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It is shown that a topological group G is topologically isomorphic to the isometry group of a (complete) metric space if and only if G coincides with its $\mathscr{G}_\delta$-closure in the Raĭkov completion of G (resp. if G is Raĭkov-complete). It is also shown that for every Polish (resp. compact Polish; locally compact Polish) group G there is a complete (resp. proper) metric d on X inducing the topology of X such that G is isomorphic to $\mathrm{Iso}(X, d)$, where $X = \ell_2$ (resp. $X = [0, 1]^\omega$; $X = [0, 1]^\omega \setminus \{\text{point}\}$). It is demonstrated that there are a separable Banach space E and a nonzero vector $e \in E$ such that G is isomorphic to the group of all (linear) isometries of E which leave the point e fixed. Similar results are proved for arbitrary Raĭkov-complete topological groups.

1. Introduction

Gao and Kechris [2003] proved that every Polish group is isomorphic to the (full) isometry group of some separable complete metric space. Melleray [2008] and Malicki and Solecki [2009] improved this result in the context of compact and, respectively, locally compact Polish groups by showing that every such group is isomorphic to the isometry group of a compact and, respectively, a proper metric space. (A metric space is *proper* if and only if each closed ball in this space is compact). All their proofs were complicated and based on the techniques of the so-called *Katětov* maps. In [Niemiec 2012] we introduced a new method to characterize groups of homeomorphisms of a locally compact Polish space which coincide with the isometry groups of the space with respect to some proper metrics. As a consequence, we showed that every (separable) Lie group is isomorphic to the isometry group of another Lie group equipped with some proper metric and that every finite-dimensional [locally] compact Polish group is isomorphic to the isometry group of a finite-dimensional [proper locally] compact metric space. One

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of the aims of this paper is to give the results of Gao and Kechris, Melleray, and Malicki and Solecki a more "explicit" and unified form:

Theorem 1.1. *Let G be a Polish group.*

(a) *There is a complete compatible metric d on ℓ_2 such that G is isomorphic to* $\mathrm{Iso}(\ell_2, d)$.
(b) *If G is compact, there is a compatible metric d on the Hilbert cube Q such that G is isomorphic to* $\mathrm{Iso}(Q, d)$.
(c) *If G is locally compact, there is a proper compatible metric d on $Q \setminus \{\text{point}\}$ such that G is isomorphic to* $\mathrm{Iso}(Q \setminus \{\text{point}\}, d)$.

We shall also prove the following:

Theorem 1.2. *For every Polish group G there exist a separable real Banach space E and a nonzero vector $e \in E$ such that G is isomorphic to the group of all linear isometries of E (endowed with the pointwise convergence topology) which leave the point e fixed.*

Our methods can be adapted to general settings and give a characterization of topological groups which are isomorphic to isometry groups of complete as well as incomplete metric spaces. To this end, we recall that a topological group G is *Raĭkov-complete* (or *upper-complete*) if and only if it is complete with respect to the upper uniformity, by [Arhangel'skii and Tkachenko 2008, §3.6] or [Roelcke and Dierolf 1981] (see also the remarks on page 1581 in [Uspenskij 2008]). In other words, G is upper-complete if every net $\{x_\sigma\}_{\sigma \in \Sigma} \subset G$ satisfying the following condition is convergent in G:

(C) For every neighborhood U of the neutral element of G there is $\sigma_0 \in \Sigma$ such that both $x_\sigma x_{\sigma'}^{-1}$ and $x_\sigma^{-1} x_{\sigma'}$ belong to U for any $\sigma, \sigma' \geqslant \sigma_0$.

Equivalently, the net $\{x_\sigma\}_{\sigma \in \Sigma}$ satisfies (C) if both the nets $\{x_\sigma\}_{\sigma \in \Sigma}$ and $\{x_\sigma^{-1}\}_{\sigma \in \Sigma}$ are fundamental with respect to the left uniformity of G. We call Raĭkov-complete groups briefly *complete*, following [Uspenskij 2008]. The class of all complete topological groups coincides with the class of all absolutely closed topological groups (a topological group is *absolutely closed* if it is closed in every topological group containing it as a topological subgroup). It is well-known that for every topological group G there exists a unique (up to topological isomorphism) complete topological group containing G as a dense subgroup (see, e.g., [Roelcke and Dierolf 1981; Arhangel'skii and Tkachenko 2008, §3.6]). This complete group is called the *Raĭkov completion* of G and we shall denote it by $\overline{G}$.

Less classical are topological groups, which we call $\mathscr{G}_\delta$-complete. To define them, let us agree with the following general convention: Whenever τ is a topology on a set X, τ_δ stands for the topology on X whose base is formed by all $\mathscr{G}_\delta$-sets (with respect to τ) in X. (In particular, $\mathscr{G}_\delta$-sets in (X, τ) are open in (X, τ_δ).) Subsets of X which are closed or dense in the topology τ_δ are called $\mathscr{G}_\delta$-closed and $\mathscr{G}_\delta$-dense, respectively (see, for example, [Arhangel'skii 2002; Arhangel'skii and Tkachenko 2008, page 268]).

It may be easily verified that if (G, τ) is a topological group, so is (G, τ_δ).

Definition 1.3. A topological group G is *$\mathscr{G}_\delta$-complete* if (G, τ_δ) is a complete topological group (where τ is the topology of G).

Equivalently, a topological group G is $\mathscr{G}_\delta$-complete if and only if G is $\mathscr{G}_\delta$-closed in $\overline{G}$. The class of all $\mathscr{G}_\delta$-complete groups is huge (see Proposition 4.3 below) and contains all complete as well as metrizable topological groups (more detailed discussion on this class is included in Section 4). (It is worth noting here that, according to the Birkhoff–Kakutani theorem, a topological group is metrizable if and only if it is first-countable, that is, if it has a countable base of neighborhoods of the neutral element. For a proof see, for example, Theorem 3.3.12 in [Arhangel'skii and Tkachenko 2008].) However, there are topological groups which are not $\mathscr{G}_\delta$-complete (see Example 4.5 below).

$\mathscr{G}_\delta$-complete groups turn out to characterize isometry groups of metric spaces:

Theorem 1.4. *Let G be a topological group*:

(A) *The following conditions are equivalent*:

  (A1) *There exists a metric space (X, d) such that G is isomorphic to* $\mathrm{Iso}(X, d)$.

  (A2) *G is $\mathscr{G}_\delta$-complete.*

  Moreover, *if G is $\mathscr{G}_\delta$-complete*, *the space X witnessing* (A1) *may be chosen so that $w(X) = w(G)$.*

(B) *The following conditions are equivalent*:

  (B1) *There exists a complete metric space (X, d) such that G is isomorphic to* $\mathrm{Iso}(X, d)$.

  (B2) *G is complete.*

  Moreover, *if G is complete*, *the space X witnessing* (B1) *may be chosen so that $w(X) = w(G)$.*

(By $w(X)$ we denote the topological weight of a topological space X.)

One concludes from Theorem 1.4 that the isometry group of an arbitrary metric space is always Dieudonné-complete (see Corollary 4.4 below). This solves a problem posed, for example, by Arhangel'skii and Tkachenko [2008, Open Problem 3.5.4 on page 181].

A generalization of Theorems 1.1 and 1.2 has the following form:

Proposition 1.5. *Let G be a complete topological group of topological weight not greater than $\beta \geqslant \aleph_0$.*

(a) *There is a complete compatible metric* ϱ *on* $\mathscr{H}_\beta$ *such that* G *is isomorphic to* $\mathrm{Iso}(\mathscr{H}_\beta, \varrho)$, *where* $\mathscr{H}_\beta$ *is a real Hilbert space of* (*Hilbert space*) *dimension equal to* β.

(b) *There are an infinite-dimensional real Banach space* E *of topological weight* β *and a nonzero vector* $e \in E$ *such that* G *is isomorphic to the group of all linear isometries of* E *which leave the point* e *fixed.*

As an immediate consequence of Theorem 1.4 and Proposition 1.5 we obtain:

Corollary 1.6. *Let* $\mathscr{H}$ *be a Hilbert space of Hilbert space dimension* $\beta \geqslant \aleph_0$ *and let*

$$\mathscr{G} = \{\mathrm{Iso}(\mathscr{H}, \varrho) \mid \varrho \text{ is a complete compatible metric on } \mathscr{H}\}.$$

Then, up to isomorphism, $\mathscr{G}$ *consists precisely of all complete topological groups of topological weight not exceeding* β.

The paper is organized as follows: In Section 2 we give a new proof of the Gao–Kechris theorem mentioned above. We consider our proof more transparent, more elementary, and less complicated. The techniques of this part are adapted in Section 3, where we demonstrate that every closed subgroup of the isometry group of a (complete) metric space (X, d) is actually (isomorphic to) the isometry group of a certain (complete) metric space, closely "related" to (X, d). This theorem is applied in Section 4, where we establish basic properties of the class of all $\mathscr{G}_\delta$-complete groups and prove Theorem 1.4. Section 5 contains proofs of Theorem 1.2, Proposition 1.5, Theorem 1.1(a), and Corollary 1.6. In Section 6 we study topological groups isomorphic to isometry groups of completely metrizable metric spaces. Section 7 is devoted to the proofs of points (b) and (c) of Theorem 1.1.

Notation and terminology. In this paper $\mathbb{N} = \{0, 1, 2, \dots\}$ (and it is equipped with the discrete topology). All isomorphisms between topological groups are topological, all topological groups are Hausdorff, and all isometries between metric spaces are, by definition, bijective. All normed vector spaces are assumed to be real. The topological weight of a topological space X is denoted by $w(X)$ and it is understood as an infinite cardinal number. Isometry groups (and all their subsets) of metric as well as normed vector spaces are endowed with the pointwise convergence topology, which makes them topological groups. A *Polish* space (resp. group) is a completely metrizable separable topological space (resp. group). A metric on a topological space is *compatible* if and only if it induces the topology of the space. It is *proper* if all closed balls with respect to this metric are compact (in the topology induced by this metric). Whenever (X, d) is a metric space, $a \in X$ and $r > 0$, $B_X(a, r)$ and $\bar{B}_X(a, r)$ stand for, respectively, the open and the closed d-balls with center at a and of radius r. The Hilbert cube, that is, the countable infinite Cartesian power of [0, 1], is denoted by Q and ℓ_2 stands for the separable Hilbert space. A *map* means a continuous function.

2. The Gao–Kechris theorem revisited

This part is devoted to the proof of the Gao–Kechris theorem [2003] mentioned in the introductory part and stated below. Another proof may be found in [Melleray 2008].

Theorem 2.1. *Every Polish group is isomorphic to the isometry group of a certain separable complete metric space.*

For the purpose of this and the next section, let us agree with the following conventions: For every nonempty collection $\{X_s\}_{s\in S}$ of topological spaces, $\bigsqcup_{s\in S} X_s$ denotes the topological disjoint union of these spaces. In particular, whenever the notation $\bigsqcup_{s\in S} X_s$ appears, the sets $X_s (s \in S)$ are assumed to be pairwise disjoint (the same rule for the symbol "$\sqcup$"). For a function $f : X \to X$ and an integer $n \geqslant 1$, we denote by $f^{ⓝ} : X^n \to X^n$ the n-th Cartesian power of f, given by $f^{ⓝ}(x_1, \dots, x_n) = (f(x_1), \dots, f(x_n))$; and by $f \times w$, for an arbitrary point w, we denote the map $X \times \{w\} \to X \times \{w\}$ that sends (x, w) to $(f(x), w)$ for any $x \in X$. Similarly, if d is a metric on X, we denote by $d^{ⓝ}$ the maximum metric on X^n induced by d; that is,

$$d^{ⓝ}((x_1, \dots, x_n), (y_1, \dots, y_n)) = \max_{j=1,\dots,n} d(x_j, y_j),$$

and $d \times w$ is the metric on $X \times \{w\}$ such that $(d \times w)((x, w), (y, w)) = d(x, y)$. Finally, for a topological space V and a map $v : V \to V$ we put $\widehat{V} = (V \times \mathbb{N}) \sqcup \left(\bigsqcup_{n=2}^{\infty} V^n\right) \sqcup \mathbb{N}$ and define $\hat{v} : \widehat{V} \to \widehat{V}$ by the following rules: $\hat{v}(x, m) = (v(x), m)$, $\hat{v}|_{V^n} = v^{ⓝ}$, and $\hat{v}(m) = m$ for any $x \in V$, $m \in \mathbb{N}$, and $n \in \mathbb{N}\setminus\{0\}$. To avoid repetitions, for a metric space (X, d) and arbitrary sets $A, B \subset \mathbb{N}$, and $C \subset \mathbb{N} \setminus \{0, 1\}$, let us say a metric ϱ on $(X \times A) \sqcup \left(\bigsqcup_{j\in C} X^j\right) \sqcup B$ $(\subset \widehat{X})$ *respects* d if and only if the following three conditions are satisfied:

(AX1) ϱ coincides with $d \times m$ on $X \times \{m\}$ for each $m \in A$.

(AX2) ϱ coincides with $d^{ⓝ}$ on X^n for each $n \in C$.

(AX3) $\varrho(x, y) \geqslant 1$ whenever x and y belong to distinct members of the collection $\{X \times \{m\} \mid m \in A\} \cup \{X^n \mid n \in C\} \cup \{\{k\} \mid k \in B\}$.

Observe that (AX1)–(AX3) imply that

(AX4) if ϱ respects d then ϱ is compatible; if, moreover, d is complete, so is ϱ.

The main result of this section is the following:

Proposition 2.2. *Let* (X, d) *be a separable bounded complete metric space and let* G *be a closed subgroup of* $\operatorname{Iso}(X, d)$. *There exists a metric* ϱ *on* $\widehat{X}$ *such that* ϱ *respects* d *and the function*

$$G \ni u \mapsto \hat{u} \in \operatorname{Iso}(\widehat{X}, \varrho)$$

is a well-defined isomorphism of topological groups.

The proof of Proposition 2.2 will be preceded by a few auxiliary results. The first of them is a kind of folklore and we leave its (simple) proof to the reader.

Lemma 2.3. *Let* $\{(X_s, d_s)\}_{s\in S}$ *be a nonempty family of metric spaces such that for* $A = \bigcap_{s\in S} X_s$ *we have*:

- $X_s \cap X_{s'} = A$ *and* $d_s|_{A\times A} = d_{s'}|_{A\times A}$ *for any two distinct indices* s *and* s' *of* S.
- A *is nonempty and closed in* (X_s, d_s) *for each* $s \in S$.

Let $X = \bigcup_{s\in S} X_s$ *and let* $d : X \times X \to [0, \infty)$ *be given by the rules*:

- d *coincides with* d_s *on* $X_s \times X_s$ *for every* $s \in S$.
- $d(x, y) = \inf\{d_s(x, a) + d_{s'}(a, y) \mid a \in A\}$ *whenever* $x \in X_s \setminus X_{s'}$ *and* $y \in X_{s'} \setminus X_s$ *for distinct indices* s *and* s'.

Then d *is a well-defined metric on* X *with the following property*: *Whenever* $f_s \in \operatorname{Iso}(X_s, d_s)$ $(s \in S)$ *are maps such that* $f_s|_A = f_{s'}|_A$ *and* $f_s(A) = A$ *for any* $s, s' \in S$ *then their union* $f := \bigcup_{s\in S} f_s$ (*that is,* $f = f_s$ *on* X_s) *is a well-defined function such that* $f \in \operatorname{Iso}(X, d)$.

The above result will be the main tool for constructing the metric ϱ appearing in Proposition 2.2.

In the next two results, (X, p) is a complete nonempty metric space with

$$p < 1. \tag{1}$$

Lemma 2.4. *Let* $J \subset \mathbb{N} \setminus \{0\}$ *be a finite set such that* $n = \operatorname{card}(J) > 1$. *There is a metric* λ *on* $F := [X \times (J \cup \{0\})] \sqcup X^n$ *with the following properties*:

(a) λ *respects* p *and* $\lambda \leqslant 5$.

(b) *For every* $u \in \operatorname{Iso}(X, p)$, $\hat{u}|_F \in \operatorname{Iso}(F, \lambda)$.

(c) *If* $g \in \operatorname{Iso}(F, \lambda)$ *is such that* $g(X \times \{j\}) = X \times \{j\}$ *for each* $j \in J \cup \{0\}$ *then* $g = \hat{u}|_F$ *for some* $u \in \operatorname{Iso}(X, p)$.

