ON THE STEINBERG CHARACTER OF A SEMISIMPLE $p$-ADIC GROUP

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Dedicated to Robert Steinberg on the occasion of his ninetieth birthday.

We show that the character of the Steinberg representation of a split semisimple $p$-adic group at a very regular element is given (up to sign) by a power of $q$, the number of elements in the residue field. We also show that (under an assumption on the characteristic) the character of an Iwahori-spherical representation at a split very regular element is given by a trace in the corresponding Hecke algebra module.

1. Introduction

1.1. Let $K$ be a nonarchimedean local field and let $\overline{K}$ be a maximal unramified field extension of $K$. Let $\mathcal{O}$ be the ring of integers of $K$ and let $p$ be the maximal ideal of $\mathcal{O}$; the counterparts for $\overline{K}$ are denoted by $\overline{\mathcal{O}}$ and $\overline{p}$. Let $\overline{K}^* = \overline{K} - \{0\}$. We write $\mathcal{O}/p = F_q$, a finite field with $q$ elements of characteristic $p$.

Let $G$ be a semisimple almost simple algebraic group defined and split over $K$ with a given $\mathcal{O}$-structure compatible with the $K$-structure.

If $V$ is an admissible representation of $G(K)$ of finite length, we denote by $\phi_V$ the character of $V$ in the sense of Harish-Chandra, viewed as a $\mathbb{C}$-valued function on the set $G(K)_{rs} := G_{rs} \cap G(K)$. (Here, $G_{rs}$ is the set of regular semisimple elements of $G$, and $\mathbb{C}$ is the field of complex numbers.)

In this paper we study the restriction of the function $\phi_V$ to:

(a) a certain subset $G(K)_{vr}$ of $G(K)_{rs}$, namely, the set of very regular elements in $G(K)$ (see 1.2) in the case where $V$ is the Steinberg representation of $G(K)$, and

(b) a certain subset $G(K)_{svr}$ of $G(K)_{vr}$, namely, the set of split very regular elements in $G(K)$ (see 1.2) in the case where $V$ is an irreducible admissible representation of $G(K)$ with nonzero vectors fixed by an Iwahori subgroup.

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In case (a), we show that $\phi_Y(g)$ with $g \in G(K)_{rs}$ is of the form $\pm q^n$ with $n \in \{0, -1, -2, \ldots \}$ (see Corollary 3.4) with more precise information when $g \in G(K)_{s_{sr}}$ (see Theorem 2.2) or when $g \in G(K)_{c_{ur}}$ (see Theorem 3.2). In case (b) we show (with some restriction on characteristic) that $\phi_Y(g)$ with $g \in G(K)_{s_{sr}}$ can be expressed as the trace of a certain element of an affine Hecke algebra on an irreducible module (see Theorem 4.3).

Note that the Steinberg representation $\mathbf{S}$ is an irreducible admissible representation of $G(K)$ with a one-dimensional subspace invariant under an Iwahori subgroup on which the corresponding affine Hecke algebra acts through the “sign” representation; see [Matsumoto 1969; Shalika 1970]. This is a $p$-adic analogue of the Steinberg representation [Steinberg 1951] of a reductive group over $F_p$-adic representation; see [Matsumoto 1969; Shalika 1970]. This is a $p$-adic analogue of the Steinberg representation [Steinberg 1951] of a reductive group over $F$. In [Rodier 1986], it is proven that $\phi_S(g) \neq 0$ for any $g \in G(K)_{rs}$.

1.2. Let $g \in G_{rs} \cap G(K)$. Let $T' = T'_g$ be the maximal torus of $G$ that contains $g$. We say that $g$ is very regular if $T'$ is split over $K$ and for any root $\alpha$ with respect to $T'$ viewed as a homomorphism $T'(K) \to K^*$ we have $\alpha(g) \notin (1 + p)$. If, in addition, $\alpha(g) \in \mathcal{O}$, we say that $g$ is compact very regular.

Let $G(K)_{vr}$ be the set of elements in $G(K)$ that are very regular, and $G(K)_{c_{ur}}$ the set of compact very regular ones. We write $G(K)_{vr} = G(K)_{vr} \cap G(K)$ and $G(K)_{c_{ur}} = G(K)_{c_{ur}} \cap G(K)$. Let $G(K)_{s_{ur}}$ be the set of all $g \in G(K)_{vr}$ such that $T'_g$ is split over $K$.

