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In the study of line arrangements, searching for minimal examples of line arrangements whose fundamental groups are not combinatorially invariant is a very interesting and hard problem. It is known that such a minimal arrangement must have at least 9 lines. In this paper, we extend the number to 10 by a new method. We classify arrangements of 9 projective lines according to the irreducibility of their moduli spaces and show that fundamental groups of complements of arrangements of 9 projective lines are combinatorially invariant. The idea and results have been used to classify arrangements of 10 projective lines.

1. Introduction

A *hyperplane arrangement* $\mathscr{A} = \{L_1, L_2, \dots, L_n\}$ in $\mathbb{CP}^r$ is a finite collection of hyperplanes. We call $M(\mathscr{A}) = \mathbb{CP}^r \setminus (\bigcup_{L\in\mathscr{A}} L)$ the complement of $\mathscr{A}$. The set $L(\mathscr{A}) = \left\{\bigcap_{i\in S} L_i \mid S \subseteq \{1, 2, \dots, n\}\right\}$ partially ordered by reverse inclusion is called the *intersection lattice* of $\mathscr{A}$. Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two arrangements of n hyperplanes. We say that intersection lattices $L(\mathscr{A}_1)$ and $L(\mathscr{A}_2)$ are isomorphic, denoted by $L(\mathscr{A}_1) \sim L(\mathscr{A}_2)$, if there is a permutation ϕ of the set $\{1, 2, \dots, n\}$ such that

$$\dim \bigcap_{\substack{i\in S\\ G_i\in\mathscr{A}_1}} G_i = \dim \bigcap_{\substack{j\in\phi(S)\\ H_j\in\mathscr{A}_2}} H_j$$

for any nonempty subset $S \subseteq \{1, 2, \dots, n\}$. Two arrangements are *lattice isomorphic* if their lattices are isomorphic. In this paper, we only consider line arrangements in $\mathbb{CP}^2$.

An essential topic in hyperplane arrangements theory is to study the interaction between topology of complements and combinatorics of intersection lattices. Naturally enough, one may ask how close topology and combinatorics of a given arrangement

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are related. Two arrangements, $\mathscr{A}_1$ and $\mathscr{A}_2$, are *homeomorphic equivalent* if there is a homeomorphism between their complements. A more concrete question is: how close are lattice isomorphism and homeomorphic equivalence to being in one-to-one correspondence?

The deepest theorem in the theory of line arrangement in projective 2-dimensional space is that of Jiang and Yau [1998], which asserts that the intersection lattice of the line arrangement is a topological invariant. It is natural to ask to what extent the converse of the Jiang–Yau theorem is true. Jiang and Yau [1994], and subsequently Wang and Yau [2005], have shown that the converse statement is also true for a large class of line arrangements. Therefore, the Jiang–Yau theorem initiates a new research direction: Can one find a Zariski pair of line arrangements; that is, a pair of arrangements which are lattice isomorphic but not homeomorphic equivalent.

A pair of arrangements which are lattice isomorphisms but not homeomorphic equivalent is called a *Zariski pair*. Our definition is stronger than the definition introduced by Artal [1994], which we shall call *weak Zariski pairs* (see [Artal et al. 2008] for a survey on Zariski pairs). The first Zariski pair of arrangements was constructed by Rybnikov [2011]. Each arrangement in Rybnikov's example consists of 13 lines and 15 triple points. Artal et al. [2005] provided another (weak) Zariski pair of two arrangements $\mathscr{H}^+ := \mathscr{C}^+ \cup \{N^+\}$ and $\mathscr{H}^- := \mathscr{C}^- \cup \{N^-\}$, where $\mathscr{C}^+$, $\mathscr{C}^-$ are arrangements (Figure 7) extending Falk–Sturmfels arrangements (Figure 2), and N^+, N^- are lines passing through a triple point and a double point of $\mathscr{C}^\pm$. The proof is based on the observation that there is no order-preserving homeomorphism between $(\mathbb{P}^2, \mathscr{C}^+)$ and $(\mathbb{P}^2, \mathscr{C}^-)$. In the contrary direction, Garber, Teicher, and Vishne [Garber et al. 2003] proved that there is no Zariski pair of arrangements of up to 8 real lines which covered the result of Fan [1997] on arrangements of 6 lines. This result was recently generalized to arrangements of 8 complex lines by Nazir and Yoshinaga [2012].

A natural question is: what is the minimal number of lines of a Zariski pair of line arrangements?

On the other hand, it was Jiang and Yau [1994] who first observed that the statement "two lattice isotopy line arrangements (that is, they are connected by a one-parameter family with constant intersection lattice) have diffeomorphic complements" follows trivially from Teissier's numerical characterization of the Whitney condition. In [Jiang and Yau 1994] and [Wang and Yau 2005], the authors found large classes of line arrangements, called *nice arrangements* and *simple arrangements*, whose intersection lattices determine the topology of the complements. Nazir and Yoshinaga [2012] found new classes of line arrangements whose intersection lattices determine the topology of the complements. Unlike nice and simple arrangements whose intersection lattices have special properties, Nazir and Yoshinaga's new classes require that all intersection points with multiplicity at least 3 be in special positions. This makes their results more useful for studying arrangements of a few lines. Indeed, in their paper they classify arrangements of 8 lines and give a list of classes of arrangements of 9 lines.

