ON THE STEINBERG CHARACTER OF A SEMISIMPLE $p$-ADIC GROUP

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Dedicated to Robert Steinberg on the occasion of his ninetieth birthday.

We show that the character of the Steinberg representation of a split semisimple $p$-adic group at a very regular element is given (up to sign) by a power of $q$, the number of elements in the residue field. We also show that (under an assumption on the characteristic) the character of an Iwahori-spherical representation at a split very regular element is given by a trace in the corresponding Hecke algebra module.

1. Introduction

1.1. Let $K$ be a nonarchimedean local field and let $\mathcal{K}$ be a maximal unramified field extension of $K$. Let $\mathcal{O}$ be the ring of integers of $K$ and let $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}$; the counterparts for $\mathcal{K}$ are denoted by $\mathcal{O}$ and $\mathfrak{p}$. Let $\mathcal{K}^* = \mathcal{K} \setminus \{0\}$. We write $\mathcal{O}/\mathfrak{p} = F_q$, a finite field with $q$ elements of characteristic $p$.

Let $G$ be a semisimple almost simple algebraic group defined and split over $K$ with a given $\mathcal{O}$-structure compatible with the $K$-structure.

If $V$ is an admissible representation of $G(K)$ of finite length, we denote by $\phi_V$ the character of $V$ in the sense of Harish-Chandra, viewed as a $\mathbb{C}$-valued function on the set $G(K)_{rs} := G_{rs} \cap G(K)$. (Here, $G_{rs}$ is the set of regular semisimple elements of $G$, and $\mathbb{C}$ is the field of complex numbers.)

In this paper we study the restriction of the function $\phi_V$ to:

(a) a certain subset $G(K)_{vr}$ of $G(K)_{rs}$, namely, the set of very regular elements in $G(K)$ (see 1.2) in the case where $V$ is the Steinberg representation of $G(K)$, and

(b) a certain subset $G(K)_{s, vr}$ of $G(K)_{vr}$, namely, the set of split very regular elements in $G(K)$ (see 1.2) in the case where $V$ is an irreducible admissible representation of $G(K)$ with nonzero vectors fixed by an Iwahori subgroup.

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In case (a), we show that $\phi_V(g)$ with $g \in G(K)_{rs}$ is of the form $\pm q^n$ with $n \in \{0, -1, -2, \ldots\}$ (see Corollary 3.4) with more precise information when $g \in G(K)_{sur}$ (see Theorem 2.2) or when $g \in G(K)_{cur}$ (see Theorem 3.2). In case (b) we show (with some restriction on characteristic) that $\phi_V(g)$ with $g \in G(K)_{sur}$ can be expressed as the trace of a certain element of an affine Hecke algebra on an irreducible module (see Theorem 4.3).

Note that the Steinberg representation $S$ is an irreducible admissible representation of $G(K)$ with a one-dimensional subspace invariant under an Iwahori subgroup on which the corresponding affine Hecke algebra acts through the “sign” representation; see [Matsumoto 1969; Shalika 1970]. This is a $p$-adic analogue of the Steinberg representation [Steinberg 1951] of a reductive group over $F_q$. In [Rodier 1986], it is proven that $\phi_S(g) \neq 0$ for any $g \in G(K)_{rs}$.

1.2. Let $g \in G_{rs} \cap G(K)$. Let $T' = T'_g$ be the maximal torus of $G$ that contains $g$. We say that $g$ is very regular if $T'$ is split over $K$ and for any root $\alpha$ with respect to $T'$ viewed as a homomorphism $T'(K) \to K^*$ we have $\alpha(g) \notin (1 + p)$. If, in addition, $\alpha(g) \in \mathcal{O}$, we say that $g$ is compact very regular.

Let $G(K)_{vr}$ be the set of elements in $G(K)$ that are very regular, and $G(K)_{cur}$ the set of compact very regular ones. We write $G(K)_{vr} = G(K)_{ur} \cap G(K)$ and $G(K)_{cur} = G(K)_{ur} \cap G(K)$. Let $G(K)_{sur}$ be the set of all $g \in G(K)_{ur}$ such that $T'_g$ is split over $K$.

1.3. Notation. Let $K^* = K - \{0\}$, and let $v : K^* \to \mathbb{Z}$ be the unique (surjective) homomorphism such that $v(p^n - p^{n+1}) = n$ for any $n \in \mathbb{N}$. For $a \in K^*$ we set $|a| = q^{-v(a)}$.