Proof. With no loss of generality, we may assume that $J = \{1, \dots, n\}$. Let $A = \{(x_1, \dots, x_n) \in X^n \mid x_1 = \dots = x_n\}$ and let λ_0' be the metric on ${X_0}' := (X \times \{0\}) \sqcup A$ that coincides with $p \times 0$ on $X \times \{0\}$, with $p^{ⓝ}$ on A, and such that $\lambda_0'((x, 0), (a, \dots, a)) = 1 + p(x, a)$ for $x \in X$ and $(a, \dots, a) \in A$. Now apply Lemma 2.3 for $\{({X_0}', \lambda_0'), (X^n, p^{ⓝ})\}$ to obtain a metric λ_0 on $X_0 := (X \times \{0\}) \sqcup X^n$

which extends both λ_0' and $p^{ⓝ}$. Observe that λ_0 respects p, that $\lambda_0 \leqslant 3$ (by (1)), that $\hat{u}|_{X_0} \in \mathrm{Iso}(X_0, \lambda_0)$ for each $u \in \mathrm{Iso}(X, p)$, and that, for arbitrary $x, x_1, \dots, x_n \in X$,

$$\lambda_0((x, 0), (x_1, \dots, x_n)) = 1 \iff x_1 = \cdots = x_n = x. \tag{2}$$

Further, for $j \in J$ let λ_j be the metric on $X_j := (X \times \{j\}) \sqcup X^n$ that coincides with $p \times j$ on $X \times \{j\}$, with $p^{ⓝ}$ on X^n, and such that $\lambda_j((x, j), (x_1, \dots, x_n)) = 1 + p(x, x_j)$ for any $x, x_1, \dots, x_n \in X$ (λ_j is indeed a metric thanks to (1)). Similarly, as before, notice that λ_j respects p, $\lambda_j \leqslant 2$ and for any $x, x_1, \dots, x_n \in X$:

$$\lambda_j((x, j), (x_1, \dots, x_n)) = 1 \iff x_j = x. \tag{3}$$

Now again apply Lemma 2.3 for the family $\{(X_j, \lambda_j) \mid j \in J \cup \{0\}\}$ to obtain a metric λ on F which extends each of $\lambda_j (j \in J \cup \{0\})$. It follows from the construction and Lemma 2.3 that points (a) and (b) are satisfied. We turn to (c). Let g be as specified there. Let $u : X \to X$ be such that $u \times 0 = g|_{X \times \{0\}}$ and, similarly, for $j \in J$ let $u_j : X \to X$ be such that $u_j \times j = g|_{X \times \{j\}}$. Finally, put $f = g|_{X^n} : X^n \to X^n$. Since λ respects p, $u \in \mathrm{Iso}(X, p)$. So we only need to check that $u_1 = \dots u_n = u$ and $f = u^{ⓝ}$. Let $\pi_j : X^n \to X$ be the projection onto the j-th coordinate ($j = 1, \dots, n$). For any $x = (x_1, \dots, x_n) \in X^n$ and $j \in J$ we have, by (3),

$$1 = \lambda((\pi_j(x), j), x) = \lambda(g(\pi_j(x), j), g(x)) = \lambda(((u_j \circ \pi_j)(x), j), f(x))$$

and therefore, again by (3), $u_j \circ \pi_j = \pi_j \circ f$. Consequently, $f(x_1, \dots, x_n) = (u_1(x_1), \dots, u_n(x_n))$. Finally, for any $z \in X$ we have, by (2),

$$1 = \lambda((z, 0), (z, \dots, z)) = \lambda(g(z, 0), g(z, \dots, z)) = \lambda((u(z), 0), (u_1(z), \dots, u_n(z)))$$

and hence, again by (2), $u_1(z) = \cdots = u_n(z) = u(z)$. □

Lemma 2.5. *Let G be a subgroup of* $\mathrm{Iso}(X, p)$ *and let $z \in X^n$ and $J \subset \mathbb{N} \setminus \{0\}$ be such that* $\mathrm{card}(J) = n > 1$. *Let D denote the closure (in X^n) of the set $\{u^{ⓝ}(z) \mid u \in G\}$. There exists a metric μ on $F := [X \times (J \cup \{0\})] \sqcup X^n \sqcup \{n-1\}$ which has the following properties*:

(a) *μ respects p and $\mu \leqslant 11$.*

(b) *$\hat{u}|_F \in \mathrm{Iso}(F, \mu)$ for every $u \in G$.*

(c) *For any $g \in \mathrm{Iso}(F, \mu)$ there is $u \in \mathrm{Iso}(X, p)$ such that $g = \hat{u}|_F$ and $u^{ⓝ}(z) \in D$.*

Proof. Without loss of generality, we may assume that $J = \{1, \dots, n\}$. Let λ be as in Lemma 2.4 (so, λ is a metric on $F \setminus \{n-1\}$). Let $c_0, \dots, c_{n+1}$ be such that

$$5 < c_0 < c_1 < \cdots < c_{n+1} < 6. \tag{4}$$

Put $A = [X \times (J \cup \{0\})] \sqcup D$ and denote by μ_0 the metric on $A \sqcup \{n-1\}$ such that μ_0 coincides with λ on A, $\mu_0((x, j), n-1) = c_j$ for $x \in X$ and $j = 0, \dots, n$, and $\mu_0(y, n-1) = c_{n+1}$ for $y \in D$ (μ_0 is a metric thanks to Lemma 2.4(a), (AX3), and (4)). Now apply Lemma 2.3 for the family $\{(A \cup \{n-1\}, \mu_0), (F \setminus \{n-1\}, \lambda)\}$ to obtain a metric μ which extends both μ_0 and λ. We infer the validity of (a) from (4) and Lemma 2.4(a). Further, since $u^{\textcircled{n}}(D) = D$ for each $u \in G$ and thanks to of Lemma 2.4(b), condition (b) is fulfilled as well (see Lemma 2.3). We turn to (c).

Let $g \in \mathrm{Iso}(F, \mu)$. Since $n-1$ is a unique point $q \in F$ such that $\mu(q, x) = c_0$ and $\mu(q, y) = c_1$ for some $x, y \in F$ (since $\lambda \leqslant 5 < c_0 < c_1$), we conclude that $g(n-1) = n-1$. Further, observe that for each $x \in X^n$, $\mu(x, n-1) \geqslant c_{n+1}$ because of (AX3) and (4). Consequently, $X \times \{j\} = \{x \in F \mid \mu(x, n-1) = c_j\}$ for $j = 0, 1, \dots, n$. Thus, we see that $g(X \times \{j\}) = X \times \{j\}$ for such j's. Since $g|_{F \setminus \{n-1\}} \in \mathrm{Iso}(F \setminus \{n-1\}, \lambda)$ and $g(n-1) = n-1$, point (c) of Lemma 2.4 implies that there is $u \in \mathrm{Iso}(X, p)$ such that $g = \hat{u}|_F$. Finally, $g(z) = u^{\textcircled{n}}(z) \in X^n$, and for $y \in X^n$, $\mu(y, n-1) = c_{n+1}$ if and only if $y \in D$ (by (4) and (AX3)), which gives $u^{\textcircled{n}}(z) \in D$. □

Proof of Proposition 2.2. Let $r \geqslant 1$ be such that $d < r$. Put $p = d/r < 1$ and notice that $\mathrm{Iso}(X, p) = \mathrm{Iso}(X, d)$. Let $X_0 = \{x_n \mid n \geqslant 1\}$ be a dense subset of X. Let $J_1, J_2, \dots$ be pairwise disjoint sets such that $\bigcup_{n=1}^{\infty} J_n = \mathbb{N} \setminus \{0\}$ and $\mathrm{card}(J_n) = n+1$. For each $n \geqslant 2$ put $z_n = (x_1, \dots, x_n) \in X^n$, $F_n = [X \times (J_{n-1} \cup \{0\})] \sqcup X^n \sqcup \{n-1\}$, and let D_n be the closure (in X^n) of $\{u^{\textcircled{n}}(z_n) \mid u \in G\}$. Further, let μ_n be a metric on F_n obtained from Lemma 2.5 (applied for z_n and J_{n-1}). Now apply Lemma 2.3 for the collection $\{(F_n, \mu_n) \mid n \geqslant 2\}$ to get a metric λ_0 on $\widehat{X} \setminus \{0\}$ which extends each of μ_n ($n \geqslant 2$). In particular, λ_0 respects p and $\lambda_0 \leqslant 22$. Finally, we extend the metric λ_0 to a metric λ on $\widehat{X}$ in such a way that for $k \geqslant 0$, $\lambda(x, 0) = c_{k,1}$ for $x \in X \times \{k\}$, $\lambda(x, 0) = c_{k,2}$ for $x \in X^{k+2}$, and $\lambda(k+1, 0) = c_{k,3}$, where

$$(5) \qquad c_{0,1}, c_{0,2}, c_{0,3}, c_{1,1}, c_{1,2}, c_{1,3}, \dots \text{ are all different numbers,}$$

are greater than 22, and smaller than 23 (λ is a metric thanks to (AX3)). It follows from Lemma 2.3 and Lemma 2.5(b) that $\hat{u} \in \mathrm{Iso}(\widehat{X}, \lambda)$ for any $u \in G$. It is clear that the function $G \ni u \mapsto \hat{u} \in \mathrm{Iso}(\widehat{X}, \lambda)$ is a group homomorphism and a topological embedding. We shall now show that it is also surjective.

Let $g \in \mathrm{Iso}(\widehat{X}, \lambda)$. Since 0 is a unique point $q \in \widehat{X}$ such that $\lambda(q, x) = c_{0,1}$ and $\lambda(q, y) = c_{0,2}$ for some $x, y \in \widehat{X}$, we see that $g(0) = 0$. Consequently, $g(X \times \{k\}) = X \times \{k\}$, $g(X^{k+2}) = X^{k+2}$, and $g(k+1) = k+1$ for each $k \geqslant 0$, by (5). So, taking into account that $g|_{F_n} \in \mathrm{Iso}(F_n, \mu)$, point (c) of Lemma 2.5 yields that there is $u \in \mathrm{Iso}(X, p)$ such that $g = \hat{u}$ and $u(z_n) \in D_n$. The latter condition implies that there are elements $u_1, u_2, \dots$ of G which converge pointwise to u on X_0. We now infer from the density of X_0 in X that $u = \lim_{n\to\infty} u_n$, and in fact $u \in G$ by the closedness of G.

To end the proof, it suffices to put $\varrho = r\lambda$. □

Proof of Theorem 2.1. Let $(H, \cdot)$ be a Polish group. First we introduce a standard argument used, for example, by Melleray [2008] in his proof of this theorem: Take a left-invariant metric $d_0 \leqslant 1$ on H and denote by (X, d) the completion of (H, d_0). Then, of course, X is separable and for every $h \in H$ there is a unique $u_h \in \mathrm{Iso}(X, d)$ such that $u_h(x) = hx$ for $x \in H$. Observe that the function $H \ni h \mapsto u_h \in \mathrm{Iso}(X, d)$ is a group homomorphism as well as a topological embedding. Therefore, its image G is isomorphic to H. Since G is a Polish subgroup of a Polish group, G is closed in $\mathrm{Iso}(X, d)$. Now it suffices to apply Proposition 2.2 and to use (AX4) to deduce the completeness of the metric obtained by that result. □

3. Closed subgroups of isometry groups

In this section we generalize the ideas of the previous part to the context of all isometry groups. Our aim is to show that a closed subgroup of the isometry group of a metric space is isomorphic to the isometry group of another metric space. We have decided to discuss the separable case separately, because in that case the proofs are more transparent and easier. Actually all tools were prepared in the previous section, except the following one:

Lemma 3.1. *Let X be a set with* $\mathrm{card}(X) \neq 2$ *and $I \subset (0, \infty)$ be a nondegenerate interval. There is a metric $d : X \times X \to I \cup \{0\}$ such that the identity map of X is a unique member of* $\mathrm{Iso}(X, d)$.

We shall prove a stronger version of Lemma 3.1 at the end of the section. Now we generalize the concepts in Section 3. Let β be an infinite cardinal number and let D_β denote a fixed discrete topological space of cardinality β. For a metrizable space X and a function $f : X \to X$ let X^0 be a one-point space and $f^{⓪} : X^0 \to X^0$ denote the identity map. Further, we put $T(X) = \bigsqcup_{n\in\mathbb{N}} X^n$ (recall that $0 \in \mathbb{N}$). Finally, denote by $\widehat{X}_\beta$ and $\hat{f}_\beta$ (resp.) the product $T(X) \times D_\beta$ and the function of $\widehat{X}_\beta$ into itself such that $\hat{f}_\beta = f^{ⓝ} \times \xi$ on $X^n \times \{\xi\}$ for any $n \in \mathbb{N}$ and $\xi \in D_\beta$. (Notice that $w(\widehat{X}_\beta) = \beta$ provided $\beta \geqslant w(X)$.) For any $J \subset \mathbb{N}$ and a collection $\{A_n\}_{n\in J}$ of subsets of D_β, we say a metric ϱ on $\bigsqcup_{n\in J}(X^n \times A_n) \subset \widehat{X}_\beta$ *respects* a compatible metric d on X if and only if the following two conditions are fulfilled:

(PR1) ϱ coincides with $d^{ⓝ} \times \xi$ on $X^n \times \{\xi\}$ for any $n \in J \setminus \{0\}$ and $\xi \in A_n$.

(PR2) $\varrho(x, y) \geqslant 1$ whenever x and y belong to distinct members of the collection

$$\{X^n \times \{\xi\} \mid n \in J,\ \xi \in A_n\}.$$

As before, we see that (AX4) is satisfied.

A counterpart of Proposition 2.2 in the general case is the following:

Theorem 3.2. *Let β be an infinite cardinal number and (X, d) be a nonempty bounded metric space such that $w(X) \leqslant \beta$. For any closed subgroup G of $\operatorname{Iso}(X, d)$ there exists a metric ϱ on $\widehat{X}_\beta$ such that ϱ respects d and the function*

$$G \ni u \mapsto \hat{u}_\beta \in \operatorname{Iso}(\widehat{X}_\beta, \varrho) \tag{6}$$

is a well-defined isomorphism of topological groups.

Proof. In what follows, we shall (naturally) identify $X^0 \times D_\beta$ with D_β. It follows from the proof of Proposition 2.2 that we may assume $d < 1$. Let Z be a dense set in X such that $\operatorname{card}(Z) \leqslant \beta$. Fix arbitrary $\theta \in D_\beta$ and write $D_\beta \setminus \{\theta\}$ in the form $\bigcup_{n=0}^\infty S_n$, where $\operatorname{card}(S_n) = \beta$ for any n and

$$S_n \cap S_m = \varnothing \quad (n \neq m). \tag{7}$$

The set S_0 and the point θ will be employed in the last part of the proof. For simplicity, put $S_* = \bigcup_{n=1}^\infty S_n$ and $X_* := \widehat{X}_\beta \setminus (S_0 \cup \{\theta\}) = \big(\bigsqcup_{n\geqslant 1}(X^n \times D_\beta)\big) \sqcup S_*$. It follows from (7) that for any $\xi \in D_\beta \setminus \{\theta\}$ there is a unique number $n(\xi) \in \mathbb{N}$ such that $\xi \in S_{n(\xi)-1}$. Further, for every $n \geqslant 2$ there are a surjection $\kappa_n : S_{n-1} \to Z^n$ and a bijection $\tau_n : S_{n-1} \to D_\beta$. Take a collection $\{J_\xi \mid \xi \in S_*\}$ such that for any $\xi, \eta \in S_*$

(S1) $\operatorname{card}(J_\xi) = n(\xi) (\geqslant 2)$;

(S2) $J_\xi \cap J_\eta = \varnothing$ whenever $\xi \neq \eta$;

(S3) $\bigcup_{\zeta \in S_*} J_\zeta = D_\beta \setminus \{\theta\}$.

We deduce from (S1) and Lemma 2.5 that for each $\xi \in S_*$ there exists a metric μ_ξ on $F_\xi := [X \times (J_\xi \cup \{\theta\})] \sqcup (X^{n(\xi)} \times \{\tau_{n(\xi)}(\xi)\}) \sqcup \{\xi\}$ which has the following properties:

(D1) μ_ξ respects d and $\mu_\xi \leqslant 11$.

(D2) $\hat{u}_\beta|_{F_\xi} \in \operatorname{Iso}(F_\xi, \mu_\xi)$ for every $u \in G$.

(D3) For any $g \in \operatorname{Iso}(F_\xi, \mu_\xi)$ there is $u \in \operatorname{Iso}(X, d)$ with $g = \hat{u}_\beta|_{F_\xi}$, and $u^{ⓝ}(\kappa_n(\xi))$ belongs to the closure B_ξ of $\{f^{ⓝ}(\kappa_n(\xi)) \mid f \in G\}$ in X^n, where $n = n(\xi)$.