1.3. Notation. Let $K^* = K - \{0\}$, and let $v : K^* \to \mathbb{Z}$ be the unique (surjective) homomorphism such that $v(p^n - p^{n+1}) = n$ for any $n \in \mathbb{N}$. For $a \in K^*$ we set $|a| = q^{-v(a)}$.

We fix a maximal torus $T$ of $G$ defined and split over $K$. Let $Y$ (resp. $X$) be the group of cocharacters (resp. characters) of the algebraic group $T$. Let $\langle , , \rangle : Y \times X \to \mathbb{Z}$ be the obvious pairing. Let $R \subset X$ be the set of roots of $G$ with respect to $T$, let $R^+$ be a set of positive roots for $R$, and let $\Pi$ be the set of simple roots of $R$ determined by $R^+$. We write $\Pi = \{\alpha_i : i \in I\}$. Let $R^- = R - R^+$. Let $Y^+$ (resp. $Y^{++}$) be the set of all $y \in Y$ such that $\langle y, \alpha \rangle \geq 0$ (resp. $\langle y, \alpha \rangle > 0$) for all $\alpha \in R^+$. We define $2\rho \in X$ by $2\rho = \sum_{\alpha \in R^+} \alpha$.

We have canonically $T(K) = K^* \otimes Y$; we define a homomorphism $\chi : T(K) \to Y$ by $\chi(\lambda \otimes y) = v(\lambda)y$ for any $\lambda \in K^*$, $y \in Y$. For any $y \in Y$, we set $T(K)_y = \chi^{-1}(y)$. For $y \in Y$, let $T(K)_y^* = T(K)_y \cap G(K)_{s_{ur}}$. Note that if $y \in Y^{++}$ then $T(K)_y^* = T(K)_y$.

For each $\alpha \in R$ let $U_\alpha$ be the corresponding root subgroup of $G$.

2. Calculation of $\phi_S$ on $G(K)_{s_{ur}}$

2.1. Let $\mathcal{W} \subset \text{Aut}(T)$ be the Weyl group of $G$ regarded as a Coxeter group; for $i \in I$, let $s_i$ be the simple reflection in $\mathcal{W}$ determined by $\alpha_i$. We can also view
We show that \( \varphi_J \) where for any \( J \subset I \), let \( \mathcal{W}_J \) be the subgroup of \( \mathcal{W} \) generated by \( \{ s_i : i \in J \} \) and let \( R_J = R \cap \sum_{i \in J} \mathbb{Z}a_i \). Let
\[
R_J^+ = R_J \cap R^+ \quad \text{and} \quad R_J^- = R_J - R_J^+.
\]

Let \( \mathfrak{g} \) be the Lie algebra of \( G \), and let \( \mathfrak{t} \subset \mathfrak{g} \) be the Lie algebra of \( T \). For any \( J \subset I \), let \( I_J \) be the Lie subalgebra of \( \mathfrak{g} \) spanned by \( \mathfrak{t} \) and the root spaces corresponding to the roots in \( R_J \). Let \( n_J \) be the Lie subalgebra of \( \mathfrak{g} \) spanned by the root spaces corresponding to roots in \( R^+ - R_J^+ \).

According to [Casselman 1973], \( \phi_S \) is an alternating sum of characters of representations induced from one-dimensional representations of various parabolic subgroups of \( G \) defined over \( K \). From this, one can deduce that if \( t \in T(K) \cap G(K)_{rs} \) then
\[
\phi_S(t) = \sum_{J \subset I} (-1)^{|J|} \sum_{w \in J^\mathcal{W}} \delta_J(w(t))^{1/2} D_{I,J}(w(t))^{-1/2},
\]
where for any \( J \subset I \) and \( t' \in T(K) \cap G(K)_{rs} \), we set
\[
D_{I,J}(t') = \left| \det(1 - \text{Ad}(t')|_{\mathfrak{g}/I_J}) \right|,
\]
\[
\delta_J(t') = \left| \det(\text{Ad}(t')|_{n_J}) \right|,
\]
and \( J^\mathcal{W} \) is the set of representatives of minimal length for the cosets \( \mathcal{W}_J \backslash \mathcal{W} \). Here for a real number \( a \geq 0 \) we denote by \( a^{1/2} \) or \( \sqrt{a} \) the nonnegative square root of \( a \).