We fix a maximal torus $T$ of $G$ defined and split over $K$. Let $Y$ (resp. $X$) be the group of cocharacters (resp. characters) of the algebraic group $T$. Let $\langle \ , \ \rangle : Y \times X \to \mathbb{Z}$ be the obvious pairing. Let $R \subset X$ be the set of roots of $G$ with respect to $T$, let $R^+$ be a set of positive roots for $R$, and let $\Pi$ be the set of simple roots of $R$ determined by $R^+$. We write $\Pi = \{\alpha_i : i \in I\}$. Let $R^- = R - R^+$. Let $Y^+$ (resp. $Y^{++}$) be the set of all $y \in Y$ such that $\langle y, \alpha \rangle \geq 0$ (resp. $\langle y, \alpha \rangle > 0$) for all $\alpha \in R^+$. We define $2\rho \in X$ by $2\rho = \sum_{a \in R^+} \alpha$.

We have canonically $T(K) = K^* \otimes Y$; we define a homomorphism $\chi : T(K) \to Y$ by $\chi(\lambda \otimes y) = v(\lambda)y$ for any $\lambda \in K^*$, $y \in Y$. For any $y \in Y$, we set $T(K)_y = \chi^{-1}(y)$. For $y \in Y$, let $T(K)^*_y = T(K)_y \cap G(K)_{sur}$. Note that if $y \in Y^{++}$ then $T(K)^*_y = T(K)_y$.

For each $\alpha \in R$ let $U_\alpha$ be the corresponding root subgroup of $G$.

2. Calculation of $\phi_S$ on $G(K)_{sur}$

2.1. Let $\mathcal{W} \subset \text{Aut}(T)$ be the Weyl group of $G$ regarded as a Coxeter group; for $i \in I$, let $s_i$ be the simple reflection in $\mathcal{W}$ determined by $\alpha_i$. We can also view
\( \mathcal{W} \) as a subgroup of \( \text{Aut}(Y) \) or \( \text{Aut}(X) \). Let \( w = w_0 \) be the longest element of \( \mathcal{W} \). For any \( J \subset I \), let \( \mathcal{W}_J \) be the subgroup of \( \mathcal{W} \) generated by \( \{s_i : i \in J\} \) and let \( R_J = R \cap \sum_{i \in J} \mathbb{Z} \alpha_i \). Let

\[
R_J^+ = R_J \cap R^+ \quad \text{and} \quad R_J^- = R_J - R_J^+.
\]

Let \( \mathfrak{g} \) be the Lie algebra of \( G \), and let \( t \subset \mathfrak{g} \) be the Lie algebra of \( T \). For any \( J \subset I \), let \( \mathfrak{l}_J \) be the Lie subalgebra of \( \mathfrak{g} \) spanned by \( t \) and the root spaces corresponding to the roots in \( R_J \). Let \( \mathfrak{n}_J \) be the Lie subalgebra of \( \mathfrak{g} \) spanned by the root spaces corresponding to roots in \( R^+ - R_J^+ \).

According to [Casselman 1973], \( \phi_S \) is an alternating sum of characters of representations induced from one-dimensional representations of various parabolic subgroups of \( G \) defined over \( K \). From this, one can deduce that if \( t \in T(K) \cap G(K)_{rs} \) then

\[
\phi_S(t) = \sum_{J \subset I} (-1)^{\sharp J} \sum_{w \in J \mathcal{W}} \delta_J(w(t))^{1/2} D_{I,J}(w(t))^{-1/2},
\]

where for any \( J \subset I \) and \( t' \in T(K) \cap G(K)_{rs} \) we set

\[
D_{I,J}(t') = |\det(1 - \text{Ad}(t')|_{\mathfrak{g}/\mathfrak{l}_J})|,
\]

\[
\delta_J(t') = |\det(\text{Ad}(t')|_{\mathfrak{n}_J})|,
\]

and \( J \mathcal{W} \) is the set of representatives of minimal length for the cosets \( \mathcal{W}_J \setminus \mathcal{W} \). Here for a real number \( a \geq 0 \) we denote by \( a^{1/2} \) or \( \sqrt{a} \) the nonnegative square root of \( a \).