Observe that (S2)–(S3), (D1), and the bijectivity of the τ_n imply that

$$F_\xi \cap F_\eta = X \times \{\theta\} \tag{8}$$

for distinct $\xi, \eta \in S_*$, and $X \times \{\theta\}$ is closed in (F_ξ, μ_ξ), by (PR2). Moreover, it follows from (D1) that we may apply Lemma 2.3 for the family $\{(F_\xi, \mu_\xi) \mid \xi \in S_*\}$. Let μ be the metric on $\bigcup_{\xi \in S_*} F_\xi = X_*$ obtained by that lemma. Then

$$\hat{u}_\beta|_{X_*} \in \operatorname{Iso}(X_*, \mu) \quad (u \in G) \tag{9}$$

(see (D2) and the last claim in Lemma 2.3) and

$$\mu \text{ respects } d \quad \text{and} \quad \mu \leqslant 22 \tag{10}$$

(by (D1)). What is more,

($\star$) if $g \in \mathrm{Iso}(X_*, \mu)$ is such that $g(A) = A$ for any

$$A \in \mathcal{S} := \{X^n \times \{\xi\} \mid n \geqslant 1,\ \xi \in D_\beta\} \cup \{\{\xi\} \mid \xi \in S_*\},$$

then $g = \hat{u}_\beta|_{X_*}$ for some $u \in G$.

Let us briefly show ($\star$). If g is as specified there then $g(F_\xi) = F_\xi$ for any $\xi \in S_*$. So we infer from (D3) that $g = \widehat{(u_\xi)}_\beta$ on F_ξ for some $u_\xi \in \mathrm{Iso}(X, d)$ with

$$(u_\xi)^{ⓝ}(\kappa_n(\xi)) \in B_\xi, \tag{11}$$

where $n = n(\xi)$. We conclude from (8) that $u := u_\xi$ is independent of the choice of $\xi \in S_*$. Consequently, $g = \hat{u}_\beta|_{X_*}$. To end the proof of ($\star$), it remains to check that $u \in G$. Since the κ_n are surjective, (11) yields that $(u(z_1), \ldots, u(z_n))$ belongs to the closure (in X^n) of $\{(f(z_1), \ldots, f(z_n)) \mid f \in G\}$ for any $n \geqslant 2$ and $z_1, \ldots, z_n \in Z$. But this, combined with the fact that the function $\mathrm{Iso}(X, d) \ni f \mapsto f|_Z \in X^Z$ is an embedding (when X^Z is equipped with the pointwise convergence topology), yields that u belongs to the closure of G in $\mathrm{Iso}(X, d)$. But G is a closed subgroup, so we are done.

By Lemma 3.1, there is a metric

$$\lambda : S_0 \times S_0 \to \{0\} \cup [1, 2] \tag{12}$$

for which $\mathrm{Iso}(S_0, \lambda) = \{\mathrm{id}_{S_0}\}$ (id_{S_0} is the identity map on S_0). Let $\mathcal{S}$ be as in ($\star$). Since $\mathrm{card}(\mathcal{S}) = \beta = \mathrm{card}(S_0)$, there is a one-to-one function $v : \mathcal{S} \to \{11, 12\}^{S_0}$. We define a metric ϱ on $X_* \sqcup S_0 = \widehat{X}_\beta \setminus \{\theta\}$ by the rules:

- $\varrho = \mu$ on $X_* \times X_*$.
- $\varrho = \lambda$ on $S_0 \times S_0$.
- $\varrho(\xi, \eta) = \varrho(\eta, \xi) = [v(A)](\eta)$ for $\xi \in X_*$ and $\eta \in S_0$, where $A \in \mathcal{S}$ is such that $\xi \in A$ (such A is unique).

That ϱ is indeed a metric follows from (12), (10), axiom (PR2) for μ, and the fact that for any $\eta \in S_0$, $\varrho(\,\cdot\,, \eta)$ is constant on each member of $\mathcal{S}$. Finally, we extend the metric ϱ to $\widehat{X}_\beta$ by putting $\varrho(\xi, \theta) = 22$ for $\xi \in X_*$ and $\varrho(\xi, \theta) = 23$ for $\xi \in S_0$. Direct calculations show that ϱ is indeed a metric on $\widehat{X}_\beta$ and that ϱ respects d. It remains to check that (6) is a well-defined surjection (compare the proof of Proposition 2.2). We infer from (9) and the fact that $\varrho(\,\cdot\,, \eta)$ is constant on each member of $\mathcal{S}$ for any $\eta \in S_0 \cup \{\theta\}$ that the function (6) is well-defined. Now let $g \in \mathrm{Iso}(\widehat{X}_\beta, \varrho)$. Since θ is a unique point $\omega \in \widehat{X}_\beta$ such that $\mathrm{card}(\{\xi \in \widehat{X}_\beta \mid \varrho(\xi, \omega) = 23\}) > 1$, we obtain $g(\theta) = \theta$. Consequently, $g(X_*) = X_*$ and $g(S_0) = S_0$. The latter yields that

$$g|_{S_0} = \mathrm{id}_{S_0} \tag{13}$$

(because ϱ extends λ). Now if $\xi, \eta \in X_*$ are arbitrary, the injectivity of v and the definition of ϱ imply that $\varrho(\xi, \cdot) = \varrho(\eta, \cdot)$ on S_0 if and only if ξ and η belong to a common member of $\mathcal{S}$. But this, combined with (13), allows us to conclude that $g(A) = A$ for any $A \in \mathcal{S}$. Now an application of ($\star$) (recall that ϱ extends μ) provides us the existence of $u \in G$ for which $g = \hat{u}_\beta$ on X_*. Since $g(\xi) = \xi$ for $\xi \notin X_*$, we see that $g = \hat{u}_\beta$, which finishes the proof. $\square$

Remark 3.3. Under the notation and the assumptions of Theorem 3.2, if $M \geqslant 1$ is such that $d \leqslant M$ and $\varepsilon > 0$ is arbitrary, the metric ϱ appearing in the assertion of that theorem may be chosen so that $\varrho \leqslant M + \varepsilon$. Indeed, the above proof provides us the existence of a bounded metric ϱ, say $\varrho \leqslant C$, where $M < C < \infty$. Now if ε is small enough (that is, if $M + \varepsilon \leqslant C$), it suffices to replace ϱ by $\omega \circ \varrho$, where $\omega : [0, C] \to [0, M + \varepsilon]$ is affine on $[0, M]$ and $[M, C]$, and $\omega(0) = 0$, $\omega(M) = M$, and $\omega(C) = M + \varepsilon$.

Let (X, d) be a nonempty metric space and (Y, ϱ) denote the completion of (X, d). Since every isometry of (X, d) extends to a unique isometry of (Y, ϱ), the topological group $\operatorname{Iso}(X, d)$ may naturally be identified with the subgroup $\{u \in \operatorname{Iso}(Y, \varrho) \mid u(X) = X\}$ of $\operatorname{Iso}(Y, \varrho)$. If we follow this idea, Theorem 3.2 may be strengthened as follows:

Proposition 3.4. *Let (X, d) be a nonempty bounded metric space and (Y, ϱ) denote its completion. Let β be an infinite cardinal not less than $w(X)$. Further, let G be a closed subgroup of* $\operatorname{Iso}(X, d) \subset \operatorname{Iso}(Y, \varrho)$ *and $\bar{G}$ denote its closure in* $\operatorname{Iso}(Y, \varrho)$. *There are a complete metric λ on $\widehat{Y}_\beta$ respecting ϱ and a dense set $X_\beta \subset \widehat{Y}_\beta$ such that $(\widehat{Y}_\beta \setminus X_\beta, \lambda)$ is isometric to $(Y \setminus X, \varrho)$, and the function*

$$\bar{G} \ni u \mapsto \hat{u}_\beta \in \operatorname{Iso}(\widehat{Y}_\beta, \lambda) \tag{14}$$

is a well-defined isomorphism of topological groups which transforms G onto the group of all $u \in \operatorname{Iso}(\widehat{Y}_\beta, \lambda)$ with $u(X_\beta) = X_\beta$.

Proof. Fix $\theta \in D_\beta$ and put $X_\beta = \widehat{Y}_\beta \setminus [(Y \setminus X) \times \{\theta\}]$. By Theorem 3.2, there is a metric λ on $\widehat{Y}_\beta$ which respects ϱ and for which (14) is a well-defined isomorphism. Note that then λ is complete (see (AX4)), X_β is dense in $\widehat{Y}_\beta$, and $(\widehat{Y}_\beta \setminus X_\beta, \lambda)$ is isometric to $(Y \setminus X, \varrho)$ (since λ respects ϱ). Finally, if $u \in \bar{G}$ then $\hat{u}_\beta(X_\beta) = X_\beta$ if and only if $u(X) = X$ (which follows from the formulas for $\hat{u}_\beta$ and X_β). Equivalently, $\hat{u}_\beta(X_\beta) = X_\beta$ if and only if $u \in \operatorname{Iso}(X, d) \cap \bar{G} = G$, by the closedness of G in $\operatorname{Iso}(X, d)$. This shows the last claim of the theorem. $\square$

This proposition will be applied in Section 6, which is devoted to isometry groups of completely metrizable metric spaces.

To complete the proof of Theorem 3.2, we need to show Lemma 3.1. But the latter result immediately follows from the following much stronger result:

Proposition 3.5. *Let a and b be two reals such that*

$$0 < a < b \leqslant 2a. \tag{15}$$

For every set X having more than 5 points there is a metric $d : X \times X \to \{0, a, b\}$ such that

$$\mathrm{Iso}(X, d) = \{\mathrm{id}_X\}. \tag{16}$$

Proof. First of all, observe that any function $d : X \times X \to \{0, a, b\}$ which is symmetric and vanishes precisely on the diagonal of X is automatically a complete metric, which follows from (15). So we only need to take care of (16). For the same reason, we may (and do) assume, with no loss of generality, that $a = 1$ and $b = 2$. We shall make use of transfinite induction with respect to $\beta = \mathrm{card}(X) > 5$. Everywhere below in this proof, for $x \in X$, by $S(x)$ we denote the set of all $y \in X$ with $d(x, y) = 1$. Since we have to define a metric taking values in $\{0, 1, 2\}$, it is readily seen that it suffices to describe the sets $S(x)$ $(x \in X)$.

First assume $\beta = n \geqslant 6$ is finite. We may assume that $X = \{1, \dots, n\}$. Our metric d is defined by the following rules: $S(1) = \{2\}$, $S(2) = \{1, 3, 4, 5\}$, $S(3) = \{2, 4\}$, $S(4) = \{2, 3, 5\}$, $S(5) = \{2, 4, 6\}$, $S(n) = \{n - 1\}$, and $S(j) = \{j - 1, j + 1\}$ if $5 < j < n$. Take $g \in \mathrm{Iso}(X, d)$ and observe that:

- $g(2) = 2$, since 2 is the only point $j \in X$ such that $\mathrm{card}(S(j)) = 4$.
- $g(1) = 1$, because 1 is the unique point $j \in X$ for which $S(j) = \{2\}$.
- $g(3) = 3$, since 3 is the only point $j \in X$ such that $\mathrm{card}(S(j)) = 2$ and $2 \in S(j)$.
- $g(4) = 4$, because 4 is the unique point $j \in X$ for which $2 \neq j \in S(3)$.
- $g(5) = 5$, since 5 is the only point $j \in X$ such that $j \in S(4) \setminus \{2, 3\}$.

Now it is easy to check, using induction, that $g(j) = j$ for $j = 6, \dots, n$.

When $\beta = \aleph_0$, we may assume $X = \mathbb{N}$. Define a metric $d : \mathbb{N} \times \mathbb{N} \to \{0, 1, 2\}$ by $d(n, m) = \min(|m - n|, 2)$. It is left to the reader that $\mathrm{Iso}(\mathbb{N}, d) = \{\mathrm{id}_{\mathbb{N}}\}$ (use induction to show that $g(n) = n$ for any $n \in \mathbb{N}$ and $g \in \mathrm{Iso}(\mathbb{N}, d)$). Below we assume that $\beta > \aleph_0$ is such that for every infinite $\alpha < \beta$ the proposition holds for an arbitrary set X of cardinality α. For simplicity, for any uncountable cardinal γ we denote by I_γ the set of all cardinals α for which $\aleph_0 \leqslant \alpha < \gamma$. To get the assertion, we consider three cases.

First assume β is not limit; that is, β is the immediate successor of an infinite cardinal α. We may assume that X is the union of three pairwise disjoint sets X', $X' \times Y$, and $\{a\}$, where $\mathrm{card}(X') = \alpha$ and $\mathrm{card}(Y) = \beta$. It follows from the transfinite induction hypothesis that there exists a metric $d' : X' \times X' \to \{0, 1, 2\}$ such that $\mathrm{Iso}(X', d') = \{\mathrm{id}_{X'}\}$. Since $\beta \leqslant 2^\alpha$, there exists a one-to-one function

$\mu : X' \times Y \to \{1, 2\}^{X'}$ such that

$$[\mu(x, y)](x) = 1 \quad (x \in X', y \in Y) \tag{17}$$

(such a function μ may easily be constructed by transfinite induction with respect to an initial well-order on $X' \times Y$). We now define a metric d on X (with values in $\{0, 1, 2\}$) by the rules:

(d1) $d = d'$ on $X' \times X'$.

(d2) $d((x, y), (x', y')) = 1$ if (x, y) and (x', y') are distinct elements of $X' \times Y$.

(d3) $d((x, y), x') = [\mu(x, y)](x')$ if $x, x' \in X'$ and $y \in Y$.

(d4) $d(x, a) = 1$ and $d((x, y), a) = 2$ for any $x \in X'$ and $y \in Y$.

Observe that $S(x) \supset \{x\} \times Y$ and $S(x, y) \supset (X' \times Y) \setminus \{(x, y)\}$ for any $x \in X'$ and $y \in Y$ (thanks to (17) and (d2)–(d3)), and

$$S(a) = X' \tag{18}$$

(by (d4)). We infer from these facts that a is a unique point $x \in X$ such that $\mathrm{card}(S(x)) = \alpha$. Consequently, if $g \in \mathrm{Iso}(X, d)$ then $g(a) = a$ and $g(X') = X'$ (because of (18)). So $g|_{X'} \in \mathrm{Iso}(X', d')$ and therefore $g(x) = x$ for any $x \in X'$. Finally, if $x, x' \in X'$ and $y \in Y$ are arbitrary then $g(x, y) \notin X'$ and $d(g(x, y), x') = d((x, y), x')$, which yields that $\mu(g(x, y)) = \mu(x, y)$. So $g(x, y) = (x, y)$ (since μ is one-to-one) and we are done.

Now we assume that β is limit and $\mathrm{card}(I_\beta) < \beta$. For simplicity, put $I = I_\beta$. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of pairwise disjoint sets such that

$$X_\alpha \cap I = \varnothing \quad \text{and} \quad \mathrm{card}(X_\alpha) = \alpha < \beta \quad (\alpha \in I). \tag{19}$$

Note that the set $X_* = \bigsqcup_{\alpha \in I} X_\alpha$ is of cardinality β and therefore we may assume $X = \{\omega\} \sqcup I \sqcup X_*$ (recall that this notation means that $\omega \notin I \cup X_*$). It follows from the transfinite induction hypothesis that there are metrics $d_I : I \times I \to \{0, 1, 2\}$ and $d_\alpha \mid X_\alpha \times X_\alpha \to \{0, 1, 2\}$ $(\alpha \in I)$ for which the groups $\mathrm{Iso}(I, d_I)$ and $\mathrm{Iso}(X_\alpha, d_\alpha)$ are trivial. We define a metric d on X as follows:

(d1′) $d = d_I$ on $I \times I$ and $d = d_\alpha$ on $X_\alpha \times X_\alpha$ for any $\alpha \in I$.

(d2′) $d(x, y) = 1$ if x and y belong to different members of the collection $\{X_\alpha\}_{\alpha \in I}$.

(d3′) $d(\alpha, x) = 2$ for $x \in X_\alpha$ and $d(\alpha, x) = 1$ for $x \in X_* \setminus X_\alpha$ $(\alpha \in I)$.

(d4′) $d(\alpha, \omega) = 1$ and $d(x, \omega) = 2$ for any $\alpha \in I$ and $x \in X_*$.

Observe that for any $\alpha \in I$ and $x \in X_\alpha$, we have $S(\alpha) \supset X_* \setminus X_\alpha$ by (d3′), and $S(x) \supset X_* \setminus X_\alpha$, by (d2′). At the same time, $S(\omega) = I$, by (d4′), and hence ω is the unique point $x \in X$ such that $\mathrm{card}(S(x)) < \beta$ (see (19)). Consequently, if $g \in \mathrm{Iso}(X, d)$ then $g(\omega) = \omega$, $g(I) = I$, and $g(X_*) = X_*$. Then $g|_I \in \mathrm{Iso}(I, d_I)$

(see (d1′)) and hence $g(\alpha) = \alpha$ for each $\alpha \in I$. Now use (d3′) to conclude that $g(X_\alpha) = X_\alpha$ for any $\alpha \in I$. So, according to (d1′), $g|_{X_\alpha} \in \mathrm{Iso}(X_\alpha, d_\alpha)$ for every $\alpha \in I$ and consequently $g(x) = x$ for all $x \in \bigcup_{\alpha \in I} X_\alpha = X_*$, and we are done.