Writing \( \phi \) instead of \( \phi_S \), we have:

**Theorem 2.2.** Let \( y \in Y^+ \) and let \( t \in T(K)_y^\bullet \). Then \( \phi(t) = q^{-(y, 2\rho)} \).

2.3. More generally, let \( t \in T(K)_y^\bullet \), where \( y \in Y \). By a standard property of Weyl chambers, there exists \( w \in \mathcal{W} \) such that \( w(y) \in Y^+ \). Let \( t_1 = w(t) \). Then the theorem is applicable to \( t_1 \), and we have \( \phi(t) = \phi(t_1) = q^{-(w(y), 2\rho)} \).

2.4. Let \( y' = w_0(y) \), \( t' = w_0(t) \). We have \( \phi_S(t) = \phi_S(t'), t' \in T(K)_{y'}^\bullet, -y' \in Y^+ \).

We show that
\[
v(1 - \beta(t')) = \begin{cases} 
  v(\beta(t')) & \text{if } \beta \in R^+, \\
  0 & \text{if } \beta \in R^-.
\end{cases}
\]

Assume first that \( \beta \in R^+ \). If \( v(\beta(t')) \neq 0 \) then \( v(\beta(t')) < 0 \) (since \( \langle y', \beta \rangle \neq 0 \) and \( \langle y', \beta \rangle \leq 0 \)); hence, \( v(1 - \beta(t')) = v(\beta(t')) \). If \( v(\beta(t')) = 0 \) then \( \beta(t') - 1 \in O - p \); hence, \( v(1 - \beta(t')) = 0 = v(\beta(t')) \) as required.

Assume next that \( \beta \in R^- \). If \( v(\beta(t')) \neq 0 \) then \( v(\beta(t')) > 0 \) (since \( \langle y', \beta \rangle \neq 0 \) and \( \langle y', \beta \rangle \geq 0 \)); hence, \( v(1 - \beta(t')) = 0 \). If \( v(\beta(t')) = 0 \) then \( \beta(t') - 1 \in O - p \); hence, \( v(1 - \beta(t')) = 0 \) as required.
For any $w \in \mathcal{W}$, $J \subset I$ we have

$$D_{t, J}(w(t')) = \prod_{\alpha \in R^- R_J} q^{-v(1-\alpha(w(t')))} = \prod_{\alpha \in R^- R_J} q^{-v(\alpha(w(t')))} = \prod_{\alpha \in R^- R_J} q^{-(\alpha', w^{-1} \alpha)}$$

and

$$\delta_J(w(t')) = \prod_{\alpha \in R^+ - R^+_J} q^{-v(\alpha(w(t')))} = \prod_{\alpha \in R^+ - R^+_J} q^{-(\alpha', w^{-1} \alpha)}.$$

(We have used (1) with $\beta = w^{-1}(\alpha)$.) We see that

$$\phi(t) = \phi(t') = \sum_{J \subset I} (-1)^{\#J} \sum_{w \in J \mathcal{W}} \sqrt{q}^{-\langle \alpha', x_w \rangle},$$

where for $w \in J \mathcal{W}$ we have

$$x_{w, J} = \sum_{\alpha \in R^+ - R^+_J} w^{-1} \alpha - \sum_{\alpha \in R^- R_J} w^{-1} \alpha$$

$$= \sum_{\alpha \in R^+ - R^+_J} w^{-1} \alpha - \sum_{\alpha \in R^- R_J} w^{-1} \alpha$$

$$= 2 \sum_{\alpha \in R^+ - R^+_J} w^{-1} \alpha \in X.$$

For $w \in J \mathcal{W}$, we have $\alpha \in R^+_J \Rightarrow w^{-1} \alpha \in R^+$; hence,

$$\sum_{\alpha \in R^+ - R^+_J} w^{-1} \alpha = \sum_{\alpha \in R^+} w^{-1} \alpha,$$

