a hole of Σ_1 and an arc α_1 connecting them. This arc is disjoint from the

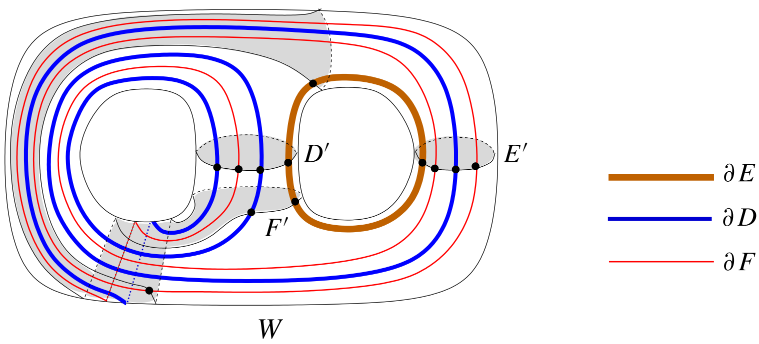

Figure 5. The primitive triple $\{D', E', F'\}$ of W in $L(3, 1)$ with the boundary circles ∂D, ∂E, and ∂F of the disks in the primitive triple of V.

arcs $y \cap \Sigma_1$, otherwise ∂E_2 represents a word containing yxy^{-1}; that is, E_2 is not primitive. Consequently, ∂E_2 is uniquely determined and it represents a word of the form $xyxy^{p-1}$, and so it is primitive if and only if $p = 3$. Thus, only when $p = 3$, we have the unique primitive triple $\{E, E_1, E_2\}$ containing the pair $\{E, E_1\}$. □

Remark 3.5. For any primitive triple $\{D, E, F\}$ of V in $L(3, 1)$, by Theorem 3.3, there exist unique common dual disks D', E', and F' of the disks in the pairs $\{E, F\}$, $\{F, D\}$, and $\{D, E\}$ respectively. In fact, the disks D', E', and F' form a primitive triple of W. Furthermore, we have $|\partial D' \cap \partial D| = |\partial E' \cap \partial E| = |\partial F' \cap \partial F| = 2$. Figure 5 illustrates the triple $\{D', E', F, \}$ of W together with the boundary circles of D, E and F in $\partial W = \Sigma$.

4. The complex of primitive disks

Let M be an irreducible 3-manifold with compressible boundary. The *disk complex* of M is a simplicial complex defined as follows: The vertices of the disk complex are isotopy classes of essential disks in M, and a collection of $k + 1$ vertices spans a k-simplex if and only if it admits a collection of representative disks which are pairwise disjoint. In particular, if M is a handlebody of genus $g \geq 2$, then the disk complex is $(3g - 4)$-dimensional and is not locally finite. The following is a key property of a disk complex:

Theorem 4.1. *If $\mathcal{K}$ is a full subcomplex of the disk complex satisfying the following condition, then $\mathcal{K}$ is contractible*:

Let E and D be disks in M representing vertices of $\mathcal{K}$. If E and D intersect transversely and minimally, then at least one of the disks from surgery on E along an outermost subdisk of D cut up by $D \cap E$ represents a vertex of $\mathcal{K}$.

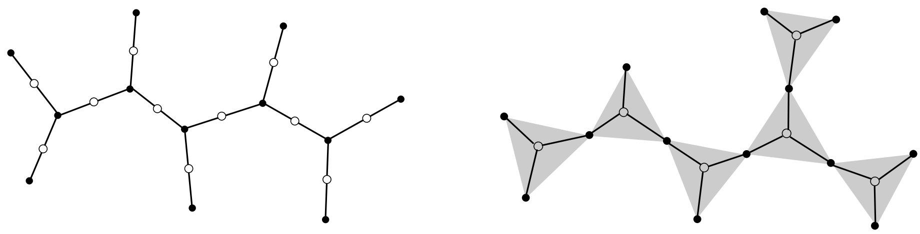

Figure 6. Small portions of primitive disk complexes $\mathcal{P}(V)$ for $p \neq 3$ (left) and $p = 3$ (right).