In this paper, we introduce new ideas to classify arrangements of lines. We prove that Nazir and Yoshinaga's list on the classification of arrangements of 9 lines is complete. As a corollary, we conclude that there is no Zariski pair of arrangements of 9 lines. The idea and results of this paper have been used to classify moduli spaces of arrangements of 10 projective lines (see [Amram et al. 2012]).

The paper is organized as follows: In Section 2, we recall results in Nazir and Yoshinaga. In Section 3, we prove that their list of classes of arrangements of 9 lines is complete. In Section 4, we consider the example of arrangements of 10 lines $\mathscr{C}^{\pm}$ and give an explicit diffeomorphism between the complements $M(\mathscr{C}^{\pm})$.

2. Simple $C_{\leq 3}$ line arrangements

Consider the dual space $(\mathbb{CP}^2)^*$ of the projective space $\mathbb{CP}^2$. A line arrangement $\mathscr{A} = \{L_1, L_2, \ldots, L_n\}$ can be viewed as an n-tuple of points $(L_1^*, L_2^*, \ldots, L_n^*)$ in the product of the dual spaces $((\mathbb{CP}^2)^*)^n$. We define the moduli space of arrangements with the fixed lattice $L(\mathscr{A})$ as

$$\mathcal{M}_{\mathscr{A}} = \frac{\{\mathscr{B} \in ((\mathbb{CP}^2)^*)^n \mid L(\mathscr{B}) \sim L(\mathscr{A})\}}{\mathrm{PGL}_3(\mathbb{C})} \subseteq \frac{((\mathbb{CP}^2)^*)^n}{\mathrm{PGL}_3(\mathbb{C})}.$$

We say that a singular point P of $L_1 \cup L_2 \cup \cdots \cup L_n$ is a *multiple point* of $\mathscr{A}$ if the multiplicity of P is at least 3.

The following definition is a combination of Nazir and Yoshinaga's original definitions of C_1, C_2, and simple C_3 arrangements.

Definition 2.1. A line arrangement is called $C_{\leq 3}$ if all the multiple points are on at most three lines; say, L_1, L_2, and L_3. A line arrangement is called simple $C_{\leq 3}$ if it is $C_{\leq 3}$, and one of the following condition holds:

(i) $L_1 \cap L_2 \cap L_3 \neq \varnothing$, or

(ii) one of L_1, L_2 and L_3 contains at most one more multiple point apart from the possible multiple points $L_1 \cap L_2$, $L_2 \cap L_3$, and $L_1 \cap L_3$.

Here are some examples of arrangements which are not simple $C_{\leq 3}$:

Example 2.2. A *Mac Lane arrangement* (see Figure 1) consists of eight lines and eight triple points such that each line passes through exactly three triple points. It is not hard to check that the moduli space of Mac Lane arrangements consists of two points. Representatives of the two points can be defined by the equation

$$xy(x-z)(y-z)(x-y)(x-\varepsilon^{\pm}z)(y-\varepsilon^{\pm}z)(-\varepsilon^{\mp}x-y+z)=0,$$

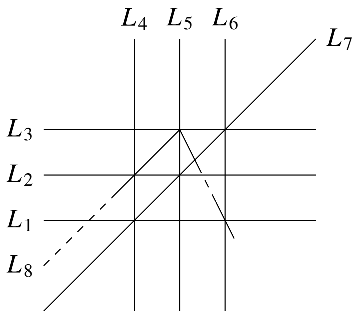

Figure 1. A Mac Lane arrangement.

where $\varepsilon^{\pm} = \frac{1}{2}(1 \pm \sqrt{-3})$ are the roots of $x^2 - x + 1 = 0$.

Since each line passes through three triple points, there are at most seven triple points on three lines. Thus, Mac Lane arrangements cannot be simple $C_{\leq 3}$.

Example 2.3. *Falk–Sturmfels arrangements* are the arrangements of nine lines with one quadruple point, eight triple points, and one line passing through four triple points (Figure 2). We denote them by

$$FS^{\pm} = \{L_i^{\pm}, K_i^{\pm}, H_9^{\pm}, \ i = 1, 2, 3, 4\},$$

where the lines are defined by

$$\begin{aligned}
&L_1^{\pm} : x = 0, \quad L_2^{\pm} : x = \gamma_{\pm}(y - z), \quad L_3^{\pm} : y = z, \quad L_4^{\pm} : x + y = z,\\
&K_1^{\pm} : x = z, \quad K_2^{\pm} : x = \gamma_{\pm} y, \qquad\quad K_3^{\pm} : y = 0, \quad K_4^{\pm} : x + y = (\gamma_{\pm} + 1)z,\\
&H_9^{\pm} : z = 0,
\end{aligned}$$

with $\gamma_{\pm} = \frac{1}{2}(1 \pm \sqrt{5})$ the roots of $x^2 - x - 1 = 0$. It is known (see [Nazir and