Writing \( \phi \) instead of \( \phi_S \), we have:

**Theorem 2.2.** Let \( y \in Y^+ \) and let \( t \in T(K)_y^\bullet \). Then \( \phi(t) = q^{-(y,2\rho)} \).

2.3. More generally, let \( t \in T(K)_y^\bullet \), where \( y \in Y \). By a standard property of Weyl chambers, there exists \( w \in \mathcal{W} \) such that \( w(y) \in Y^+ \). Let \( t_1 = w(t) \). Then the theorem is applicable to \( t_1 \), and we have \( \phi(t) = \phi(t_1) = q^{-(w(y),2\rho)} \).

2.4. Let \( y' = w_0(y), t' = w_0(t) \). We have \( \phi_S(t) = \phi_S(t'), t' \in T(K)_{y'}^\bullet, -y' \in Y^+ \).

We show that

\[
v(1 - \beta(t')) = \begin{cases} 
v(\beta(t')) & \text{if } \beta \in R^+, \\
0 & \text{if } \beta \in R^-.
\end{cases}
\]

Assume first that \( \beta \in R^+ \). If \( v(\beta(t')) \neq 0 \) then \( v(\beta(t')) < 0 \) (since \( \langle y', \beta \rangle \neq 0 \) and \( \langle y', \beta \rangle \leq 0 \)); hence, \( v(1 - \beta(t')) = v(\beta(t')) \). If \( v(\beta(t')) = 0 \) then \( \beta(t') - 1 \in \mathcal{O} - p \); hence, \( v(1 - \beta(t')) = 0 = v(\beta(t')) \) as required.

Assume next that \( \beta \in R^- \). If \( v(\beta(t')) \neq 0 \) then \( v(\beta(t')) > 0 \) (since \( \langle y', \beta \rangle \neq 0 \) and \( \langle y', \beta \rangle \geq 0 \)); hence, \( v(1 - \beta(t')) = 0 \). If \( v(\beta(t')) = 0 \) then \( \beta(t') - 1 \in \mathcal{O} - p \); hence, \( v(1 - \beta(t')) = 0 \) as required.
For any \( w \in \mathcal{W}, J \subset I \) we have

\[
D_{I,J}(w(t')) = \prod_{\alpha \in R^- R_J} q^{-v(1-\alpha(w(t')))} = \prod_{\alpha \in R^- R_J} q^{-v(\alpha(w(t')))} = \prod_{\alpha \in R^- R_J} q^{-(y', w^{-1} \alpha)}
\]

and

\[
\delta_J(w(t')) = \prod_{\alpha \in R^+ R_J^+} q^{-v(\alpha(w(t')))} = \prod_{\alpha \in R^+ R_J^+} q^{-(y', w^{-1} \alpha)}.
\]

(We have used (1) with \( \beta = w^{-1}(\alpha) \).) We see that

\[
\phi(t) = \phi(t') = \sum_{J \subset I} (-1)^{|J|} \sum_{w \in J} \sqrt{q}^{-\langle y', x_w, J \rangle},
\]

where for \( w \in J \mathcal{W} \) we have

\[
x_{w,J} = \sum_{\alpha \in R^+ R_J^+} w^{-1} \alpha - \sum_{\alpha \in R^- R_J^-} w^{-1} \alpha
\]

\[
= \sum_{\alpha \in R^+ R_J^+} w^{-1} \alpha - \sum_{\alpha \in R^- R_J^-} w^{-1} \alpha
\]

\[
= 2 \sum_{\alpha \in R^+ R_J^+} w^{-1} \alpha \in X.
\]

For \( w \in J \mathcal{W} \), we have \( \alpha \in R_J^+ \Rightarrow w^{-1} \alpha \in R^+ \); hence,

\[
\sum_{\alpha \in R^+ R_J^+} w^{-1} \alpha = \sum_{\alpha \in R^+} w^{-1} \alpha,
\]

so that \( x_{w,J} = x_w \), where

\[
x_w = 2 \sum_{\alpha \in R^+} w^{-1} \alpha \in X.
\]

Thus, we have

\[
\phi(t) = \sum_{J \subset I} (-1)^{|J|} \sum_{w \in J} \sqrt{q}^{-\langle y', x_w \rangle} = \sum_{w \in \mathcal{W}} c_w \sqrt{q}^{-\langle y', x_w \rangle},
\]

where for \( w \in \mathcal{W} \) we set

\[
c_w = \sum_{J \subset I, w \in J} (-1)^{|J|}.
\]

