GENERALIZED EIGENVALUE PROBLEMS OF NONHOMOGENEOUS ELLIPTIC OPERATORS AND THEIR APPLICATION

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We consider the equation $-\text{div}(a(x, |\nabla u|) \nabla u) = \lambda |u|^{p-2}u$ (whose special case $a(x, t) = t^{p-2}$ is the $p$-Laplace equation) on a bounded domain $\Omega \subset \mathbb{R}^N$ with $C^2$ boundary, with null boundary condition. We prove that there are $\lambda \in \mathbb{R}$ for which the equation has a nontrivial solution. As an application, by variational methods, we present the existence of a positive solution to $-\text{div}(a(x, |\nabla u|) \nabla u) = f(x, u)$ in $\Omega$, where $f$ is asymptotically $(p-1)$-linear near zero and $\infty$, considering the nonresonant, resonant, and doubly resonant cases. We show that, generally, the spectrum of the operator $-\text{div}(a(x, |\nabla u|) \nabla u)$ on $W^{1,p}_0(\Omega)$ is not discrete.

1. Introduction

Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $C^2$ boundary $\partial \Omega$. We are interested in values of $\lambda \in \mathbb{R}$ such that a nontrivial solution exists to the equation

$$\begin{cases}
-\text{div} A(x, \nabla u) = \lambda |u|^{p-2}u & \text{in } \Omega, \\
 u = 0 & \text{on } \partial \Omega;
\end{cases}$$

such a $\lambda$ is called an eigenvalue for $A$. Here $A: \bar{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is a map that is strictly monotone in the second variable and satisfies the regularity conditions in Assumption A below.

The $p$-Laplace equation is the special case of $(\text{EV}; \lambda)$ with $A(x, y) = |y|^{p-2}y$, and in this case the eigenvalues for $A$ are the usual eigenvalues of the $p$-Laplacian. However, we do not suppose that $A$ is $(p-1)$-homogeneous in the second variable. Instead, these are the assumptions we make on the map $A$:

**Assumption A.** $A(x, y) = a(x, |y|)y$, where $a(x, t) > 0$ for all $x \in \bar{\Omega}$ and all $t \in (0, +\infty)$; furthermore:

(i) $A \in C^0(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N) \cap C^1(\overline{\Omega} \times (\mathbb{R}^N \setminus \{0\}), \mathbb{R}^N)$. 

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(ii) There exists $C_1 > 0$ such that
\[ |D_y A(x, y)| \leq C_1 |y|^{p-2} \quad \text{for every } x \in \bar{\Omega} \text{ and } y \in \mathbb{R}^N \setminus \{0\}.
\]

(iii) There exists $C_0 > 0$ such that
\[ D_y A(x, y) \xi \cdot \xi \geq C_0 |y|^{p-2} |\xi|^2 \quad \text{for every } x \in \bar{\Omega}, \ y \in \mathbb{R}^N \setminus \{0\} \text{ and } \xi \in \mathbb{R}^N;
\]

(iv) there exists $C_2 > 0$ such that
\[ |D_x A(x, y)| \leq C_2 (1 + |y|^{p-1}) \quad \text{for every } x \in \bar{\Omega} \text{ and } y \in \mathbb{R}^N \setminus \{0\}.
\]

(v) There exist $C_3 > 0$ and a positive $t_0 \leq 1$ such that
\[ |D_x A(x, y)| \leq C_3 |y|^{p-1} (-\log |y|)
\]
for every $x \in \bar{\Omega}$, $y \in \mathbb{R}^N$ with $0 < |y| < t_0$.

*From now on, we assume that $C_0 \leq p - 1 \leq C_1$ which leads to no loss of generality, as can be seen from Assumption A(ii)–(iii).*

A similar hypothesis to Assumption A is considered in the study of quasi-linear elliptic problems; see [Motreanu and Papageorgiou 2011, Example 2.2; Damascelli 1998; Motreanu et al. 2011; Miyajima et al. 2012; Tanaka 2012a]. We also refer to [García-Huidobro et al. 1995; Kim 2009; Kim and Kim 2010; Fukagai and Narukawa 2007; Prado and Ubilla 1998; Robinson 2004] for generalized $p$-Laplace operators. In particular, when $A(x, y) = |y|^{p-2} y$ — that is, when $\text{div} A(x, \nabla u)$ is the usual $p$-Laplacian $\Delta_p u$ — we can take $C_0 = C_1 = p - 1$ in Assumption A. Conversely, if $C_0 = C_1 = p - 1$ in Assumption A, the inequalities in Remark 1(ii)–(iii) below show that $a(x, t) = |t|^{p-2}$, whence $A(x, y) = |y|^{p-2} y$.

In the $p$-Laplace case, the first eigenvalue $\lambda_1$ is obtained by the Rayleigh quotient: $\lambda_1 = \inf\{ \int_{\Omega} |\nabla u|^p \ dx / \|u\|_p^p : u \neq 0 \}$. But since our operator is nonhomogeneous, $\inf\{ \lambda \in \mathbb{R} : \lambda \text{ is an eigenvalue of } A \}$ is in general not obtained by such a Rayleigh quotient corresponding to $A$. In Section 3, since the Rayleigh quotient plays an important role, we study its behavior as $\|u\|_p \to 0$ or $\|u\|_p \to \infty$ under an additional condition describing an asymptotic $(p-1)$-homogeneity. For example, we can consider
\[
\text{div } A(x, \nabla u) = \text{div} \left( (a_0(x)|\nabla u|^{p-2} + c_{\infty}(x)|\nabla u|^{q-2}) (1 + |\nabla u|^q)^{(p-q)/q} \nabla u \right)
\]
for $1 < p \leq q < \infty$, $a_0, c_{\infty} \in C^1(\bar{\Omega})$ with $\min_{\Omega} a_0 > 0$ and $\min_{\Omega} c_{\infty} > 0$. This satisfies
\[ A(x, y) - a_0(x) |y|^{p-2} y = o(|y|^{p-1}) \quad \text{as } |y| \to 0,
\]
\[ A(x, y) - c_{\infty}(x) |y|^{p-2} y = o(|y|^{p-1}) \quad \text{as } |y| \to \infty.
\]
Under these conditions (see (AH0) and (AH) in Section 3), we shall prove
that
\[
\min \left\{ \int_\Omega \int_0^{\left| \nabla u(x) \right|} a(x, t) \frac{t}{r^p} \, dt \, dx : \|u\|_p = r \right\}
\]
approaches \( \lambda_1(a_0)/p \) as \( r \to +0 \) and \( \lambda_1(a_\infty)/p \) as \( r \to +\infty \); here
\[
\lambda_1(a_0) = \min \left\{ \int_\Omega a_0(x) |\nabla u|^p \, dx : \|u\|_p = 1 \right\},
\]
\[
\lambda_1(a_\infty) = \min \left\{ \int_\Omega a_\infty(x) |\nabla u|^p \, dx : \|u\|_p = 1 \right\}.
\]

Concerning the eigenvalue problem for a nonhomogeneous operator, we can refer to [Robinson 2004; Tanaka 2012b] under the Neumann boundary condition.

In Section 4, as an application of Section 3, we present the existence of a positive solution for the quasilinear elliptic equation
\[
(P) \quad \begin{cases} -\text{div} A(x, \nabla u) = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases}
\]
where \( f \) satisfies the following assumption.

**Assumption (f).** \( f \) is a Carathéodory function on \( \Omega \times \mathbb{R} \) with \( f(x, 0) = 0 \) for a.e. \( x \in \Omega \), \( f \) is bounded on bounded sets and \( f \) is asymptotically \((p-1)\)-linear near \(+0\) and \(+\infty\) in the following sense:

(i) \( \lim_{u \to +0} \frac{f(x, u)}{u^{p-1}} = \alpha_0 \) uniformly in a.e. \( x \in \Omega \),

(ii) \( \lim_{u \to +\infty} \frac{f(x, u)}{u^{p-1}} = \alpha \) uniformly in a.e. \( x \in \Omega \),

for some constants \( \alpha_0 \) and \( \alpha \).

Regarding the existence of a positive solution under the Dirichlet boundary condition, we can refer to [Fukagai and Narukawa 2007; Prado and Ubilla 1998] for nonhomogeneous operators. However, we can not apply these results to our nonlinear term which is only asymptotically \((p-1)\)-linear near \(+0\) and \(+\infty\), and furthermore with possibly different weights. In [García-Huidobro et al. 1995], it is proved the existence of a positive radial solution for nonhomogeneous operators.

For the \( p \)-Laplace equation, it is well known that if \( (\alpha - \lambda_1)(\alpha_0 - \lambda_1) < 0 \) (where \( \lambda_1 \) denotes the first eigenvalue of \(-\Delta_p\) under a Dirichlet boundary condition),
\[
-\Delta_p u = f(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,
\]
has a positive solution (see [Dancer and Perera 2001]). One of our main purposes is to extend this existence result from the \( p \)-Laplace equation to the corresponding problem involving our nonhomogeneous operator \( A \). This is done in Theorem 25. We mention that in the special case of \( A(x, y) = A(y) \), the result in [Kyritsi
et al. 2010] provides the existence of a positive solution if \( \alpha < \lambda_1 C_0 / (p - 1) \) and \( \lambda_1 C_1 / (p - 1) < \alpha_0 \) hold (note that we can apply this result only to the case where \( \alpha < \alpha_0 \)). We emphasize that, for our general operator, the case \( \lambda_1(a_0) \neq \lambda_1(a_1) \) can occur. Note that in such a situation, contrary to the \( p \)-Laplacian case, we can still apply our theorem when \( \alpha_0 = \alpha \) provided this number is between \( \lambda_1(a_0) \) and \( \lambda_1(a_1) \). The known result for the \( p \)-Laplacian case is obtained from our theorem simply by setting \( a_0 \equiv 1 \) and \( a_\infty \equiv 1 \).

In particular, our theorem implies that if \( \lambda_1(a_0) \neq \lambda_1(a_\infty) \), then every \( \lambda \) between \( \lambda_1(a_0) \) and \( \lambda_1(a_\infty) \) is an eigenvalue of \( A \) (see Corollary 26) and has a positive eigenfunction. This shows that, generally, the spectrum of the operator \(-\text{div} A(x, \nabla \cdot)\) on \( W^{1,p}_0(\Omega) \) is not discrete.

In the final part of the paper, we treat the one side resonant and doubly resonant cases under additional conditions on \( f \). For the \( p \)-Laplace equation, we refer to [Tanaka 2009] for the resonant and doubly resonant cases. Our Theorem 31 provides the existence of a positive solution in all cases of resonance for problem (P) with a nonhomogeneous operator in the principal part.

2. The properties of the map \( A \)

In what follows, the norm on \( W^{1,p}_0(\Omega) \) is given by

\[
\|u\|^p := \|\nabla u\|^p_p,
\]

where \( \|u\|_q \) denotes the usual norm of \( L^q(\Omega) \) for \( u \in L^q(\Omega) \) \((1 \leq q \leq \infty)\). Setting

\[
G(x, y) := \int_0^{|y|} a(x, t)t \, dt,
\]

we can easily see that

\[
\nabla_y G(x, y) = A(x, y) \quad \text{and} \quad G(x, 0) = 0
\]

for every \( x \in \Omega \); see [Motreanu et al. 2011] for details.

Remark 1. The following assertions hold under Assumption A:

(i) For all \( x \in \Omega \), \( A(x, y) \) is maximal monotone and strictly monotone in \( y \).

(ii) \( |A(x, y)| \leq \frac{C_1}{p - 1} |y|^{p-1} \) for every \( (x, y) \in \Omega \times \mathbb{R}^N \).

(iii) \( A(x, y)y \geq \frac{C_0}{p - 1} |y|^p \) for every \( (x, y) \in \Omega \times \mathbb{R}^N \).

(iv) \( G(x, y) \) is strictly convex in \( y \) for all \( x \) and satisfies the inequalities

\[
|A(x, y)|y \geq G(x, y) \geq \frac{C_0}{p(p - 1)} |y|^p \quad \text{and} \quad G(x, y) \leq \frac{C_1}{p(p - 1)} |y|^p
\]

for every \( (x, y) \in \Omega \times \mathbb{R}^N \).
The following result is important for the proof of the Palais–Smale condition for the functionals related to our problem.

**Proposition 2** [Motreanu et al. 2011, Proposition 1]. Let \( V : W^{1,p}_0(\Omega) \to W^{1,p}_0(\Omega)^* \) be the map defined by

\[
\langle V(u), v \rangle = \int_{\Omega} A(x, \nabla u) \nabla v \, dx
\]

for \( u, v \in W^{1,p}_0(\Omega) \). Then any sequence \( \{u_m\} \) that converges weakly to \( u \) and satisfies \( \limsup_{m \to \infty} \langle V(u_m), u_m - u \rangle \leq 0 \) also converges strongly to \( u \).

**Remark 3.** (i) If \( u \in W^{1,p}_0(\Omega) \) is a solution of (P), then \( u \in C^{1,\alpha}(\overline{\Omega}) \) for some \( 0 < \alpha < 1 \).

(ii) If \( u \in W^{1,p}_0(\Omega) \) is a nontrivial solution of (P) such that \( u \geq 0 \), then \( u > 0 \) in \( \Omega \) and \( \partial u / \partial \nu < 0 \) on \( \partial \Omega \), where \( \nu \) denotes the outward unit normal vector on \( \partial \Omega \).

**Sketch of proof.** (i) Let \( u \in W^{1,p}_0(\Omega) \) be a solution of (P). Then, because \( u \in L^\infty(\Omega) \) as shown by using the Moser iteration process (cf. [Miyajima et al. 2012, Appendix]), we see that \( u \in C^{1,\alpha}(\overline{\Omega}) \) \( (0 < \alpha < 1) \) by the regularity result in [Lieberman 1988].

(ii) Let \( u \in W^{1,p}_0(\Omega) \) be a solution of (P) satisfying \( u \geq 0 \) and \( u \not\equiv 0 \). Then, by Assumption (f), we obtain a constant \( \lambda > 0 \) satisfying

\[
-\text{div} A(x, \nabla u) + \lambda u^{p-1} \geq 0 \quad \text{in} \ \Omega.
\]

Noting that \( u \in C^{1,\alpha}(\overline{\Omega}) \) \( (0 < \alpha < 1) \) by (i), we have \( u(x) > 0 \) for every \( x \in \Omega \) by [Miyajima et al. 2012, Appendix, Theorem B]. In addition, using the strong maximum principle [ibid., Appendix, Theorem A], we easily see that \( \partial u(x) / \partial \nu < 0 \) for every \( x \in \partial \Omega \).

**Proposition 4.** Let \( f_n : \Omega \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function satisfying

\[
|f_n(x, t)| \leq D(1 + |t|^{r-1}) \quad \text{for every} \ x \in \Omega, \ t \in \mathbb{R}
\]

with some positive constant \( D \) independent of \( n \) and \( r \in [p, p^*) \), where \( p^* = \infty \) if \( N \leq p \) and \( p^* = pN/(N - p) \) if \( N > p \). Assume that \( A_n : \Omega \times \mathbb{R}^N \to \mathbb{R}^N \) is a map satisfying parts (i)–(iv) of Assumption A with positive constants \( C_1', C_0', C_2' \) independent of \( n \). If \( u_n \) is a solution for

\[
-\text{div} A_n(x, \nabla u) = f_n(x, u) \quad \text{in} \ \Omega, \ u = 0 \quad \text{on} \ \partial \Omega
\]

and \( \{u_n\} \) is bounded in \( W^{1,p}_0(\Omega) \), then there exist a subsequence \( \{u_{n_l}\} \) of \( \{u_n\} \) and \( u_0 \in C_0^1(\overline{\Omega}) \) such that \( u_{n_l} \to u_0 \) in \( C_0^1(\overline{\Omega}) \) as \( l \to \infty \).