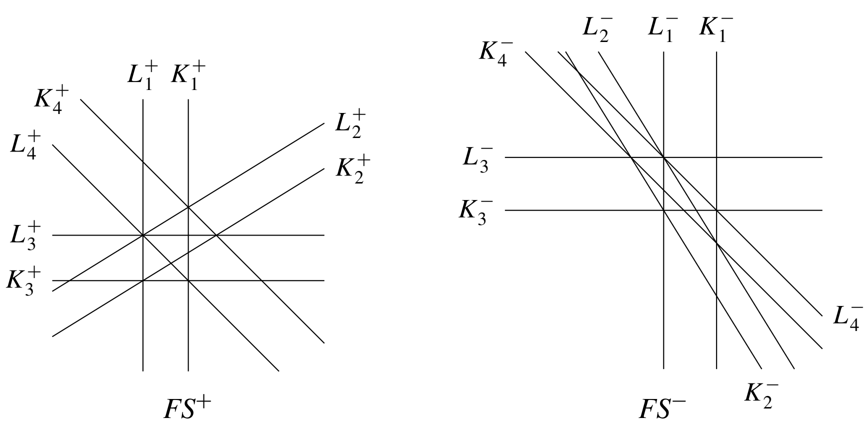

Figure 2. Falk–Sturmfels arrangements.

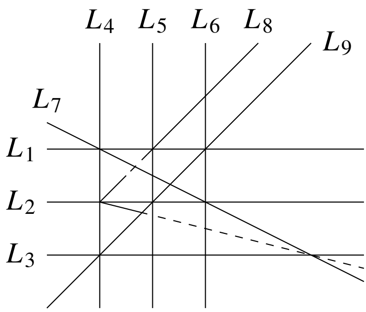

Figure 3. The arrangements $\mathscr{A}^{\pm\sqrt{-1}}$.

Yoshinaga 2012, Example 5.2], for instance) that the moduli space $\mathcal{M}_{L(FS^{\pm})}$ consists of 2 points, $\{FS^+, FS^-\}$.

Example 2.4 [Nazir and Yoshinaga 2012, Example 5.3]. The arrangements $\mathscr{A}^{\pm\sqrt{-1}}$ consist of nine lines and ten triple points such that there are three lines which do not intersect at a point and have four triple points on each. Moreover, each of the other six lines contains exactly three triple points. Those arrangements (see Figure 3) can be defined by the equation

$$xy(x-z)(y-z)(x\mp \mathrm{i}z)(y\mp \mathrm{i}z)(x-y)\big((\pm\mathrm{i}-1)x\pm\mathrm{i}y+z\big)\big((1\mp\mathrm{i})x+y-z\big)=0.$$

Lemma 2.5 [Nazir and Yoshinaga 2012, Lemma 4.4]. *If a line arrangement is not simple $C_{\leq 3}$, then it has 6 lines $L_1, L_2, \dots, L_6$ such that $L_1\cap L_2\cap L_3\neq\varnothing$, $L_4\cap L_5\cap L_6\neq\varnothing$, and $(L_1\cup L_2\cup L_3)\cap(L_4\cup L_5\cup L_6)$ consists of 9 distinct double points.*

Let $\mathscr{A}_s=\{L_1, L_2, \dots, L_6\}$ be the arrangement which has two triple points $L_1\cap L_2\cap L_3$ and $L_4\cap L_5\cap L_6$, and nine double points $Q_{ij}=L_i\cap L_{j+3}$, where $i, j\in\{1,2,3\}$.

Using Lemma 2.5, one can easily prove that an arrangement of 7 lines is simple $C_{\leq 3}$. It is also not hard to prove the following result:

Proposition 2.6 [Nazir and Yoshinaga 2012, Proposition 4.6]. *An arrangement of eight lines is either a simple $C_{\leq 3}$ line arrangement or a Mac Lane arrangement.*

More generally:

Theorem 2.7 [Nazir and Yoshinaga 2012, Theorem 3.5]. *The moduli space $\mathcal{M}_{\mathscr{A}}$ of simple $C_{\leq 3}$ line arrangements with the fixed intersection lattice $L(\mathscr{A})$ is irreducible.*

Let $\mathscr{A}=\{L_1, L_2, \dots, L_n\}$ be a line arrangement, and $\mathscr{A}'=\{L_1, L_2, \dots, L_{n-1}\}$ be a subarrangement. The following lemma shows when the irreducibility of the moduli space $\mathcal{M}_{\mathscr{A}'}$ will be inherited:

Lemma 2.8 [Nazir and Yoshinaga 2012, Lemma 2.4]. *Assume that the line L_n passes through at most two multiple points of the arrangement $\mathscr{A}$. Then the moduli space $\mathcal{M}_{\mathscr{A}}$ is a fiber bundle over the moduli space of $\mathcal{M}_{\mathscr{A}'}$. In particular, the moduli space $\mathcal{M}_{\mathscr{A}}$ is irreducible if $\mathcal{M}_{\mathscr{A}'}$ is irreducible.*

Applying this lemma to arrangements of 9 lines, we have the following corollary:

Corollary 2.9. *Let $\mathscr{A}$ be an arrangement of* 9 *lines. If there is a line in $\mathscr{A}$ which passes through at most two multiple points of $\mathscr{A}$, then either $\mathscr{A}$ contains a Mac Lane arrangement as a subarrangement, or the moduli space $\mathcal{M}_{\mathscr{A}}$ is irreducible.*

Proof. The conclusion follows directly from Proposition 2.6 and Lemma 2.8. □

3. Classification of arrangements of 9 lines

For a line arrangement $\mathscr{A}$, we denote by $m_{\mathscr{A}}$ the highest multiplicity of a multiple point of $\mathscr{A}$. We will divide the classification of arrangements of 9 lines into three cases according to the value of $m_{\mathscr{A}}$.

Let n_r be the number of multiple points of multiplicity r. We first recall two well-known results on the number of multiple points.

Theorem 3.1 [Hirzebruch 1986]. *Let $\mathscr{A}$ be an arrangement of t lines in $\mathbb{CP}^2$. Assume that $n_t = n_{t-1} = n_{t-2} = 0$. Then,*

$$n_2 + \tfrac{3}{4}n_3 \geq t + \sum_{r\geq 5}(2r-9)n_r.$$

Lemma 3.2 (see, for instance, [Hirzebruch 1986]). *Let $\mathscr{A}$ be a line arrangement of n lines in $\mathbb{CP}^2$. We have the intersection formula*

$$\frac{n(n-1)}{2} = \sum_{r\geq 2}\left(n_r \cdot \frac{r(r-1)}{2}\right).$$

3A. *The case $m_{\mathscr{A}} \geq 5$.*

Proposition 3.3. *Let $\mathscr{A}$ be an arrangement of* 9 *lines. If $\mathscr{A}$ has multiple points of multiplicity* (*at least* 5), *then the moduli space $\mathcal{M}_{\mathscr{A}}$ is irreducible.*

Proof. Assume that $L_1 \cap L_2 \cap \cdots \cap L_5 \neq \varnothing$. There are at most 6 double points in $L_6 \cup L_7 \cup L_8 \cup L_9$. Then, there are at most 7 multiple points in $L_1 \cup L_2 \cup \cdots \cup L_5$. So, at least one of the five lines $L_1, L_2, \ldots, L_5$ contains only two multiple points. By Corollary 2.9, the moduli space $\mathcal{M}_{\mathscr{A}}$ is irreducible. □

3B. ***The case $m_{\mathscr{A}} = 4$.*** Let $\mathscr{A}$ be an arrangement of 9 lines. In this subsection, we assume that multiple points of $\mathscr{A}$ are at most quadruple points.

Proposition 3.4. *Assume that each line of $\mathcal{A}$ passes through at least three multiple points, and $n_4 \geq 1$. Then either $\mathcal{M}_{\mathcal{A}}$ is irreducible, or $\mathcal{A}$ is lattice isomorphic to a Falk–Sturmfels arrangement.*

Proof. We will first show that $n_4 = 1$.

Let $L_1 \cap L_2 \cap L_3 \cap L_4$ be a quadruple point of $\mathcal{A}$. Since each line passes through at least three multiple points, L_1, L_2, L_3 and L_4 should pass through two more multiple points besides the quadruple point $L_1 \cap L_2 \cap L_3 \cap L_4$. Then, there will be at least 9 multiple points on those four lines. Since multiple points of $\mathcal{A}$ are at most quadruple points, there are n_4 quadruple points. Therefore, there should be at least $9 - n_4$ triple points on those four lines such that each line passes through at least 3 multiple points. By Theorem 3.1 and Lemma 3.2, we have

$$36 = 6n_4 + 3n_3 + n_2 \geq 6n_4 + \tfrac{9}{4}n_3 + 9 \geq 6n_4 + \tfrac{9}{4}(9 - n_4) + 9.$$

Solving the inequality, we obtain that $n_4 \leq \frac{9}{5} < 2$. Therefore, by the assumption, we have $n_4 = 1$.

Now we claim that all triple points should be on the lines passing through the quadruple point.

Let $L_1 \cap L_2 \cap L_3 \cap L_4$ be the quadruple. Suppose, contrary to our claim, that $L_5 \cap L_6 \cap L_7$ is a triple point which is not on $L_1 \cup L_2 \cup L_3 \cup L_4$. Note that there are at most 7 double points on $L_5 \cup L_6 \cup L_7 \cup L_8 \cup L_9$. Then the intersection set $(L_1 \cup L_2 \cup L_3 \cup L_4) \cap (L_5 \cup L_6 \cup L_7 \cup L_8 \cup L_9)$ will contain at most 7 triple points which are on $L_1 \cup L_2 \cup L_3 \cup L_4$. However, there should be at least 8 triple points so that each of the four lines L_1, L_2, L_3 and L_4 passes through at least three multiple points. Therefore, by the assumption, all triple points must be on the lines passing through the quadruple point.