For \( w \in \mathcal{W} \), let \( \mathcal{L}(w) = \{i \in I : s_i w > w\} \), where \( > \) refers to the standard partial
order on \( \mathcal{W} \). For \( J \subset I \), we have \( w \in J \mathcal{W} \) if and only if \( J \subset \mathcal{L}(w) \); thus,
\[
c_w = \sum_{J \subset \mathcal{L}(w)} (-1)^{|J|},
\]
and this is 0 unless \( \mathcal{L}(w) = \emptyset \) (that is \( w = w_0 \)), in which case \( c_w = 1 \). Note also that \( x_{w_0} = -4p \); thus, we have
\[
\phi(t) = c_{w_0} \sqrt{q^{-\langle y', x_{w_0} \rangle}} = q^{\langle y', 2p \rangle} = q^{-(y, 2p)}.
\]

Theorem 2.2 is proved. \( \square \)

2.5. Assume now that \( \tau \in T(K) \) satisfies the following condition: for any \( \alpha \in R \) we have \( \alpha(\tau) - 1 \in p - \{0\} \) so that \( \alpha(\tau) - 1 \in p^n - p^{n+1} \) for a well defined integer \( n_{\alpha} \geq 1 \). Note that \( n_{-\alpha} = n_\alpha \) and \( v(1 - \alpha(\tau)) = n_\alpha \geq 1 \) for all \( \alpha \in R \); hence,
\[
\phi(\tau) = \sum_{J \subset I} (-1)^{|J|} \sum_{w \in J \mathcal{W}} q^{\sum_{\alpha \in R} n_\alpha/2 - \sum_{J} n_{w-1(\alpha)}/2}.
\]
Thus,
\[
\phi(\tau) = \sharp(\mathcal{W}) q^{\sum_{\alpha \in R} n_\alpha/2} + \text{strictly smaller powers of } q.
\]

In the case where \( K \) is the field of power series over \( F_q \), the leading term in (2) is equal to \( \sharp(\mathcal{W}) q^m \), where \( m \) is the dimension of the “variety” of Iwahori subgroups of \( G(K) \) that contain the topologically unipotent element \( \tau \) (see [Kazhdan and Lusztig 1988]).

3. Calculation of \( \phi_S \) on \( G(K)_{cr} \)

3.1. We will again write \( \phi \) instead of \( \phi_S \). In this section we assume that we are given \( \gamma \in G(K)_{cr} \). Let \( T' = T'_\gamma \). Note that \( T' \) is defined over \( K \); let \( A' \) be the largest \( K \)-split torus of \( T' \). For any parabolic subgroup \( P \) of \( G \) defined over \( K \) such that \( \gamma \in P \), we set \( \delta_P(\gamma) = |\text{det}(\text{Ad}(\gamma))|_n \), where \( n \) is the Lie algebra of the unipotent radical of \( P \).

Let \( \mathcal{X} \) be the set of all pairs \( (P, A) \), where \( P \) is a parabolic subgroup of \( G \) defined over \( K \) and \( A \) is the unique maximal \( K \)-split torus in the center of some Levi subgroup of \( P \) defined over \( K \). Then that Levi subgroup is uniquely determined by \( A \) and is denoted by \( M_A \). Let \( \mathcal{X'} = \{(P, A) \in \mathcal{X} : A \subset A'\} \). According to [Harish-Chandra 1973], we have
\[
\phi(\gamma) = (-1)^{\dim T} \sum_{(P, A) \in \mathcal{X'}} (-1)^{\dim A} \delta_P(\gamma)^{1/2} D_{G/M_A}(\gamma)^{-1/2},
\]
where \( D_{G/M_A}(\gamma) = |\text{det}(1 - \text{Ad}(\gamma)|_{l/1}| \) (we denote by \( l \) the Lie algebra of \( M_A \)).

Theorem 3.2. Assume in addition that \( \gamma \in G(K)_{cr} \). Then \( \phi(\gamma) = (-1)^{\dim T - \dim A'} \).
Proof. From our assumptions we see that \( \delta_P(\gamma) = 1 = D_{G/M_A}(\gamma) \) for all \((P, A) \in \mathcal{X}'\); hence, (3) becomes

\[
\phi(\gamma) = (-1)^{\dim T} \sum_{(P, A) \in \mathcal{X}'} (-1)^{\dim A}.
\]