**Proof.** Since \( \{u_n\} \) is bounded in \( W^{1,p}_0(\Omega) \), we may assume that \( u_n \) converges weakly to some \( u_0 \) in \( W^{1,p}_0(\Omega) \) by choosing a subsequence. We can show that there exists a \( C > 0 \) depending only on \( |\Omega|, p, N, D, C_0', C_1', \) and the embedding constant of
We introduce a function $J : W^{1,p}_0(\Omega) \to \mathbb{R}$ by
\[
J(u) = \int_{\Omega} G(x, \nabla u) \, dx \quad \text{for all } u \in W^{1,p}_0(\Omega).
\]

It is clear that $J$ is of class $C^1$. We also note that
\[
rS := \{ u \in W^{1,p}_0(\Omega) : \|u\|_p = r \} \quad \text{for } r > 0
\]
is a $C^1$ Finsler manifold (cf. [Deimling 1985, Sections 27.4 and 27.5]) because $r$ is a regular value of the function $u \mapsto \|u\|_p$ on $W^{1,p}_0(\Omega)$. Hence the norm of the derivative at $u \in (rS)$ of the restriction $\tilde{J}$ of $J$ to $rS$ is defined by
\[
\|\tilde{J}'(u)\|_* := \min \{ \|J'(u) - r\Phi'(u)\|_{W^{1,p}_0(\Omega)*} : t \in \mathbb{R} \}
= \sup \{ \langle J'(u), v \rangle : v \in T_u(rS), \|v\| = 1 \},
\]
where $\Phi(u) := (1/p)\|u\|_p^p$ and $T_u(rS)$ denotes the tangent space of $rS$ at $u$, that is, $T_u(rS) = \{ v \in W^{1,p}_0(\Omega) : \int_{\Omega} |u|^p - 2uv \, dx = 0 \}$. It follows that the restriction $\tilde{J} = J|_{rS}$ is a $C^1$-function on $rS$ in the sense of manifolds.

**Proposition 5.** For $r > 0$, the infimum
\[
\mu_1(A, r) = \inf_{u \in (rS)} \int_{\Omega} G(x, \nabla u) \, dx
\]
is attained at points $\pm \hat{u}_r \in (rS)$ with $\hat{u}_r \in C^{1,\alpha}(\overline{\Omega})$ and $\hat{u}_r > 0$ in $\Omega$. Moreover, $\pm \hat{u}_r$ are solutions of (EV; $\lambda$) with $\lambda = \lambda_1(A, \hat{u}_r)/r^p$, where
\[
\lambda_1(A, \hat{u}_r) = \int_{\Omega} A(x, \nabla \hat{u}_r) \nabla \hat{u}_r \, dx \geq \frac{C_0}{p-1} \lambda_1 r^p.
\]

**Proof.** Let $\{u_n\} \subset (rS)$ be a minimizing sequence for (5). Using (2), it follows that $\{u_n\}$ is bounded in $W^{1,p}_0(\Omega)$, so along a relabeled subsequence we have $u_n \rightharpoonup u$ in $W^{1,p}_0(\Omega)$ and $u_n \to u$ in $L^p(\Omega)$ for some $u \in W^{1,p}_0(\Omega)$, thus $u \in (rS)$. Since
$G(x, \cdot)$ is convex and continuous for all $x \in \Omega$, $J$ is weakly lower semicontinuous on $W^{1,p}_0(\Omega)$ [Mawhin and Willem 1989, Theorem 1.2]. Therefore, we derive that

$$\mu_1(A, r) \leq \int_\Omega G(x, \nabla u) \, dx \leq \liminf_{n \to \infty} \int_\Omega G(x, \nabla u_n) \, dx,$$

which yields

$$\mu_1(A, r) = \int_\Omega G(x, \nabla u) \, dx.$$

The fact that the functional $J$ is even implies that $|u|$ is also a global minimizer of $\tilde{J}_r$. Consequently, we may assume that $u \geq 0$. On the other hand, the Lagrange multiplier rule leads to the existence of $t \in \mathbb{R}$ such that

$$J = \int_\Omega A(x, \nabla u) \nabla v \, dx = t \int_\Omega u^{p-1} v \, dx \quad \text{for all } v \in W^{1,p}_0(\Omega).$$

Inserting $v = u$ in (7) entails

$$\text{tr}^p = \int_\Omega A(x, \nabla u) \nabla u \, dx \geq \frac{C_0}{p-1} \|\nabla u\|_p^p \geq \frac{C_0 \lambda_1}{p-1} \|u\|_p^p = \frac{C_0 \lambda_1}{p-1} r^p.$$

Therefore, we have

$$t = \frac{\lambda_1(A, u)}{r^p} \geq \frac{C_0 \lambda_1}{p-1}.$$

From (7), it follows that $u$ is a solution of $(\text{EV}; \lambda)$ with $\lambda = t = \lambda_1(A, u)/r^p$. According to Remark 3 with $f(x, u) = t|u|^{p-2}u$, it follows that $u \in C^{1,\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$ and $u > 0$ in $\Omega$. Since $J$ is even and $\lambda_1(A, u) = \lambda_1(A, -u)$, we have that $J(-u) = J(u) = \mu_1(A, r)$ and $-u$ is a negative solution of $(\text{EV}; \lambda)$ with $\lambda = t = \lambda_1(A, u)/r^p$. The result is thus established with $\hat{u}_r = u$.}

We define

$$K_1(A, r) := \{ u \in (rS) : J(u) = \mu_1(A, r) \}.$$

Then it follows from Proposition 5 that $K_1(A, r)$ is not empty for each $r > 0$.

Because we do not know whether the minimizers of $\tilde{J}_r$ are only $\pm \hat{u}_r$, we introduce the following:

$$\lambda_1(A, r) := \inf \left\{ \int_\Omega A(x, \nabla u) \nabla u \, dx : u \in K_1(A, r) \right\},$$

$$\tilde{\lambda}_1(A, r) := \sup \left\{ \int_\Omega A(x, \nabla u) \nabla u \, dx : u \in K_1(A, r) \right\}.$$

**Lemma 6.** For every $r > 0$, $\lambda_1(A, r)$ and $\tilde{\lambda}_1(A, r)$ are attained.

**Proof.** We only deal with $\lambda_1(A, r)$ because $\tilde{\lambda}_1(A, r)$ can be treated similarly. Fix any $r > 0$. Let $u_n \in K_1(A, r)$ satisfy $\lambda_1(A, u_n) \to \lambda_1(A, r)$ as $n \to \infty$. Then we
see that \( \| \nabla u_n \|_p \) is bounded from the inequality
\[
\frac{C_0}{p(p-1)} \| \nabla u_n \|_p^p \leq \int_{\Omega} G(x, \nabla u_n) \, dx = \mu_1(A, r) \leq \int_{\Omega} G(x, \nabla w) \, dx
\]
for \( w \in rS \), where we use the definition of \( \mu_1(A, r) \) and (2). Recall that each \( u_n \) is a solution of (EV; \( \lambda \)) with \( \lambda = \lambda_1(A, u_n)/r^p \). Moreover, we have
\[
\frac{C_0}{p-1} \lambda_1 r^p \leq \lambda_1(A, u_n) \leq \frac{C_1}{p-1} \| \nabla u_n \|_p^p
\]
by Remark 1(ii) (see (6) for the first inequality), whence \( \lambda_1(A, u_n)/r^p \) is bounded. As a result, due to Proposition 4, we may assume that there exists \( u_0 \in W_0^{1,p}(\Omega) \) such that \( u_n \to u_0 \) in \( C_0^1(\overline{\Omega}) \) by choosing a subsequence if necessary. Since \( J \) and \( \lambda_1(A, \cdot) \) are continuous in \( W_0^{1,p}(\Omega) \), we see that \( J(u_0) = \lim_{n \to \infty} J(u_n) = \mu_1(A, r) \), \( u_0 \in K_1(A, r) \), and \( \lambda_1(A, u_0) = \lim_{n \to \infty} \lambda_1(A, u_n) = \lambda_1(A, r) \). Thus, our conclusion holds.

Define
\[
\lambda_1(A) := \inf_{u \neq 0} \int_{\Omega} \frac{A(x, \nabla u) \nabla u}{\| u \|_p^p} \, dx \quad \text{and} \quad \mu_1(A) := \inf_{u \neq 0} \int_{\Omega} \frac{G(x, \nabla u)}{\| u \|_p^p} \, dx.
\]

**Lemma 7.**
\[
\frac{C_0}{p-1} \lambda_1 \leq \lambda_1(A) \leq \min \left\{ \inf_{r > 0} \frac{\lambda_1(A, r)}{r^p}, \frac{C_1}{p-1} \lambda_1 \right\} \quad \text{and} \quad \mu_1(A) = \inf_{r > 0} \frac{\mu_1(A, r)}{r^p}.
\]

**Proof.** First, we consider \( \lambda_1(A) \). For every \( 0 \neq u \in W_0^{1,p}(\Omega) \), we have
\[
(9) \quad \frac{C_0}{p-1} \| \nabla u \|_p^p \leq \int_{\Omega} \frac{A(x, \nabla u) \nabla u}{\| u \|_p^p} \, dx \leq \frac{C_1}{p-1} \| \nabla u \|_p^p
\]
by Remark 1(ii)–(iii). Thus, \( (C_0/(p-1)) \lambda_1 \leq \lambda_1(A) \leq (C_1/(p-1)) \lambda_1 \) by taking the infimum with respect to \( u \).

Here we fix any \( \varepsilon > 0 \). Then there exists an \( r_\varepsilon > 0 \) such that \( \lambda_1(A, r_\varepsilon)/r_\varepsilon^p \leq \inf_{r > 0} (\lambda_1(A, r)/r^p) + \varepsilon \). By Lemma 6, we can choose \( u_\varepsilon \in (r_\varepsilon S) \) such that \( \lambda_1(A, u_\varepsilon) = \lambda_1(A, r_\varepsilon) \), that is, \( \int_{\Omega} A(x, \nabla u_\varepsilon) \nabla u_\varepsilon \, dx = \lambda_1(A, r_\varepsilon) \). By the definition of \( \lambda_1(A) \), we obtain
\[
\lambda_1(A) \leq \int_{\Omega} \frac{A(x, \nabla u_\varepsilon) \nabla u_\varepsilon}{\| u_\varepsilon \|_p^p} \, dx = \frac{\lambda_1(A, r_\varepsilon)}{r_\varepsilon^p} \leq \inf_{r > 0} \frac{\lambda_1(A, r)}{r^p} + \varepsilon.
\]
Because \( \varepsilon > 0 \) is arbitrary, we have \( \lambda_1(A) \leq \inf_{r > 0} (\lambda_1(A, r)/r^p) \).

Next we treat \( \mu_1(A) \). Fix any \( \varepsilon > 0 \). Then there exists an \( r_\varepsilon > 0 \) such that \( \mu_1(A, r_\varepsilon)/r_\varepsilon^p \leq \inf_{r > 0} (\mu_1(A, r)/r^p) + \varepsilon \). On the other hand, because \( \mu_1(A, r_\varepsilon) \) is
attained at some \( u_\varepsilon \in (r_\varepsilon S) \), we have

\[
\inf_{u \neq 0} \int_\Omega \frac{G(x, \nabla u)}{\|u\|_p} \, dx \leq \int_\Omega \frac{G(x, \nabla u_\varepsilon)}{\|u_\varepsilon\|_p} \, dx = \frac{\mu_1(A, r_\varepsilon)}{r_\varepsilon^p} \leq \inf_{r > 0} \frac{\mu_1(A, r)}{r^p} + \varepsilon.
\]

Because \( \varepsilon > 0 \) is arbitrary, this yields that \( \mu_1(A) \leq \inf_{r > 0} (\mu_1(A, r)/r^p) \).

For any \( \varepsilon > 0 \), we take \( v_\varepsilon \neq 0 \) such that \( \int_\Omega (G(x, \nabla v_\varepsilon)/\|v_\varepsilon\|_p) \, dx \leq \mu_1(A) + \varepsilon \). Then \( r_\varepsilon := \|v_\varepsilon\|_p > 0 \) and so

\[
\frac{\mu_1(A, r_\varepsilon)}{r_\varepsilon^p} \leq \int_\Omega \frac{G(x, \nabla v_\varepsilon)}{\|v_\varepsilon\|_p} \, dx \leq \mu_1(A) + \varepsilon.
\]

This leads to \( \mu_1(A) \geq \inf_{r > 0} (\mu_1(A, r)/r^p) \). \( \square \)

**Proposition 8.** If \( \lambda < \lambda_1(A) \), \((\text{EV}; \lambda)\) has no nontrivial solutions.

**Proof.** Let \( u \) be a nontrivial solution of \((\text{EV}; \lambda)\) with \( \lambda < \lambda_1(A) \). Then we have

\[
\lambda_1(A) \leq \int_\Omega \frac{A(x, \nabla u) \nabla u}{\|u\|_p^p} \, dx = \lambda
\]

by the definition of \( \lambda_1(A) \). This is a contradiction. \( \square \)

Set

\[
A_p := \frac{C_1}{p-1} \left( \frac{C_1}{C_0} \right)^{p-1} \geq 1,
\]

which is equal to 1 exactly in the case of \( A(x, y) = |y|^{p-2}y \) (that is, the special case of the \( p \)-Laplacian ) because we can choose \( C_0 = C_1 = p - 1 \).

**Lemma 9** [Tanaka 2012a, Lemma 16]. Let \( \varepsilon > 0 \). For every

\[
u, \varphi \in W^{1,p}(\Omega) \cap C^1(\Omega) \cap L^\infty(\Omega)
\]

with \( u \geq 0 \) and \( \varphi \geq 0 \) in \( \Omega \), we have

\[
\int_\Omega A(x, \nabla u) \nabla \left( \frac{\varphi^p}{(u + \varepsilon)^{p-1}} \right) \, dx \leq A_p \|\nabla \varphi\|_p^p.
\]

**Proposition 10.** Any nontrivial solution of \((\text{EV}; \lambda)\) with \( \lambda > A_p \lambda_1 \) changes sign.

**Proof.** By way of contradiction, assume there is a solution \( u \) that does not change sign. Then we may suppose that \( u \geq 0 \) because \( A \) is odd. Due to the strong maximum principle and the regularity theorem (see Remark 3), it follows that \( u \in C^1_0(\overline{\Omega}) \) and \( u > 0 \) in \( \Omega \). Let \( \varphi_1 \) be the positive eigenfunction of \( -\Delta_p \) corresponding to \( \lambda_1 \) such that \( \|\varphi_1\|_p = 1 \). According to Lemma 9, we obtain

\[
A_p \lambda_1 = A_p \|\nabla \varphi_1\|_p^p \geq \int_\Omega A(x, \nabla u) \nabla \left( \frac{\varphi_1^p}{(u + \varepsilon)^{p-1}} \right) \, dx = \lambda \int_\Omega \left( \frac{u}{u + \varepsilon} \right)^{p-1} \varphi_1^p \, dx
\]

for every \( \varepsilon > 0 \). By taking \( \varepsilon \downarrow 0 \), we have \( \lambda \leq A_p \lambda_1 \). This is a contradiction. \( \square \)
Proposition 11. Assume \( A_p \lambda_1 < C_0 \lambda_2/(p - 1) \), where \( \lambda_2 > \lambda_1 \) is the second eigenvalue of \(-\Delta_p\). If \( A_p \lambda_1 < \lambda < C_0 \lambda_2/(p - 1) \), (EV; \( \lambda \)) has no nontrivial solutions.