If $\mathcal{A}$ is simple $C_{\leq 3}$, then the moduli space $\mathcal{M}_{\mathcal{A}}$ is irreducible. We only need to consider the case that $\mathcal{A}$ is not simple $C_{\leq 3}$. By Lemma 2.5, we know that the arrangement $\mathcal{A}$ has a subarrangement $\mathcal{A}_s$. It is not hard to see that the quadruple point should be Q_{ij}, where $i, j \in \{1, 2, 3\}$.

Up to a lattice isomorphism, we may assume that the only quadruple point is $L_1 \cap L_4 \cap L_7 \cap L_8 = Q_{11}$.

Since all triple points should be on $L_1 \cup L_4 \cup L_7 \cup L_8$, then all possible triple points on L_7 and L_8 should be in the following set of points:

$$\{Q_{22}, Q_{23}, Q_{32}, Q_{33}, L_7 \cap L_9, L_8 \cap L_9\}.$$

The following figure is an example, but an excluding one, for L_6 passes through only one triple point.

Hence, each of the lines L_7 and L_8 will have at least one Q_{ij}, where $i, j \in \{2, 3\}$.

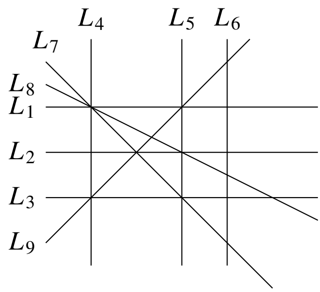

Figure 4. An excluding arrangement.

(1) Assume that each of the lines L_7 and L_8 passes through exactly one of the points $\{Q_{22}, Q_{23}, Q_{32}, Q_{33}\}$.

If those two Q_{ij} are on the same line, then one of the four lines L_2, L_3, L_5 and L_6 will have at most two multiple points. For example, in Figure 4, the line L_6 passes through only one multiple point, $L_4 \cap L_5 \cap L_6$.

Assume that they are not on the same line. Up to switching labels between L_2 and L_3, correspondingly L_5 and L_6, we may assume that $Q_{32} \in L_7$ and $Q_{23} \in L_8$. Then, either $\{Q_{31}, Q_{13}\} \subset L_9$ or $\{Q_{21}, Q_{12}\} \subset L_9$. Correspondingly, $\{L_2 \cap L_7, L_5 \cap L_8\} \subset L_9$ or $\{L_3 \cap L_7, L_6 \cap L_8\} \subset L_9$. By switching the labels between L_2 and L_3, L_5 and L_6, and L_7 and L_8, we see that those two arrangements are lattice isomorphic. Moreover, one can check that both arrangements (see Figure 5, left) are lattice isomorphic to Falk–Sturmfels arrangements.

(2) Assume that either the line L_7 or L_8 passes through two points out of the four points Q_{22}, Q_{23}, Q_{32} and Q_{33}, but the other one passes through only one point out of the four points Q_{22}, Q_{23}, Q_{32} and Q_{33}.

Up to a lattice isomorphism, we may assume that $\{Q_{11}, Q_{22}, Q_{33}\} \subset L_7$, and $\{Q_{11}, Q_{32}\} \subset L_8$. Then either $L_2 \cap L_8 \in L_9$, or $L_6 \cap L_8 \in L_9$. Otherwise, L_8 will have only two multiple points. Correspondingly, $\{Q_{31}, Q_{13}\} \subset L_9$, or $\{Q_{21}, Q_{12}\} \subset L_9$.

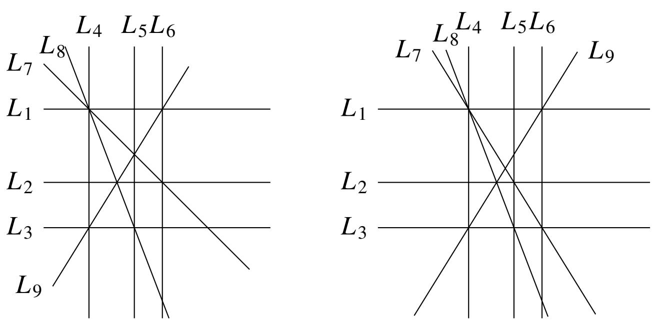

Figure 5. Falk–Sturmfels arrangements 1 and 2.

By first switching the labels between L_1 and L_4, L_2 and L_5, and L_3 and L_6, then switching the labels between L_2 and L_3, and L_5 and L_6, we see that those two arrangements are lattice isomorphic. Moreover, we check that $\mathcal{A}$ (see Figure 5, right) is also lattice isomorphic to Falk–Sturmfels arrangements.