In [Cho 2008], the above theorem is proved in the case that M is a handlebody, but the proof is still valid for an arbitrary irreducible manifold with compressible boundary. From the theorem, we see that the disk complex itself is contractible.

Now consider the genus-two Heegaard splitting $(V, W; \Sigma)$ of a lens space $L(p, 1)$ with $p \geq 2$. We define the *primitive disk complex*, denoted by $\mathcal{P}(V)$, to be the full subcomplex of the disk complex spanned by the vertices of primitive disks in V. We already know that every primitive disk is a member of infinitely many primitive pairs, and so every vertex of $\mathcal{P}(V)$ has infinite valency. The following is our main theorem, a direct consequence of Theorems 3.2, 3.4 and 4.1:

Theorem 4.2. *Let* $(V, W; \Sigma)$ *be the genus-two Heegaard splitting of the lens space* $L = L(p, 1)$ *with* $p \geq 2$. *The primitive disk complex* $\mathcal{P}(V)$ *is contractible. In particular, if* $p \neq 3$ *it is a tree, and if* $p = 3$ *it is* 2*-dimensional and every edge is contained in a unique* 2*-simplex.*

Figure 6 illustrates portions of the primitive disk complexes. The black vertices are the vertices of $\mathcal{P}(V)$ while the white ones are the barycenters of edges when $p \neq 3$ and of 2-simplices when $p = 3$. Observe that the 2-dimensional $\mathcal{P}(V)$ deformation retracts to a tree in its barycentric subdivision, as in the figure.

5. Genus-two Goeritz groups of lens spaces $L(p, 1)$

In this section, we give explicit presentation of the genus-two Goeritz group $\mathcal{G}$ of each lens space $L(p, 1)$. From Theorem 4.2, if $p \neq 3$, the primitive disk complex $\mathcal{P}(V)$ is a tree, and if $p = 3$, then $\mathcal{P}(V)$ is 2-dimensional but deformation retracts to a tree. We simply denote by $\mathcal{T}$ the barycentric subdivision of the tree $\mathcal{P}(V)$ if $p \neq 3$ and the deformation retract of $\mathcal{P}(V)$ if $p = 3$. Each of the trees $\mathcal{T}$ is bipartite, as in Figure 6, with the black vertices of (countably) infinite valence, and the white vertices of valence 2 if $p \neq 3$ and of valence 3 if $p = 3$.

Each black vertex of $\mathcal{T}$ is represented by a primitive disk, while each white vertex is represented by a primitive pair if $p \neq 3$ and by a primitive triple if $p = 3$.

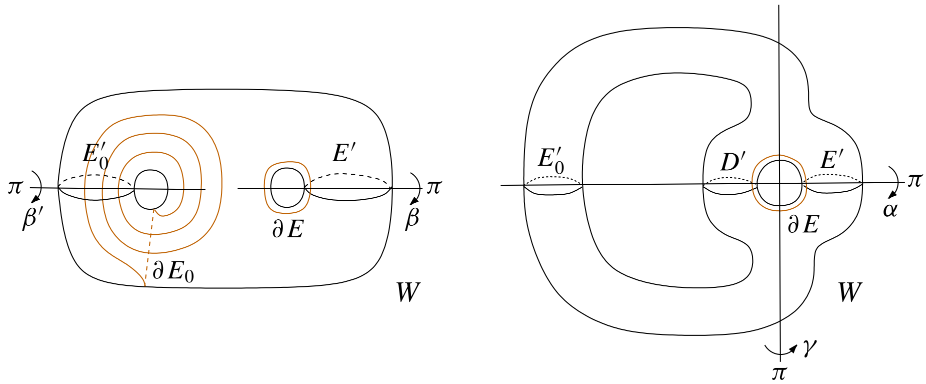

Figure 7. Generators of the stabilizer subgroup $\mathcal{G}_{\{E\}}$.