so that $x_{w, J} = x_w$, where

$$x_w = 2 \sum_{\alpha \in R^+} w^{-1} \alpha \in X.$$

Thus, we have

$$\phi(t) = \sum_{J \subset I} (-1)^{\#J} \sum_{w \in J \mathcal{W}} \sqrt{q}^{-\langle \alpha', x_w \rangle} = \sum_{w \in \mathcal{W}} c_w \sqrt{q}^{-\langle \alpha', x_w \rangle},$$

where for $w \in \mathcal{W}$ we set

$$c_w = \sum_{J \subset I} (-1)^{\#J}.$$

For $w \in \mathcal{W}$, let $\mathcal{L}(w) = \{ i \in I : s_i w > w \}$, where $>$ refers to the standard partial
order on \( \mathcal{W} \). For \( J \subset I \), we have \( w \in J^I \mathcal{W} \) if and only if \( J \subset \mathcal{L}(w) \); thus,

\[
c_w = \sum_{J \subset \mathcal{L}(w)} (-1)^{|J|},
\]

and this is 0 unless \( \mathcal{L}(w) = \emptyset \) (that is \( w = w_0 \)), in which case \( c_w = 1 \). Note also that \( x_{w_0} = -4p \); thus, we have

\[
\phi(t) = c_{w_0} \sqrt{q}^{-\langle y', x_{w_0} \rangle} = q^{\langle y', 2p \rangle} = q^{-\langle y, 2p \rangle}.
\]

Theorem 2.2 is proved. \( \square \)

2.5. Assume now that \( \tau \in T(K) \) satisfies the following condition: for any \( \alpha \in R \) we have \( \alpha(\tau) - 1 \notin p - \{0\} \) so that \( \alpha(\tau) - 1 \in p^{n_\alpha} - p^{n_\alpha+1} \) for a well defined integer \( n_\alpha \geq 1 \). Note that \( n_{-\alpha} = n_\alpha \) and \( v(1 - \alpha(\tau)) = n_\alpha \geq 1 \) for all \( \alpha \in R \); hence,

\[
\phi(\tau) = \sum_{J \subset I} (-1)^{|J|} \sum_{w \in J^I \mathcal{W}} q^{\sum_{\alpha \in R} n_\alpha / 2 - \sum_{\alpha \in R} n_{-\alpha} / 2}.
\]

Thus,

\[\phi(\tau) = \sharp(\mathcal{W})q^{\sum_{\alpha \in R} n_\alpha / 2} + \text{strictly smaller powers of } q.\]

In the case where \( K \) is the field of power series over \( \mathbb{F}_q \), the leading term in (2) is equal to \( \sharp(\mathcal{W})q^m \), where \( m \) is the dimension of the “variety” of Iwahori subgroups of \( G(K) \) that contain the topologically unipotent element \( \tau \) (see [Kazhdan and Lusztig 1988]).

3. Calculation of \( \phi_s \) on \( G(K)_{vr} \)

3.1. We will again write \( \phi \) instead of \( \phi_s \). In this section we assume that we are given \( \gamma \in G(K)_{vr} \). Let \( T' = T'_{\gamma} \). Note that \( T' \) is defined over \( K \); let \( A' \) be the largest \( K \)-split torus of \( T' \). For any parabolic subgroup \( P \) of \( G \) defined over \( K \) such that \( \gamma \in P \), we set \( \delta_P(\gamma) = |\det(\text{Ad}(\gamma)|_n)| \), where \( n \) is the Lie algebra of the unipotent radical of \( P \).

Let \( X' \) be the set of all pairs \( (P, A) \), where \( P \) is a parabolic subgroup of \( G \) defined over \( K \) and \( A \) is the unique maximal \( K \)-split torus in the center of some Levi subgroup of \( P \) defined over \( K \). Then that Levi subgroup is uniquely determined by \( A \) and is denoted by \( M_A \). Let \( X' = \{(P, A) \in X : A \subset A' \} \). According to [Harish-Chandra 1973], we have

\[
\phi(\gamma) = (-1)^{\dim T} \sum_{(P, A) \in X'} (-1)^{\dim A} \delta_P(\gamma)^{1/2} D_{G/M_A}(\gamma)^{-1/2},
\]

where \( D_{G/M_A}(\gamma) = |\det(1 - \text{Ad}(\gamma)|_{g/i})| \) (we denote by \( I \) the Lie algebra of \( M_A \)).