Finally, assume β is limit and $\mathrm{card}(I_\beta) = \beta$. Then we may assume $X = I_\beta$. Since

$$\mathrm{card}(I_\alpha) \leqslant \alpha < \beta \tag{20}$$

whenever $\alpha \in X$, for every $\alpha \in X$ there is a cardinal $\gamma(\alpha) \in X$ such that

$$\mathrm{card}(\{\xi \mid \alpha < \xi \leqslant \gamma(\alpha)\}) = \alpha. \tag{21}$$

Now define a metric $d : X \times X \to \{0, 1, 2\}$ by the following rule: If $\aleph_0 \leqslant \alpha_1 < \alpha_2 < \beta$ then $d(\alpha_1, \alpha_2) = 1$ if and only if $\alpha_2 \leqslant \gamma(\alpha_1)$. It is easy to check that then $\mathrm{card}(S(\alpha)) = \alpha$ for any $\alpha \in X$ (thanks to (20) and (21)) and hence the identity map is a unique isometry on (X, d). □

4. Models for $\mathscr{G}_\delta$-complete groups

We begin this section with a useful characterization of $\mathscr{G}_\delta$-complete groups.

Proposition 4.1. *For a topological group G all conditions stated below are equivalent*:

(I) *G is $\mathscr{G}_\delta$-complete.*

(II) *G is isomorphic to a $\mathscr{G}_\delta$-closed subgroup of a complete topological group.*

(III) *G is $\mathscr{G}_\delta$-closed in every topological group which contains G as a topological subgroup.*

(IV) *Every net $\{x_\sigma\}_{\sigma \in \Sigma}$ of elements of G satisfying the following condition is convergent in G*:

(CC) *For every sequence $U_1, U_2, \ldots$ of neighborhoods of the neutral element of G there exist points $y, z \in G$ and a sequence $\sigma_1, \sigma_2, \dots \in \Sigma$ such that both $x_\sigma^{-1} y$ and $x_\sigma z^{-1}$ belong to U_n whenever $n \geqslant 1$ and $\sigma \geqslant \sigma_n$.*

(V) *Every net $\{x_\sigma\}_{\sigma \in \Sigma}$ of elements of G satisfying the following condition is convergent in G*:

(CC′) *For every continuous left-invariant pseudometric d on G there are points $y, z \in G$ such that* $\lim_{\sigma \in \Sigma} d(x_\sigma, y) = \lim_{\sigma \in \Sigma} d(x_\sigma^{-1}, z^{-1}) = 0$.

Proof. Everywhere below, τ is the topology of G and e is its neutral element.

First assume G is a $\mathscr{G}_\delta$-closed subgroup of a complete group H. We want to show that G is $\mathscr{G}_\delta$-complete. Let $\{x_\sigma\}_{\sigma \in \Sigma} \subset G$ be a net which satisfies condition (C) with respect to the topology τ_δ. It then satisfies this condition with respect to τ as well. Since H is complete, there is $y \in H$ such that $\lim_{\sigma \in \Sigma} x_\sigma = y$. It suffices to check that y belongs to the $\mathscr{G}_\delta$-closure of G in H. Take a $\mathscr{G}_\delta$-subset A of H

containing y and write Ay^{-1} in the form $Ay^{-1} = \bigcap_{n=1}^{\infty} U_n$, where each U_n is open in H and contains e. Using the regularity of the space H, we may find a sequence $V_1, V_2, \dots$ of open (in H) neighborhoods of e such that the closure (in H) of V_n is contained in $V_{n-1} \cap U_n$ for each n, where $V_0 = H$. Then $F := \bigcap_{n=1}^{\infty} V_n$ is a closed $\mathscr{G}_\delta$-subset of H and $e \in F \subset Ay^{-1}$. It follows from our assumption about the net that there is $\sigma_0 \in \Sigma$ such that $x_{\sigma_0} x_\sigma^{-1} \in F$ for any $\sigma \geqslant \sigma_0$. We now infer from the closedness of F in H that $x_{\sigma_0} y^{-1} \in F$ as well, and consequently, $x_{\sigma_0} \in A$, which shows that y belongs to the $\mathscr{G}_\delta$-closure of G. This proves that (II) is followed by (I). Conversely, if G is a $\mathscr{G}_\delta$-complete subgroup of a topological group K and $\mathbb{O}$ denotes the topology of K then τ_δ coincides with the topology (on G) of a subspace inherited from $(K, \mathbb{O}_\delta)$. It now follows from the completeness of (G, τ_δ) that G is closed in $(K, \mathbb{O}_\delta)$ or, equivalently, that G is $\mathscr{G}_\delta$-closed in K, which proves that (III) is implied by (I). Since (II) obviously follows from (III), in this way we have shown that conditions (I), (II), and (III) are equivalent. We shall now show that (II) is equivalent to (IV) and then that (IV) is equivalent to (V).

If (II) is fulfilled then G is $\mathscr{G}_\delta$-closed in $\overline{G}$. Let $\{x_\sigma\}_{\sigma\in\Sigma}$ be a net of elements of G which satisfies condition (CC). Then it fulfills condition (C) as well and hence there is $w \in \overline{G}$ such that $\lim_{\sigma\in\Sigma} x_\sigma = w$. It suffices to check that $w \in G$ or, equivalently, that w belongs to the $\mathscr{G}_\delta$-closure of G. To this end, take any $\mathscr{G}_\delta$-subset A of $\overline{G}$ which contains w. Write $Aw^{-1} = \bigcap_{n=1}^{\infty} V_n$, where $V_1, V_2, \dots$ are open neighborhoods of e. For each $n \geqslant 1$ take a neighborhood U_n of e with $U_n = U_n^{-1}$ and $U_n \cdot U_n \subset V_n$. Now let y, z, and $\sigma_1, \sigma_2, \dots$ be as in (CC), applied for the sequence

$$U_1 \cap G, U_2 \cap G, \dots.$$

Fix for a moment $n \geqslant 1$. Choose $\sigma \geqslant \sigma_n$ such that $x_\sigma w^{-1} \in U_n$. Then $zw^{-1} = (x_\sigma z^{-1})^{-1}(x_\sigma w^{-1}) \subset U_n^{-1} \cdot U_n \subset V_n$. So $zw^{-1} \in \bigcap_{n=1}^{\infty} V_n = Aw^{-1}$, which implies that $w \in A$. Consequently, $A \cap G \neq \varnothing$ and we are done.

The converse implication goes similarly: When (IV) is satisfied, we show that G is $\mathscr{G}_\delta$-closed in $\overline{G}$. Let $w \in \overline{G}$ belong to the $\mathscr{G}_\delta$-closure of G. Then, of course, w is in the closure of G and thus there is a net $\{x_\sigma\}_{\sigma\in\Sigma} \subset G$, which converges to w. To prove that $w \in G$, it is enough to verify that (CC) is fulfilled. To this end, fix a sequence $U_1, U_2, \dots$ of neighborhoods of e and choose its open symmetric neighborhoods $V_1, V_2, \dots$ such that $V_n \cdot V_n \subset U_n$ $(n \geqslant 1)$. We conclude from the fact that w is in the $\mathscr{G}_\delta$-closure of G that there is $y \in G$ such that $y \in \bigcap_{n=1}^{\infty}(V_n w \cap w V_n)$. Fix $n \geqslant 1$. There is $\sigma_n \in \Sigma$ such that both $x_\sigma w^{-1}$ and $w^{-1} x_\sigma$ belong to V_n for $\sigma \geqslant \sigma_n$. Then, for such σ's, $x_\sigma y^{-1} = (x_\sigma w^{-1})(yw^{-1})^{-1} \subset V_n \cdot V_n^{-1} \subset U_n$ and $x_\sigma^{-1} y = (w^{-1} x_\sigma)^{-1}(w^{-1} y) \subset V_n^{-1} \cdot V_n \subset U_n$ as well. This shows that (CC) is satisfied for $z = y$, and we are done.

Point (V) is easily implied by (IV) (for a fixed continuous left-invariant pseudometric d and a net satisfying (CC$'$) apply (CC) for $U_n = \{x \in G \mid d(x, e) < 2^{-n}\}$).

The converse implication follows from the well-known fact that for an arbitrary sequence $U_1, U_2, \dots$ of neighborhoods of e there exists a left-invariant pseudometric d on G such that $\{x \in G \mid d(x, e) < 2^{-n}\} \subset U_n$ for every $n \geqslant 1$ (see, for example, the proof of the Kakutani–Birkhoff theorem on the metrizability of topological groups presented in [Berberian 1974, Theorem 6.3]; or use Markov's theorem [Arhangel'skii and Tkachenko 2008, Theorem 3.3.9] to deduce this property). $\square$

Remark 4.2. The proof of Proposition 4.1 shows that points (IV) and (V) of that result may be weakened by assuming that every net satisfying condition (CC) or (CC$'$) with $z = y$ is convergent. However, to prove Theorem 1.4, we need (IV) in its present form.

Now we can give many examples of $\mathscr{G}_\delta$-complete groups. We inform that by the Cartesian product of a family $\{G_s\}_{s\in S}$ of topological groups we mean the "full" Cartesian product $\prod_{s\in S} G_s$ of them and by the direct product of this family we mean the topological subgroup $\bigoplus_{s\in S} G_s$ of $\prod_{s\in S} G_s$ consisting of all its finitely supported elements.

Proposition 4.3. *Each of the following topological groups is $\mathscr{G}_\delta$-complete*:

(a) *A $\mathscr{G}_\delta$-closed subgroup of a $\mathscr{G}_\delta$-complete group. A complete group.*

(b) *The Cartesian as well as the direct product of arbitrary collection of $\mathscr{G}_\delta$-complete groups.*

(c) *A topological group which is the countable union of its subgroups each of which is $\mathscr{G}_\delta$-complete.*

(d) *A topological group which is σ-compact as a topological space. In particular, all countable topological groups are $\mathscr{G}_\delta$-complete.*

(e) *A topological group in which singletons are $\mathscr{G}_\delta$. In particular, metrizable groups are $\mathscr{G}_\delta$-complete.*

(f) $G = \mathrm{Iso}(X, d)$ *for an arbitrary metric space* (X, d). *Moreover*, $w(G) \leqslant w(X)$, *and G is complete provided* (X, d) *is complete.*

Proof. In each point we invoke Proposition 4.1.

To prove point (a), use the equivalence between conditions (I) and (III) in Proposition 4.1. We turn to (b). Let $\{G_s\}_{s\in S}$ be a nonempty collection of $\mathscr{G}_\delta$-complete groups and let $G = \prod_{s\in S} G_s$. Let $x_\sigma = (x_\sigma^{(s)})_{s\in S} \in G$ $(\sigma \in \Sigma)$ be a net satisfying condition (CC). It remains to check that for any $t \in S$, the net $\{x_\sigma^{(t)}\}_{\sigma\in\Sigma} \subset G_t$ satisfies condition (CC) as well (because then it will be convergent), which is immediate: If $V_1, V_2, \dots$ is a sequence of neighborhoods of the neutral element of G_t, apply (CC) for the sequence $U_1, U_2, \dots$ with $U_j := \{(x^{(s)})_{s\in S} \in G \mid x^{(t)} \in V_j\}$ $(j \geqslant 1)$ to obtain two points $y, z \in G$ and then use their t-coordinates. Now to prove that $H := \bigoplus_{s\in S} G_s$ is also $\mathscr{G}_\delta$-complete, it suffices to check that it is

$\mathscr{G}_\delta$-closed in G (by (a)). But if $y = (y_s)_{s\in S} \in G \setminus H$, there is a countable (infinite) set $S' \subset S$ such that $y_s \neq e_s$ for any $s \in S'$, where e_s is the neutral element of G_s. Then the set $A := \{(z_s)_{s\in S} \in G \mid z_s \neq e_s \text{ for all } s \in S'\}$ is a $\mathscr{G}_\delta$-subset of G which contains y and is disjoint from H, which finishes the proof of (b).

Since the proofs of points (c) and (d) are similar, we shall show only (c). Let $G = \bigcup_{n=1}^{\infty} G_n$, where G_n is $\mathscr{G}_\delta$-complete for any n. Let $y \in \overline{G} \setminus G$. Then $y \notin G_n$ and G_n is $\mathscr{G}_\delta$-closed in $\overline{G}$. Consequently, there are $\mathscr{G}_\delta$-subsets $A_1, A_2, \dots$ of $\overline{G}$ containing y such that $A_n \cap G_n = \varnothing$. Then $A := \bigcap_{n=1}^{\infty} A_n$ is also a $\mathscr{G}_\delta$-subset of $\overline{G}$ containing y, and $A \cap G = \varnothing$. This shows that y is not in the $\mathscr{G}_\delta$-closure of G and we are done.

Further, if all singletons are $\mathscr{G}_\delta$ in G then τ_δ is discrete and hence G is $\mathscr{G}_\delta$-complete. This proves (e).

Finally, we turn to (f). The second and the third claims of (f) are well-known, but for the sake of completeness we shall prove them too. Let (X, d) be a metric space and $G = \mathrm{Iso}(X, d)$. Let $\{u_\sigma\}_{\sigma\in\Sigma} \subset G$ be a net satisfying condition (CC$'$). Fix $x \in X$ and put $\varrho : G \times G \ni (u, v) \mapsto d(u(x), v(x)) \in [0, \infty)$. Observe that ϱ is a left-invariant continuous pseudometric on G. It follows from (CC$'$) that there are $f, g \in G$ such that $\lim_{\sigma\in\Sigma} d(u_\sigma(x), f(x)) = \lim_{\sigma\in\Sigma} d(u_\sigma^{-1}(x), g(x)) = 0$. We conclude that both the nets $\{u_\sigma(x)\}_{\sigma\in\Sigma}$ and $\{u_\sigma^{-1}(x)\}_{\sigma\in\Sigma}$ converge in X. So we may define $u, v : X \to X$ by $u(x) = \lim_{\sigma\in\Sigma} u_\sigma(x)$ and $v(x) = \lim_{\sigma\in\Sigma} u_\sigma^{-1}(x)$ $(x \in X)$. It is readily seen that both u and v are isometric. What is more, a standard argument proves that $u \circ v = v \circ u = \mathrm{id}_X$ and hence $u \in G$ and $\lim_{\sigma\in\Sigma} u_\sigma = u$. So G is $\mathscr{G}_\delta$-complete. Furthermore, if D is a dense subset of X such that $\mathrm{card}(D) = w(X)$ then the function $G \ni g \mapsto g|_D \in X^D$ is a topological embedding (when X^D is equipped with the pointwise convergence topology) and therefore $w(G) \leqslant w(X^D) \leqslant w(X)$. Finally, if (X, d) is in addition complete and $\{u_\sigma\}_{\sigma\in\Sigma} \subset G$ is a net satisfying (C), similar argument to that above shows that then for any $x \in X$ the nets $\{u_\sigma(x)\}_{\sigma\in\Sigma}$ and $\{u_\sigma^{-1}(x)\}_{\sigma\in\Sigma}$ are fundamental in (X, d) and hence converge. It now follows from the previous part of the proof that $\{u_\sigma\}_{\sigma\in\Sigma}$ is convergent in G, which finishes the proof. □

For the purpose of the next result, recall that a topological space X is *Dieudonné-complete* if and only if there is a complete uniformity on X inducing the topology of X (see, for example, [Engelking 1989, Chapter 8]). Accordingly, a topological group is *Dieudonné-complete* if and only if it is Dieudonné-complete as a topological space ([Arhangel'skii and Tkachenko 2008]).

Corollary 4.4. *For every metric space* (X, d) *the topological group* $\mathrm{Iso}(X, d)$ *is Dieudonné-complete.*

Proof. By Proposition 4.3, $\mathrm{Iso}(X, d)$ is $\mathscr{G}_\delta$-complete and hence it is $\mathscr{G}_\delta$-closed in $\overline{G}$ thanks to Proposition 4.1. Consequently, $\mathrm{Iso}(X, d)$ is Dieudonné-complete (since

$\overline{G}$ is such and $\mathscr{G}_\delta$-closed subsets of Dieudonné-complete spaces are Dieudonné-complete as well — see [Dieudonné 1939] or Problem 8.5.13(f) on page 465 in [Engelking 1989]; see also the proof of Proposition 6.5.2 on page 366 in [Arhangel'skii and Tkachenko 2008]). □

The above result gives a full answer to the question of when the isometry group of a metric space is Dieudonné-complete, posed by Arhangel'skii and Tkachenko [2008, Open Problem 3.5.4 on page 181].