An element of the group $\mathcal{G}$ can be considered a simplicial automorphism of $\mathcal{T}$. The tree $\mathcal{T}$ is invariant under the action of $\mathcal{G}$ for each $L(p, 1)$. In particular, $\mathcal{G}$ acts transitively on the set of black vertices and on the set of white vertices, and hence the quotient of $\mathcal{T}$ by the action of $\mathcal{G}$ is a single edge of which one end vertex is black and another one white. Thus, by the theory of groups acting on trees due to Bass and Serre (see [Serre 1980]), the group $\mathcal{G}$ can be expressed as the free product of the stabilizer subgroups of two end vertices with the amalgamated stabilizer subgroup of the edge.

First, we find a presentation of the stabilizer subgroup of a black vertex of $\mathcal{T}$; that is, of (the isotopy class of) a primitive disk in V. For convenience, we will not distinguish disks (pairs and triples of disks) and homeomorphisms from their isotopy classes in their notations. Throughout the section, $\mathcal{G}_{\{A_1, A_2, \ldots, A_k\}}$ will denote the subgroup of $\mathcal{G}$ of elements preserving $A_1, A_2, \ldots, A_k$ setwise, where A_i will be (isotopy classes of) disks or unions of disks in V or in W.

Lemma 5.1. *Let E be a primitive disk in V. The stabilizer subgroup $\mathcal{G}_{\{E\}}$ of E has the presentation $\langle \alpha \mid \alpha^2 = 1 \rangle \oplus \langle \beta, \gamma \mid \gamma^2 = 1 \rangle$, where the generators α, β and γ are described in Figure 7.*

Proof. Let $\mathcal{P}'(W)$ be the full subcomplex of the primitive disk complex $\mathcal{P}(W)$ for W spanned by the vertices of dual disks of E. There is a unique semiprimitive disk E'_0 in W disjoint from ∂E, and it is easy to show that any dual disk of E is disjoint from E'_0. Thus $\mathcal{P}'(W)$ is 1-dimensional and further, by a similar argument used for $\mathcal{P}(V)$, we have that $\mathcal{P}'(W)$ is a tree whose vertices have infinite valence. That is, when two dual disks of E intersect each other, one of the two disks from the surgery construction is E'_0 and the other one is again a dual disk of E. Denote by $\mathcal{T}'$ the barycentric subdivision of $\mathcal{P}'(W)$. The tree $\mathcal{T}'$ is invariant under the action of the stabilizer subgroup $\mathcal{G}_{\{E\}}$, and the quotient of $\mathcal{T}'$ by the action is a single edge. One vertex of this edge corresponds to a dual disk E' of E, and the other one to a primitive pair $\{E', D'\}$ of dual disks of E. Thus $\mathcal{G}_{\{E\}}$ can be expressed as the free product of the stabilizer subgroups $\mathcal{G}_{\{E,E'\}} * \mathcal{G}_{\{E,E'\cup D'\}}$ amalgamated by $\mathcal{G}_{\{E,E',D'\}}$.

Consider the subgroup $\mathcal{G}_{\{E,E'\}}$ first. Any element of $\mathcal{G}_{\{E,E'\}}$ also preserves the disks E_0 and E'_0, which are unique meridian disks disjoint from $E \cup E'$ in V and in W respectively. Since V cut up by $E \cup E_0$ and W cut up by $E' \cup E'_0$ are all 3-balls, the group $\mathcal{G}_{\{E,E'\}}$ is identified with the group of isotopy classes of orientation-preserving homeomorphisms of $\Sigma = \partial V = \partial W$ which preserve ∂E, $\partial E'$, ∂E_0, and $\partial E'_0$. This group has a presentation $\langle \beta, \beta' \mid (\beta\beta')^2 = 1, \beta\beta' = \beta'\beta \rangle$, where the generators β and β' are π-rotations (half Dehn twists) described in Figure 7, left.