Proof. By way of contradiction, we assume that (EV; \( \lambda \)) has a nontrivial solution \( u \). Then it follows from Proposition 10 that \( u \) changes sign. Moreover, by taking \( u_\pm \) as a test function in (EV; \( \lambda \)), we have

\[
\frac{C_0}{p - 1} \|\nabla u_\pm\|^p_p \leq \int_\Omega A(x, \nabla u)(\pm \nabla u_\pm) \, dx = \lambda \|u_\pm\|^p_p,
\]

whence

\[
(11) \quad \|\nabla u_\pm\|^p_p < \lambda_2 \|u_\pm\|^p_p.
\]

This inequality guarantees the existence of a continuous path \( \gamma_0 \) on \( S \) such that \( \gamma_0(0) = \varphi_1, \gamma_0(1) = -\varphi_1 \) and \( \max_{r \in [0,1]} \|\nabla \gamma_0(r)\|^p_p < \lambda_2 \): refer to [Cuesta et al. 1999, Lemma 5.3]. This contradicts the equality

\[
\lambda_2 = \inf_{r \in [0,1]} \max_{\gamma \in \Sigma} \Phi(\gamma(t)),
\]

where \( \Phi(u) := \|\nabla u\|^p_p \) and \( \Sigma := \{\gamma \in C([0,1], S) : \gamma(0) = \varphi_1, \gamma(1) = -\varphi_1\} \); see [Anane 1987; Cuesta et al. 1999]. This contradiction proves our result.

For the reader’s convenience, we give the sketch of the construction of a path \( \gamma_0 \) as required above. Define paths as follows:

\[
\gamma_1(t) := \frac{tu + (1 - t)u_+}{\|tu + (1 - t)u_+\|^p_p} = \frac{u_+ - tu_-}{\|u_+ - tu_-\|^p_p}, \quad \gamma_2(t) := \frac{tu_+ + (1 - t)u_-}{\|tu_+ + (1 - t)u_-\|^p_p},
\]

\[
\gamma_3(t) := \frac{(1 - t)u - tu_-}{\|(1 - t)u - tu_-\|^p_p} = \frac{(1 - t)u_+ - u_-}{\|(1 - t)u_+ - u_-\|^p_p}
\]

for \( t \in [0,1] \). Then, setting \( \tilde{\Phi} := \Phi|_S \), we obtain by (11)

\[
\max_{t \in [0,1]} \tilde{\Phi}(\gamma_i(t)) < \lambda_2, \quad \text{for } i = 1, 2, 3.
\]

We recall that any component of \( \mathcal{C}(r) := \{u \in S : \tilde{\Phi}(u) < r\} \) contains at least one critical point of \( \tilde{\Phi} \), where \( r > 0 \) [Cuesta et al. 1999, Lemma 3.6]. Note that \( \mathcal{C}(\lambda_2) \) contains just two critical points \( \varphi_1 \) and \( -\varphi_1 \) because a critical value \( c \) of \( \tilde{\Phi} \) corresponds to the eigenvalue \( c \) of the negative \( p \)-Laplacian. Since any component of \( \mathcal{C}(\lambda_2) \) is path connected [ibid., Lemma 3.5], there exists a path \( \gamma_4 \) joining from \( u_-/\|u_-\|^p_p \) to \( \varphi_1 \) or \( -\varphi_1 \) in \( \mathcal{C}(\lambda_2) \). Thus, by noting that \( \Phi \) is even, we can construct a path \( \gamma_0 \in \Sigma \) such that \( \max_{r \in [0,1]} \Phi(\gamma_0(r)) < \lambda_2 \) by considering \( \gamma_1^{-1} \cdot \gamma_2 \cdot \gamma_1 \cdot \gamma_3 \cdot (-\gamma_4) \) or its inverse, where \( \gamma_i^{-1}(t) := \gamma_i(1 - t) \) and \( \gamma_i \cdot \gamma_j \) denotes the path defined by \( \gamma_i(2t) \) if \( 0 \leq t \leq \frac{1}{2} \) and \( \gamma_i(2t - 1) \) if \( \frac{1}{2} < t \leq 1 \).
3.1. Asymptotically homogeneous case near zero. We now consider the case where $A$ is asymptotically $(p-1)$-homogeneous near zero in the following sense.

(AH0) There exist a positive function $a_0 \in C^1(\overline{\Omega}, \mathbb{R})$ and a continuous function $\tilde{a}_0(x, t)$ on $\overline{\Omega} \times [0, +\infty)$ such that

$$A(x, y) = a_0(x)|y|^{p-2}y + \tilde{a}_0(x, |y|)y$$

for every $x \in \Omega$, $y \in \mathbb{R}^N$, where

$$\lim_{t \to +0} \frac{\tilde{a}_0(x, t)}{t^{p-2}} = 0 \quad \text{uniformly in } x \in \overline{\Omega}.$$

For this weight function $a_0$, we define

$$\lambda_1(a_0) := \inf \left\{ \int_{\Omega} a_0(x)|\nabla u|^p \, dx : \|u\|_p = 1 \right\}.$$

Because $0 < \min_{x \in \overline{\Omega}} a_0(x) \leq \max_{x \in \overline{\Omega}} a_0(x) < \infty$, by the same argument as the one for the first eigenvalue of the negative $p$-Laplacian, we can prove that $\lambda_1(a_0)$ is the first eigenvalue of

$$-\text{div}(a_0(x)|\nabla u|^{p-2}\nabla u) = \lambda|u|^{p-2}u \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial \Omega.$$

Moreover, $\lambda_1(a_0)$ has a positive eigenfunction $\varphi_{a_0} \in C^1(\overline{\Omega})$ and it is simple. It is proved that (13) has no constant sign solutions other than $0$ provided $\lambda \neq \lambda_1(a_0)$.

**Theorem 12.** Assume (AH0). For every $\varepsilon > 0$ there exists $r_0 > 0$ such that equation (EV; $\lambda$) has no nontrivial solutions in $B_p(r_0) := \{v \in W_0^{1,p}(\Omega) : \|v\|_p < r_0\}$ provided $\lambda < \lambda_1(a_0) - \varepsilon$.

**Proof.** We argue by contradiction. Thus we assume that there exist $\varepsilon_0 > 0$, $\{\lambda_n\}$ and $\{u_n\}$ such that $\lambda_n < \lambda_1(a_0) - \varepsilon_0$, $u_n \in B_p(1/n)$ and $u_n$ is a nontrivial solution of (EV; $\lambda_n$). By taking $u_n$ as a test function in (EV; $\lambda_n$), we have

$$\frac{C_0}{p-1}\|\nabla u_n\|_p^p \leq \int_{\Omega} A(x, \nabla u_n) \nabla u_n \, dx = \lambda_n\|u_n\|_p^p \leq (\lambda_1(a_0) - \varepsilon_0)/n^p \to 0$$

as $n \to \infty$. Therefore, $u_n \to 0$ in $W_0^{1,p}(\Omega)$. In addition, by noting that $u_n$ is a nontrivial solution of (EV; $\lambda_n$) and $0 \leq \lambda_n < \lambda_1(a_0) - \varepsilon_0$, Proposition 4 yields that $u_n$ converges to $0$ in $C^1(\overline{\Omega})$.

Set $v_n := u_n/\|u_n\|_p$. Then it follows from (14) and the boundedness of $\{\lambda_n\}$ that $\{v_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Hence, by choosing a subsequence, we may assume that $v_n$ converges to some $v_0$ weakly in $W_0^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$. Again by taking $u_n/\|u_n\|_p$ as a test function in (EV; $\lambda_n$), we obtain
\[ \lambda_1(a_0) - \varepsilon_0 > \lambda_n = \int_{\Omega} a_0(x) \frac{|\nabla u_n|^p}{\|u_n\|_p^p} \, dx + \int_{\Omega} \frac{\tilde{a}_0(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^p} \, dx \]
\[ = \int_{\Omega} a_0(x) |\nabla v_n|^p \, dx + \int_{\Omega} \frac{\tilde{a}_0(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^p} \]
\[ \geq \lambda_1(a_0) + \int_{\Omega} \frac{\tilde{a}_0(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^p} =: \lambda_1(a_0) + I \]

because of the characterization of \( \lambda_1(a_0) \). Hypothesis (AH0) guarantees that for every \( \delta > 0 \) there exists \( \rho_0 > 0 \) such that \( |\tilde{a}_0(x, t)| \leq \delta |t|^{p-2} \) if \( |t| \leq \rho_0 \). Since \( \|u_n\|_{C^1(\overline{\Omega})} \to 0 \) and in view of (14), we can get
\[ |I| \leq \delta \int_{\Omega} |\nabla v_n|^p \, dx \leq \frac{\delta(p-1)}{C_0} \lambda_n \leq \frac{\delta(p-1)}{C_0} (\lambda_1(a_0) - \varepsilon_0) \]

for sufficiently large \( n \). As a result, by taking a sufficiently small \( \delta > 0 \), we have a contradiction for sufficiently large \( n \).

\[ \square \]

**Theorem 13.** Assume (AH0). For every \( \varepsilon > 0 \) there exists \( r_1 > 0 \) such that \( (EV; \lambda) \) has no constant sign solutions in \( B_p(r_1) \setminus \{0\} \) provided \( \lambda > \lambda_1(a_0) + \varepsilon \).

**Proof.** By way of contradiction, we assume that there exist \( \varepsilon_0 > 0 \), \( \{\lambda_n\} \) and \( \{u_n\} \) such that \( \lambda_n > \lambda_1(a_0) + \varepsilon_0 \), \( 0 \neq u_n \in B_p(1/n) \) and \( u_n \) is a constant sign solution of \( (EV; \lambda_n) \). Because \( A \) is odd, we may suppose that \( u_n \geq 0 \) by considering \(-u_n\) if necessary. Thus, by Remark 3(i)–(ii), \( u_n \in C^1(\overline{\Omega}) \) and \( u_n > 0 \) in \( \Omega \). We note that \( \lambda_n \leq A_p \lambda_1(-\Delta_p) \) by Proposition 10, where \( \lambda_1(-\Delta_p) \) denotes the first eigenvalue of \(-\Delta_p\) (see (10) for the definition of \( A_p \)), and so \( \{\lambda_n\} \) is bounded. Therefore, we may assume that \( \lambda_n \) converges to some \( \lambda_0 \) by choosing a subsequence. In addition, by the same argument as in Theorem 12, we can show that \( u_n \to 0 \) in \( C^1(\overline{\Omega}) \).

Set \( A_n(x, y) := A(x, \|u_n\|_p y)/\|u_n\|_p^{p-1} \) and \( f_n(x, t) := \lambda_n|t|^{p-2}t \). Then \( A_n \) satisfies Assumption A(i)–(iv) with the same constants \( C_0 \), \( C_1 \), and \( C_2 \). Moreover, \( |f_n(x, t)| \leq \lambda_n|t|^{p-1} \leq A_p \lambda_1(-\Delta_p)|t|^{p-1} \) for every \( t \in \mathbb{R} \), a.e. \( x \in \Omega \). Note also that we have the boundedness of \( \|v_n\| \) due to the inequality \( C_0 \|\nabla u_n\|_p^p/(p-1) \leq \int_{\Omega} A(x, \nabla u_n) \nabla u_n \, dx = \lambda_n\|u_n\|_p^p \). Since \( v_n := u_n/\|u_n\|_p \) is a positive solution of
\[ -\text{div}(A_n(x, \nabla u)) = f_n(x, u) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega, \]

Proposition 4 guarantees that \( \{v_n\} \) has a convergent subsequence in \( C^1(\overline{\Omega}) \). By choosing a subsequence, we may suppose that \( v_n \to v_0 \neq 0 \) in \( C^1(\overline{\Omega}) \) (note that \( \|v_0\|_p = 1 \)). Using that we obtain, for every \( w \in W_0^{1,p}(\Omega) \), that
\[ \int_{\Omega} \frac{\tilde{a}_0(x, |\nabla u_n|) \nabla u_n \nabla w}{\|u_n\|_p^{p-1}} \, dx = \int_{\Omega} \frac{\tilde{a}_0(x, |\nabla u_n|) \nabla u_n \nabla w |\nabla v_n|^{p-1}}{|\nabla u_n|^{p-1}} \, dx \to 0 \]
as \( n \to \infty \) in view of (AH0) and the convergence \( u_n \to 0 \). As a result, letting
\[ n \to \infty \text{ in the equality} \]
\[ \int_\Omega a_0(x)|\nabla v_n|^p \nabla v_n \nabla w \, dx + \int_\Omega \tilde{a}_0(x, |\nabla u_n|) \nabla u_n \nabla w \, dx = \lambda_n \int_\Omega |v_n|^{p-2} v_n w \, dx \]

for each \( w \in W^{1,p}_0(\Omega) \), we see that \( v_0 \neq 0 \) is a positive solution of (13) with \( \lambda = \lambda_0 \) (see Remark 3(ii) for \( \lambda_0 > 0 \)). This yields that \( \lambda_0 = \lambda_1(a_0) \), because (13) has no positive solutions other that \( \lambda = \lambda_1(a_0) \). Therefore we have a contradiction, because \( \lambda_0 = \lim_{n \to \infty} \lambda_n \geq \lambda_1(a_0) + \epsilon_0 \).

**Proposition 14.** Assume (AH0). Then, for every \( \epsilon > 0 \), there exists \( r_0 > 0 \) such that
\[ \frac{\lambda_1(A, r)}{r^p} \geq \lambda_1(a_0) - \epsilon \quad \text{for every } 0 < r < r_0. \]

**Proof.** Assume that there exist \( \epsilon > 0 \) and \( r_n > 0 \) such that \( r_n \to 0 \) as \( n \to \infty \) and \( \lambda_1(A, r_n)/r_n^p < \lambda_1(a_0) - \epsilon \) for every \( n \in \mathbb{N} \). Because of Proposition 5 and Lemma 6 (note that \( A \) is odd in the second variable), we can choose a positive function \( u_n \in (r_n S) \cap C^1(\overline{\Omega}) \) satisfying
\[ \int_\Omega A(x, \nabla u_n) \nabla u_n \, dx = \lambda_1(A, r_n), \quad \min_{v \in r_n S} \int_\Omega G(x, \nabla v) \, dx = \int_\Omega G(x, \nabla u_n) \, dx. \]

Note that
\[ \frac{C_0}{p-1} \|\nabla u_n\|_p^p \leq \int_\Omega A(x, \nabla u_n) \nabla u_n \, dx = \lambda_1(A, r_n) < (\lambda_1(a_0) - \epsilon)r_n^p \to 0, \]
and so \( u_n \to 0 \) in \( W^{1,p}_0(\Omega) \). Because \( u_n \) is a solution of (EV; \( \lambda \)) with \( \lambda = \lambda_1(A, r_n)/r_n^p \) (see Proposition 5), by combining the inequality
\[ \lambda_1(a_0) - \epsilon > \frac{\lambda_1(A, r_n)}{r_n^p} = \int_\Omega a_0(x)|\nabla v_n|^p \, dx + \int_\Omega \tilde{a}_0(x, |\nabla u_n|) |\nabla u_n|^2 \, dx \]
and an argument as in Theorem 12 with \( \lambda_n = \lambda_1(A, r_n)/r_n^p \), we have a contradiction.