Example 4.5. As we announced in the introductory part, not every topological group is absolutely $\mathscr{G}_\delta$-closed. Let us briefly justify our claim. Let S be an uncountable set and for each $s \in S$ let G_s be a nontrivial complete group with the neutral element e_s. Then $G := \prod_{s \in S} G_s$ is a complete group as well and

$$G_0 = \{(x_s)_{s\in S} \in G \mid \operatorname{card}(\{s \in S \mid x_s \neq e_s\}) \leqslant \aleph_0\}$$

is a proper subgroup of G which is $\mathscr{G}_\delta$-dense in G (and thus G_0 is not $\mathscr{G}_\delta$-closed in G). Indeed, if $y = (y_s)_{s\in S} \in G$ and A is a $\mathscr{G}_\delta$-subset of G containing y then there is a countable set $S_0 \subset S$ such that $\{(x_s)_{s\in S} \in G \mid x_s = y_s \text{ for all } s \in S_0\} \subset A$; then, $z \in G_0 \cap A$, where $z_s = y_s$ for $s \in S_0$ and $z_s = e_s$ otherwise.

We are almost ready to prove Theorem 1.4. For the purpose of its proof and the nearest result, let us introduce auxiliary notations and terminology. Whenever d and d' are two bounded pseudometrics on a common nonempty set X, we put

$$\|d - d'\|_\infty := \sup_{x,y\in X} |d(x, y) - d'(x, y)|.$$

Further, the relation $R := \{(x, y) \in X \times X \mid d(x, y) = 0\}$ is an equivalence on X. Let $\pi : X \to X/R$ be the canonical projection. The function $D : (X/R) \times (X/R) \ni (\pi(x), \pi(y)) \mapsto d(x, y) \in [0, \infty)$ is a well-defined metric on X/R. We call a triple $(Y, \varrho; \xi)$ a *metric space associated with* (X, d) if (Y, ϱ) is a metric space isometric to $(X/R, D)$ and ξ is a function of X onto Y such that there is an isometry $g : (X/R, D) \to (Y, \varrho)$ for which $\xi = g \circ \pi$. Observe that then $\varrho(\xi(x), \xi(y)) = d(x, y)$ for any $x, y \in X$.

With use of the following result we shall take care of condition $w(X) = w(G)$ in Theorem 1.4(A):

Lemma 4.6. *Let G be a topological group and $\{\varrho_s\}_{s\in S}$ be a collection of bounded continuous left-invariant pseudometrics on G. For each $s \in S$, let $(X_s, d_s; \pi_s)$ be a metric space associated with (G, ϱ_s) chosen so that the sets X_s are pairwise disjoint. There exists a metric d on $X := \bigcup_{s\in S} X_s$ with the following properties*:

(DD1) $\frac{1}{2}\sqrt[3]{d_s(x, y)} \leqslant d(x, y) \leqslant \sqrt[3]{d_s(x, y)}$ *for any $x, y \in X_s$ and $s \in S$.*

(DD2) $d(\pi_s(a), \pi_t(a)) \leqslant \sqrt[3]{\|\varrho_s - \varrho_t\|_\infty}$ *for any $a \in G$ and $s, t \in S$.*

(DD3) *Each of the sets* X_s $(s \in S)$ *is closed in* (X, d).

(DD4) $d(\pi_s(ag), \pi_t(ah)) = d(\pi_s(g), \pi_t(h))$ *for any* $a, g, h \in G$ *and* $s, t \in S$.

Proof. To simplify arguments, for each $x \in X$ denote by $\kappa(x)$ the unique index $s \in S$ such that $x \in X_s$. Define a function $v : X \times X \to [0, \infty)$ as follows:

$$v(x, y) = \|\varrho_{\kappa(x)} - \varrho_{\kappa(y)}\|_\infty + \inf\{d_{\kappa(x)}(x, \pi_{\kappa(x)}(g)) + d_{\kappa(y)}(\pi_{\kappa(y)}(g), y) \mid g \in G\}.$$

Observe that:

- $v(x, x) = 0$ and $v(y, x) = v(x, y)$ for any $x, y \in X$.
- $v(x, y) = d_s(x, y)$ for any $x, y \in X_s$ and $s \in S$.
- $v(\pi_s(g), \pi_t(g)) = \|\varrho_s - \varrho_t\|_\infty$ whenever $s, t \in S$ and $g \in G$.
- $v(\pi_s(g), \pi_t(h)) \geqslant \|\varrho_s - \varrho_t\|_\infty$ for all $s, t \in S$ and $g, h \in G$.
- $v(\pi_s(ag), \pi_t(ah)) = v(\pi_s(g), \pi_t(h))$ for any $a, g, h \in G$ and $s, t \in S$.

Let us now show that for any $x_0, x_1, x_2, x_3 \in X$ and each $\varepsilon > 0$

$$\max_{j=1,2,3} v(x_{j-1}, x_j) < \varepsilon \implies v(x_0, x_3) < 8\varepsilon. \tag{22}$$

Assume $v(x_{j-1}, x_j) < \varepsilon$ $(j = 1, 2, 3)$. This means that there are $a_1, a_2, a_3 \in G$ for which

$$\|\varrho_{\kappa(x_{j-1})} - \varrho_{\kappa(x_j)}\|_\infty + d_{\kappa(x_{j-1})}(x_{j-1}, \pi_{\kappa(x_{j-1})}(a_j)) + d_{\kappa(x_j)}(\pi_{\kappa(x_j)}(a_j), x_j) < \varepsilon. \tag{23}$$

In particular, $\|\varrho_{\kappa(x_{j-1})} - \varrho_{\kappa(x_j)}\|_\infty < \varepsilon$ for $j = 1, 2, 3$ and thus

$$\|\varrho_{\kappa(x_0)} - \varrho_{\kappa(x_3)}\|_\infty < 3\varepsilon. \tag{24}$$

For simplicity, for $j \in \{0, 1, 2, 3\}$ put $s_j = \kappa(x_j)$ and take $b_j \in G$ such that $\pi_{s_j}(b_j) = x_j$. Recall that $d_s(\pi_s(g), \pi_s(h)) = \varrho_s(g, h)$ for any $s \in S$ and $g, h \in G$. Therefore we have

$$\begin{aligned} \varrho_{s_j}(b_{j-1}, b_j) &\leqslant \varrho_{s_j}(b_{j-1}, a_j) + \varrho_{s_j}(a_j, b_j) \\ &\leqslant \|\varrho_{s_{j-1}} - \varrho_{s_j}\|_\infty + \varrho_{s_{j-1}}(b_{j-1}, a_j) + \varrho_{s_j}(a_j, b_j) < \varepsilon \end{aligned}$$

(by (23)) and consequently

$$\begin{aligned} \varrho_{s_2}(b_0, b_2) &\leqslant \varrho_{s_2}(b_0, b_1) + \varrho_{s_2}(b_1, b_2) \\ &\leqslant \|\varrho_{s_2} - \varrho_{s_1}\|_\infty + \varrho_{s_1}(b_0, b_1) + \varrho_{s_2}(b_1, b_2) < 3\varepsilon. \end{aligned}$$

Similarly,

$$\begin{aligned} \varrho_{s_3}(b_0, b_3) &\leqslant \varrho_{s_3}(b_0, b_2) + \varrho_{s_3}(b_2, b_3) \\ &\leqslant \|\varrho_{s_3} - \varrho_{s_2}\|_\infty + \varrho_{s_2}(b_0, b_2) + \varrho_{s_3}(b_2, b_3) < 5\varepsilon. \end{aligned} \tag{25}$$

Finally, by (24) and (25) we obtain

$$\begin{aligned} v(x_0, x_3) &\leqslant \|\varrho_{s_0} - \varrho_{s_3}\|_\infty + d_{s_0}(\pi_{s_0}(b_0), \pi_{s_0}(b_0)) + d_{s_3}(\pi_{s_3}(b_0), \pi_{s_3}(b_3)) \\ &< 3\varepsilon + \varrho_{s_3}(b_0, b_3) \\ &< 8\varepsilon, \end{aligned}$$

and the proof of (22) is complete. Now let $f : X \times X \to [0, \infty)$ be given by $f(x, y) = \sqrt[3]{v(x, y)}$. Below we collect all properties established for v and translated to the case of the function f:

(F1) $f(x, x) = 0$ and $f(x, y) = f(y, x) > 0$ for any distinct points x and y of X.

(F2) If $\varepsilon > 0$ and $\{f(x, y), f(y, z), f(z, w)\} \subset [0, \varepsilon]$ for some $x, y, z, w \in X$ then $f(x, w) \leqslant 2\varepsilon$, thanks to (22).

(F3) $f(x, y) = \sqrt[3]{d_s(x, y)}$ and $f(\pi_s(g), \pi_t(g)) = \sqrt[3]{\|\varrho_s - \varrho_t\|_\infty}$ whenever $x, y \in X_s$, $g \in G$, and $s, t \in S$.

(F4) $f(\pi_s(g), \pi_t(h)) \geqslant \sqrt[3]{\|\varrho_s - \varrho_t\|_\infty}$ for all $g, h \in G$ and $s, t \in S$.

(F5) $f(\pi_s(ag), \pi_t(ah)) = f(\pi_s(g), \pi_t(h))$ for any $a, g, h \in G$ and $s, t \in S$.

Finally, we define $d : X \times X \to [0, \infty)$ as follows:

$$d(x, y) = \inf\left\{\sum_{j=1}^{n} f(z_{j-1}, z_j) \;\middle|\; n \geqslant 1, z_0, \dots, z_n \in X, z_0 = x, z_n = y\right\}.$$

Lemma 6.2 of [Berberian 1974] asserts that for any function $f : X \times X \to [0, \infty)$ satisfying conditions (F1)–(F2) the function d defined above is a metric on X such that

$$\tfrac{1}{2} f(x, y) \leqslant d(x, y) \leqslant f(x, y) \quad (x, y \in X). \tag{26}$$

It follows from (F5) and the formula of d that (DD4) is fulfilled, while (DD1) and (DD2) may easily be deduced from (F3) and (26). Finally, (DD3) is a consequence of (F4) and (26). □

Proof of Theorem 1.4. The implications (A1) ⇒ (A2) and (B1) ⇒ (B2) follow from Proposition 4.3. It remains to show the converse implications.

First assume that G is complete (in this case the proof is much shorter). By a well-known result (see, for example, [Uspenskij 2008, Theorem 2.1]), there is a bounded metric space (Y, ϱ) such that $w(G) = w(Y)$ and G is isomorphic to a subgroup H of $\mathrm{Iso}(Y, \varrho)$. Since $\mathrm{Iso}(Y, \varrho)$ is naturally isomorphic to a subgroup of $\mathrm{Iso}(\bar{Y}, \bar{\varrho})$, where $(\bar{Y}, \bar{\varrho})$ is the completion of (Y, ϱ), we may assume that (Y, ϱ) is a complete metric space. Since G is complete, H is a closed subgroup of (Y, ϱ). Now Theorem 3.2 implies that H is isomorphic to $\mathrm{Iso}(X, d)$, where $X = \widehat{Y}_\beta$ with $\beta = w(Y)$, and d is a metric which respects ϱ. Notice that d is complete (by (AX4)),

$w(X) = w(G)$ (because $\beta = w(Y) = w(G)$) and G is isomorphic to $\mathrm{Iso}(X, d)$ (being isomorphic to H). This proves the remainder of point (B).

We now turn to (A). Assume G is $\mathcal{G}_\delta$-complete. Thanks to Theorem 3.2, it suffices to show that G is isomorphic to a closed subgroup of $\mathrm{Iso}(X, d)$ for a metric space (X, d) of topological weight equal to $w(G)$ (see the previous part of the proof). We shall do this employing Lemma 4.6 and improving a classical argument, presented, for example, in the first proof of [Uspenskij 2008, Theorem 2.1]. (That proof shows that every topological group is isomorphic to a topological subgroup of $\mathrm{Iso}(X, d)$ for some metric space (X, d). However, this fact is insufficient for us, not only because the topological weight of X is out of control. A much more difficult problem is to provide the closedness of the subgroup of $\mathrm{Iso}(X, d)$ isomorphic to a given $\mathcal{G}_\delta$-complete group.)

Let $\mathcal{B}$ be a base of open neighborhoods of the neutral element e of G such that $\mathrm{card}(\mathcal{B}) \leqslant w(G)$. Let S be the set of all finite and all infinite sequences of members of $\mathcal{B}$. For any $U \in \mathcal{B}$ there exists a continuous left-invariant pseudometric λ_U on G bounded by 1 such that

$$\{x \in G \mid \lambda_U(x, e) < 1\} \subset U. \tag{27}$$

We leave it as a simple exercise that the family $\{\lambda_U\}_{U \in \mathcal{B}}$ determines the topology of G. Now for any $s = (U_j)_{j=1}^N \in S$ (where N is finite or $N = \infty$) let

$$\varrho_s := \sum_{j=1}^{N} \frac{1}{2^j} \lambda_{U_j}. \tag{28}$$

Notice that ϱ_s is a continuous left-invariant pseudometric on G bounded by 1. What is more,

(T) the family $\{\varrho_s\}_{s \in S}$ determines the topology of G

(since $\varrho_s = \lambda_U$ for $s = (U, U, \dots) \in S$). Now we apply Lemma 4.6 for the family $\{\varrho_s\}_{s \in S}$. Let $(X_s, d_s; \pi_s)$ $(s \in S)$ and (X, d) be as stated there. That is, $(X_s, d_s; \pi_s)$ is a metric space associated with (G, ϱ_s), the sets X_s are pairwise disjoint, $X = \bigcup_{s \in S} X_s$, and d is a metric on X satisfying conditions (DD1)–(DD4). For each $s \in S$ let $\tilde{\varrho}_s : G \times G \to [0, \infty)$ be given by $\tilde{\varrho}_s(g, h) = d(\pi_s(g), \pi_s(h))$. It is clear that $\tilde{\varrho}_s$ is a pseudometric on G. What is more, it is left-invariant, thanks to (DD4), and moreover

$$\tfrac{1}{2}\sqrt[3]{\varrho_s} \leqslant \tilde{\varrho}_s \leqslant \sqrt[3]{\varrho_s} \quad (s \in S), \tag{29}$$

by (DD1). Consequently, each of the pseudometrics $\tilde{\varrho}_s$ is continuous and

(T′) the family $\{\tilde{\varrho}_s\}_{s \in S}$ determines the topology of G

(see (T)). We infer from the continuity of $\tilde{\varrho}_s$ that π_s, as a function of G into (X, d), is continuous as well. We claim that $w(X) \leqslant w(G)$. To see this, let S_f consists of all finite sequences of members of $\mathscr{B}$, and let D be a dense subset of G such that $\operatorname{card}(D) \leqslant w(G)$. Observe that $Z := \bigcup_{s \in S_f} \pi_s(D)$ has cardinality not exceeding $w(G)$. We will now show that Z is dense in X. First of all, note that $\pi_s(D)$ is dense in X_s, since π_s is continuous. In particular, the closure of Z contains all points of $\bigcup_{s \in S_f} X_s$. Fix $s \notin S_f$ and $a \in G$. Then s is of the form $s = (U_j)_{n=1}^{\infty} \in S$. Put $s_n := (U_j)_{j=1}^{n} \in S_f$ and observe that, by (DD2) and (28),

$$d(\pi_{s_n}(a), \pi_s(a)) \leqslant \sqrt[3]{\|\varrho_{s_n} - \varrho_s\|_\infty} \to 0 \quad (n \to \infty).$$

So, since $\pi_s(G) = X_s$, the above argument shows that Z is indeed dense in X.