Next, consider the subgroup $\mathcal{G}_{\{E,E'\cup D'\}}$. Any element of this group preserves $E' \cup D'$ in W, and further it preserves E and $E_0 \cup D_0$ in V where E_0 and D_0 are unique meridian disks in V disjoint from $E \cup E'$ and $E \cup D'$ respectively. Thus $\mathcal{G}_{\{E,E'\cup D'\}}$ is generated by two elements α and γ, where α is the hyperelliptic involution, and γ is the element of order 2 exchanging E' and D' described in Figure 7, right. Thus $\mathcal{G}_{\{E,E'\cup D'\}}$ has the presentation $\langle \alpha \mid \alpha^2 = 1 \rangle \oplus \langle \gamma \mid \gamma^2 = 1 \rangle$. Similarly, $\mathcal{G}_{\{E,E',D'\}}$ has the presentation $\langle\, \alpha \mid \alpha^2 = 1 \rangle$. Observing that α satisfies $\beta\beta' = \alpha$, we have the desired presentation of $\mathcal{G}_{\{E\}}$. □

Thus the stabilizer subgroups of black vertices have the same presentation for each $p \geq 2$, but for white vertices, we have the following cases depending on p:

Lemma 5.2. *A white vertex of $\mathcal{T}$ corresponds to a primitive pair if $p \neq 3$ and to a primitive triple if $p = 3$.*

(1) *Let $\{D, E\}$ be a primitive pair of V in $L(p, 1)$. Then the stabilizer subgroup $\mathcal{G}_{\{D\cup E\}}$ has the presentation $\langle \rho, \gamma \mid \rho^4 = \gamma^2 = (\rho\gamma)^2 = 1 \rangle$ if $p = 2$, and $\langle \alpha \mid \alpha^2 = 1 \rangle \oplus \langle\, \sigma \mid \sigma^2 = 1 \rangle$ if $p \geq 3$, where the generators are described in Figures 8 and 9.*

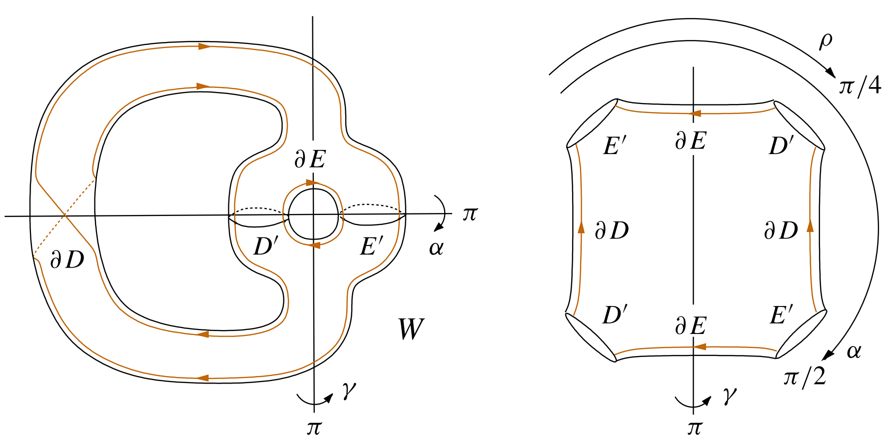

Figure 8. Generators of the stabilizer subgroup $\mathcal{G}_{\{D\cup E\}}$ for $L(2, 1)$.

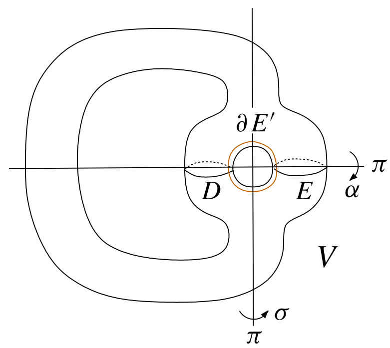

Figure 9. Generators of the stabilizer subgroup $\mathcal{G}_{\{D\cup E\}}$ for $L(p,1)$, with $p \geq 3$.