Let \( \mathcal{Y} \) be the group of cocharacters of \( A' \) and let \( \mathfrak{h} = \mathcal{Y} \otimes \mathbb{R} \). The real vector space \( \mathfrak{h} \) can be partitioned into facets \( F_{P, A} \) indexed by \((P, A) \in \mathcal{X}'\) such that \( F_{P, A} \) is homeomorphic to \( \mathbb{R}^{\dim A} \). Note that the Euler characteristic with compact support of \( F_{P, A} \) is \((-1)^{\dim A} \), and the Euler characteristic with compact support of \( \mathfrak{h} \) is \((-1)^{\dim A} \mathfrak{h} = (-1)^{\dim A} \). Using the additivity of the Euler characteristic with compact support we see that \( \sum_{(P, A) \in \mathcal{X}'} (-1)^{\dim A} = (-1)^{\dim A'} \); thus, \( \phi(\gamma) = (-1)^{\dim T - \dim A'} \), as required. \( \square \)

3.3. In the setup of 3.1, let \( P_\gamma \) be the parabolic subgroup of \( G \) associated to \( \gamma \) as in [Casselman 1977]. Note that \( P_\gamma \) is defined over \( K \). The following result can be deduced by combining Theorem 3.2 with the results in [Casselman 1977] and with Proposition 2 in [Rodier 1986].

Corollary 3.4. We have \( \phi(\gamma) = (-1)^{\dim T - \dim A'} \delta_{P_\gamma}(\gamma) \).

The corollary provides another proof of Theorem 2.2.

4. Iwahori spherical representations: split elements

4.1. Let \( B \) be the subgroup of \( G(K) \) generated by

\[
\{ U_\alpha(O) : \alpha \in R^+ \} \cup \{ U_\alpha(p) : \alpha \in R^- \} \cup T(K)_0.
\]

(The subgroups \( U_\alpha(O) \), \( U_\alpha(p) \) of \( U_\alpha \) are defined by the \( O \)-structure of \( G \).) Then \( B \) is an Iwahori subgroup of \( G(K) \). For any \( \alpha \in R \) we choose an isomorphism \( x_\alpha : K \sim U_\alpha(K) \) (the restriction of an isomorphism of algebraic groups from the additive group to \( U_\alpha \), which carries \( O \) onto \( U_\alpha(O) \) and \( p \) onto \( U_\alpha(p) \). We set

\[
W := Y \cdot \mathcal{W}
\]

with \( \mathcal{W} \) normal in \( W \) (recall that \( \mathcal{W} \) acts naturally on \( Y \)). Let \( Y' \) be the subgroup of \( Y \) generated by the coroots. Then \( W' := Y' \cdot \mathcal{W} \) is naturally a subgroup of \( W \). According to [Iwahori and Matsumoto 1965], \( W \) is an extended Coxeter group (the semidirect product of the Coxeter group \( W' \) with the finite abelian group \( Y/Y' \)) with length function

\[
l(yw) = \sum_{\alpha \in R^+} \| \langle y, \alpha \rangle \| + \sum_{\alpha \in R^+} \| \langle y, \alpha \rangle \| - 1,
\]

where \( \|a\| = a \) if \( a \geq 0 \) and \( \|a\| = -a \) if \( a < 0 \). From the same reference we know that the set of double cosets \( B \backslash G(K)/B \) is in bijection with \( W \); to \( yw \) (where \( y \in Y \), \( w \in \mathcal{W} \)) corresponds the double coset \( \Omega_{yw} \) containing \( T(K)_y \hat{w} \) (here \( \hat{w} \) is
an element in $G(O)$ which normalizes $T(K)_0$ and acts on it in the same way as $w$; moreover, $\sharp(\Omega_{yw}/B) = \sharp(B \setminus \Omega_{yw}) = q^{l(yw)}$ for any $y \in Y$, $w \in W$. For example, if $y \in Y^+$ then $l(y) = \langle y, 2\rho \rangle$.

Let $H$ be the algebra of $B$-biinvariant functions $G(K) \to \mathbb{C}$ with compact support with respect to convolution (we use the Haar measure $dg$ on $G(K)$ for which $\text{vol}(B) = 1$). For $y, w$ as above, let $\mathcal{I}_{yw} \in H$ be the characteristic function of $\Omega_{yw}$. Then the functions $\mathcal{I}_w$, $w \in W$ form a $\mathbb{C}$-basis of $H$, and according to [Iwahori and Matsumoto 1965], we have

$$\mathcal{I}_w \mathcal{I}_w' = \mathcal{I}_{ww'} \quad \text{for } w, w' \in W \text{ with } l(ww') = l(w) + l(w'),$$

$$(\mathcal{I}_w + 1)(\mathcal{I}_w - q) = 0 \quad \text{for } w \in W \text{ with } l(w) = 1.$$  

In other words, $H$ is what one now calls the Iwahori–Hecke algebra of the (extended) Coxeter group $W$ with parameter $q$.