It remains to check that G is isomorphic to a closed subgroup of $\operatorname{Iso}(X, d)$. For $g \in G$ let $u_g : X \to X$ be such that $u_g(\pi_s(x)) = \pi_s(gx)$ for any $s \in S$ and $x \in G$. Then $\Phi : G \ni g \mapsto u_g \in \operatorname{Iso}(X, d)$ is a well-defined (by (DD4)) group homomorphism as well as a topological embedding (thanks to (T′)). So it follows from Proposition 4.3(f) that $w(X) \geqslant w(G)$ and hence $w(X) = w(G)$. We shall check that $\Phi(G)$ is closed, which will finish the proof. Assume $\{x_\sigma\}_{\sigma \in \Sigma}$ is a net in G such that the net $\{u_{x_\sigma}\}_{\sigma \in \Sigma}$ converges in $\operatorname{Iso}(X, d)$ to some $u \in \operatorname{Iso}(X, d)$. It is enough to prove that the net $\{x_\sigma\}_{\sigma \in \Sigma}$ is convergent in G. Since G is $\mathscr{G}_\delta$-complete, actually it suffices to verify condition (CC) (see Proposition 4.1). To this end, let $V_1, V_2, \dots$ be a sequence of neighborhoods of e. For any $n \geqslant 1$ choose $U_n \in \mathscr{B}$ for which $U_n \subset V_n$. Now for $s := (U_j)_{j=1}^{\infty} \in S$ we have

$$\lim_{\sigma \in \Sigma} d(\pi_s(x_\sigma), u(\pi_s(e))) = \lim_{\sigma \in \Sigma} d(u_{x_\sigma}(\pi_s(e)), u(\pi_s(e))) = 0,$$
$$\lim_{\sigma \in \Sigma} d(\pi_s(x_\sigma^{-1}), u^{-1}(\pi_s(e))) = \lim_{\sigma \in \Sigma} d(u_{x_\sigma}^{-1}(\pi_s(e)), u^{-1}(\pi_s(e))) = 0.$$

We infer from (DD3) and the above convergences that there are $y, z \in G$ such that $\lim_{\sigma \in \Sigma} d(\pi_s(x_\sigma), \pi_s(y)) = \lim_{\sigma \in \Sigma} d(\pi_s(x_\sigma^{-1}), \pi_s(z^{-1})) = 0$. But for any $a, b \in G$, $d(\pi_s(a), \pi_s(b)) = \tilde{\varrho}_s(a, b)$ and thus, thanks to (29),

$$\lim_{\sigma \in \Sigma} \varrho_s(x_\sigma, y) = \lim_{\sigma \in \Sigma} \varrho_s(x_\sigma^{-1}, z^{-1}) = 0.$$

For each $n \geqslant 1$ let $\sigma_n \in \Sigma$ be such that both the numbers $\varrho_s(x_\sigma, y) = \varrho_s(x_\sigma^{-1} y, e)$ and $\varrho_s(x_\sigma^{-1}, z^{-1}) = \varrho_s(x_\sigma z^{-1}, e)$ are less than 2^{-n} for any $\sigma \geqslant \sigma_n$. We deduce from the formula of s, (28), and (27) that $\{x_\sigma^{-1} y, x_\sigma z^{-1}\} \subset U_n (\subset V_n)$ for all $\sigma \geqslant \sigma_n$. $\square$

The above proof provides the existence of a metric space (namely, $\widehat{X}_\beta$) whose isometry group is isomorphic to a given $\mathscr{G}_\delta$-complete group G. This metric space is highly disconnected (since it contains a clopen discrete set whose cardinality is equal to the topological weight of the whole space). In the next section we shall improve Theorem 1.4 by showing that G is isomorphic to the isometry group of a contractible open set in a normed vector space of the same topological weight as G.

Corollary 4.7. (A) *Let G be a topological group and β be an infinite cardinal number. There exists a* (*complete*) *metric space* (X, d) *such that* $w(X) = \beta$ *and* $\operatorname{Iso}(X, d)$ *is isomorphic to G if and only if G is $\mathscr{G}_\delta$-complete* (*resp. complete*) *and* $\beta \geqslant w(G)$.

(B) *A topological group is isomorphic to the isometry group of some separable metric space if and only if it is second-countable.*

Proof. Both points (A) and (B) follow from Theorems 1.4 and 3.2, (AX4) and, respectively, points (f) and (e) of Proposition 4.3. □

We call a metrizable space X *zero-dimensional* if and only if X has a base consisting of clopen (that is, simultaneously open and closed) sets; X is *strongly zero-dimensional* if the covering dimension of X equals 0.

Corollary 4.8. *Let G be an infinite metrizable topological group and $\beta = w(G)$. There exists a compatible metric ϱ on $Y := \widehat{G}_\beta$ such that G is isomorphic to* $\operatorname{Iso}(Y, \varrho)$. *In particular*:

(A) *If G is discrete*, *there is a complete compatible* (*possibly nonleft-invariant*) *metric d on G such that G is isomorphic to* $\operatorname{Iso}(G, d)$.

(B) *If G is countable and nondiscrete*, *there is a compatible metric d on $F := \mathbb{Q} \sqcup \mathbb{Z}$ such that G is isomorphic to* $\operatorname{Iso}(F, d)$.

(C) *If G is totally disconnected* (*zero-dimensional*; *strongly zero-dimensional*) *then there is a metric space* (X, d) *such that X is totally disconnected* (*resp. zero-dimensional*; *strongly zero-dimensional*) *as well*, $w(X) = w(G)$, *and* $\operatorname{Iso}(X, d)$ *is isomorphic to G*.

Proof. Let p be a bounded left-invariant compatible metric on G (if G is discrete, we may additionally assume that p is complete). It is an easy exercise (and a well-known fact) that all left translations on G form a closed subgroup of $\operatorname{Iso}(G, p)$. Consequently, by Theorem 3.2, G is isomorphic to $\operatorname{Iso}(Y, \varrho)$ for some metric ϱ which respects p. Note that if G is discrete, Y is homeomorphic to G, which proves (A). Further, if G is countable and nondiscrete, it is homeomorphic to the space of all rationals (for example, by Sierpiński's theorem [1920] that every countable metrizable topological space without isolated points is homeomorphic to $\mathbb{Q}$; see point (d) of Exercise 6.2.A on page 370 in [Engelking 1989]) and hence Y is homeomorphic to F, from which we deduce (B). Finally, (C) simply follows from the fact that, if G is totally disconnected, or zero-dimensional, or strongly zero-dimensional, then Y has the same property: see Theorems 1.3.6, 4.1.25, and 4.1.3 in [Engelking 1978]. □

Corollary 4.8(A) may be generalized to the context of so-called nonarchimedean complete topological groups. Recall that a topological group is *nonarchimedean* if and only if it has a base of neighborhoods of the neutral element consisting of open subgroups. Nonarchimedean Polish groups play an important role, for example, in model theory; see §1.5 in [Becker and Kechris 1996]. The equivalence between points (i) and (ii) of the following result is taken from this book (it was formulated there only for Polish groups, but the proof works in the general case):

Corollary 4.9. *Let G be a topological group and $\beta = w(G)$. Let D be a discrete topological space of cardinality β and S_β be the symmetric group of the set D* (*that is*, *S_β consists of all permutations of D*). *The following conditions are equivalent*:

(i) *G is nonarchimedean and complete.*

(ii) *G is isomorphic to a closed subgroup of S_β.*

(iii) *For arbitrary $\varepsilon > 0$, G is isomorphic to $\mathrm{Iso}(D, \varrho)$ for some metric ϱ on D such that $\delta_D \leqslant \varrho \leqslant (1+\varepsilon)\delta_D$, where δ_D is the discrete metric on D.*

Proof. First note that $S_\beta = \mathrm{Iso}(D, \delta)$. Thus, if (ii) is satisfied, Theorem 3.2 implies that there is a metric ϱ on $X = \widehat{D}_\beta$ which respects δ_D such that G is isomorphic to $\mathrm{Iso}(X, \varrho)$. Since ϱ respects δ_D, we readily have $\varrho \geqslant \delta_X$. Use Remark 3.3 to improve the metric ϱ and take care of the inequality $\varrho \leqslant (1+\varepsilon)\delta_X$. Finally, noticing that X is homeomorphic to D gives (iii).

Now assume (iii) is satisfied. Since ϱ is complete, so is G, by Proposition 4.3. Furthermore, the sets $U_A = \{u \in \mathrm{Iso}(D, \varrho) \mid u(a) = a \text{ for any } a \in A\}$, where A runs over all finite subsets of D, are open (because $\varrho \geqslant 1$) subgroups of $\mathrm{Iso}(D, \varrho)$ which form a base of neighborhoods of the identity map and thus G is nonarchimedean.

Finally, assume (i) is fulfilled. Let $\mathscr{B}$ be a base of neighborhoods of the neutral element e_G of G which consists of open subgroups and has cardinality β. For any $H \in \mathscr{B}$, the size of the collection $\{gH \mid g \in G\}$ does not exceed β and hence $\mathrm{card}(\mathscr{U}) = \beta = \mathrm{card}(D)$ for $\mathscr{U} = \{gH \mid g \in G, H \in \mathscr{B}\}$. Hence we may and do identify the set D with $\mathscr{U}$. For any $g \in G$ put $\pi_g : \mathscr{U} \ni U \mapsto gU \in \mathscr{U}$. Under the above identification, we readily have $\pi_g \in S_\beta$. What is more, the function $\pi : G \ni g \mapsto \pi_g \in S_\beta$ is easily seen to be a group homomorphism (possibly discontinuous). It suffices to check that π is an embedding (because then $\pi(G)$ is closed, thanks to the completeness of G). Since $\bigcap \mathscr{B} = \{e_G\}$, π is one-to-one. To complete the proof, observe that for any net $\{g_\sigma\}_{\sigma\in\Sigma} \subset G$ one has

$$\begin{aligned}
\lim_{\sigma\in\Sigma} g_\sigma = e_G &\iff \forall g \in G : \lim_{\sigma\in\Sigma} g^{-1} g_\sigma g = e_G \\
&\iff \forall g \in G \ \ \forall H \in \mathscr{B} \ \ \exists \sigma_0 \in \Sigma \ \ \forall \sigma \geqslant \sigma_0 : \pi_{g_\sigma}(gH) = gH \\
&\iff \lim_{\sigma\in\Sigma} \pi_{g_\sigma} = \pi_{e_G}
\end{aligned}$$

(recall that $\mathscr{U}$, as identified with D, has discrete topology and that for any $x \in G$ and $H \in \mathscr{B}$, $x \in H$ if and only if $xH = H$). □

Using Proposition 3.4, we may easily strengthen Theorem 1.4:

Proposition 4.10. *For any $\mathscr{G}_\delta$-complete group G there are a complete metric space (X, d) with $w(X) = w(G)$, a dense set $X' \subset X$, and an isomorphism $\Phi : \overline{G} \to \operatorname{Iso}(X, d)$ such that $\Phi(G) = \{u \in \operatorname{Iso}(X, d) \mid u(X') = X'\} (= \operatorname{Iso}(X', d))$.*

Proof. First use Theorem 1.4 to get the isomorphism between G and $\operatorname{Iso}(Y, \varrho)$ for some metric space (Y, ϱ) with $w(Y) = w(G)$ and then apply Proposition 3.4 to conclude the whole assertion (recall that the isometry group of a complete metric space is complete). □

Example 4.11. Let (X, d) be an arbitrary metric space and G be a subgroup of $\operatorname{Iso}(X, d)$. It follows from Theorem 1.4 and Proposition 4.1 that G is isomorphic to the isometry group of some metric space if and only if G is $\mathscr{G}_\delta$-closed in $\operatorname{Iso}(X, d)$. Let us briefly show that the $\mathscr{G}_\delta$-closure of G coincides with the set of all $u \in \operatorname{Iso}(X, d)$ such that for any separable subspace A of X there is $v \in G$ which agrees with u on A. Indeed, there is a countable set $D \subset A$ which is dense in A. Then the set $F(u, A) := \{v \in \operatorname{Iso}(X, d) : v|_A = u|_A\}$ coincides with $\{v \in \operatorname{Iso}(X, d) : v|_D = u|_D\}$. This implies that $F(u, A)$ is $\mathscr{G}_\delta$ in $\operatorname{Iso}(X, d)$. Consequently, if u belongs to the $\mathscr{G}_\delta$-closure of G then necessarily $G \cap F(u, A) \neq \varnothing$. Conversely, it may be easily shown that for every $\mathscr{G}_\delta$-set P containing u there is a countable set A such that $F(u, A) \subset P$ and hence to conclude that u belongs to the $\mathscr{G}_\delta$-closure of G it is sufficient that $F(u, A) \cap G \neq \varnothing$ for all countable A.

According to the above remark, Theorem 3.2 may now be generalized as follows: a subgroup G of $\operatorname{Iso}(X, d)$ is isomorphic to the isometry group of some metric space (of the same topological weight as X) if and only if G satisfies the following condition: Whenever $u \in \operatorname{Iso}(X, d)$ is such that for any separable subspace A of X there exists $v \in G$ which agrees with u on A then $u \in G$.

We end the section with the concept of $\mathscr{G}_\delta$-completions. Similarly, as in the case of Raĭkov completion, any topological group G has a unique $\mathscr{G}_\delta$-completion; that is, G may be embedded in a $\mathscr{G}_\delta$-complete group as a $\mathscr{G}_\delta$-dense subgroup in a unique way, as shown by:

Proposition 4.12. *Let G and K be $\mathscr{G}_\delta$-complete groups and H be a $\mathscr{G}_\delta$-dense subgroup of G:*

(a) *Every continuous homomorphism of H into K extends uniquely to a continuous homomorphism of G into K.*
(b) *If $f : H \to K$ is a group homomorphism as well as a topological embedding and $\widetilde{G}$ denotes the $\mathscr{G}_\delta$-closure of $f(H)$ in K then there is a unique (topological) isomorphism $F : G \to \widetilde{G}$ that extends f.*

Proof. Let $f : H \to K$ be a continuous homomorphism. Since H is dense in G, there is a unique continuous group homomorphism $F : G \to \overline{K}$ which extends f. It suffices to show that $F(G) \subset K$. But this follows from the fact that the preimage of a $\mathscr{G}_\delta$-closed set under a continuous function is $\mathscr{G}_\delta$-closed too. This proves (a). If in addition f is a topological embedding, it follows from the above argument that there is a continuous group homomorphism $\widetilde{F} : \widetilde{G} \to G$ which extends f^{-1}. We then readily see that both $\widetilde{F} \circ F$ and $F \circ \widetilde{F}$ are the identity maps and hence F is an isomorphism (and $\widetilde{F} = F^{-1}$), which shows (b). □

Definition 4.13. Let G be a topological group. The *$\mathscr{G}_\delta$-completion* of G is a $\mathscr{G}_\delta$-complete group which contains G as a $\mathscr{G}_\delta$-dense (topological) subgroup. It follows from Proposition 4.12 that the $\mathscr{G}_\delta$-completion is unique up to isomorphism fixing the points of G. It is also obvious that any group has the $\mathscr{G}_\delta$-completion.

5. Hilbert spaces as underlying topological spaces

Our first aim of this section is to prove Theorem 1.2 and Proposition 1.5(b). To this end, we recall a classical construction due to Arens and Eells [1956] (see also [Weaver 1999, Chapter 2]).

Definition 5.1. Let (X, d) be a nonempty complete metric space. For every $p \in X$ let $\chi_p : X \to \{0, 1\}$ be such that $\chi_p(x) = 1$ if $x = p$ and $\chi_p(x) = 0$ otherwise. A *molecule* of X is any function $m : X \to \mathbb{R}$ which is supported on a finite set and satisfies $\sum_{p\in X} m(p) = 0$. Denote by $\mathrm{AE}_0(X)$ the real vector space of all molecules of X, and for $m \in \mathrm{AE}_0(X)$ put

$$\|m\|_{AE} = \inf\left\{\sum_{j=1}^{n} |a_j| d(p_j, q_j) \;\middle|\; m = \sum_{j=1}^{n} a_j(\chi_{p_j} - \chi_{q_j})\right\}.$$

Then $\|\cdot\|_{AE}$ is a norm and the completion of $(\mathrm{AE}_0(X), \|\cdot\|_{AE})$ is called the *Arens–Eells* space of (X, d) and denoted by $(\mathrm{AE}(X), \|\cdot\|_{AE})$. Moreover, $w(\mathrm{AE}(X)) = w(X)$.

It is an easy observation that every isometry $u : X \to Y$ between complete metric spaces X and Y induces a unique linear isometry $\mathrm{AE}(u) : \mathrm{AE}(X) \to \mathrm{AE}(Y)$ such that $\mathrm{AE}(u)(\chi_p - \chi_q) = \chi_{u(p)} - \chi_{u(q)}$ for any $p, q \in X$. The following result is surely well-known, but probably nowhere explicitly stated. Therefore, for the reader's convenience, we give its short proof:

Lemma 5.2. *For every complete metric space (X, d), the function*

$$\Psi : \mathrm{Iso}(X, d) \ni u \mapsto \mathrm{AE}(u) \in \mathrm{Iso}(\mathrm{AE}(X), \|\cdot\|_{AE}) \tag{30}$$

is both a group homomorphism and a topological embedding.