(2) *Let $\{D, E, F\}$ be a primitive triple of V in $L(3,1)$. The stabilizer subgroup $\mathcal{G}_{\{D\cup E\cup F\}}$ has the presentation $\langle \alpha \mid \alpha^2 = 1\rangle \oplus \langle \delta, \gamma \mid \delta^3 = \gamma^2 = (\gamma\delta)^2 = 1\rangle$, where the generators are described in Figure 10.*

Proof. (1) First, let $\{D, E\}$ be a primitive pair of V in $L(2,1)$. Then, by Theorem 3.3, there is a unique primitive pair $\{D', E'\}$ of W such that D' and E' are common dual disks of D and E. Any element of $\mathcal{G}_{\{D\cup E\}}$ preserves $D'\cup E'$, and hence $\mathcal{G}_{\{D\cup E\}}$ is identified with the stabilizer subgroup $\mathcal{G}_{\{D\cup E, D'\cup E'\}}$. Since $D\cup E$ and $D'\cup E'$ cut up V and W into 3-balls, the group $\mathcal{G}_{\{D\cup E, D'\cup E'\}}$ is identified with the group of isotopy classes of orientation-preserving homeomorphisms of $\Sigma = \partial V = \partial W$ which preserve $\partial D \cup \partial E$ and $\partial D' \cup \partial E'$. This is the dihedral group D_8 of order 8 with generators ρ and γ described in Figure 8. The 3-ball in Figure 8, right, is obtained by cutting up W along $D'\cup E'$. Figure 8 gives two descriptions of the

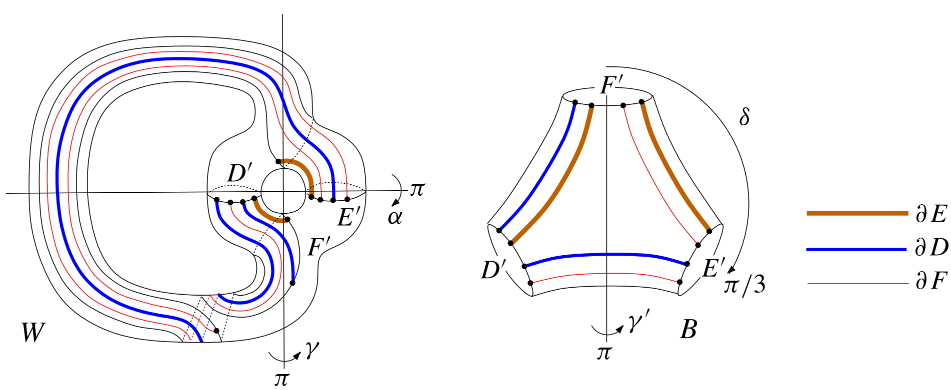

Figure 10. Left: The primitive triple $\{D', E', F'\}$ of W and the arcs $(\partial D\cup\partial E\cup\partial F)\cap\partial B$. Right: The 3-ball B.

elements α and γ. Thus we have the presentation $\langle \rho, \gamma \mid \rho^4 = \gamma^2 = (\rho\gamma)^2 = 1 \rangle$. We remark that the hyperelliptic involution α equals ρ^2.

Next, let $\{D, E\}$ be a primitive pair of V in $L(p, 1)$ with $p \geq 3$. There is a unique common dual disk E' of D and E by Theorem 3.3, and hence $\mathcal{G}_{\{D \cup E\}}$ is identified with the stabilizer subgroup $\mathcal{G}_{\{D \cup E, E'\}}$. As in the case of $\mathcal{G}_{\{E, E' \cup D'\}}$ in the proof of Lemma 5.1, this group is generated by two elements: One is the hyperelliptic involution α, and the other one is the element, denoted by σ, of order 2 exchanging D and E described in Figure 9. Thus we have the presentation $\langle \alpha \mid \alpha^2 = 1 \rangle \oplus \langle \sigma \mid \sigma^2 = 1 \rangle$.