**Proposition 15.** Assume (AH0). Then, for every \( \epsilon > 0 \), there exists \( r_1 > 0 \) such that
\[ \frac{\lambda_1(A, r)}{r^p} \leq \lambda_1(a_0) + \epsilon \quad \text{for every } 0 < r < r_1. \]

**Proof.** Assume that there exist \( \epsilon_0 > 0 \) and \( r_n > 0 \) such that \( r_n \to 0 \) as \( n \to \infty \) and \( \lambda_1(A, r_n)/r_n^p > \lambda_1(a_0) + \epsilon_0 \) for every \( n \in \mathbb{N} \). According to Lemma 6 and Proposition 5, we can take a positive function \( u_n \in (r_n S) \cap C^1(\overline{\Omega}) \) satisfying
\[ \int_\Omega A(x, \nabla u_n) \nabla u_n \, dx = \lambda_1(A, r_n), \quad \min_{v \in r_n S} \int_\Omega G(x, \nabla v) \, dx = \int_\Omega G(x, \nabla u_n) \, dx. \]

Noting that, with \( \varphi_{a_0} \) the positive eigenfunction corresponding to \( \lambda_1(a_0) \) satisfying
we can prove that
$\|\varphi_{a_0}\|_p = 1$, we have
$$\frac{C_0}{p(p-1)} \|\nabla u_n\|_p^p \leq \int_{\Omega} G(x, \nabla u_n) \, dx \leq \int_{\Omega} G(x, r_n \nabla \varphi_{a_0}) \, dx \leq \frac{C_1 r_n^p}{p(p-1)} \|\nabla \varphi_{a_0}\|_p^p,$$
we see that $u_n \to 0$ in $C^1(\overline{\Omega})$ due to Proposition 4, because $u_n$ is a positive solution
of (EV; $\lambda$) with $\lambda = \bar{\lambda}_1(A, r_n)/r_n^p$ and $(\bar{\lambda}_1(a_0) + \varepsilon_0 <) \bar{\lambda}_1(A, r_n)/r_n^p \leq A_p \bar{\lambda}_1(-\Delta_p)$
by Proposition 10, where $\bar{\lambda}_1(-\Delta_p)$ denotes the first eigenvalue of $-\Delta_p$ (see (10)
for the definition of $A_p$). Therefore, by the same argument as in Theorem 13 with
$\lambda_n = \bar{\lambda}_1(A, r_n)/r_n^p$, we have a contradiction. \hfill \Box

The following result follows from Propositions 14 and 15, (note $\bar{\lambda}_1(A, r) \leq \bar{\lambda}_1(A, r)$ for every $r > 0$).

**Corollary 16.** **Under (AH0), we have**
$$\lim_{r \to +0} \frac{\bar{\lambda}_1(A, r)}{r^p} = \lim_{r \to +0} \frac{\bar{\lambda}_1(A, r)}{r^p} = \lambda_1(a_0).$$

**Proposition 17.** **Under (AH0), we have**
$$\lim_{r \to +0} \frac{\mu_1(A, r)}{r^p} = \frac{\lambda_1(a_0)}{p}.$$

**Proof.** Due to Proposition 5, for every $r > 0$, there exists a positive solution
$u_r \in (rS) \cap C^1(\overline{\Omega})$ of (EV; $\lambda$) with $\lambda = \lambda_1(A, u_r)/r^p$ and $\mu_1(A, r) = J(u_r).$ Then
we can prove that $u_r \to 0$ in $C^1(\overline{\Omega})$ as $r \to +0$ and $u_r/\|u_r\|_p$ is bounded in $W^{1, p}_0(\Omega)$
as $r \to +0$ by a similar reason to the one in Proposition 15 (note that $\lambda_1(A, u_r)/r^p$
is bounded as $r \to +0$ by the inequality below and Corollary 16).

Set $\tilde{G}_0(x, y) := \int_0^{\|y\|} \tilde{a}_0(x, t) \, dt$ for $y \in \mathbb{R}^N$. We point out that
$$\bar{\lambda}_1(A, r) \leq \lambda_1(A, u_r) \leq \bar{\lambda}_1(A, r)$$
and
$$\mu_1(A, r) = \int_{\Omega} G(x, \nabla u_r) \, dx = \frac{1}{p} \int_{\Omega} a_0(x) |\nabla u_r|^p \, dx + \int_{\Omega} \tilde{G}_0(x, \nabla u_r) \, dx \leq \frac{\lambda_1(A, u_r)}{p} + \frac{1}{p} \int_{\Omega} \tilde{a}_0(x, |\nabla u|) |\nabla u_r|^2 \, dx + \int_{\Omega} \tilde{G}_0(x, \nabla u_r) \, dx.$$

Thus, by Corollary 16 and $r = \|u_r\|_p$, it suffices to prove
$$\lim_{r \to +0} \int_{\Omega} \frac{\tilde{a}_0(x, |\nabla u|) |\nabla u_r|^2}{\|u_r\|_p^p} \, dx = 0 \quad \text{and} \quad \lim_{r \to +0} \int_{\Omega} \frac{\tilde{G}_0(x, \nabla u_r)}{\|u_r\|_p^p} \, dx = 0.$$

Now we fix any $\varepsilon > 0$. Then, by (AH0), there exists $\delta > 0$ such that
$$|\tilde{a}_0(x, t)| \leq \varepsilon t^{p-2} \quad \text{and} \quad |\tilde{G}_0(x, y)| \leq \varepsilon |y|^p / p \quad \text{for every } 0 < t \leq \delta, \ |y| \leq \delta.$$
Because \( u_r \to 0 \) in \( C^1(\bar{\Omega}) \) as \( r \to +0 \), we may assume that \( \|u_r\|_{C^1(\bar{\Omega})} \leq \delta \) for sufficiently small \( r > 0 \). Therefore, we obtain

\[
\left| \int_{\Omega} \tilde{a}_0(x, |\nabla u|)|\nabla u_r|^2 \over \|u_r\|_p^p \right| \leq \varepsilon \|\nabla u_r\|_p^p, \quad \left| \int_{\Omega} \tilde{G}_0(x, \nabla u) \over \|u_r\|_p^p \right| \leq \varepsilon \|\nabla u_r\|_p^p.
\]

Since \( \|\nabla u_r\|_p^p/\|u_r\|_p^p \) is bounded as \( r \to +0 \) and \( \varepsilon > 0 \) is arbitrary, our conclusion holds.

3.2. Asymptotically homogeneous case near \( \infty \). In this subsection, we consider the case where \( A \) is asymptotically \((p-1)\)-homogeneous near \( \infty \) in the following sense.

(AH) There exist a positive function \( a_\infty \in C^1(\bar{\Omega}, \mathbb{R}) \) and a continuous function \( \tilde{a}(x, t) \) on \( \bar{\Omega} \times \mathbb{R} \) such that

\[
A(x, y) = a_\infty(x)|y|^{p-2}y + \tilde{a}(x, |y|)y \quad \text{for every } x \in \Omega, \ y \in \mathbb{R}^N,
\]

where

\[
\lim_{t \to +\infty} \frac{\tilde{a}(x, t)}{t^{p-2}} = 0 \quad \text{uniformly in } x \in \bar{\Omega}.
\]

For the weight function \( a_\infty \), we define

\[
\lambda_1(a_\infty) := \inf \left\{ \int_{\Omega} a_\infty(x)|\nabla u|^p \over \|u\|_p = 1 \right\}.
\]

Because \( 0 < \min_{x \in \bar{\Omega}} a_\infty(x) \leq \max_{x \in \bar{\Omega}} a_\infty(x) < \infty \), by the same argument as for the first eigenvalue of \(-\Delta_p\), we can prove the following elementary results:

(i) \( \lambda_1(a_\infty) \) is the first eigenvalue of

\[
-\text{div}(a_\infty(x)|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u \quad \text{in } \Omega, \ u = 0 \quad \text{on } \partial \Omega.
\]

(ii) \( \lambda_1(a_\infty) \) has a positive eigenfunction \( \varphi_{a_\infty} \in C^1(\bar{\Omega}) \) with \( \|\varphi_{a_\infty}\|_p = 1 \) and it is simple.

(iii) If \( \lambda \neq \lambda_1(a_\infty) \), then (17) has no constant sign solutions other than 0.

Theorem 18. Assume (AH). For every \( \varepsilon > 0 \) there exists \( R_0 > 0 \) such that equation (EV; \( \lambda \)) has no solutions in \( W^{1,p}_0(\Omega) \setminus B_p(R_0) \) provided \( \lambda < \lambda_1(a_\infty) - \varepsilon \).

To prove the theorem, we need the following result.

Lemma 19. Assume (AH) and let \( \{u_n\} \subset W^{1,p}_0(\Omega) \) be a sequence satisfying \( \|u_n\|_p \to \infty \) as \( n \to \infty \). If \( v_n := u_n/\|u_n\|_p \) is bounded in \( W^{1,p}_0(\Omega) \), the following assertions hold:

(i) \( \lim_{n \to \infty} \int_{\Omega} \tilde{a}(x, |\nabla u_n|)|\nabla u_n|^2 \over \|u_n\|_p^p \) \( dx = 0 \).
(ii) For every \( w \in W^{1,p}_0(\Omega) \),
\[
\lim_{n \to \infty} \int_\Omega \frac{\tilde{a}(x, |\nabla u_n|) \nabla u_n \nabla w}{\|u_n\|_p^{p-1}} \, dx = 0.
\]

(iii) \( \lim_{n \to \infty} \int_\Omega \frac{\tilde{G}(x, \nabla u_n)}{\|u_n\|_p^p} \, dx = 0 \), where \( \tilde{G}(x, y) := \int_0^{|y|} \tilde{a}(x, t) \, dt \) for \( y \in \mathbb{R}^N \).

Proof. (i) Fix any \( \varepsilon > 0 \). By the property of the function \( \tilde{a} \), there exist \( R > 0 \) and \( C > 0 \) such that
\[
|\tilde{a}(x, t)| \leq \varepsilon |t|^{p-2} \text{ if } t \geq R \quad \text{and} \quad |\tilde{a}(x, t)| \leq C \text{ if } 0 \leq t \leq R.
\]

Therefore, we obtain
\[
\left| \int_\Omega \frac{\tilde{a}(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^p} \, dx \right| \leq \int_{|\nabla u_n| > R} \varepsilon |\nabla u_n|^p \, dx + \int_{|\nabla u_n| \leq R} \frac{C|\nabla u_n|^2}{\|u_n\|_p^p} \, dx
\]
\[
\leq \varepsilon \|\nabla u_n\|_p^p + \frac{C R^2 |\Omega|}{\|u_n\|_p^p} \leq \varepsilon D^p + \frac{C R^2 |\Omega|}{\|u_n\|_p^p}
\]
by (18), where \( D := \sup_n \|\nabla v_n\|_p \). Letting \( n \to \infty \), we have
\[
\limsup_{n \to \infty} \left| \int_\Omega \frac{\tilde{a}(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^p} \, dx \right| \leq \varepsilon D^p,
\]
because \( \|u_n\|_p \to \infty \) as \( n \to \infty \). Thus, since \( \varepsilon > 0 \) is arbitrary, our conclusion holds.

(ii) For any \( \varepsilon > 0 \) and \( w \in W^{1,p}_0(\Omega) \), we have
\[
\left| \int_\Omega \frac{\tilde{a}(x, |\nabla u_n|) \nabla u_n \nabla w}{\|u_n\|_p^{p-1}} \, dx \right|
\]
\[
\leq \int_{|\nabla u_n| > R} \varepsilon |\nabla u_n|^{p-1} |\nabla w| \, dx + \int_{|\nabla u_n| \leq R} \frac{C |\nabla u_n||\nabla w|}{\|u_n\|_p^{p-1}} \, dx
\]
\[
\leq \varepsilon \|\nabla u_n\|_p^{p-1} \|\nabla w\|_p + \frac{C R \|\nabla w\|_p |\Omega|^{(p-1)/p}}{\|u_n\|_p^{p-1}}
\]
by Hölder’s inequality and (18). By combining this inequality and a similar argument to that used in (i), our conclusion is shown.

(iii) It is easily shown that, for every \( \varepsilon > 0 \), there exists \( C > 0 \) such that
\[
|\tilde{G}(x, y)| \leq \varepsilon |y|^p + C \quad \text{for every } y \in \mathbb{R}^N.
\]

Therefore, \( \left| \int_\Omega \tilde{G}(x, \nabla u_n) \, dx \right| \leq \varepsilon \|\nabla u_n\|_p^p + C |\Omega| \). This implies our conclusion. \( \square \)
Proof of Theorem 18. By way of contradiction, we assume that there exist \( \varepsilon_0 > 0, \{ \lambda_n \}, \) and \( \{ u_n \} \) such that \( \lambda_n < \lambda_1(a_\infty) - \varepsilon_0, \lim_{n \to \infty} \| u_n \|_p = \infty, \) and \( u_n \) is a solution of \((\text{EV}; \lambda_n)\). By taking \( u_n \) as a test function in \((\text{EV}; \lambda_n)\), we have

\[
\frac{C_0}{p-1} \| \nabla u_n \|_p^p \leq \int_\Omega A(x, \nabla u_n) \nabla u_n \, dx = \lambda_n \| u_n \|_p^p \leq (\lambda_1(a_\infty) - \varepsilon_0) \| u_n \|_p^p;
\]

refer to Remark 1(iii). Therefore, \( v_n := u_n/\| u_n \|_p \) is bounded in \( W^{1,p}_0(\Omega) \).

Again by taking \( u_n/\| u_n \|_p \) as a test function in \((\text{EV}; \lambda_n)\), we obtain

\[
\lambda_1(a_\infty) - \varepsilon_0 > \lambda_n = \int_\Omega a_\infty(x) \| \nabla u_n \|_p^p \, dx + \int_\Omega \tilde{a}(x, |\nabla u_n|) |\nabla u_n|^2 \, dx
\]

\[
= \int_\Omega a_\infty(x) |\nabla v_n|_p^2 \, dx + \int_\Omega \tilde{a}(x, |\nabla u_n|) |\nabla u_n|^2 \, dx
\]

\[
\geq \lambda_1(a_\infty) + o(1),
\]

using the definition of \( \lambda_1(a_\infty) \) and Lemma 19(i). This is a contradiction. \( \square \)

Theorem 20. Assume (AH). For every \( \varepsilon > 0 \) there exists \( R_1 > 0 \) such that \((\text{EV}; \lambda)\) has no constant sign solutions in \( W^{1,p}_0(\Omega) \setminus B_p(R_1) \) provided \( \lambda > \lambda_1(a_\infty) + \varepsilon \).