(3) Assume that L_7 and L_8 each contain two of $\{Q_{22}, Q_{23}, Q_{32}, Q_{33}\}$, then L_9 will contain at most two multiple points.

Therefore, we conclude that either $\mathcal{M}_{\mathcal{A}}$ is irreducible or $\mathcal{A}$ is lattice isomorphic to a Falk–Sturmfels arrangement. □

3C. *The case $m_{\mathcal{A}} = 3$.* Now we consider the last case in which all multiple points are triple points. We will first investigate possible values of n_3 such that each line has at least three triple points. Notice that n_3 should be no less than 9. On the other hand, we observe the following result:

Lemma 3.5. *Let $\mathcal{A}$ be an arrangement of* 9 *lines, all of whose multiple points are triple points. Assume that $\mathcal{A}$ does not contain a Mac Lane arrangement as a subarrangement and is not simple $C_{\leq 3}$. Then $\mathcal{A}$ has at most* 10 *triple points.*

Proof. By Lemma 3.2, to show that $n_3 \leq 10$, it is enough to show that $n_2 \geq 4$.

Since $\mathcal{A}$ does not contain a Mac Lane arrangement, at most one of the lines L_7, L_8, and L_9 passes through three Q_{ij}, where $i, j \in \{1, 2, 3\}$ (defined as above). We may assume that each of the lines L_7 and L_8 passes through at most two Q_{ij}. By our assumption and Lemma 2.5, the arrangement $\mathcal{A}$ has a subarrangement $\mathcal{A}_s$.

Let x be the number of Q_{ij} which are not in $L_7 \cup L_8 \cup L_9$. It is clear that $x \geq 2$. Let y and z be the number of double points of $\mathcal{A}$ which are in $L_7 \cap (L_1 \cup L_2 \cup \cdots \cup L_6)$ and $L_8 \cap (L_1 \cup L_2 \cup \cdots \cup L_6)$ respectively. If $y + z \geq 2$, then we have $n_2 \geq x + (y + z) \geq 4$.

Assume that $y + z \leq 1$. Then each of the lines L_7 and L_8 should pass through exactly two Q_{ij}. Moreover, $L_7 \cap L_8$ must be a triple point in $L_1 \cup L_2 \cup \cdots \cup L_6$. We see now the subarrangement $\mathcal{A}' = \{L_1, L_2, \ldots, L_8\}$ has 7 double points. Without a loss of generality, we assume that $L_7 \cap L_8$ is on L_2. It is not hard to see that the 7 double points of $\mathcal{A}'$ are all on $L_4 \cup L_5 \cup L_6$. The line L_9 can only pass through at most three double points of $\mathcal{A}'$. Therefore, the arrangement $\mathcal{A}$ still has at least 4 double points. □

Remark 3.6. By Theorem 2.15 in [Csima and Sawyer 1993], if our arrangements are *real arrangements*, that is, if the coefficients of the defining equations of the lines are real numbers, then there are at least $\frac{60}{13} > 4$ double points. Hence, there should be at most 10 triple points. However, there seems to be no similar result for complex line arrangements.

Proposition 3.7. *Let $\mathcal{A}$ be an arrangement of* 9 *lines with* 9 *triple points. Assume that all multiple points of $\mathcal{A}$ are triple points, and each line passes through exactly three triple points. Then, the moduli space $\mathcal{M}_{\mathcal{A}}$ is irreducible.*

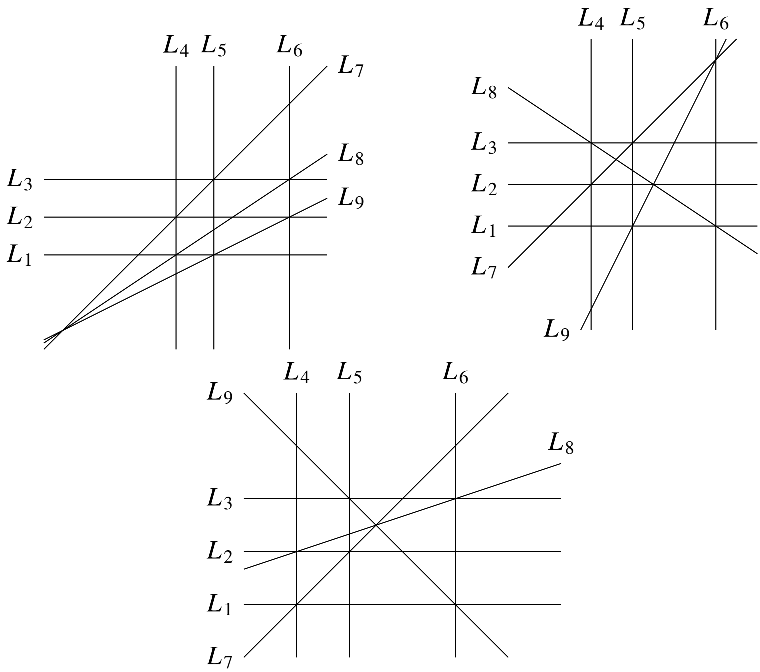

Figure 6. 9_3 arrangements.