(2) Let $\{D, E, F\}$ be a primitive triple of V in $L(3, 1)$. Then there exists a unique primitive triple $\{D', E', F'\}$ of W as described in Remark 3.5 and Figure 5. Thus the stabilizer subgroup $\mathcal{G}_{\{D \cup E \cup F\}}$ is identified with $\mathcal{G}_{\{D \cup E \cup F, D' \cup E' \cup F'\}}$. The union of three disks $D' \cup E' \cup F'$ cuts up W into two 3-balls. One of them, say B, is shown in Figure 10, right. Consider the group of isotopy classes of orientation-preserving homeomorphisms of B which preserve $D' \cup E' \cup F'$ and $(\partial D \cup \partial E \cup \partial F) \cap \partial B$ on the boundary. This group is the dihedral group $D_6 = \langle \delta, \gamma' \mid \delta^3 = \gamma'^2 = (\gamma'\delta)^2 = 1 \rangle$ of order 6 with generators δ and γ' in Figure 10, right. The element γ in Figure 10, left, is different from γ', since γ exchanges the two 3-balls. But they are related by $\gamma = \alpha\gamma'$, where α is the hyperelliptic involution exchanging the two 3-balls as described in Figure 10, left. Thus we see that the relation $(\gamma'\delta)^2 = 1$ in D_6 is equivalent to $(\gamma\delta)^2 = 1$. Since the elements α, γ and δ extend to elements of $\mathcal{G}_{\{D \cup E \cup F, D' \cup E' \cup F'\}}$, this group can be considered as the extension of D_6 by $\langle \alpha \mid \alpha^2 = 1 \rangle$ with relations $\alpha\gamma\alpha = \gamma$ and $\alpha\delta\alpha = \delta$. Thus we have the desired presentation of $\mathcal{G}_{\{D \cup E \cup F\}}$. □

Finally, the stabilizer subgroups of an edge are calculated in a similar way.

Lemma 5.3. *An edge of $\mathcal{T}$ corresponds to the pair of end vertices.*

(i) *Let $\{D, E\}$ be a primitive pair of V in $L(p, 1)$. Then $\mathcal{G}_{\{E, D \cup E\}} = \mathcal{G}_{\{E, D\}}$ has a presentation $\langle \alpha \mid \alpha^2 = 1 \rangle \oplus \langle \gamma \mid \gamma^2 = 1 \rangle$ if $p = 2$, and a presentation $\langle \alpha \mid \alpha^2 = 1 \rangle$ if $p \geq 3$.*

(ii) *Let $\{D, E, F\}$ be a primitive triple of V in $L(3, 1)$. Then $\mathcal{G}_{\{E, D \cup E \cup F\}} = \mathcal{G}_{\{E, D \cup F\}}$ has a presentation $\langle \alpha \mid \alpha^2 = 1 \rangle \oplus \langle \gamma \mid \gamma^2 = 1 \rangle$.*

Combining Lemmas 5.1, 5.2, and 5.3, we obtain the main result.

Theorem 5.4. *The genus-2 Goeritz group $\mathcal{G}$ of a lens space $L(p, 1)$ with $p \geq 2$ has the following presentations*:

(i) $\langle \beta, \rho, \gamma \mid \rho^4 = \gamma^2 = (\gamma\rho)^2 = \rho^2\beta\rho^2\beta^{-1} = 1 \rangle$ *if* $p = 2$.

(ii) $\langle \alpha \mid \alpha^2 = 1 \rangle \oplus \langle \beta, \delta, \gamma \mid \delta^3 = \gamma^2 = (\gamma\delta)^2 = 1 \rangle$ *if* $p = 3$.

(iii) $\langle \alpha \mid \alpha^2 = 1 \rangle \oplus \langle \beta, \gamma, \sigma \mid \gamma^2 = \sigma^2 = 1 \rangle$ *if* $p \geq 4$.

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