Theorem 3.2. Assume in addition that \( \gamma \in G(K)_{cvr} \). Then \( \phi(\gamma) = (-1)^{\dim T - \dim A'} \).
Proof. From our assumptions we see that \( \delta_P(\gamma) = 1 = D_{G/M_A}(\gamma) \) for all \((P, A) \in \mathcal{X}'\); hence, (3) becomes

\[
\phi(\gamma) = (-1)^{\dim T} \sum_{(P, A) \in \mathcal{X}'} (-1)^{\dim A}.
\]

Let \( \mathcal{Y} \) be the group of cocharacters of \( A' \) and let \( \mathcal{S}_Y = \mathcal{Y} \otimes \mathbb{R} \). The real vector space \( \mathcal{S}_Y \) can be partitioned into facets \( F_{P, A} \) indexed by \((P, A) \in \mathcal{X}'\) such that \( F_{P, A} \) is homeomorphic to \( \mathbb{R}^{\dim A} \). Note that the Euler characteristic with compact support of \( F_{P, A} \) is \((-1)^{\dim A} \), and the Euler characteristic with compact support of \( \mathcal{S}_Y \) is \((-1)^{\dim A} \mathcal{S}_Y = (-1)^{\dim A}' \). Using the additivity of the Euler characteristic with compact support we see that \( \sum_{(P, A) \in \mathcal{X}'} (-1)^{\dim A} = (-1)^{\dim A}' \); thus, \( \phi(\gamma) = (-1)^{\dim T - \dim A'} \), as required. \( \square \)

3.3. In the setup of 3.1, let \( P_T \) be the parabolic subgroup of \( G \) associated to \( \gamma \) as in [Casselman 1977]. Note that \( P_T \) is defined over \( K \). The following result can be deduced by combining Theorem 3.2 with the results in [Casselman 1977] and with Proposition 2 in [Rodier 1986].

Corollary 3.4. We have \( \phi(\gamma) = (-1)^{\dim T - \dim A'} \delta_{P_T}(\gamma) \).

The corollary provides another proof of Theorem 2.2.

4. Iwahori spherical representations: split elements

4.1. Let \( B \) be the subgroup of \( G(K) \) generated by

\[
\{ U_{\alpha}(O) : \alpha \in R^+ \} \cup \{ U_{\alpha}(p) : \alpha \in R^- \} \cup T(K)_0.
\]

(The subgroups \( U_{\alpha}(O) \), \( U_{\alpha}(p) \) of \( U_{\alpha} \) are defined by the \( O \)-structure of \( G \).) Then \( B \) is an Iwahori subgroup of \( G(K) \). For any \( \alpha \in R \) we choose an isomorphism \( x_{\alpha} : K \rightarrow U_{\alpha}(K) \) (the restriction of an isomorphism of algebraic groups from the additive group to \( U_{\alpha} \)), which carries \( O \) onto \( U_{\alpha}(O) \) and \( p \) onto \( U_{\alpha}(p) \). We set \( W := Y \cdot \mathcal{W} \) with \( Y \) normal in \( W \) (recall that \( \mathcal{W} \) acts naturally on \( Y \)). Let \( Y' \) be the subgroup of \( Y \) generated by the coroots. Then \( W' := Y' \cdot \mathcal{W} \) is naturally a subgroup of \( W \). According to [Iwahori and Matsumoto 1965], \( W \) is an extended Coxeter group (the semidirect product of the Coxeter group \( W' \) with the finite abelian group \( Y/Y' \)) with length function

\[
I(y w) = \sum_{\alpha \in R^+} \| \langle y, \alpha \rangle \| + \sum_{\alpha \in R^+, w^{-1}(\alpha) \in R^-} \| \langle y, \alpha \rangle - 1 \|,
\]