Proof. By way of contradiction, we assume that there exist \( \varepsilon_0 > 0, \{ \lambda_n \}, \) and \( \{ u_n \} \) such that \( \lambda_n > \lambda_1(a_\infty) + \varepsilon_0, \lim_{n \to \infty} \| u_n \|_p = \infty, \) and \( u_n \) is a constant sign solution of \((\text{EV}; \lambda_n)\). Because \( A \) is odd, we may suppose that \( u_n \geq 0 \) by considering \( -u_n \) if necessary. Thus, by Remark 3, \( u_n \in C^{1}(\bar{\Omega}) \) and \( u_n > 0 \) in \( \Omega \). Here we note that \( \lambda_n \leq A_p\lambda_1(-\Delta_p) \) by Proposition 10, where \( \lambda_1(-\Delta_p) \) denotes the first eigenvalue of \( -\Delta_p \) (see (10) for the definition of \( A_p \)), and so \( \{ \lambda_n \} \) is bounded. Hence we may assume, by taking a subsequence, that \( \lambda_n \) converges to some

\[
\lambda_0 \in [\lambda_1(a_\infty) + \varepsilon_0, A_p\lambda_1(-\Delta_p)].
\]

In addition, we know that \( v_n := u_n/\| u_n \|_p \) is bounded in \( W^{1,p}_0(\Omega) \)

\[
\frac{C_0}{p-1} \| \nabla u_n \|_p^p \leq \int_\Omega A(x, \nabla u_n) \, dx = \lambda_n \| u_n \|_p^p,
\]

where we take \( u_n \) as a test function in \((\text{EV}; \lambda_n)\). Thus, by choosing a subsequence, we may suppose that \( v_n \) converges to some \( v \) weakly in \( W^{1,p}_0(\Omega) \) and strongly in \( L^p(\Omega) \).

We claim that \( v \) is a positive solution of

\[
-\text{div}(a_\infty(x)|\nabla v|^{p-2} \nabla v) = \lambda_0 |v|^{p-2} v \quad \text{in} \ \Omega, \quad v = 0 \quad \text{on} \ \partial \Omega,
\]

(19) that is, \( v \) is a positive eigenfunction corresponding to \( \lambda_0 \). If our claim holds, then \( \lambda_0 = \lambda_1(a_\infty) \) occurs because (17) has no positive solutions in the case of \( \lambda \neq \lambda_1(a_\infty) \). Hence this contradicts \( \lambda_1(a_\infty) + \varepsilon_0 \leq \lim_{n \to \infty} \lambda_n = \lambda_0 \).
We now prove our claim. First, we show that $v_n$ converges to $v$ strongly in $W^{1,p}_0(\Omega)$. Indeed, by taking $(v_n - v)/\|u_n\|^{p-1}_p$ as a test function in $(EV; \lambda_n)$, we have
\[
\lambda_n \int_\Omega v_n^{p-1}(v_n - v) \, dx \\
= \int_\Omega a_\infty(x)|\nabla v_n|^{p-2}\nabla v_n \nabla (v_n - v) \, dx + \int_\Omega \tilde{a}(x, |\nabla u_n|) \nabla u_n \nabla (v_n - v) \, dx \\
= \int_\Omega a_\infty(x)|\nabla v_n|^{p-2}\nabla v_n \nabla (v_n - v) \, dx + o(1)
\]
as $n \to \infty$ due to Lemma 19(i)–(ii). Since $v_n \to v$ in $L^p(\Omega)$, this implies that $\int_\Omega a_\infty(x)|\nabla v_n|^{p-2}\nabla v_n \nabla (v_n - v) \, dx$ converges to 0 as $n \to \infty$. Noting that
\[
o(1) = \int_\Omega a_\infty(x)(|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v) \nabla (v_n - v) \, dx \\
\geq \min_{\Omega} a_\infty \int_\Omega (|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v) \nabla (v_n - v) \, dx \\
\geq \min_{\Omega} a_\infty(\|\nabla v_n\|^{p-1}_p - \|\nabla v\|^{p-1}_p)(\|\nabla v_n\|_p - \|\nabla v\|_p) \geq 0,
\]
we have $v_n \to v$ in $W^{1,p}_0(\Omega)$ (note $0 < \min_{\Omega} a_\infty \leq \max_{\Omega} a_\infty < \infty$). As a result, $v$ is a solution of (19) by letting $n \to \infty$ in the equality
\[
\int_\Omega a_\infty(x)|\nabla v_n|^{p-2}\nabla v_n \nabla w \, dx + \int_\Omega \tilde{a}(x, |\nabla u_n|) \nabla u_n \nabla w \, dx = \lambda_n \int_\Omega v_n^{p-1}w \, dx
\]
for every $w \in W^{1,p}_0(\Omega)$; note that, by Lemma 19(ii), the second term converges to zero. Since $v_n = u_n/\|u_n\|_p > 0$ in $\Omega$, $v$ is nonnegative, and so $v$ is positive by Remark 3(i) and $\|v\|_p = 1$. Thus our claim is shown. \hfill \Box

**Proposition 21.** Assume (AH). Then, for every $\varepsilon > 0$, there exists $R_0 > 0$ such that
\[
\frac{\lambda_1(A, r)}{r^p} \geq \lambda_1(a_\infty) - \varepsilon \quad \text{for every } r > R_0.
\]

**Proof.** Assume that there exist $\varepsilon_0 > 0$ and $r_n > 0$ such that $r_n \to \infty$ as $n \to \infty$ and $\lambda_1(A, r_n)/r_n^p < \lambda_1(a_\infty) - \varepsilon_0$ for every $n \in \mathbb{N}$. Because of Proposition 5 and Lemma 6, we can choose a positive function $u_n \in (r_n S) \cap C^1(\overline{\Omega})$ satisfying
\[
\int_\Omega A(x, \nabla u_n) \nabla u_n \, dx = \dot{\lambda}_1(A, r_n), \quad \min_{v \in r_n S} \int_\Omega G(x, \nabla v) \, dx = \int_\Omega G(x, \nabla u_n) \, dx.
\]
Note that
\[
\frac{C_0}{p-1} \|\nabla u_n\|_p^p \leq \int_\Omega A(x, \nabla u_n) \nabla u_n \, dx = \dot{\lambda}_1(A, r_n) < (\lambda_1(a_\infty) - \varepsilon_0)r_n^p,
\]
and so \( u_n/r_n = u_n/\|u_n\|_p \) is bounded in \( W^{1,p}_0(\Omega) \). Because \( u_n \) is a solution of \((\text{EV}; \lambda)\) with \( \lambda = \tilde{\lambda}_1(A, r_n)/r_n^p \) (see Proposition 5), by the same argument as in Theorem 18 with \( \lambda_n = \tilde{\lambda}_1(A, r_n)/r_n^p \), we have a contradiction. \(\square\)

**Proposition 22.** Assume (AH). Then, for every \( \varepsilon > 0 \), there exists \( R_1 > 0 \) such that

\[
\frac{\tilde{\lambda}_1(A, r)}{r^p} \leq \lambda_1(a_{\infty}) + \varepsilon \quad \text{for every } r > R_1.
\]

**Proof.** Assume that there exist \( \varepsilon_0 > 0 \) and \( r_n > 0 \) such that \( r_n \to \infty \) as \( n \to \infty \) and \( \tilde{\lambda}_1(A, r_n)/r_n^p > \lambda_1(a_{\infty}) + \varepsilon_0 \) for every \( n \in \mathbb{N} \). According to Lemma 6 and Proposition 22, we can take a positive function \( u \in (r_n, S) \cap C^1(\Omega) \) satisfying

\[
\int_{\Omega} A(x, \nabla u_n) \nabla u_n \, dx = \tilde{\lambda}_1(A, r_n), \quad \min_{v \in \mathcal{R}_n} \int_{\Omega} G(x, \nabla v) \, dx = \int_{\Omega} G(x, \nabla u_n) \, dx.
\]

Note that, with \( \phi_{a_{\infty}} \) as in item (ii) of page 165, we have

\[
\frac{C_0}{p(p-1)} \|\nabla u_n\|_p^p \leq \int_{\Omega} G(x, \nabla u_n) \, dx \leq \int_{\Omega} \min_{v \in \mathcal{R}_n} \int_{\Omega} G(x, r_n \nabla \phi_{a_{\infty}}) \, dx \leq \frac{C_1 r_n^p}{p(p-1)} \|\nabla \phi_{a_{\infty}}\|_p^p.
\]

Hence \( u_n/r_n = u_n/\|u_n\|_p \) is bounded in \( W^{1,p}_0(\Omega) \). Since \( u_n \) is a positive solution of \((\text{EV}; \lambda)\) with \( \lambda = \tilde{\lambda}_1(A, r_n)/r_n^p \), by the same argument as in Theorem 20 with \( \lambda_n = \tilde{\lambda}_1(A, r_n)/r_n^p \), we have a contradiction. \(\square\)

By Propositions 21 and 22, we have the following result.

**Corollary 23.** Under (AH), we have

\[
\lim_{r \to +\infty} \frac{\tilde{\lambda}_1(A, r)}{r^p} = \lim_{r \to +\infty} \frac{\lambda_1(A, r)}{r^p} = \lambda_1(a_{\infty}).
\]

**Proposition 24.** Under (AH), we have

\[
\lim_{r \to +\infty} \frac{\mu_1(A, r)}{r^p} = \frac{\lambda_1(a_{\infty})}{p}.
\]

**Proof.** Due to Proposition 5, for every \( r > 0 \), there exists a positive solution \( u_r \in (r S) \cap C^1(\Omega) \) of \((\text{EV}; \lambda)\) with \( \lambda = \lambda_1(A, u_r)/r^p \) and \( \mu_1(A, r) = J(u_r) \). Then \( u_r/\|u_r\|_p = u_r/r \) is bounded in \( W^{1,p}_0(\Omega) \), as seen from

\[
\frac{C_0}{p(p-1)} \|\nabla u_r\|_p^p \leq \int_{\Omega} G(x, \nabla u_r) \, dx \leq \int_{\Omega} G(x, r \nabla w) \, dx \leq r^p \frac{C_1}{p(p-1)} \|\nabla w\|_p^p
\]

for any \( w \in W^{1,p}_0(\Omega) \) with \( \|w\|_p = 1 \).

Set

\[
\tilde{G}(x, y) := \int_0^{\|y\|} \tilde{a}(x, t) \, dt \quad \text{for } y \in \mathbb{R}^N.
\]

Note that

\[
\tilde{\lambda}_1(A, r) \leq \lambda_1(A, u_r) \leq \tilde{\lambda}_1(A, r)
\]
and
\[
\mu_1(A, r) = \int_\Omega G(x, \nabla u_r) \, dx = \frac{1}{p} \int_\Omega a_\infty(x)|\nabla u_r|^p \, dx + \int_\Omega \tilde{G}(x, \nabla u_r) \, dx
\]
\[
= \frac{\lambda_1(A, u_r)}{p} - \frac{1}{p} \int_\Omega \tilde{a}(x, |\nabla u|)|\nabla u_r|^2 \, dx + \int_\Omega \tilde{G}(x, \nabla u_r) \, dx.
\]
According to Corollary 23 and Lemma 19(i) and (iii) (note \(\|u_r\|_p = r \to +\infty\)), our conclusion is achieved. 
\[
\square
\]

4. Existence of a positive solution

In this section, we provide the existence of a positive solution to the equation

\[(P) \begin{cases} -\text{div} A(x, \nabla u) = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases}\]

where the nonlinear term \(f\) satisfies Assumption \((f)\).

**Theorem 25.** Assume \((AH_0), (AH),\) and \((f)\). Let \(\lambda_1(a_0)\) and \(\lambda_1(a_\infty)\) be the first eigenvalues of, respectively, (13) and (17) (see the discussion there). If one of the following conditions holds, \((P)\) has at least one positive solution.

(i) \(\alpha_0 > \lambda_1(a_0)\) and \(\alpha < \lambda_1(a_\infty)\).

(ii) \(\alpha_0 < \lambda_1(a_0)\) and \(\alpha > \lambda_1(a_\infty)\).

This addresses the existence of an eigenvalue for our operator because we can apply Theorem 25 to \(f(x, u) = \lambda |u|^{p-2}u\).

**Corollary 26.** Assume \((AH_0), (AH),\) and \(\lambda_1(a_0) \neq \lambda_1(a_\infty)\). Then, for every \(\lambda\) between \(\lambda_1(a_0)\) and \(\lambda_1(a_\infty)\), \((\text{EV}; \lambda)\) has a nontrivial (positive) solution. Therefore \(\lambda\) is an eigenvalue of \(A\).

To show the existence of a positive solution, we define a \(C^1\) functional \(I\) on \(W_0^{1,p}(\Omega)\) by

\[I(u) := \int_\Omega G(x, \nabla u) \, dx - \int_\Omega F_+(x, u) \, dx \quad \text{for } u \in W_0^{1,p}(\Omega),\]

where \(F_+(x, u) := \int_0^u f_+(x, u) \, dx\), with \(f_+(x, t)\) given by \(f(x, t)\) if \(t \geq 0\) and 0 if \(t \leq 0\).

**Remark 27.** If \(u \in W_0^{1,p}(\Omega)\) is a nontrivial critical point of \(I\), then \(u\) is a positive solution of \((P)\).

Indeed, by taking \(-u_-\) as a test function, we obtain

\[
0 = \langle I'(u), -u_- \rangle = \int_\Omega A(x, \nabla u)(-\nabla u_-) \, dx - \int_\Omega f_+(x, u)(-u_-) \, dx
\]
\[
= \int_\Omega A(x, \nabla u)(-\nabla u_-) \, dx \geq \frac{C_0}{p-1} \|\nabla u_-\|_p^p.
\]
Thus $u \geq 0$. By Remark 3(ii) (note that $u \neq 0$), we see that $u$ is a positive solution of $(P)$ (note that $f_+(x, u) = f(x, u)$).

**Convention.** From now on, let Assumption $(f)$ be satisfied.

**Lemma 28.** If $\alpha \neq \lambda_1(a_\infty)$, then $I$ satisfies the Palais–Smale condition.

**Proof.** Let $\{u_n\}$ be a Palais–Smale sequence of $I$, which means that

$$I(u_n) \to c \quad \text{and} \quad \|I'(u_n)\|_{W_0^{1,p}(\Omega)^*} \to 0 \quad \text{as} \quad n \to \infty$$

for some $c \in \mathbb{R}$. In view of Proposition 2 and the compactness of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, it is sufficient to prove the boundedness of $\{u_n\}$ in $W_0^{1,p}(\Omega)$. Then, in view of the inequality

$$\frac{C_0}{p(p-1)} \| \nabla u_n \|^p_p \leq \int_{\Omega} G(x, \nabla u_n) \, dx = I(u_n) + \int_{\Omega} F_+(x, u_n) \, dx \leq I(u_n) + C \| u_n \|^p_p,$$

it is sufficient to prove the boundedness of $\{u_n\}$ in $L^p(\Omega)$. By way of contradiction we may assume that $\| u_n \|^p_p \to \infty$ as $n \to \infty$ by choosing a subsequence if necessary. Let $v_n := u_n / \| u_n \|^p_p$. The inequality (20) ensures that $\{v_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Hence, by choosing a subsequence, we may suppose that $v_n \rightharpoonup v_0$ in $W_0^{1,p}(\Omega)$ and $v_n \to v_0$ in $L^p(\Omega)$ for some $v_0$.

First, we see that $v_0 \geq 0$ for a.e. $x \in \Omega$. Indeed, by taking $-(u_n)_-$ as a test function, we have

$$o(1) \| \nabla (u_n)_- \|^p_p = \langle I'(u_n), -(u_n)_- \rangle = - \int_{\Omega} A(x, \nabla u_n)(-\nabla (u_n)_-) \, dx \geq \frac{C_0}{p-1} \| \nabla (u_n)_- \|^p_p.$$

Because $p > 1$, we have $\| \nabla (u_n)_- \|^p_p \to 0$ as $n \to \infty$. Thus $(v_n)_- \to 0$ in $W_0^{1,p}(\Omega)$, and hence $(v_0)_- = 0$ for a.e. $x \in \Omega$.