Proof. By Theorem 2.2.1 in [Grünbaum 2009] that $\mathscr{A}$ is lattice isomorphic to one of the three arrangements appearing in Figure 6.

One can check that the moduli space $\mathcal{M}_{\mathscr{A}}$ is irreducible in each case. □

Proposition 3.8. *Let $\mathscr{A}$ be an arrangement of 9 lines with 10 triple points. Assume that all multiple points of $\mathscr{A}$ are triple points and each line passes through at least three triple points. If $\mathscr{A}$ is not simple $C_{\leq 3}$, then it is isomorphic to $\mathscr{A}^{\pm\sqrt{-1}}$* (Figure 3).

Proof. Let a be the number of lines that pass through 4 triple points and b the number of lines that pass through 3 triple points. Then $a+b=9$ and $4a+3b=30$. We have $a=3$ and $b=6$.

If the three lines with 4 triple points on each of them intersect at a triple point, then all 10 triples should be on them. Consequently, the arrangement is simple $C_{\leq 3}$.

Assume that L_1, L_2 and L_4 are the three lines with 4 triple points on each of them and $L_1 \cap L_2 \cap L_4 = \varnothing$. Then, at least two of $\{L_1 \cap L_2, L_1 \cap L_4, L_2 \cap L_4\}$ are triple points. Otherwise there should be at least 11 triple points so that L_1, L_2, and L_4 will have 4 triple points. So, we may assume that $L_1 \cap L_2 \cap L_3$ and $L_1 \cap L_4 \cap L_7$ are triple points. Let $L_4 \cap L_5 \cap L_6$ be a triple point which is not on $L_1 \cup L_2 \cup L_3$. Then, L_7 must pass through $L_2 \cap L_5$ or $L_2 \cap L_6$. Otherwise, L_2 will have at most 3 triples. By switching labels of L_5 and L_6, we may assume that

$L_2 \cap L_6 \cap L_7 \neq \varnothing$. Then the two points Q_{21} and Q_{22} must be on $L_8 \cup L_9$ so that L_2 will pass through 4 triple points. We may assume that $Q_{21} \in L_8$ and $Q_{22} \in L_9$. Since the line L_4 also passes through 4 triple points, then Q_{31} should be on L_9. Similarly, since the line L_1 passes through 4 triple points, then Q_{13} should be on L_9 and Q_{12} should be on L_8. Now we have 9 triple points. The last triple point must be $L_3 \cap L_7 \cap L_8$ so that L_7 will pass through three triple points. The arrangements with such intersection lattices are just $\mathscr{A}^{\pm\sqrt{-1}}$ (see Figure 3). □

3D. *Classification and applications.* We summarize Section 3 so far as follows:

Theorem 3.9. *Any arrangement of nine lines in $\mathbb{CP}^2$ belongs to one of the following classes*:

(i) *arrangements whose moduli spaces are irreducible*;

(ii) *arrangements containing Mac Lane arrangements* (Example 2.2);

(iii) *Falk–Sturmfels arrangements* (Example 2.3);

(iv) $\mathscr{A}^{\pm\sqrt{-1}}$ *arrangements* (Example 2.4).

Proof. The classification simply follows from Corollary 2.9 and Propositions 3.3, 3.4, 3.7, and 3.8. □

As an application, we obtain the following result which generalizes a result of Theorem 8.3 in [Garber et al. 2003].

Theorem 3.10. *The fundamental group of the complement of an arrangement of* 9 *lines is determined by the intersection lattice.*

Proof. If the moduli space is irreducible, then the fundamental group is determined by the lattice according to the lattice-isotopy theorem.

It follows from Example 5.2 in [Nazir and Yoshinaga 2012] (see also Section 7.5 in [Cohen and Suciu 1997]) that the fundamental groups $\pi_1(M(FS^+))$ and $\pi_1(M(FS^-))$ are isomorphic. Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two arrangements containing Mac Lane arrangements. Then, either they are in the same connected component of the moduli spaces, or $\mathscr{A}_1$ and the conjugate of $\mathscr{A}_2$ are in the same connected component. By Theorem 3.9 in [Cohen and Suciu 1997], the fundamental groups of $\mathscr{A}_1$ and $\mathscr{A}_2$ are isomorphisms. According to the same theorem, the fundamental groups of $\mathscr{A}^{+\sqrt{-1}}$ and $\mathscr{A}^{-\sqrt{-1}}$ are isomorphic too. □

4. Arrangements of 10 lines: an example

We have seen that there is no Zariski pair of arrangements of 9 lines, but we do not know if there is a Zariski pair of arrangements of 10 lines. To get a Zariski pair, a naive idea is to add lines to those arrangements whose moduli spaces are disconnected. In general, it is very hard to determine if the resulting pair of arrangements is a Zariski pair. The following example is a trial:

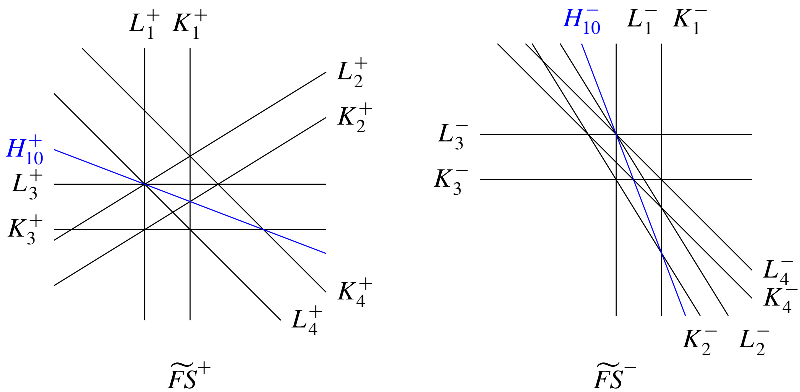

Figure 7. Extended Falk–Sturmfels arrangement.

Example 4.1. Starting from the Falk–Sturmfels arrangements (see Example 2.3), we will construct new arrangements of 10 lines such that the moduli space is disconnected.

We define two line arrangements of 10 lines, called extended Falk–Sturmfels arrangements (see Figure 7):

$$\widetilde{FS}^{\pm} = \{L_i^{\pm}, K_i^{\pm}, H_9^{\pm}, H_{10}^{\pm}, \ i = 1, 2, 3, 4\}$$

by adding lines:

$$H_{10}^{\pm} : y = \left(\frac{1}{\gamma_{\pm}} - 1\right)x + z$$

to $FS^{\pm}$ respectively.

Notice that $\widetilde{FS}^{\pm}$ are both fiber-type line arrangements according to Theorem 3.12 in [Jiang et al. 2001] .

It is not hard to see that $\mathcal{M}_{\widetilde{FS}^{\pm}} \cong \mathcal{M}_{FS^{\pm}}$. In fact, the line H_{10}^+ (respectively, H_{10}^-) is always passing through three points of $L(FS^{\pm})$: $L_1^+ \cap L_2^+$, $K_1^+ \cap K_2^+$ and $K_3^+ \cap K_4^+$ (respectively, $K_2^- \cap K_4^-$, $K_3^- \cap K_4^-$ and $K_1^- \cap K_2^-$).

This pair of arrangements has been studied by Artal, Carmona, Cogolludo, and Marco. They show (Theorem 4.19 in [Artal et al. 2005]) that there is no order-preserving homeomorphism between the pairs $(\mathbb{P}^2, \widetilde{FS}^+)$ and $(\mathbb{P}^2, \widetilde{FS}^-)$. Here, we present an explicit diffeomorphism between the complements $M(\widetilde{FS}^+)$ and $M(\widetilde{FS}^-)$. In fact, by Example 5.2 in [Nazir and Yoshinaga 2012], we know that there is an automorphism $A \in \mathrm{PGL}(\mathbb{C}^3)$ of $\mathbb{CP}^2$,

$$A := \begin{pmatrix} -\gamma_- & -1 & 0 \\ -\gamma_- & 0 & 0 \\ \gamma_- & 1 & 1 \end{pmatrix},$$

acting from the right (via matrix multiplication) on points $[x, y, z]$ in the projective space $\mathbb{P}^2$, which sends

$$\begin{array}{llll} L_1^+ \mapsto L_3^-, & L_2^+ \mapsto L_4^-, & L_3^+ \mapsto L_2^-, & L_4^+ \mapsto L_1^-, \\ K_1^+ \mapsto K_3^-, & K_2^+ \mapsto K_4^-, & K_3^+ \mapsto K_2^-, & K_4^+ \mapsto K_1^-, \\ H_9^+ \mapsto H_9^-. & & & \end{array}$$

To see that A induces a diffeomorphism between $M(\widetilde{FS}^+)$ and $M(\widetilde{FS}^-)$, it suffices to show that the automorphism A sends H_{10}^+ to H_{10}^-.

Recall that $\gamma_\pm = \frac{1}{2}(1 \pm \sqrt{5})$. One can check that for any point

$$P := [x, (1/\gamma_+ - 1)x + z, z]$$

on H_{10}^+, the image $P \cdot A$ is a point on H_{10}^-. In fact,

$$\begin{pmatrix} x & (1/\gamma_+ - 1)x + z & z \end{pmatrix} \cdot A \cdot \begin{pmatrix} 1/\gamma_- - 1 \\ -1 \\ 1 \end{pmatrix} \equiv 0.$$

Therefore, the pair $(\widetilde{FS}^+, \widetilde{FS}^-)$ is not a Zariski pair.

From this example, we see that moduli spaces of fiber-type projective line arrangements do not have to be connected. In fact, we can produce infinitely many fiber-type projective line arrangements whose moduli spaces are disconnected. On the other hand, we do not know if fundamental groups of complements of fiber-type projective line arrangements are determined by intersection lattices.

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