where \( \| a \| = a \) if \( a \geq 0 \) and \( \| a \| = -a \) if \( a < 0 \). From the same reference we know that the set of double cosets \( B \mathcal{G}(K)/B \) is in bijection with \( W \); to \( y w \) (where \( y \in Y, w \in \mathcal{W} \)) corresponds the double coset \( \Omega_{y w} \) containing \( T(K)_y \dot{w} \) (here \( \dot{w} \) is
an element in $G(\mathcal{O})$ which normalizes $T(K)_0$ and acts on it in the same way as $w$; moreover, $\sharp(\Omega_{yw}/B) = \sharp(B\setminus \Omega_{yw}) = q^{l(yw)}$ for any $y \in Y, w \in \mathcal{W}$. For example, if $y \in Y^{++}$ then $l(y) = (y, 2\rho)$.

Let $H$ be the algebra of $B$-biinvariant functions $G(K) \to \mathbb{C}$ with compact support with respect to convolution (we use the Haar measure $dg$ on $G(K)$ for which $\text{vol}(B) = 1$). For $y, w$ as above, let $\mathcal{I}_yw \in H$ be the characteristic function of $\Omega_{yw}$. Then the functions $\mathcal{I}_yw, w \in W$ form a $\mathbb{C}$-basis of $H$, and according to [Iwahori and Matsumoto 1965], we have

$$
\hat{\mathcal{I}}_w \hat{\mathcal{I}}_{w'} = \mathcal{I}_{ww'} \quad \text{for } w, w' \in W \text{ with } l(ww') = l(w) + l(w'),
$$

$$(\mathcal{I}_w + 1)(\mathcal{I}_w - q) = 0 \quad \text{for } w \in W' \text{ with } l(w) = 1.
$$

In other words, $H$ is what one now calls the Iwahori–Hecke algebra of the (extended) Coxeter group $W$ with parameter $q$.

4.2. Let $C^\infty_c(G(K))$ be the vector space of locally constant functions with compact support from $G(K)$ to $\mathbb{C}$. Let $(V, \sigma)$ be an irreducible admissible representation of $G(K)$ such that the space $V^B$ of $B$-invariant vectors in $V$ is nonzero. If $f \in C^\infty_c(G(K))$ then there is a well defined linear map $\sigma_f : V \to V$ such that for any $x \in V$ we have $\sigma_f(x) = \int_G f(g)\sigma(g)(x) \, dg$. This linear map has finite rank; hence, it has a well defined trace $\text{tr}(\sigma_f) \in \mathbb{C}$. From the definitions we see that for $f, f' \in C^\infty_c(G(K))$ we have $\sigma_{ff'} = \sigma_f \sigma_{f'} : V \to V$ where $*$ denotes convolution. If $f \in H$ then $\sigma_f$ maps $V$ into $V^B$ and $\text{tr}(\sigma_f) = \text{tr}(\sigma_f|_{V^B})$. (Recall that $\dim V^B < \infty$.) We see that the maps $\sigma_f|_{V^B}$ define a (unital) $H$-module structure on $V^B$. It is known that the $H$-module $V^B$ is irreducible [Borel 1976]. Moreover, for $w \in W$ we have $\text{tr}(\sigma_{\mathcal{I}_w}) = \text{tr}(\mathcal{I}_w)$, where the trace in the right side is taken in the $H$-module $V^B$.

**Theorem 4.3.** Assume that $K$ has characteristic zero and that $p$ is sufficiently large. Let $y \in Y^+$ and $t \in T(K)_y^\bullet$. We have

$$
\phi_V(t) = q^{-(y, 2\rho)} \text{tr}(\mathcal{I}_y),
$$

where the trace in the right side is taken in the irreducible $H$-module $V^B$.

An equivalent statement is that

$$
\phi_V(t) = \text{tr}(\sigma_{\mathcal{I}_y})/\text{vol}(\Omega_y).
$$

(Recall that $\mathcal{I}_y$ on the right side is the characteristic function of $\Omega_y = BT(K)_y B$.)

The assumption on characteristic in the theorem is needed only to be able to use a result from [Adler and Korman 2007]; see (5) below. We expect that the theorem holds without that assumption.