Now we prove that

$$\lim_{n \to \infty} \frac{\| f_+(\cdot, u_n) - \alpha (u_n)^{p-1} \|_{L^p}^p}{\| u_n \|_{L^p}^{p-1}} = 0,$$

where $p' = \frac{p}{p-1}$. Fix an arbitrary $\varepsilon > 0$. It follows from condition (ii) of Assumption $(f)$ that there exists a $C_\varepsilon > 0$ such that

$$|f(x, u) - \alpha u^{p-1}| \leq \varepsilon |u|^{p-1} + C_\varepsilon \quad \text{for every} \quad u \geq 0, \text{ a.e.} \ x \in \Omega.$$

Then we obtain

$$\int_{\Omega} |f_+(x, u_n) - \alpha (u_n)^{p-1} |^{p'} \, dx \leq 2^{p'-1}(\varepsilon^{p'-1} |(u_n)_+|^p + C_\varepsilon^{p'-1} |\Omega|).$$
Since we are assuming that \( \|u_n\|_p \to \infty \) as \( n \to \infty \), this shows that
\[
\lim_{n \to \infty} \| f_+ (\cdot, u_n) - \alpha (u_n)^{p-1}_+ \|_p / \| u_n \|_p^{p-1} = 0,
\]
because \( \varepsilon > 0 \) is arbitrary.

Here we recall the following result proved in Lemma 19:
\[
(22) \lim_{n \to \infty} \int_{\Omega} \frac{\tilde{a}(x, |\nabla u_n|) \nabla u_n \nabla (v_n - v_0)}{\| u_n \|_p^{p-1}} \, dx = \lim_{n \to \infty} \int_{\Omega} \frac{\tilde{a}(x, |\nabla u_n|) \nabla u_n \nabla \varphi}{\| u_n \|_p^{p-1}} \, dx = 0
\]
for every \( \varphi \in W_0^{1,p} (\Omega) \). Thus, by considering
\[
o(1) = \frac{\langle I' (u_n), v_n - v_0 \rangle}{\| u_n \|_p^{p-1}} = \int_{\Omega} a_\infty (x) |\nabla v_n|^{p-2} \nabla v_n \nabla (v_n - v_0) \, dx + o(1),
\]
and using Proposition 2, we see that \( v_n \) converges strongly to \( v_0 \) in \( W_0^{1,p} (\Omega) \). Hence, by passing to the limit in \( o(1) = \langle I' (u_n), \varphi \rangle / \| u_n \|_p^{p-1} \) for any \( \varphi \in W_0^{1,p} (\Omega) \) and by noting (21) and (22), we infer that \( v_0 \) is a nontrivial solution of
\[
- \text{div} (a_\infty |\nabla u|^{p-2} \nabla u) = \alpha |u|^{p-2} u \quad \text{in} \, \Omega, \quad u = 0 \quad \text{on} \, \partial \Omega
\]
(note that \( \| v_0 \|_p = 1 \) and \( v_0 \geq 0 \) for a.e. \( x \in \Omega \)). Since \( v_0 \geq 0 \) for a.e. \( x \in \Omega \), \( v \) is a positive solution of (17) with \( \lambda = \alpha \) (see Remark 3). This implies that \( \alpha = \lambda_1 (a_\infty) \), because (17) has no positive solutions if \( \lambda \neq \lambda_1 (a_\infty) \). It contradicts the hypothesis \( \alpha \neq \lambda_1 (a_\infty) \). Hence \( \| u_n \|_p \) is bounded, which completes the proof. \( \square \)

Lemma 29. Assume (AH) and \( \alpha < \lambda_1 (a_\infty) \). Then \( I \) is coercive, bounded from below and weakly lower semicontinuous (wls) on \( W_0^{1,p} (\Omega) \).

Proof. Because \( \alpha < \lambda_1 (a_\infty) \), we can take sufficiently small constants \( \varepsilon > 0 \) and \( 0 < \delta < 1 \) satisfying
\[
(23) \quad (1 - \delta) (\lambda_1 (a_\infty) - \varepsilon) > \alpha + \varepsilon.
\]

By condition (ii) of Assumption (f), there exists a \( C > 0 \) such that
\[
|F_+ (x, u) | \leq (\alpha + \varepsilon) \frac{u^p}{p} + C
\]
for every \( u \geq 0 \) and a.e. \( x \in \Omega \). Due to Proposition 24 and the definition of \( \mu_1 (A, r) \), there exists an \( R > 0 \) such that, for every \( u \in W_0^{1,p} (\Omega) \) with \( \| u \|_p \geq R \),
\[
(24) \quad \int_{\Omega} G (x, \nabla u) \, dx \geq \mu_1 (A, \| u \|_p) \geq \frac{\lambda_1 (a_\infty) - \varepsilon}{p} \| u \|_p^p.
\]
Hence, for every \( u \in W_0^{1,p} (\Omega) \) with \( \| u \|_p \geq R \), we obtain
According to Proposition 17, there exists an $r$ satisfying

$$
\int_\Omega G(x, \nabla u) \, dx + \delta C_0 \|\nabla u\|_p^p \geq \frac{\delta C_0}{\rho(p-1)} \|\nabla u\|_p^p - \frac{\alpha + \varepsilon}{p} \|u_+\|_p^p - C|\Omega|
$$

by (2), (23), and (24), where $u_+ := \max\{0, u\}$. This yields that $I$ is coercive. Moreover, because $I$ is bounded from below on $B_\rho(R)$, we see that $I$ is bounded from below on $W_0^{1,p}(\Omega)$. Since $J$ is wslc (see the proof of Proposition 5) and $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, $I$ is wslc on $W_0^{1,p}(\Omega)$. □

**Lemma 30.** Assume (AH0) and $\alpha_0 < \lambda_1(a_0)$. Let $p < q \leq p^*$, where $p^* = Np/(N - p)$ if $N > p$ and $p^* = +\infty$ if $N \leq p$. Then there exists $\rho_0 > 0$ such that

$$
\inf \{I(u) : \|u\|_q = \rho \} > 0 \quad \text{for every } 0 < \rho < \rho_0.
$$

**Proof.** Because $\alpha_0 < \lambda_1(a_0)$, we can take some sufficiently small $\varepsilon > 0$ and $0 < \delta < 1$ satisfying

$$
(1 - \delta)(\lambda_1(a_0) - \varepsilon) > \alpha_0 + \varepsilon.
$$

According to Proposition 17, there exists an $r_0 > 0$ such that

$$
\frac{\mu_1(A, r)}{r^p} \geq \frac{\lambda_1(a_0) - \varepsilon}{p} \quad \text{for every } 0 < r < r_0.
$$

In addition, Assumption (f) guarantees the existence of $D_q > 0$ satisfying

$$
F_+(x, u) \leq \frac{\alpha_0 + \varepsilon}{p} u^p + D_q u^q \quad \text{for every } u \geq 0, \ \text{a.e. } x \in \Omega.
$$

Because $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous, we can take a positive constant $C_q$ such that $\|u\|_q \leq C_q \|\nabla u\|_p$ for every $W_0^{1,p}(\Omega)$. We choose a positive constant $\rho$ satisfying

$$
\rho < \min \left\{ r_0 \|\Omega\|^{1/q - 1/p}, \left( \frac{\delta C_0}{2p(p-1)D_q C_q^p} \right)^{1/(q-p)} \right\} =: \rho_0.
$$

Note that $\|u\|_p < r_0$ if $\|u\|_q = \rho$, by Hölder’s inequality and (28). Therefore, for every $\|u\|_q = \rho$, we have

$$
I(u) = (1 - \delta) \int_\Omega G(x, \nabla u) \, dx + \delta \int_\Omega G(x, \nabla u) \, dx - \int_\Omega F_+(x, u) \, dx
\geq (1 - \delta) \frac{\mu_1(A, \|u\|_p)}{\|u\|_p^p} \|u\|_p^p + \frac{\delta C_0}{p(p-1)} \|\nabla u\|_p^p - \frac{\alpha_0 + \varepsilon}{p} \|u_+\|_p^p - D_q \|u_+\|_q^q
\geq \frac{1}{p} \left\{ (1 - \delta)(\lambda_1(a_0) - \varepsilon) - \alpha_0 - \varepsilon \right\} \|u\|_p^p + \left( \frac{\delta C_0}{p(p-1)C_q^p} - D_q \|u\|_q^{q-p} \right) \|u\|_q^p
$$
\[ \geq \frac{\delta C_0}{2p(p-1)c_q^p} \rho^p, \]

by the definition of \( \mu_1(A, r) \), (2), (27), (26), (25), and (28). This ensures our conclusion. \( \square \)

**Proof of Theorem 25.**

(i) Lemma 29 guarantees the existence of a global minimizer of \( I \). Thus it suffices to prove that \( \min_{W^{1,p}_0(\Omega)} I < 0 \) to show the existence of a nontrivial critical point of \( I \). Choose a positive constant \( \varepsilon > 0 \) such that \( \alpha_0 > \lambda_1(a_0) + 2\varepsilon \). Let \( \varphi_{a_0} \in C^1(\bar{\Omega}) \) be a positive eigenfunction corresponding to \( \lambda_1(a_0) \) with \( \|\varphi_{a_0}\|_p = 1 \) (refer to the text below (13) and note that (13) is a homogeneous equation). It is easily seen that \( \int_{\Omega} \tilde{G}_0(x, r \nabla \varphi_{a_0}) \, dx / r^p \to 0 \) as \( r \to +0 \) (refer to the proof of Proposition 17 with \( \|r \varphi_{a_0}\|_p = r \)). Hence there exists \( r_0 > 0 \) such that

\[
(29) \quad \int_{\Omega} G(x, r \nabla \varphi_{a_0}) \, dx = \frac{r^p}{p} \int_{\Omega} a_0(x) |\nabla \varphi_{a_0}|^p \, dx + r^p \int_{\Omega} \frac{\tilde{G}_0(x, r \nabla \varphi_{a_0})}{r^p} \, dx
\leq \frac{\lambda_1(a_0) + \varepsilon}{p} r^p = \frac{\lambda_1(a_0) + \varepsilon}{p} \|r \varphi_{a_0}\|_p
\]

for every \( 0 < r < r_0 \). On the other hand, it follows from part (i) of Assumption (f) that there exists a \( \delta > 0 \) such that

\[
(30) \quad F_+(x, u) \geq \frac{\alpha_0 - \varepsilon}{p} u^p \quad \text{for every } u \in [0, \delta], \text{ a.e. } x \in \Omega.
\]

Therefore, for every \( 0 < r < \min\{r_0, \delta / \|\varphi_{a_0}\|_\infty \} \), we have

\[
I(ru_0) \leq \frac{r^p}{p} (\lambda_1(a_0) + 2\varepsilon - \alpha_0) \|\varphi_{a_0}\|_p < 0,
\]

by (29) and (30) (note \( \lambda_1(a_0) + 2\varepsilon - \alpha_0 < 0 \)), whence \( \min_{W^{1,p}_0(\Omega)} I < 0 \).

(ii) Let \( p < q \leq p^* \). Then, by Lemma 30, we obtain \( \rho > 0 \) satisfying

\[
\delta_0 := \inf \{ I(u) : \|u\|_q = \rho \} > 0.
\]

Now we claim the existence of \( w \in W^{1,p}_0(\Omega) \) such that

\[
(31) \quad \|w\|_q > \rho \quad \text{and} \quad I(w) < \delta_0.
\]

Admitting this claim, we define

\[
c := \inf_{\gamma \in \Gamma} \max_{r \in [0, 1]} I(\gamma(r)), \quad \Gamma := \{ \gamma \in C([0, 1], W^{1,p}_0(\Omega)) : \gamma(0) = 0, \gamma(1) = w \}.
\]

It is obvious that \( \Gamma \neq \emptyset \) and \( \gamma([0, 1]) \cap \{u \in W^{1,p}_0(\Omega) : \|u\|_q = \rho \} \neq \emptyset \) for every \( \gamma \in \Gamma \), since \( W^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega) \) is continuous. Thus the mountain pass theorem guarantees that \( c(\geq \delta_0) \) is a nontrivial critical value of \( I \) because \( I \) satisfies the Palais–Smale condition by Lemma 28.
Finally, we prove the existence of \( w \) satisfying (31). Because \( \alpha > \lambda_1(a_\infty) \), we can choose a positive constant \( \varepsilon_0 > 0 \) such that

\[
\alpha > \lambda_1(a_\infty) + 2\varepsilon_0.
\]

Using item (ii) on page 165, we can take \( \varphi_{a_\infty} \in C^1(\Omega) \) be a positive eigenfunction corresponding to \( \lambda_1(a_\infty) \) with \( \|\varphi_{a_\infty}\|_p = 1 \). It follows from Lemma 19(iii) that

\[
\int_\Omega \tilde{G}(x, r \nabla \varphi_{a_\infty}) dx / r^p \to 0
\]
as \( r \to +\infty \) (note that \( \|r \varphi_{a_\infty}\|_p = r \)). Hence there exists \( R_0 > 0 \) such that

\[
\int_\Omega G(x, r \nabla \varphi_{a_\infty}) dx = \frac{r^p}{p} \int_\Omega a_\infty(x)|\nabla \varphi_{a_\infty}|^p dx + \frac{r^p}{p} \int_\Omega \frac{\tilde{G}_0(x, r \nabla \varphi_{a_\infty})}{r^p} dx \leq \frac{\lambda_1(a_\infty) + \varepsilon_0}{p} r^p = \frac{\lambda_1(a_\infty) + \varepsilon_0}{p} \|r \varphi_{a_\infty}\|_p^p
\]

for every \( r \geq R_0 \). In addition, it follows from condition (ii) of Assumption (f) that there exists \( D > 0 \) such that

\[
F_+(x, u) \geq \frac{\alpha - \varepsilon_0}{p} u^p - D
\]

for every \( u \geq 0 \), a.e. \( x \in \Omega \).

Consequently, by (32), (33), and (34), we obtain

\[
I(r \varphi_{a_0}) \leq \frac{r^p}{p} (\lambda_1(a_\infty) + 2\varepsilon_0 - \alpha)\|\varphi_{a_0}\|_p^p + D|\Omega| \to -\infty
\]
as \( t \to +\infty \). This implies the existence of \( w \) satisfying (31). \( \square \)

4.1. Resonant cases. To consider the resonant cases, we introduce the following hypotheses for

\[
\tilde{G}(x, y) := \int_0^{|y|} \tilde{a}(x, t) t \, dt \quad \text{and} \quad \tilde{G}_0(x, y) := \int_0^{|y|} \tilde{a}_0(x, t) t \, dt,
\]

where \( \tilde{a} \) and \( \tilde{a}_0 \) are as in (AH) and (AH0).