**4.2.** Let $C_0^\infty(G(K))$ be the vector space of locally constant functions with compact support from $G(K)$ to $\mathbb{C}$. Let $(V, \sigma)$ be an irreducible admissible representation of $G(K)$ such that the space $V^B$ of $B$-invariant vectors in $V$ is nonzero. If $f \in C_0^\infty(G(K))$ then there is a well defined linear map $\sigma_f : V \to V$ such that for any $x \in V$ we have $\sigma_f(x) = \int_G f(g)\sigma(x)(g) dg$. This linear map has finite rank; hence, it has a well defined trace $\text{tr}(\sigma_f) \in \mathbb{C}$. From the definitions we see that for $f, f' \in C_0^\infty(G(K))$ we have $\sigma_{f \ast f'} = \sigma_f \sigma_f' : V \to V$ where $\ast$ denotes convolution. If $f \in H$ then $\sigma_f$ maps $V$ into $V^B$ and $\text{tr}(\sigma_f) = \text{tr}(\sigma_f|_{V^B})$. (Recall that $\dim V^B < \infty$.) We see that the maps $\sigma_f|_{V^B}$ define a (unital) $H$-module structure on $V^B$. It is known that the $H$-module $V^B$ is irreducible [Borel 1976]. Moreover, for $w \in W$ we have $\text{tr}(\sigma_{\mathcal{I}_w}) = \text{tr}(\mathcal{I}_w)$, where the trace in the right side is taken in the $H$-module $V^B$.

**Theorem 4.3.** Assume that $K$ has characteristic zero and that $p$ is sufficiently large. Let $y \in Y^+$ and $t \in T(K)^\bullet$. We have

$$\phi_V(t) = q^{-(y, 2\rho)}\text{tr}(\mathcal{I}_y),$$

where the trace in the right side is taken in the irreducible $H$-module $V^B$.

An equivalent statement is that

$$\phi_V(t) = \text{tr}(\sigma_{\mathcal{I}_y})/\text{vol}(\Omega_y).$$

(Recall that $\mathcal{I}_y$ on the right side is the characteristic function of $\Omega_y = BT(K)_y B$.)

The assumption on characteristic in the theorem is needed only to be able to use a result from [Adler and Korman 2007]; see (5) below. We expect that the theorem holds without that assumption.

In the case where $y = 0$, the theorem becomes

$$t \in T(K) \cap G_{cvr} \Rightarrow \phi_V(t) = \dim(V^B).$$

(4)
As pointed out to us by R. Bezrukavnikov and S. Varma, in the special case where $y \in Y^+$, Theorem 4.3 can be deduced from results in [Casselman 1977].

4.4. In the case where $V = S$ (see 1.1), for any $y \in Y^+$, $\Sigma_y$ acts on the one-dimensional vector space $V^B$ as the identity map, so that $\phi_V(t) = q^{(y, 2\rho)}$ for all $t \in T(K)_y$. We thus recover Theorem 2.2 (which holds in any characteristic).

5. Proof of Theorem 4.3

5.1. Let $B = B_0, B_1, B_2, \ldots$ be the strictly decreasing Moy–Prasad [1994] filtration of $B$. This is a sequence associated to a point $x$ in the building such that $B = G_{x,0}$. Each $B_i/B_{i+1}$ is abelian. Let $T_n := T(K) \cap B_n$. Applying [Adler and Korman 2007, Corollary 12.11] to $\phi_V$, we conclude that

\begin{equation}
\phi_V \text{ is constant on the } \text{Ad}(G)\text{-orbit } G(tT_1) \text{ of } tT_1.
\end{equation}

Lemma 5.2. Let $n \geq 1$. For any $t' \in T(K)_y^\bullet$ and $z \in B_n$, there exist $g \in B_n, t'' \in T_n$, and $z' \in B_{n+1}$ such that $\text{Ad}(g)(t'z) = t''t'z'$.