In the case where $y = 0$, the theorem becomes

$$
(4) \quad \quad t \in T(K) \cap G_{crf} \implies \phi_V(t) = \dim(V^B).
$$
As pointed out to us by R. Bezrukavnikov and S. Varma, in the special case where \( y \in Y^{++} \), Theorem 4.3 can be deduced from results in [Casselman 1977].

4.4. In the case where \( V = S \) (see 1.1), for any \( y \in Y^+ \), \( \Sigma_y \) acts on the one-dimensional vector space \( V^B \) as the identity map, so that \( \phi_V(t) = q^{-y,2p} \) for all \( t \in T(K)_y^\bullet \). We thus recover Theorem 2.2 (which holds in any characteristic).

5. Proof of Theorem 4.3

5.1. Let \( B = B_0, B_1, B_2, \ldots \) be the strictly decreasing Moy–Prasad [1994] filtration of \( B \). This is a sequence associated to a point \( x \) in the building such that \( B = G_{x,0} \). Each \( B_i/B_{i+1} \) is abelian. Let \( T_n := T(K) \cap B_n \). Applying [Adler and Korman 2007, Corollary 12.11] to \( \phi_V \), we conclude that

\[
\phi_V \text{ is constant on the } \text{Ad}(G)-\text{orbit } G(tT_1) \text{ of } tT_1.
\]

Lemma 5.2. Let \( n \geq 1 \). For any \( t' \in T(K)_y^\bullet \) and \( z \in B_n \), there exist \( g \in B_n \), \( t'' \in T_n \), and \( z' \in B_{n+1} \) such that \( \text{Ad}(g)(t'z) = t't''z' \).

Proof. Let \( Z = \{ \alpha \in R : U_\alpha \cap B_n \supseteq U_\alpha \cap B_{n+1} \} \). If \( Z = \emptyset \) then \( B_n = T_n B_{n+1} \); hence, \( z = t''z' \) for some \( t'' \in T_n \) and \( z' \in B_{n+1} \), and one can take \( g = 1 \). If \( Z \neq \emptyset \) then we can find \( a_\alpha \in K \) for each \( \alpha \in Z \) such that \( x_\alpha(a_\alpha) \in B_n \) and \( z = \prod_{\alpha \in Z} x_\alpha(a_\alpha) \). If \( g \in B_n \) then \( 1 - \alpha(t'(t'')) \geq 1 \) for \( y \in Y^+ \). (To show \( 1 - \alpha(t'(t'')) \geq 1 \) for \( y \in Y^+ \), we argue as for (1)). Assume first that \( \alpha \in R^+ \). If \( v(\alpha(t'(t''))) \neq 0 \) then \( v(\alpha(t'(t''))) < 0 \) (since \( \langle y, \alpha \rangle \neq 0 \), \( \langle y, \alpha \rangle \geq 0 \)); therefore, \( v(1 - \alpha(t'(t''))) = v(\alpha(t''')) < 0 \) and \( |1 - \alpha(t'')| > 1 \). If \( v(\alpha(t'')) = 0 \) then \( \alpha(t'') - 1 \notin \mathcal{O} - p \); hence, \( v(1 - \alpha(t''')) = 0 \) and \( |1 - \alpha(t'')| = 1 \) as required. Assume next that \( \alpha \in R^- \). If \( v(\alpha(t'(t''))) \neq 0 \) then \( v(\alpha(t'(t''))) > 0 \) (since \( \langle y, \alpha \rangle \neq 0 \), \( \langle y, \alpha \rangle \leq 0 \)); hence, \( v(1 - \alpha(t''')) = 0 \) and \( |1 - \alpha(t'')| = 1 \) as required. If \( v(\alpha(t'')) = 0 \) then \( \alpha(t'') - 1 \notin \mathcal{O} - p \); hence, \( v(1 - \alpha(t''')) = 0 \) and \( |1 - \alpha(t'')| = 1 \) as required.) Now, we have \( t''t'g'g^{-1} \equiv z^{-1} \) (mod \( T_n B_{n+1} \)).