(H+) There exist \( 1 \leq q < p \) and \( H_0 > 0 \) such that

\[
\lim_{|y| \to \infty} \frac{p \tilde{G}(x, y) - \tilde{a}(x, |y|)|y|^2}{|y|^q} = +\infty \quad \text{for a.e.} \ x \in \Omega,
\]

\[
p \tilde{G}(x, y) - \tilde{a}(x, |y|)|y|^2 \geq -H_0(1 + |y|^q) \quad \text{for a.e.} \ x \in \Omega, \ \text{every} \ y \in \mathbb{R}^N,
\]

\[
f(x, t)t - pF(x, t) \geq -H_0(1 + t^q) \quad \text{for a.e.} \ x \in \Omega, \ \text{every} \ t \geq 0.
\]
(H−) There exist $1 < q < p$ and $H_0 > 0$ such that

$$\lim_{|y| \to \infty} \frac{p\tilde{G}(x, y) - \tilde{a}(x, |y|)|y|^2}{|y|^q} = -\infty \quad \text{for a.e. } x \in \Omega,$$

$$p\tilde{G}(x, y) - \tilde{a}(x, |y|)|y|^2 \leq H_0(1 + |y|^q) \quad \text{for a.e. } x \in \Omega, \text{ every } y \in \mathbb{R}^N,$$

$$f(x, t)t - pF(x, t) \leq H_0(t^q + 1) \quad \text{for a.e. } x \in \Omega, \text{ every } t \geq 0.$$

(HF+) There exist $1 < q < p$ and $H_0 > 0$ such that

$$p\tilde{G}(x, y) - \tilde{a}(x, |y|)|y|^2 \geq -H_0(1 + |y|^q) \quad \text{for a.e. } x \in \Omega, \text{ every } y \in \mathbb{R}^N,$$

$$f(x, t)t - pF(x, t) \geq -H_0(1 + t^q) \quad \text{for every } t \geq 0, \text{ a.e. } x \in \Omega,$$

$$\lim_{t \to +\infty} \frac{f(x, t)t - pF(x, t)}{t^q} = +\infty \quad \text{for a.e. } x \in \Omega.$$

(HF−) There exist $1 < q < p$ and $H_0 > 0$ such that

$$p\tilde{G}(x, y) - \tilde{a}(x, |y|)|y|^2 \leq H_0(1 + |y|^q) \quad \text{for a.e. } x \in \Omega, \text{ every } y \in \mathbb{R}^N,$$

$$f(x, t)t - pF(x, t) \leq H_0(1 + t^q) \quad \text{for every } t \geq 0, \text{ a.e. } x \in \Omega,$$

$$\lim_{t \to +\infty} \frac{f(x, t)t - pF(x, t)}{t^q} = -\infty \quad \text{for a.e. } x \in \Omega.$$

(H0+) There exist $p \leq r < p^*$ and $H_0 > 0$ such that

$$\lim_{|y| \to 0} \frac{p\tilde{G}_0(x, y) - \tilde{a}_0(x, |y|)|y|^2}{|y|^r} = +\infty \quad \text{for a.e. } x \in \Omega,$$

$$p\tilde{G}(x, y) - \tilde{a}(x, |y|)|y|^2 \geq -H_0|y|^r \quad \text{for a.e. } x \in \Omega, \text{ every } |y| \leq 1,$$

$$f(x, t)t - pF(x, t) \geq -H_0t^r \quad \text{for a.e. } x \in \Omega, \text{ every } t \in [0, 1].$$

(H0−) There exist $p \leq r < p^*$ and $H_0 > 0$ such that

$$\lim_{|y| \to 0} \frac{p\tilde{G}_0(x, y) - \tilde{a}_0(x, |y|)|y|^2}{|y|^r} = -\infty \quad \text{for a.e. } x \in \Omega,$$

$$p\tilde{G}(x, y) - \tilde{a}(x, |y|)|y|^2 \leq H_0|y|^r \quad \text{for a.e. } x \in \Omega, \text{ every } |y| \leq 1,$$

$$f(x, t)t - pF(x, t) \leq H_0t^r \quad \text{for a.e. } x \in \Omega, \text{ every } t \in [0, 1].$$

(HF0+) There exist $p \leq r < p^*$ and $H_0 > 0$ such that

$$p\tilde{G}_0(x, y) - \tilde{a}_0(x, |y|)|y|^2 \geq -H_0|y|^r \quad \text{for a.e. } x \in \Omega, \text{ every } |y| \leq 1,$$

$$f(x, t)t - pF(x, t) \geq -H_0t^r \quad \text{for every } t \in [0, 1], \text{ a.e. } x \in \Omega,$$

$$\lim_{t \to +0} \frac{f(x, t)t - pF(x, t)}{t^r} = +\infty \quad \text{for a.e. } x \in \Omega.$$
There exist $p \leq r < p^*$ and $H_0 > 0$ such that
\[ p \tilde{G}_0(x, y) - \tilde{a}_0(x, |y|)|y|^2 \leq H_0 |y|^r \] for a.e. $x \in \Omega$, every $|y| \leq 1$,
\[ f(x, t)t - pF(x, t) \leq H_0 t^r \] for every $t \in [0, 1]$, a.e. $x \in \Omega$,
\[ \lim_{t \to +0} \frac{f(x, t)t - pF(x, t)}{t^r} = -\infty \] for a.e. $x \in \Omega$.

**Theorem 31.** Let Assumption (f), (AH0), and (AH) hold. If any of the following conditions is satisfied, (P) has at least one positive solution.

(i) $\alpha_0 > \alpha_1(a_0)$, $\alpha = \alpha_1(a_\infty)$, and (HF+) or (H+).

(ii) $\alpha_0 < \alpha_1(a_0)$, $\alpha = \alpha_1(a_\infty)$, and (HF-) or (H-).

(iii) $\alpha_0 = \alpha_1(a_0)$, $\alpha < \alpha_1(a_\infty)$, and (HF0+) or (H0+).

(iv) $\alpha_0 = \alpha_1(a_0)$, $\alpha > \alpha_1(a_\infty)$, and (HF0-) or (H0-).

(v) $\alpha_0 = \alpha_1(a_0)$, $\alpha = \alpha_1(a_\infty)$, (HF0+) or (H0+), and (HF+) or (H+).

(vi) $\alpha_0 = \alpha_1(a_0)$, $\alpha = \alpha_1(a_\infty)$, (HF0-) or (H0-), and (HF-) or (H-).

The rest of this section is devoted to the proof of this theorem, which involves some preparatory steps.

**The singular resonant case.** Set $f_{\pm n}(x, t) := f(x, t) + \frac{B}{n} |t|^{p-2}t$ and define approximate functionals on $W^{1, p}_0(\Omega)$ by
\[ I_{\pm n}(u) := \int_\Omega G(x, \nabla u) \, dx - \int_\Omega (F_{\pm n})_+(x, u) \, dx = I(u) + \frac{1}{n} \|u_+\|_p^p. \]

From now on, assume $f$ satisfies Assumption (f). Take first the case $\alpha = \alpha_1(a_\infty)$.

**Lemma 32.** If either (H+) or (HF+) (resp. either (H-) or (HF-)) hold and $\{u_n\}$ satisfies
\[ \sup_{n \in \mathbb{N}} I_{\pm n}(u_n) < +\infty \] and
\[ \lim_{n \to \infty} \|I_{\pm n}(u_n)\|_{W^{1, p}_0(\Omega)^*} = 0 \]
(resp. $\inf_{n \in \mathbb{N}} I_{\pm n}(u_n) > -\infty$ and
\[ \lim_{n \to \infty} \|I_{\pm n}'(u_n)\|_{W^{1, p}_0(\Omega)^*} = 0 \),

then $\{u_n\}$ is bounded in $W^{1, p}_0(\Omega)$.

**Proof.** The boundedness of $\|u_n\|_p$ guarantees that $\|u_n\|$ is bounded, since
\[ o(1)\|u_n\| = \langle I_{\pm n}'(u_n), u_n \rangle \geq C \|u_n\| - C (1 + \|u_n\|_p) \geq \frac{1}{n} \|u_+\|_p^p \]
for some $C > 0$ independent of $n$. So, by way of contradiction, we assume that $\|u_n\|_p \to \infty$ as $n \to \infty$. Then, by the same argument as in Lemma 28, we see that $v_n := u_n/\|u_n\|_p$ has a subsequence strongly converging to a positive solution $v_0$ of
\[ (35) \quad -\text{div}(a_\infty |\nabla u|^{p-2} \nabla u) = \alpha |u|^{p-2} u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega. \]
If $\alpha \neq \lambda_1(a_{\infty})$, we have a contradiction, because (35) does not have a positive solution except when $\lambda = \lambda_1(a_{\infty})$. So we may assume that $\alpha = \lambda_1(a_{\infty})$ and $v_0 = \varphi_{a_{\infty}}$ (note $\|v_0\|_p = 1$). For simplicity, we still denote the subsequence under discussion by $\{v_n\}$. Thus $u_n(x) \to \infty$ as $n \to \infty$ for a.e. $x \in \Omega$ (note $v_0 = \varphi_{a_{\infty}} > 0$ in $\Omega$).

Assume (HF+) or (HF−). We show that

(36)
$$I := \int_\Omega f_+(x, u_n)u_n - pF_+(x, u_n) \, dx \to \pm \infty,$$

where the sign on $\infty$ matches (HF±) and $q$ is a constant as in (HF±). Indeed, it follows from (HF+) that $(f_+(x, t) - pF_+(x, t))/t^q$ is bounded from below on $\Omega \times [1, +\infty)$. Therefore, since $u_n(x) \to \infty$ for a.e. $x \in \Omega$, we have (36) if (HF+) holds, by applying Fatou’s lemma to the inequality

$$I \geq \int_{u_n(x) \geq 1} \frac{f_+(x, u_n)u_n - pF_+(x, u_n)}{u_n^q} \, dx - \frac{2H_0}{\|u_n\|_p^q} |\Omega|,$$

where $H_0 > 0$ is a constant as in (HF+). The case of (HF−) is handled by the same argument, with $-f$ instead of $f$. This shows (36).

Furthermore, by Hölder’s inequality, we have

(37)
$$II := \int_\Omega \frac{p\tilde{G}(x, \nabla u_n) - \tilde{a}(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^q} \, dx$$

$$\leq H_0 \int_\Omega \left(|\nabla u_n|^q + \frac{1}{\|u_n\|_p^q} \right) \, dx \leq H_0 \|\nabla u_n\|_p^q |\Omega|^{(p-q)/p} + o(1)$$

$$\leq H_0 \|\nabla v_0\|_p^q |\Omega|^{(p-q)/p} + o(1)$$

in the case of (HF−), because $v_n \to v_0$ in $W_0^{1, p}(\Omega)$, where $q \in [1, p)$ and $H_0 > 0$ are constants as in (HF−). Similarly, we obtain

(38)
$$II \geq -H_0 \|\nabla v_0\|_p^q |\Omega|^{(p-q)/p} + o(1)$$

in the case of (HF+).

Hence we have a contradiction because of (36), (37) or (38) by taking the limit inferior or superior in the equality

$$\frac{pI_{\pm n}(u_n) - \langle I_{\pm n}'(u_n), u_n \rangle}{\|u_n\|_p^q} = II + I.$$

Assume (H+) or (H−). Because $v_0$ is a positive solution of (35), we have $|\nabla u_n(x)| \to \infty$ as $n \to \infty$ for a.e. $x \in \Omega_0 := \{x' \in \Omega : |\nabla v_0(x')| \neq 0\}$. Because $|\Omega_0| > 0$, we can show, by an argument similar to the one used for $f$, that

$$\int_\Omega \frac{p\tilde{G}(x, \nabla u_n) - \tilde{a}(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^q} \, dx \to \pm \infty,$$
where again the sign matches that of \((H\pm)\). In addition, we easily obtain that
\[
\pm \int_{\Omega} \frac{f_+(x, u_n)u_n - p F_+(x, u_n)}{\|u_n\|^p} \, dx \geq -H_0\|v_n\|^q + o(1) = -H_0\|v_0\|^q + o(1)
\]
(again, the sign matches). Hence we have a contradiction by considering the limit of \((p I_{\pm n}(u_n) - (I'_{\pm n}(u_n), u_n))/\|u_n\|^p\).

**Proof of Theorem 31(i)**. Because \(\alpha_0 > \lambda_1(a_0)\), there exists an \(n_0 \in \mathbb{N}\) such that \(\alpha_0 - p/n_0 > \lambda_1(a_0)\). Note that \(f_-n(x, t)/t^{p-1} \to \alpha_0 - p/n > \lambda_1(a_0)\) as \(t \to +0\) for \(n \geq n_0\) and \(f_-n(x, t)/t^{p-1} \to \alpha - p/n = \lambda_1(a_\infty) - p/n < \lambda_1(a_\infty)\) as \(t \to +\infty\). Hence, by using the proof of Theorem 25(i) to \(f_-n\), we can find a global minimizer \(u_n\) of \(I_{-n}\) with \(I_{-n}(u_n) < 0\) for each \(n \geq n_0\). Here we remark that \(\sup_{n \geq n_0} I_{-n}(u_n) < 0\). In fact, for every \(n \geq n_0\), we have
\[
I_{-n}(u_n) = I_0(u_0) + \frac{1}{n_0} \|u_0\|^p \leq I(0) + \frac{1}{n_0} \|u_0\|^p = I_{-n_0}(u_0) < 0,
\]
where, in the first inequality, we use the fact that \(u_n\) is a global minimizer of \(I_{-n}\). Now, due to Lemma 32, we see that \([u_n]\) is bounded in \(W_{0,\Omega}^{1, p}\). Therefore,
\[
\|I'(u_n)\|_{W_{0,\Omega}^{1, p}} = \|I'(u_n) - I'_{-n}(u_n)\|_{W_{0,\Omega}^{1, p}} \leq \frac{p}{n\lambda_1(-\Delta_p)^p} \|u_n\|^p \to 0
\]
as \(n \to \infty\), where \(\lambda_1(-\Delta_p)\) is the first eigenvalue of \(-\Delta_p\). Since \(I\) is bounded on a bounded set, we may assume that \([u_n]\) is a bounded Palais–Smale sequence of \(I\). Because \(I\) satisfies the bounded Palais–Smale condition (see Proposition 2), \(u_n\) has a subsequence converging to some \(v_0\) in \(W_{0,\Omega}^{1, p}\). It is clear that \(I(v_0) \leq \sup_{n \geq n_0} I_{-n}(u_n) = I_{-n_0}(u_0) < 0\), and so \(v_0\) is a nontrivial critical point of \(I\).

**Proof of Theorem 31(ii)**. Using Lemma 30 and \(\alpha_0 < \lambda_1(a_0)\), we can choose \(q_0 \in (p, p^*)\) and \(\rho > 0\) such that \(\inf\{I(u) : \|u\|_{q_0} = \rho\} > 0\). Since \(I_{+n}(u) \geq I(u) - \|u\|_{q_0}^{1-p/q_0}/n\) for every \(u \in W_{0,\Omega}^{1, p}\), we can take \(n_0 \in \mathbb{N}\) such that \(\alpha_0 + p/n_0 < \lambda_1(a_0)\) and \(\delta_0 := \inf\{I_{+n}(u) : \|u\|_{q_0} = \rho\} > 0\). Hence, for every \(n \geq n_0\), we have \(\inf\{I_{+n}(u) : \|u\|_{q_0} = \rho\} \geq \delta_0\), because \(I_{+n}(u) \geq I_{+n_0}(u)\) for every \(n \geq n_0\) and \(u \in W_{0,\Omega}^{1, p}\). By noting that \(f_+(x, t)/t^{p-1} \to \alpha + p/n > \alpha = \lambda_1(a_\infty)\) as \(t \to +\infty\), and applying Lemma 28 to \(f_+\) instead of \(f\), \(I_{+n}\) satisfies the Palais–Smale condition. Therefore, the proof of Theorem 25(ii) implies that, for every \(n \geq n_0\), there exists a critical point \(u_n \in W_{0,\Omega}^{1, p}\) of \(I_{+n}\) such that \(I_{+n}(u_n) \geq \delta_0\). According to Lemma 32, \([u_n]\) is bounded in \(W_{0,\Omega}^{1, p}\). Thus, because we have a bounded Palais–Smale sequence of \(I\) due to a similar reason as in the case of (i), we can obtain a nontrivial critical point of \(I\) (note that \(\inf_{n \geq n_0} I(u_n) \geq \inf_{n \geq n_0} I_{+n}(u_n) \geq \delta_0 > 0\)).