Proof. Continuity and homomorphicity of Ψ is clear (note that $\mathrm{AE}_0(X)$ is dense in $\mathrm{AE}(X)$ and $\mathrm{AE}_0(X)$ is the linear span of the set $\{\chi_p - \chi_q \mid p, q \in X\}$). Here we focus only on showing that Ψ is an embedding. We may and do assume that $\mathrm{card}(X) > 1$. Suppose $\{u_\sigma\}_{\sigma\in\Sigma} \subset \mathrm{Iso}(X, d)$ is a net such that

$$\lim_{\sigma\in\Sigma} \Psi(u_\sigma) = \Psi(u) \tag{31}$$

for some $u \in \mathrm{Iso}(X, d)$. Let $x \in X$ be arbitrary. We only need to verify that $\lim_{\sigma\in\Sigma} u_\sigma(x) = u(x)$. Let $y \in X$ be different from x. We infer from (31) that

$$\chi_{u_\sigma(x)} - \chi_{u_\sigma(y)} \to \chi_{u(x)} - \chi_{u(y)} \quad (\sigma \in \Sigma). \tag{32}$$

For $\varepsilon \in (0, d(x, y))$ let $B_\varepsilon = \{z \in X \mid d(u(x), z) \geqslant \varepsilon\}(\neq \varnothing)$ and let $v_\varepsilon : X \to [0, \infty)$ denote the distance function to B_ε, that is,

$$v_\varepsilon(z) = \inf\{d(z, b) \mid b \in B_\varepsilon\}.$$

Observe that v_ε is Lipschitz. Since the dual Banach space to $\mathrm{AE}(X)$ is naturally isomorphic to the Banach space of all Lipschitz real-valued functions on X (see, for example, Chapter 2 of [Weaver 1999]), we deduce from (32) that $\lim_{\sigma\in\Sigma}[v_\varepsilon(u_\sigma(x)) - v_\varepsilon(u_\sigma(y))] = v_\varepsilon(u(x)) - v_\varepsilon(u(y))$. But $v_\varepsilon(u(x)) - v_\varepsilon(u(y)) = v_\varepsilon(u(x)) > 0$ (because $u(y) \in B_\varepsilon$, by the isometricity of u). So there is $\sigma_0 \in \Sigma$ such that $v_\varepsilon(u_\sigma(x)) > 0$ for any $\sigma \geqslant \sigma_0$. This means that for $\sigma \geqslant \sigma_0$, $u_\sigma(x) \notin B_\varepsilon$ and consequently $d(u_\sigma(x), u(x)) < \varepsilon$. □

The homomorphism appearing in (30) is *not* surjective, unless $\mathrm{card}(X) < 3$. There is however a fascinating result, discovered by Mayer–Wolf [1981], that characterizes *all* isometries of the space $\mathrm{AE}(X)$ under some additional conditions on the metric of X. Below we formulate only a special case of it, enough for our considerations.

Theorem 5.3. *Let d be a* ***bounded*** *complete metric on a set X. Let* $\mathrm{AE}(X)$ *denote the Arens–Eells spaces of* $(X, \sqrt{d})$. *Every linear isometry of* $\mathrm{AE}(X)$ *onto itself is of the form* $\pm\,\mathrm{AE}(u)$, *where* $u \in \mathrm{Iso}(X, d)(= \mathrm{Iso}(X, \sqrt{d}))$.

Proof. All parts of this proof come from [Mayer-Wolf 1981]. Alternatively, we give references to suitable results from [Weaver 1999]. By Proposition 2.4.5 of that reference, the metric space $(X, \sqrt{d})$ is so-called concave (for the definition, see the note preceding Lemma 2.4.4 on page 51 in [Weaver 1999]). Now, Theorem 2.7.2 in that reference implies that if Φ is a linear isometry of $\mathrm{AE}(X)$ then there is $r \in \mathbb{R} \setminus \{0\}$ and a bijection $u : X \to X$ such that $\sqrt{d(u(x), u(y))} = |r|\sqrt{d(x, y)}$ and $\Phi(\chi_x - \chi_y) = \frac{1}{r}(\chi_{u(x)} - \chi_{u(y)})$ for any $x, y \in X$. The former of these relations, combined with the boundedness of d, implies that $|r| = 1$ and $u \in \mathrm{Iso}(X, d)$. So $\Phi = \pm\,\mathrm{AE}(u)$, and we are done. □

We shall also need quite an intuitive result stated below. Although its proof is not immediate, we leave it to the reader as an exercise.

Lemma 5.4. *Let X be a two-dimensional real vector space, $\|\cdot\|$ be any norm on X, and let a and b be two vectors in X:*

(a) *If $\bar{B}_X(0, 2) \subset \bar{B}_X(b, 2) \cup \bar{B}_X(a, 1)$ then $b = 0$.*

(b) *If $\|a\| = \|b\| = 2$ and $\bar{B}_X(b, 1) \subset \bar{B}_X(0, 2) \cup \bar{B}_X(a, 1)$ then $a = b$.*

Proof of Theorem 1.2 and of Proposition 1.5(b). Because Theorem 1.2 is a special case of point (b) of the proposition, we focus only on the proof of the latter result. It follows from Corollary 4.7 that there is a complete metric space (Y, ϱ) such that $w(Y) = \beta$ and $\mathrm{Iso}(Y, \varrho)$ is isomorphic to G. Since $\mathrm{Iso}(Y, \varrho) = \mathrm{Iso}(Y, \varrho/(2+2\varrho))$ and the metric $\varrho/(2+2\varrho)$ is complete (and compatible), we may and do assume that $\varrho < \frac{1}{2}$. We also assume that $Y \cap [0, 1] = \varnothing$. Let $X = Y \cup [0, 1]$. We define a metric d on X by the rules:

- $d(s, t) = |s - t|$ for $s, t \in [0, 1]$.
- $d(x, y) = \varrho(x, y)$ for $x, y \in Y$.
- $d(x, t) = d(t, x) = 1 + t$ for $x \in Y$ and $t \in [0, 1]$.

We leave it as an exercise that d is indeed a metric, that d is complete, and $w(X) = \beta$. Notice that for any $a \in X$ and $t \in [0, 1] (\subset X)$:

- $a = 1 \iff \exists b, c \in X : d(a, b) = \frac{3}{4} \wedge d(a, c) = 2$.
- $a = t \iff d(a, 1) = 1 - t$.

These equivalences imply that for every $f \in \mathrm{Iso}(X, d)$ we have $f(t) = t$ for $t \in [0, 1]$ and $f|_Y \in \mathrm{Iso}(Y, \varrho)$. It is also easy to see that each isometry of (Y, ϱ) extends (uniquely) to an isometry of (X, d). Hence the function $\mathrm{Iso}(X, d) \ni f \mapsto f|_Y \in \mathrm{Iso}(Y, \varrho)$ is a (well-defined) isomorphism. Now let $(E, \|\cdot\|) = (\mathrm{AE}(X), \|\cdot\|_{AE})$ be the Arens–Eells space of $(X, \sqrt{d})$ and let $e = \chi_1 - \chi_0$. We see that $w(E) = \beta$ and E is infinite-dimensional, since X is infinite. What is more, it follows from Theorem 5.3 that every linear isometry of E which leaves the point e fixed is of the form $\mathrm{AE}(u)$ for some $u \in \mathrm{Iso}(X, \sqrt{d})$. Since $\mathrm{AE}(u)(e) = e$ for any $u \in \mathrm{Iso}(X, d)$ (because $u(0) = 0$ and $u(1) = 1$ for such u), noticing that $\mathrm{Iso}(X, d) = \mathrm{Iso}(X, \sqrt{d})$ and Lemma 5.2 finishes the proof. $\square$

Our next aim is to show Theorem 1.1(a) and Proposition 1.5(a). We shall need the next three results.

Theorem 5.5 [Mankiewicz 1972]. *Whenever X and Y are normed vector spaces, U and V are connected open subsets of, respectively, X and Y then every isometry of U onto V extends to a unique affine isometry of X onto Y.*

We recall that a function $\Phi : X \to Y$ between real vector spaces X and Y is *affine* if $\Phi - \Phi(0)$ is linear.

The following result is a consequence, for example, of [Bessaga and Pełczyński 1975, Theorem VI.6.2] and a famous theorem of Toruńczyk's [1981; 1985], which says that every Banach space is homeomorphic to a Hilbert space:

Theorem 5.6. *Every closed convex set in an infinite-dimensional Banach space whose interior is nonempty is homeomorphic to an infinite-dimensional Hilbert space.*

Our last tool is the next result, which in the separable case was proved by Mogilski [1979]. The argument there works also in the nonseparable case. This theorem in its full generality may also be briefly concluded from the results of [Toruńczyk 1981; 1985]. For the reader's convenience, we give a sketch of its proof.

Theorem 5.7. *Let X be a metrizable space. If X is the union of its two closed subsets A and B such that each of A, B, and $A \cap B$ is homeomorphic to an infinite-dimensional Hilbert space $\mathscr{H}$ then X itself is homeomorphic to $\mathscr{H}$.*

Proof. Put $C = A \cap B$.

First assume that C is a Z-set in both A and B (for the definition of a Z-set see Section 6). Then there exist homeomorphisms $h_A : A \to \mathscr{H} \times (-\infty, 0]$ and $h_B : B \to \mathscr{H} \times [0, \infty)$ which coincide on C and $h_A(C) = h_B(C) = \mathscr{H} \times \{0\}$ (this follows from the theorem on extending homeomorphisms between Z-sets in Hilbert manifolds, see [Anderson 1967; Anderson and McCharen 1970; Chapman 1971; Bessaga and Pełczyński 1975, Chapter V]. Now it suffices to define $h : X \to \mathscr{H} \times \mathbb{R}$ as the union of h_A and h_B to obtain the homeomorphism we searched for.

Now we consider a general case. Let $X' = (A \times [-1, 0]) \cup (B \times [0, 1]) \subset X \times [-1, 1]$. Observe that $A' = A \times [-1, 0]$, $B' = B \times [0, 1]$, and $C' = C \times \{0\}$ are homeomorphic to $\mathscr{H}$ (by the assumptions of the theorem) and C' is a Z-set in both A' and B'. Thus, we infer from the first part of the proof that X' is homeomorphic to $\mathscr{H}$. Finally, note that the function $X' \ni (x, t) \mapsto (x, 0) \in X \times \{0\}$ is a proper retraction. So Toruńczyk's result [1981] implies that X is a manifold modeled on $\mathscr{H}$. Since it is contractible, it is homeomorphic to $\mathscr{H}$. □

Proof of Theorem 1.1(a) *and Proposition 1.5*(a). Again, observe that point (a) of the theorem under the question is a special case of point (a) of the proposition. Therefore we focus only on the latter result. Let E and e be as in point (b) of Proposition 1.5. Replacing, if needed, e by $2e/\|e\|$, we may assume that $\|e\| = 2$. Denote by $\mathscr{E}$ the group of all linear isometries which leave e fixed. Let $W = \bar{B}_E(0, 2) \cup \bar{B}_E(e, 1)$ be equipped with the metric p induced by the norm of E. Notice that if $V \in \mathscr{E}$ then $V(W) = W$ and $V|_W \in \mathrm{Iso}(W, p)$. Conversely, for each $g \in \mathrm{Iso}(W, p)$, $g = V|_W$ for some linear isometry $V \in \mathscr{E}$. Let us briefly justify this claim. Let $x = g(0)$ and $y = g(e)$. Then $W \subset \bar{B}_E(x, 2) \cup \bar{B}_E(y, 1)$ and consequently $\bar{B}_X(0, 2) \subset \bar{B}_X(x, 2) \cup \bar{B}_X(y, 1)$, where X is a two-dimensional linear subspace of E which contains x and y. We infer from Lemma 5.4(a) that $x = 0$. So $\|y\| = 2$ (since g is an isometry) and thus $\bar{B}_Y(e, 1) \subset \bar{B}_Y(0, 2) \cup \bar{B}_Y(y, 1)$, where Y is a two-dimensional linear subspace of E such that $e, y \in Y$. Now point (b) of Lemma 5.4 yields that $y = e$. We then have $g(B_E(0, 2) \cup B_E(e, 1)) = B_E(0, 2) \cup B_E(e, 1)$. So an application of Theorem 5.5 gives our assertion: There is a linear (since $g(0) = 0$) isometry V of E which extends g.

Having the above fact, we easily see that the function $\mathscr{E} \ni V \mapsto V|_W \in \mathrm{Iso}(W, p)$ is an isomorphism. Consequently, G is isomorphic to $\mathrm{Iso}(W, p)$. So, to finish the proof, it suffices to show that W is homeomorphic to $\mathscr{H}_\beta$. But this immediately follows from Theorems 5.6 and 5.7, since $w(E) = \beta$ and the sets $\bar{B}_E(0, 2)$, $\bar{B}_E(e, 1)$, and $\bar{B}_E(0, 2) \cap \bar{B}_E(1, 2)$ are closed, convex, and have nonempty interiors. □

Proof of Corollary 1.6. It suffices to apply Proposition 1.5 and Proposition 4.3(f). □

The arguments used in the proofs of both points of Proposition 1.5 also show the next result:

Corollary 5.8. *Let G be a $\mathscr{G}_\delta$-complete topological group of topological weight not exceeding $\beta \geqslant \aleph_0$. There are an infinite-dimensional normed vector space E of topological weight β, a contractible open set $U \subset E$, and a nonzero vector $e \in E$ such that the topological groups G,* $\mathrm{Iso}(U, d)$, *and* $\mathrm{Iso}(E|e)$ *are isomorphic, where d is the metric on U induced by the norm of E and* $\mathrm{Iso}(E|e)$ *is the group of all linear isometries of E which leave the point e fixed.*

Proof. Let (Y_0, ϱ_0) be a metric space such that $w(Y_0) = \beta$ and $\mathrm{Iso}(Y_0, \varrho_0)$ is isomorphic to G (see Theorems 1.4 and 3.2). Denote by (Y, ϱ) the completion of (Y_0, ϱ_0). Now let (X, d), $(\mathrm{AE}(X), \|\cdot\|_{AE})$, and e be as in the proof of Proposition 1.5(b). We know that the function

$$\mathrm{Iso}(X, d) \ni u \mapsto \mathrm{AE}(u) \in \mathrm{Iso}(\mathrm{AE}(X)|e)$$

is an isomorphism. Denote by E the linear span of the set $\{\chi_a - \chi_b \mid a, b \in Y_0 \cup [0, 1]\} (\subset \mathrm{AE}(X))$ (recall that $X = Y \sqcup [0, 1]$). Observe that if $u \in \mathrm{Iso}(X, d)$ is such that $\mathrm{AE}(u)(E) = E$ then $u(Y_0 \cup [0, 1]) = Y_0 \cup [0, 1]$ and consequently $u(Y_0) = Y_0$ (see the proof of point (b) of Proposition 1.5). This yields that the function

$$\mathrm{Iso}(Y_0, \varrho_0) \ni u \mapsto \mathrm{AE}(u)|_E \in \mathrm{Iso}(E|e)$$

is also an isomorphism. Now it suffices to put $U = B_E(0, \|e\|) \cup B_E(e, \frac{1}{2}\|e\|)$ and repeat the proof of Proposition 1.5(a) (involving Lemma 5.4 and Theorem 5.5) to get the whole assertion. (U is contractible as the union of two intersecting convex sets.) □

6. Isometry groups of completely metrizable metric spaces

Taking into account Corollary 1.6, the following question naturally arises: Given an infinite cardinal β, how do we characterize topological groups isomorphic to $\mathrm{Iso}(\mathcal{H}, d)$ for some compatible metric d on a Hilbert space $\mathcal{H}$ of Hilbert space dimension β? In this part we give a partial answer to this question. In fact, we will deduce our main result in this topic from the following general fact:

Proposition 6.1. *Let (S, p) be a bounded complete metric space, β an infinite cardinal not less than $w(S)$, and $\mathcal{H}$ a Hilbert space of Hilbert space dimension β. Let S' be a dense subset of S and G a closed subgroup of* $\mathrm{Iso}(S', p)$. *There exist a compatible complete metric λ on $\mathcal{H}$, a set $\mathcal{H}' \subset \mathcal{H}$, and an isomorphism $\Psi : \overline{G} \to \mathrm{Iso}(\mathcal{H}, \lambda)$ such that $\Psi(G) = \{u \in \mathrm{Iso}(\mathcal{H}, \lambda) \mid u(\mathcal{H}') = \mathcal{H}'\}$, $(\mathcal{H} \setminus \mathcal{H}', \lambda)$ is isometric to $(S \setminus S', \sqrt{p})$ and the closure of $\mathcal{H} \setminus \mathcal{H}'$ is a Z-set in $\mathcal{H}$. In particular, if S' is completely metrizable then $\mathcal{H}'$ is homeomorphic to $\mathcal{H}$.*

Recall that a closed set K in a metric space X is a *Z-set* if and only if every map of the Hilbert cube Q into X may uniformly be approximated by maps of Q into $X \setminus K$.

Proof. By Proposition 3.4, there is a bounded complete metric space (Y, ϱ), a dense set $Y' \subset Y$, and an isomorphism $F_1 : \overline{G} \to \mathrm{Iso}(Y, \varrho)$ such that $w(Y) = \beta$, $(Y \setminus Y', \varrho)$ is isometric to $(S \setminus S', p)$, and $F_1(G) = \{u \in \mathrm{Iso}(Y, \varrho) \mid u(Y') = Y'\}$. Now we shall mimic the proof of Proposition 1.5.

Replacing, if applicable, p and ϱ by $t \cdot p$ and $t \cdot \varrho$ with small enough $t > 0$ (and the final metric λ obtained from this proof by $\lambda/\sqrt{t}$), we may assume that $\varrho < \frac{1}{2}$. Now let $(X, d) \supset (Y, \varrho)$ be as in the proof of Theorem 1.2. Further, let $(E, \|\cdot\|) = (\mathrm{AE}(X), \|\cdot\|_{AE})$ be the Arens–Eells space of $(X, \sqrt{d})$ and $e = \chi_1 - \chi_0 \in E$. We denote by λ the metric on $W = \bar{B}_E(0, 1) \cup \bar{B}_E(e, \frac{1}{2})$ induced by the norm of E. The arguments used in the proofs of Theorem 1.2 and Proposition 1.5 show:

(I1) The function $F_2 : \mathrm{Iso}(X, d) \ni u \mapsto u|_Y \in \mathrm{Iso}(Y, \varrho)$ is an isomorphism and there is a dense set $X' \subset X$ such that $X \setminus X' = Y \setminus Y'$ and $F_2^{-1}(F_1(G))$ consists precisely of all $u \in \mathrm{Iso}(X, d)$ with $u(X') = X'$.