Writing \( \text{Ad}(g)(t'z) = t' \cdot (t''t'g'g^{-1}) \cdot (gzg^{-1}) \), we observe that \( gzg^{-1} \equiv z \) (mod \( B_{n+1} \)) and \( t''t'g'g^{-1}z \in T_n B_{n+1} \); hence, \( \text{Ad}(g)(t'z) \) can be written as \( t''t'z' \) with \( t'' \in T_n \) and \( z' \in B_{n+1} \). \( \square \)

Lemma 5.3. \( B_t B_1 \subset B(tT_1) \).

Proof. It is enough to show that \( tB_1 \subset B(tT_1) \). Let \( t_0z_1 \in tB_1 \) with \( t_0 = t \) and \( z_1 \in B_1 \). We will construct inductively sequences \( g_1, g_2, \ldots, t_1, t_2, \ldots, \) and \( z_1, z_2, \ldots \) such that \( \text{Ad}(g_k \cdots g_2g_1)(t_0z_1) = \text{Ad}(g_k)(t_0t_1 \cdots t_{k-1}z_k) = (t_0t_1 \cdots t_k)z_{k+1} \) with \( g_i \in B_i, t_i \in T_i, \) and \( z_i \in B_i \).
Applying Lemma 5.2 to $n = 1$, $t' = t_0$, and $z = z_1$, we find $t_1 \in T_1$ and $z_2 \in B_2$ such that $g_1 t_0 z_1 g_1^{-1} = t_0 t_1 z_2$ with $t_1 \in T_1$ and $z_2 \in B_2$. Suppose we found $g_i \in B_i$, $z_i+1 \in B_{i+1}$, and $t_i \in T_i$ for $i = 1, \ldots, k$ where $k \geq 1$. Applying Lemma 5.2 to $n = k+1$, $t' = t_0 t_1 \cdots t_k$, and $z = z_{k+1}$, we find $g_{k+1} \in B_{k+1}$, $t_{k+1} \in T_{k+1}$, and $z_{k+2} \in B_{k+2}$ so that $g_{k+1} t_0 t_1 \cdots t_k z_{k+1} g_{k+1}^{-1} \in \Ad(g_{k+1} \cdots g_2 g_1)(t_0 z_1) = t_0 t_1 \cdots t_k z_{k+1} z_{k+2}$.

Thus it remains to show (5.4). Continuing with the proof of Theorem 4.3, we note that by Lemma 5.3 and Proposition 5.5.

Proof. It is enough to show that $\sigma_{f_i}$ as linear maps $V \to V$. Restricting this equality to $B_1$, and using the fact that $\sigma(\tau)$ acts as identity on $V^B$, we obtain

(8) $\sigma_{\alpha_\tau} = \sum_{\tau \in L} \sigma_{f_i}(\tau)$ as linear maps $V^B \to V^B$. Clearly, (6) follows from (7) and this completes the proof of Theorem 4.3.

The following result will not be used in the rest of the paper:

Proposition 5.5. If $y \in Y^{++}$ and $t \in T(K)_y$ then $B \subset B^T(K)_y$.

Proof. It is enough to show that $t z \subset B^T(K)_y$ for any $z \in B$. We can write $z = t_0 z'$, where $t_0 \in T(K)_0$, $z' \in B_1$. We have $t z = t t_0 z'$, where $t t_0 \in T(K)_y = T(K)_y^\circ$ (here we use that $y \in Y^{++}$). Using Lemma 5.3, we have $t t_0 z' \subset B^T(t t_0 T_1) \subset B^T(K)_y$. 

□
5.6. In the remainder of this section we assume that $G$ is adjoint. In this case, the irreducible representations $(V, \sigma)$ as in 4.2 (up to isomorphism) are known to be in bijection with the irreducible finite-dimensional representations of the Hecke algebra $H$ (see [Borel 1976]) by $(V, \sigma) \mapsto V_B$. The irreducible finite-dimensional representations of $H$ have been classified in [Kazhdan and Lusztig 1987] in terms of geometric data; moreover, in [Lusztig 2010], an algorithm to compute the dimensions of the (generalized) weight spaces of the action of the commutative semigroup $\{\Sigma_y : y \in Y^+\}$ on any tempered $H$ module is given. In particular the right hand side of the equality in Theorem 4.3 (hence also $\phi_V(t)$ in that theorem) is computable when $V$ is tempered.

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