We next turn to the case where \(\alpha_0 = \lambda_1(a_0)\).
Lemma 33. Assume (H0−) or (HF0−) (resp. (H0+) or (HF0+)). Let $u_n \neq 0$ be an element of $W_0^{1,p}(\Omega)$ satisfying $I_{\pm n}(u_n) = 0$ for every $n \in \mathbb{N}$ and $\inf_n I_{\pm n}(u_n) \geq 0$ (resp. $\sup_n I_{\pm n}(u_n) \leq 0$). Then $\lim_{n \to \infty} \|u_n\|_p > 0$.

Proof. By way of contradiction, we assume that $\lim_{n \to \infty} \|u_n\|_p = 0$ by choosing a subsequence. Note that the boundedness of $\|u_n\|_p$ yields that $\|u_n\|$ and $\|u_n\|_p$ are bounded in view of

$$\|u_n\| = \langle I'_{\pm n}(u_n), u_n \rangle \geq \frac{C_0}{p-1} \|u_n\|^p - C(1 + \|(u_n)_+\|^p) \geq \frac{p}{n} \|u_n\|_p^p$$

for some $C > 0$ independent of $n$. Then, since $u_n$ is a positive solution of

$$-\text{div}(A(x, \nabla u)) = f_{\pm n}(x, u_n) \quad \text{in} \quad \Omega$$

(refer to Remarks 3 and 27), it follows from Proposition 4 that $u_n \to 0$ in $C^1(\overline{\Omega})$ (note that $|\{f_{\pm n}\}(x, t)| \leq C_{t+1}^{p-1}$ (see Assumption (f)) and $u_n \to 0$ in $L^p(\Omega)$). Therefore, we may assume that $\|u_n\|_{C^1(\overline{\Omega})} \leq 1$ by considering a sufficiently large $n$. Since $|f_{\pm n}(x, \|u_n\|_p t)/\|u_n\|_{C^1(\overline{\Omega})}^{p-1}| \leq C t^p$ for every $t \geq 0$, a.e. $x \in \Omega$ ($C > 0$ independent of $n$; see Assumption (f) and (39)), by a similar argument to Theorem 13, we see that $v_n := u_n/\|u_n\|_p$ has a subsequence converging to a positive solution $v_0$ in $C^1(\overline{\Omega})$.

If $\alpha_0 \neq \lambda_1(a_0)$, we have a contradiction because (13) does not have a positive solution unless $\lambda = \lambda_1(a_0)$. So we may assume that $\alpha_0 = \lambda_1(a_0)$ and $v_0 = \varphi_{a_0}$ (note $\|v_0\|_p = 1$). For simplicity, we still denote the subsequence under discussion by $\{v_n\}$.

Assume (H0+) or (H0−). Then we can prove that

$$I := \int_{\Omega} \frac{p \tilde{G}_0(x, \nabla u_n) - \tilde{a}_0(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^r} \, dx \to \pm \infty$$

(signs match), where $r \in [p, p^*)$ is a constant as in (H0+) or (H0−). Indeed, because $\|\nabla v_0\|_p > 0$, we can choose a constant $\varepsilon_0 > 0$ such that $|\{x \in \Omega : |\nabla v_0| > 2\varepsilon_0\}| > 0$. With this $\varepsilon_0$, we have under assumption (H0+)

$$I \geq \int_{|\nabla v_n| > \varepsilon_0} \frac{p \tilde{G}_0(x, \nabla u_n) - \tilde{a}_0(x, |\nabla u_n|)|\nabla u_n|^2}{|\nabla u_n|^r} \, dx - \int_{|\nabla v_n| \leq \varepsilon_0} H_0|\nabla v_n|^r \, dx$$

$$\geq \int_{|\nabla v_n| > \varepsilon_0} \frac{p \tilde{G}_0(x, \nabla u_n) - \tilde{a}_0(x, |\nabla u_n|)|\nabla u_n|^2}{|\nabla u_n|^r} \, dx - \varepsilon_0^r H_0|\Omega|,$$

where $H_0$ is a positive constant as in (H0+). Hence, applying Fatou’s lemma, our claim is shown, because the Lebesgue measure of $\{x \in \Omega : |\nabla v_0| > 2\varepsilon_0\}$ is positive. Similarly, by considering $\tilde{a}_0(x, |\nabla u_n|)|\nabla u_n|^2 - p \tilde{G}_0(x, \nabla u_n)$, we can prove (41) under (H0−).
On the other hand, by using (H0+) or (H0−), we obtain
\begin{equation}
\pm II := \pm \int_{\Omega} \frac{f_+(x, u_n)u_n - pF_+(x, u_n)}{\|u_n\|^p_p} \, dx \geq -H_0 \int_{\Omega} (v_n)^+_dx \\
\geq -H_0 \|v_n\|^r_p = -H_0\|v_0\|^r_p + o(1)
\end{equation}
(note that $\|u_n\|_{C^1(\Omega)} \leq 1$ and $v_n \to v_0$ in $C^1(\Omega)$). Now set $\Psi_n = I_{\pm n}$. Since
\begin{equation}
\pm (I + II) = \pm \frac{p\Psi_n(u_n) - \langle \Psi'_n(u_n) , u_n \rangle}{\|u_n\|^r_p} = \pm \frac{p\Psi_n(u_n)}{\|u_n\|^r_p} \leq 0
\end{equation}
if $\sup_n (\pm I_{\pm}(u_n)) \leq 0$ (where the signs match throughout), we obtain a contradiction with (41) and (42) by taking the limit superior or inferior in (43).

Assume (HF0+) or (HF0−). As in the argument for $I$ in the case of (H0±), we can show that
\[
\int_{\Omega} \frac{f_+(x, u_n)u_n - pF_+(x, u_n)}{\|u_n\|^p_p} \, dx \int_{v_n>0} \frac{f_+(x, u_n)u_n - pF_+(x, u_n)}{(u_n)^+_dx} \to \pm \infty,
\]
the sign matching that of (HF0±). Moreover, it is easily seen that
\[
\pm \int_{\Omega} \frac{p\tilde{G}_0(x, \nabla u_n) - \tilde{a}_0(x, |\nabla u_n|)\nabla u_n|^2}{\|u_n\|^r_p} \, dx \geq \mp H_0\|\nabla v_n\|^r_p = \mp H_0\|\nabla v_0\|^r_p + o(1).
\]
(Note that $\|u_n\|_{C^1(\Omega)} \leq 1$ and $v_n \to v_0$ in $C^1(\Omega)$.) Our conclusion follows from a similar argument as before.

**Proof of Theorem 31(iii).** Let $n_0 \in \mathbb{N}$ such that $\alpha + p/n_0 < \lambda_1(a_\infty)$. The proof of Theorem 25(i) guarantees that, for every $n \geq n_0$, $I_{+n}$ has a global minimizer $u_n$ such that $I_{+n}(u_n) < 0$, because $f_{+n}(x, t)/t^{p-1} \to \alpha_0 + p/n > \alpha_0 = \lambda_1(a_0)$ as $t \to +0$ and $f_{+n}(x, t)/t^{p-1} \to \alpha + p/n < \lambda_1(a_\infty)$ as $t \to +\infty$ if $n \geq n_0$. Noting that $I_{+n}(u) \geq I_{+n_0}(u)$ for every $u \in W^{1,p}_0(\Omega)$ and $n \geq n_0$, $\{u_n\}$ is bounded in $W^{1,p}_0(\Omega)$ since $I_{+n_0}$ is coercive on $W^{1,p}_0(\Omega)$ by Lemma 29. Thus $\{u_n\}$ is a bounded Palais–Smale sequence of $I$ by the same argument as in (i). Therefore, $\{u_n\}$ has a convergent subsequence to some $u_0$ in $W^{1,p}_0(\Omega)$ because $I$ satisfies the bounded Palais–Smale condition. On the other hand, Lemma 33 guarantees that $u_0 \neq 0$ (note $\sup_{n \geq n_0} I_{+n}(u_n) \leq 0$). Therefore $u_0$ is a nontrivial critical point of $I$.

**Proof of Theorem 31(iv).** Let $n_0 \in \mathbb{N}$ be such that $\alpha - p/n_0 > \lambda_1(a_\infty)$. Applying Lemma 30 to $f_{-n}$ for $n \geq n_0$ (and since $\alpha_0 - p/n < \lambda_1(a_0)$), we can choose $q_0 \in (p, p^*)$ and $\rho_n > 0$ such that $\delta_n := \inf \{I_{-n}(u) : \|u\|_{q_0} = \rho_n\} > 0$. By noting that $f_{-n}(x, t)/t^{p-1} \to \alpha - p/n > \lambda_1(a_\infty)$ as $t \to +\infty$ for every $n \geq n_0$, and applying Lemma 28 to $f_{-n}$ instead of $f$, we see that $I_{-n}$ satisfies the Palais–Smale condition. Therefore, the proof of Theorem 25(ii) implies that, for every $n \geq n_0$, there exists
a critical point \( u_n \in W^{1,p}_0(\Omega) \) of \( I_{-n} \) such that \( I_{-n}(u_n) \geq \delta_n > 0 \). By Lemma 32, \( \{u_n\} \) is bounded in \( W^{1,p}_0(\Omega) \). Thus, by arguing as in case (i), we find a subsequence \( \{u_{n_k}\} \) converging to some \( u_0 \) in \( W^{1,p}_0(\Omega) \). Also, Lemma 33 yields \( u_0 \neq 0 \) (note that \( \inf_{n \geq n_0} I_{-n}(u_{n_k}) \geq 0 \)). This shows that \( u_0 \) is a nontrivial critical point of \( I \). \[ \square \]

The doubly resonant case. Choose smooth nonnegative functions \( \varphi \) and \( \psi \) on \([0, +\infty)\) satisfying \( \varphi(t) = 1 \) if \( 0 \leq t \leq 2 \), \( \varphi(t) = 0 \) if \( t \geq 4 \), \( \psi(t) = 0 \) if \( t \leq 5 \), and \( \psi(t) = 1 \) if \( t \geq 10 \). Define approximate functionals on \( W^{1,p}_0(\Omega) \) by

\[
\bar{I}_{\pm n}(u) := I(u) \mp \frac{1}{n} \psi(\|u\|_p^p)\|u^+\|_p^p \mp \frac{1}{n} \varphi(\|u\|_p^p)\|u^+\|_p^p.
\]

Because \( \bar{I}_{\pm n}(u) = I_{\pm n}(u) \) provided \( \|u\|_p \leq 2 \), the following result can be proved by the same argument as in Lemma 33. We omit the proof.

**Lemma 34.** Assume (H0−) or (HF0−) (resp. (H0+) or (HF0+)). Let \( u_n \neq 0 \) be an element of \( W^{1,p}_0(\Omega) \) satisfying \( (\bar{I}_{\pm n})'(u_n) = 0 \) for every \( n \in \mathbb{N} \) and \( \inf_n \bar{I}_{\pm n}(u_n) \geq 0 \) (resp. \( \sup_n \bar{I}_{\pm n}(u_n) \leq 0 \)). Then \( \lim \inf_{n \to \infty} \|u_n\|_p > 0 \).

**Lemma 35.** If \( \alpha \pm p/n \neq \lambda_1(a_{\infty}) \), then \( \bar{I}_{\pm n} \) (with the matching sign) satisfies the Palais–Smale condition.

**Proof.** Let \( \{u_m\} \) be a Palais–Smale sequence of \( \bar{I}_{+n} \) or \( \bar{I}_{-n} \). If \( \|u_m\|_p \to \infty \) occurs, then \( \bar{I}_{\pm n}(u_m) = \bar{I}_{\pm n}(u_m) \) for sufficiently large \( m \). So, by applying Lemma 28 to \( f_{\pm n} \) (note that \( \alpha \pm p/n \neq \lambda_1(a_{\infty}) \)), we have a contradiction if \( \|u_m\|_p \to \infty \). Consequently, we see that \( \|u_m\|_p \) is bounded. Then, by the same reason as in Lemma 28, \( \{u_m\} \) has a convergent subsequence in \( W^{1,p}_0(\Omega) \). \[ \square \]

Because \( \bar{I}_{\pm n}(u) = I_{\pm n}(u) \) provided \( \|u\|_p \geq 10 \), the following result can be proved by the same argument as in Lemma 32. We omit the proof.

**Lemma 36.** If either (H+) or (HF+) (resp. either (H−) or (HF−)) and \( \{u_n\} \) satisfies

\[
\sup_{n \in \mathbb{N}} \bar{I}_{\pm n}(u_n) < +\infty \quad \text{and} \quad \lim_{n \to \infty} \|(\bar{I}_{\pm n})'(u_n)\|_{W^{1,p}_0(\Omega)^*} = 0
\]

(resp. \( \inf_{n \in \mathbb{N}} \bar{I}_{\pm n}(u_n) > -\infty \) and \( \lim_{n \to \infty} \|(\bar{I}_{\pm n})'(u_n)\|_{W^{1,p}_0(\Omega)^*} = 0 \)),

\( \{u_n\} \) is bounded in \( W^{1,p}_0(\Omega) \).

**Proof of Theorem 31(v).** Note that \( \bar{I}_{-n}(u) = I_{-n}(u) \) provided \( \|u\|_p \geq 10 \) and \( \bar{I}_{-n}(u) = I_{+n}(u) \) if \( \|u\|_p \leq 2 \). So, by a similar argument to that in (i), \( \bar{I}_{-n} \) has a global minimizer \( u_n \). Moreover, by a similar argument to that in (iii) (note that \( f_{+n}(x,t)/t^{p-1} \to \alpha_0 + p/n > \lambda_1(a_0) \) as \( t \to +0 \) and \( f_{-n}(x,t)/t^{p-1} \to \alpha - p/n < \lambda_1(a_{\infty}) \) as \( t \to +\infty \)), we have \( \bar{I}_{-n}(u_n) < 0 \), whence \( u_n \neq 0 \). Because Lemma 36 implies the boundedness of \( \|u_n\| \), by the same argument as in (i), we see that \( \{u_n\} \)
is a bounded Palais–Smale sequence of \( I \). Therefore, we may assume that \( u_n \) converges to some \( u_0 \) in \( W^{1,p}_0(\Omega) \) by choosing a subsequence. On the other hand, Lemma 33 yields \( \liminf_{n \to \infty} \| u_n \|_p > 0 \). Hence \( u_0 \neq 0 \). This means that \( u_0 \) is a nontrivial critical point of \( I \).

**Proof if Theorem 31(vi).** Note that \( \tilde{I}_{+n}(u) = I_{+n}(u) \) provided \( \| u \|_p \geq 10 \) and \( \tilde{I}_{+n}(u) = I_{-n}(u) \) if \( \| u \|_p \leq 2 \). So, because \( f_{-n}(x,t) / t^{p-1} \to \alpha_0 - p/n < \lambda_1(a_0) \) as \( t \to +0 \) and \( f_{+n}(x,t) / t^{p-1} \to \alpha + p/n > \lambda_1(a_\infty) \) as \( t \to +\infty \), by a similar argument to those in (ii) and (iv), for each \( n \), we have a nontrivial critical point \( u_n \) of \( \tilde{I}_{+n} \) with \( \tilde{I}_{+n}(u_n) > 0 \). As a result, by a similar reasoning as in (v), we can obtain a nontrivial critical point of \( I \).

\[ \Box \]

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