Proof. Let $Z = \{\alpha \in R : U_\alpha \cap B_n \supseteq U_\alpha \cap B_{n+1}\}$. If $Z = \emptyset$ then $B_n = T_n B_{n+1}$; hence, $z = t''z'$ for some $t'' \in T_n$ and $z' \in B_{n+1}$, and one can take $g = 1$. If $Z \neq \emptyset$ then we can find $a_\alpha \in K$ for each $\alpha \in Z$ such that $x_\alpha(a_\alpha) \in B_n$ and $z = \prod_{\alpha \in Z} x_\alpha(a_\alpha)$ (mod $T_n B_{n+1}$). Such $a_\alpha$ can be chosen independent of the order of the product since $B_n/T_n B_{n+1}$ is abelian. Take $g = \prod_{\alpha \in Z} x_\alpha((1 - \alpha(t'1))^{-1}a_\alpha)$. Then $g \in B_n$ since $|1 - \alpha(t'1)| \geq 1$ for $y \in Y^+$. (To show $|1 - \alpha(t'1)| \geq 1$ for $y \in Y^+$, we argue as for (1). Assume first that $\alpha \in R^+$. If $v(\alpha(t'1)) \neq 0$ then $v(\alpha(t'1)) < 0$ (since $\langle y, \alpha \rangle \neq 0, \langle y, \alpha \rangle \geq 0$); therefore, $v(1 - \alpha(t'1)) = v(\alpha(t'1)) < 0$ and $|1 - \alpha(t'1)| > 1$. If $v(\alpha(t'1)) = 0$ then $\alpha(t'1) - 1 \in \mathcal{O} - \mathfrak{p}$; hence, $v(1 - \alpha(t'1)) = 0$ and $|1 - \alpha(t'1)| = 1$ as required. Assume next that $\alpha \in R^-$. If $v(\alpha(t'1)) \neq 0$ then $v(\alpha(t'1)) > 0$ (since $\langle y, \alpha \rangle \neq 0, \langle y, \alpha \rangle \leq 0$); hence, $v(1 - \alpha(t'1)) = 0$ and $|1 - \alpha(t'1)| = 1$ as required. If $v(\alpha(t'1)) = 0$ then $\alpha(t'1) - 1 \in \mathcal{O} - \mathfrak{p}$; hence, $v(1 - \alpha(t'1)) = 0$ and $|1 - \alpha(t'1)| = 1$ as required.) Now, we have $t'\gamma g_{t'1}^{-1} \equiv z^{-1} \pmod{T_n B_{n+1}}$.

Writing $\text{Ad}(g)(t'z) = t' \cdot (t'1g_{t'1}^{-1}) \cdot (gzg^{-1})$, we observe that $gzg^{-1} \equiv z \pmod{B_{n+1}}$ and $t'1g_{t'1}^{-1}z \in T_n B_{n+1}$; hence, $\text{Ad}(g)(t'z)$ can be written as $t''t'z'$ with $t'' \in T_n$ and $z' \in B_{n+1}$.

Lemma 5.3. $B_1tB_1 \subset B_i(tT_1)$.

Proof. It is enough to show that $tB_1 \subset B_i(tT_1)$. Let $t_0z_1 \in tB_1$ with $t_0 = t$ and $z_1 \in B_1$. We will construct inductively sequences $g_1, g_2, \ldots, t_1, t_2, \ldots$, and $z_1, z_2, \ldots$ such that $\text{Ad}(g_k \cdots g_2g_1)(t_0z_1) = \text{Ad}(g_k)(t_0t_1 \cdots t_{k-1}z_k) = (t_0t_1 \cdots t_k)z_{k+1}$ with $g_i \in B_i, t_i \in T_i$, and $z_i \in B_i$. 

Applying Lemma 5.2 to \( n = 1, t' = t_0 \), and \( z = z_1 \), we find \( t_1 \in T_1 \) and \( z_2 \in B_2 \) such that \( g_1 t_0 z_1 g_1^{-1} = t_0 t_1 z_2 \) with \( t_1 \in T_1 \) and \( z_2 \in B_2 \). Suppose we found \( g_i \in B_i, z_{i+1} \in B_{i+1} \), and \( t_i \in T_i \) for \( i = 1, \ldots, k \) where \( k \geq 1 \). Applying Lemma 5.2 to \( n = k + 1, t' = t_0 t_1 \cdots t_k \), and \( z = z_{k+1} \), we find \( g_{k+1} = B_{k+1}, t_{k+1} \in T_{k+1} \), and \( z_{k+2} \in B_{k+2} \) so that \( g_{k+1} t_0 t_1 \cdots t_k z_{k+1} g_{k+1}^{-1} = \text{Ad}(g_{k+1} \cdots g_2 g_1)(t_0 z_1) = t_0 t_1 t_2 \cdots t_k + 1 z_{k+2} \).

(To apply Lemma 5.2 we note that \( t' \in T(K)_y \) since \( t_0 \in T(K)_y \) and \( t_1 \cdots t_k \in T_1 \), so that for any \( \alpha \in R \) we have \( \alpha(t_1 \cdots t_k) \in 1 + p \).) Taking \( g \in B_1 \) to be the limit of \( g_k \cdots g_2 g_1 \) as \( k \to \infty \), we have \( \text{Ad}(g)(t_0 z_1) \in tT_1 \).

**5.4.** Continuing with the proof of Theorem 4.3, we note that by Lemma 5.3 and (5), for the characteristic function \( f_t \) of \( B_1 t B_1 \), we have

\[
\text{tr}(\sigma_f) = \int_G f_t(g) \phi_V(g) \, dg = \int_{B_1 t B_1} \phi_V(t) \, dg = \text{vol}(B_1 t B_1) \phi_V(t).
\]

Thus it remains to show

\[
\frac{\text{tr}(\sigma_f)}{\text{vol}(B_1 t B_1)} = \frac{\text{tr}(\sigma_{\tau_y})}{\text{vol}(B t B)}.
\]

Since \( B_1 \) is normalized by \( B \), \( B \) acts on \( V^{B_1} \); moreover, since \( V \) is irreducible and \( V^B \neq 0 \), \( B \) acts trivially on \( V^{B_1} \). (Otherwise, there would exist a nonzero subspace of \( V \) on which \( B \) acts through a nontrivial character of \( B / B_1 \); since \( V^B \neq 0 \), we see that \( (V, \sigma) \) would have two distinct cuspidal supports, a contradiction.) Thus we have \( V^{B_1} = V^B \). Since \( \sigma_f \) and \( \sigma_{\tau_y} \) have images contained in \( V^{B_1} = V^B \), it is enough to show

\[
\frac{\text{tr}(\sigma_f |_{V^B})}{\text{vol}(B_1 t B_1)} = \frac{\text{tr}(\sigma_{\tau_y} |_{V^B})}{\text{vol}(B t B)}.
\]

We can find a finite subset \( L \) of \( T(K)_0 \) such that \( B t B = \bigcup_{\tau \in L} B_1 t B_1 \tau \). It follows that

\[
\text{vol}(B t B) = \text{vol}(B_1 t B_1) \sharp(L)
\]

and \( \sigma_{\tau_y} = \sum_{\tau \in L} \sigma_f, \sigma(\tau) \) as linear maps \( V \to V \). Restricting this equality to \( V^B \) and using the fact that \( \tau(\tau) \) acts as identity on \( V^B \), we obtain

\[
\sigma_{\tau_y} |_{V^B} = \sharp(L) \sigma_f |_{V^B}
\]

as linear maps \( V^B \to V^B \). Clearly, (6) follows from (7) and This completes the proof of Theorem 4.3.

The following result will not be used in the rest of the paper:

**Proposition 5.5.** If \( y \in Y^{++} \) and \( t \in T(K)_y \) then \( B t B \subset B^{i} T(K)_y \).

**Proof.** It is enough to show that \( t z \subset B^{i} T(K)_y \) for any \( z \in B \). We can write \( z = t_0 z' \), where \( t_0 \in T(K)_0, z' \in B_1 \). We have \( t z = t t_0 z' \), where \( t t_0 \in T(K)_y = T(K)_y \) (here we use that \( y \in Y^{++} \)). Using Lemma 5.3, we have \( t t_0 z' \in B^{i} (t t_0 T_1) \subset B^{i} T(K)_y \).
5.6. In the remainder of this section we assume that $G$ is adjoint. In this case, the irreducible representations $(V, \sigma)$ as in 4.2 (up to isomorphism) are known to be in bijection with the irreducible finite-dimensional representations of the Hecke algebra $H$ (see [Borel 1976]) by $(V, \sigma) \mapsto V^B$. The irreducible finite-dimensional representations of $H$ have been classified in [Kazhdan and Lusztig 1987] in terms of geometric data; moreover, in [Lusztig 2010], an algorithm to compute the dimensions of the (generalized) weight spaces of the action of the commutative semigroup $\{\Sigma_y : y \in Y^+\}$ on any tempered $H$ module is given. In particular the right hand side of the equality in Theorem 4.3 (hence also $\phi_V(t)$ in that theorem) is computable when $V$ is tempered.

References


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