(I2) The function $F_3 : \mathrm{Iso}(X, d) \ni u \mapsto \mathrm{AE}(u)|_W \in \mathrm{Iso}(W, \lambda)$ is an isomorphism and $u(0) = 0$ for any $u \in \mathrm{Iso}(X, d)$.

(I3) W is homeomorphic to $\mathcal{H}$.

Point (I3) asserts that we may identify $\mathcal{H}$ with W. Put $\Psi = F_3 \circ F_2^{-1} \circ F_1 : \overline{G} \to \mathrm{Iso}(W, \lambda)$ and $W' = W \setminus \{\chi_x - \chi_0 \mid x \in X \setminus X'\}$ (recall that $\|\chi_y - \chi_0\|_{AE} = \sqrt{d(y, 0)} = 1$ for every $y \in Y \supset X \setminus X'$) and note that Ψ is an isomorphism. What is more, we claim that $\Psi(G)$ consists of all $v \in \mathrm{Iso}(W, \lambda)$ for which $v(W') = W'$. Indeed, it follows from (I2) that each $v \in \mathrm{Iso}(W, \lambda)$ has the form $v = \mathrm{AE}(u)|_W$ for some

$u \in \mathrm{Iso}(X, d)$. So, taking into account (I1), we only need to check that $u(X') = X'$ if and only if $(\mathrm{AE}(u))(W') = W'$. But this follows from the fact that $u(0) = 0$.

Further, observe that the function $(X \setminus X', \sqrt{d}) \ni x \mapsto \chi_x - \chi_0 \in (W \setminus W', \lambda)$ is an isometry, which implies that the latter metric space is isometric to $(S \setminus S', \sqrt{p})$. To show that the closure D of $W \setminus W'$ is a Z-set in W, note that $D \subset C := \{\chi_y - \chi_0 \mid y \in Y\}$ and the maps $W \ni w \mapsto (1 - 1/n)w + e/n \in W$ converge uniformly to the identity map with respect to λ and their images are disjoint from C.

So, to complete the proof, we only need to check that if S' is completely metrizable then W' is homeomorphic to $\mathscr{H}$. But this is an immediate consequence of the fact that $W \setminus W'$ is a set isometric to $(S \setminus S', \sqrt{p})$ and contained in a Z-set in W, and a known fact on σ-Z-sets in Hilbert spaces: S', being completely metrizable, is a $\mathscr{G}_\delta$-set in S; hence, $S \setminus S'$ is $\mathscr{F}_\sigma$ in S. Consequently, $W \setminus W'$ is a countable union of sets complete in the metric λ thus an $\mathscr{F}_\sigma$-set in W. But the closure of $W \setminus W'$ is a Z-set in W, so $W \setminus W'$ is a σ-Z-set, that is, it is a countable union of Z-sets. Now the assertion follows from the well-known result that the complement of a σ-Z-set in a Hilbert space is homeomorphic to the whole space, which simply follows from Toruńczyk's characterization [1981; 1985] of Hilbert manifolds. (For the separable case one may also consult Theorem 6.3 in [Bessaga and Pełczyński 1975, Chapter V].) □

As a conclusion, we obtain:

Theorem 6.2. *Let $\mathscr{H}$ be a Hilbert space of Hilbert space dimension $\beta \geqslant \aleph_0$ and*

$$\mathscr{G} = \{\mathrm{Iso}(\mathscr{H}, \varrho) \mid \varrho \text{ is a compatible metric on } \mathscr{H}\}.$$

Then, up to isomorphism, $\mathscr{G}$ consists precisely of all topological groups G which are isomorphic to closed subgroups of $\mathrm{Iso}(X, d)$ *for some completely metrizable spaces (X, d) with $w(X) \leqslant \beta$.*

Proof. If G is a closed subgroup of $\mathrm{Iso}(X, d)$ for a completely metrizable space (X, d) with $w(X) \leqslant \beta$, we may assume that d is bounded (replacing, if necessary, d by $d/(1+d)$). Then Proposition 6.1, applied for $(S, p) =$ the completion of (X, d) and $S' = X$, yields a complete compatible metric λ on $\mathscr{H}$ and a dense set $\mathscr{H}' \subset \mathscr{H}$ homeomorphic to $\mathscr{H}$ such that G is isomorphic to $G' := \{u \in \mathrm{Iso}(\mathscr{H}, \lambda) \mid u(\mathscr{H}') = \mathscr{H}'\}$. But G' is naturally isomorphic to $\mathrm{Iso}(\mathscr{H}', \lambda)$ and we are done. □

For an infinite cardinal number α, let us denote by $\mathsf{IGH}(\alpha)$ the class of all topological groups which are isomorphic to $\mathrm{Iso}(\mathscr{H}, \varrho)$ for some compatible metric ϱ on a Hilbert space $\mathscr{H}$ of Hilbert space dimension α ("IGH" is the abbreviation of "isometry group of a Hilbert space"). Additionally, let IGH stand for the union of all classes $\mathsf{IGH}(\alpha)$. Theorem 6.2 implies that IGH is a *variety*; that is, closed subgroups as well as topological products of members of IGH belong to IGH as well.

We are mainly interested in the class $\mathsf{IGH}(\aleph_0)$. It is clear that all groups belonging to this class are second-countable. In the sequel we shall see that the axiom of second countability is insufficient for a topological group to belong to $\mathsf{IGH}(\aleph_0)$ (see Proposition 6.5 below).

As a simple consequence of Theorem 6.2 we obtain:

Corollary 6.3. *If G is a second-countable, σ-compact topological group then $G \in \mathsf{IGH}(\aleph_0)$.*

Proof. Let p be a compatible left-invariant metric on G and let $(Y, \varrho) \supset (G, p)$ denote the completion of (G, p). If the interior of G in Y is nonempty then G is locally completely metrizable and thus G is Polish. In that case the assertion follows from Theorem 1.1. On the other hand, if G is a boundary set in Y then $Y \setminus G$ is dense in Y and therefore $\mathrm{Iso}(Y \setminus G, \varrho)$ is isomorphic to $\{u \in \mathrm{Iso}(Y, \varrho) \mid u(Y \setminus G) = Y \setminus G\}$ and the latter group coincides with $\{u \in \mathrm{Iso}(Y, \varrho) \mid u(G) = G\}$, which is isomorphic to $\mathrm{Iso}(G, \varrho) = \mathrm{Iso}(G, p)$. Since G is σ-compact, $Y \setminus G$ is completely metrizable. Finally, since p is left-invariant, all left translations of G form a closed subgroup of $\mathrm{Iso}(G, p)$ isomorphic to G. So, to sum up, G is isomorphic to a closed subgroup of the isometry group of the Polish space $(Y \setminus G, \varrho)$. Now it suffices to apply Theorem 6.2. $\square$

To formulate our next result, we remind the reader that a separable metrizable space X is said to be *coanalytic* if and only if X is homeomorphic to a space of the form $Y \setminus Z$, where Y is a Polish space and $Z \subset Y$ is *analytic*, that is, Z is the continuous image of a Polish metric space. We also recall that continuous images of Borel subsets of Polish spaces are analytic.

Proposition 6.4. *Each member of $\mathsf{IGH}(\aleph_0)$ is coanalytic as a topological space.*

Proof. Let (X, ϱ) be a Polish metric space and $G = \mathrm{Iso}(X, \varrho)$. Denote by (Y, d) the completion of (X, ϱ). Since X is completely metrizable, it is a $\mathscr{G}_\delta$-set in Y. Observe that G is naturally isomorphic to the subgroup $\{u \in \mathrm{Iso}(Y, d) \mid u(X) = X\}$ of $\mathrm{Iso}(Y, d)$. Since $\mathrm{Iso}(Y, d)$ is a Polish group, it suffices to show that $A := \{u \in \mathrm{Iso}(Y, d) \mid u(X) \neq X\}$ is the continuous image of a Borel subset of a Polish metric space. To this end, notice that the set $W := \{(u, x) \in \mathrm{Iso}(Y, d) \times X \mid u(x) \in Y \setminus X\}$ is Borel in the Polish space $\mathrm{Iso}(Y, d) \times X$ and $\pi(W) = A$, where $\pi : \mathrm{Iso}(Y, d) \times X \to \mathrm{Iso}(Y, d)$ is the natural projection. $\square$

Proposition 6.5. *There exists a topological subgroup of the additive group of reals which does not belong to $\mathsf{IGH}(\aleph_0)$.*

Proof. We consider $\mathbb{R}$ as a vector space over the field $\mathbb{Q}$ of rationals. There exists a vector subspace G of $\mathbb{R}$ such that $\mathbb{Q} \cap G = \{0\}$ and $G + \mathbb{Q} = \mathbb{R}$. We claim that $G \notin \mathsf{IGH}(\aleph_0)$. To show that, it is enough to prove that G is not coanalytic (thanks to Proposition 6.4). Since analytic spaces are absolutely measurable (see, for example, Theorem A.13 in [Takesaki 1979, Appendix]), it suffices to show that G is not Lebesgue measurable. But this follows from the following two observations:

- $\mathbb{R} = \bigcup_{q\in\mathbb{Q}}(q + G)$ and hence the outer measure of G is positive.
- $G - G$ (which equals G) has empty interior and thus its inner measure is 0; this follows from [Halmos 1950, Chapter XII, §61, Theorem A, p. 266]. □

Propositions 6.4 and 6.5 make the issue of characterizing members of IGH complicated. The following problems seem to be most interesting:

Problem 6.6. If $G \in \mathsf{IGH}$, is it true that $G \in \mathsf{IGH}(w(G))$?

Problem 6.7. Does the class IGH contain all $\mathscr{G}_\delta$-complete topological groups?

Problem 6.8. Characterize members of $\mathsf{IGH}(\aleph_0)$.

7. Compact and locally compact Polish groups

This section is devoted to the proofs of points (b) and (c) of Theorem 1.1. Our main tool will be the following result, very recently shown by us in [Niemiec 2012]:

Theorem 7.1. *Let G be a locally compact Polish group, X be a locally compact Polish space, and let $G \times X \ni (g, x) \mapsto g.x \in X$ be a continuous proper action of G on X. Assume there is a point $\omega \in X$ such that the set $G.\omega = \{g.\omega \mid g \in G\}$ is nonopen and G acts freely at ω (that is, $g.\omega = \omega$ implies $g =$ the neutral element of G). Then there exists a proper compatible metric d on X such that* $\mathrm{Iso}(X, d)$ *consists precisely of all maps of the form $x \mapsto g.x$ ($g \in G$). In particular, the topological groups* $\mathrm{Iso}(X, d)$ *and G are isomorphic.*

We recall that (under the above notation) the action is *proper* if for every compact set $K \subset X$ the set $\{g \in G \mid g.K \cap K \neq \varnothing\}$ is compact as well (where $g.K = \{g.x \mid x \in K\}$).

Our next tool is the following classical result due to Keller [1931] (see also [Bessaga and Pełczyński 1975, Theorem III.3.1]):

Theorem 7.2. *Every infinite-dimensional compact convex subset of a Fréchet space is homeomorphic to the Hilbert cube.*

We recall that a Fréchet space is a completely metrizable locally convex topological vector space.

We call a function $u : (X, d) \to \mathbb{R}$ (where (X, d) is a metric space) *nonexpansive* if and only if $|u(x) - u(y)| \leqslant d(x, y)$ for all $x, y \in X$. The function u is a *Katětov map* if and only if u is nonexpansive and additionally $d(x, y) \leqslant u(x) + u(y)$ for any $x, y \in X$. Katětov maps correspond to one-point extensions of metric spaces.

Proof of Theorem 1.1(b). Let G be a compact Polish group. Take a left-invariant metric $\varrho \leqslant 1$ on G and equip the space $X = G \times [0, 1]$ with the metric d, where

$d((x, s), (y, t)) = \max(\varrho(x, y), |t - s|)$. For $g \in G$ denote by ψ_g the function $X \ni (x, t) \mapsto (g^{-1}x, t) \in X$. Notice that $\psi_g \in \mathrm{Iso}(G, X)$ for any $g \in G$. Let Δ be the space of all nonexpansive maps of (X, d) into $[0, 1]$ endowed with the supremum metric. Observe that Δ is a convex set in the Banach space of all real-valued maps on X. What is more, Δ is infinite-dimensional since X is infinite, and Δ is compact by the Ascoli type theorem. So we infer from Theorem 7.2 that Δ is homeomorphic to Q. Further, $\Phi_g(u) := u \circ \psi_g \in \Delta$ for any $g \in G$ and $u \in \Delta$ (because ψ_g is isometric). It is also easily seen that the function $G \times \Delta \ni (g, u) \mapsto \Phi_g(u) \in \Delta$ is a (proper — since both G and Δ are compact) continuous action of G on Δ. Finally, the function $\omega : X \ni (x, t) \mapsto d((x, t), (e, 1)) \in \mathbb{R}$ belongs to Δ (since $d \leqslant 1$), where e is the neutral element of G. Observe that the set $K := \{\omega \circ \psi_g \mid g \in G\}$ has empty interior in Δ, since $1/n + (1 - 1/n)\omega \circ \psi_g \in \Delta \setminus K$ for any $n \geqslant 1$. Now we apply Theorem 7.1. □

Our last aim is to prove Theorem 1.1(c). To this end, we need more information on Hilbert cube manifolds.

One of the deepest results in infinite-dimensional topology is Anderson's theorem [1967] on extending homeomorphisms between Z-sets. Below we formulate it only in the Hilbert cube settings, it holds however in a much more general context. (For the discussion on this topic consult [Bessaga and Pełczyński 1975, Chapter V]; see also [Anderson and McCharen 1970; Chapman 1971]).

Theorem 7.3. *Every homeomorphism between two Z-sets in the Hilbert cube Q is extendable to a homeomorphism of Q onto itself.*

The result stated below is a kind of folklore in Hilbert cube manifolds theory. We present its short proof because we could not find it in the literature.

Theorem 7.4. *The spaces $Q \times [0, \infty)$ and $Q \setminus \{\text{point}\}$ are homeomorphic.*

Proof. Since $Q \setminus \{\text{point}\}$ is a Hilbert cube manifold, it follows from Schori's theorem [1971] (see also [Chapman 1976]; compare with [Bessaga and Pełczyński 1975, Theorem IX.4.1]) that $(Q \setminus \{\text{point}\}) \times Q$ is homeomorphic to $Q \setminus \{\text{point}\}$. Now the assertion follows from Theorem 7.3 since $(Q \times [0, 1]) \setminus (Q \times [0, 1)) = Q \times \{1\}$ is a Z-set in $Q \times [0, 1]$ homeomorphic to the Z-set (in $Q \times Q$) $(Q \times Q) \setminus [(Q \setminus \{\text{point}\}) \times Q]$. □

Lemma 7.5. *Let (X, d) be a nonempty separable metric space and let $E(X)$ be the set of all Katětov maps on (X, d) equipped with the pointwise convergence topology*:

(i) *For any $a \in X$ and $r > 0$ the set $\{f \in E(X) \mid f(a) \leqslant r\}$ is compact* (*in $E(X)$*).

(ii) *$E(X) \times Q$ is homeomorphic to $Q \setminus \{\text{point}\}$.*

Proof. Point (i) follows from the Ascoli type theorem, since $E(X)$ consists of nonexpansive maps and for any $f \in E(X)$ and $x \in X$, $f(x) \in [0, d(x, a) + f(a)]$.

We turn to (ii). First of all, $E(X)$ is metrizable, because of the separability of X and the nonexpansivity of members of $E(X)$. Further, thanks to Theorem 7.4, it suffices to show that $E(X) \times Q$ is homeomorphic to $Q \times [0, \infty)$. Fix $a \in X$ and let $\omega \in E(X)$ be given by $\omega(x) = d(a, x)$. For each $n \geqslant 1$ let $K_n = \{f \in E(X) \mid f(a) \in [n-1, n]\}$ and $Z_{n-1} = \{f \in E(X) \mid f(a) = n-1\}$. We infer from (i) that K_n and Z_{n-1} are compact. It is also easily seen that both are convex nonempty sets ($\omega + n - 1 \in Z_{n-1} \subset K_n$). Since $K_n \times Q$ and $Z_{n-1} \times Q$ are affinely homeomorphic to convex subsets of Fréchet spaces, Theorem 7.2 yields that both these sets are homeomorphic to Q. Let $h_{n-1} : Z_{n-1} \times Q \to Q \times \{n-1\}$ be any homeomorphism. We claim that $Z_{n-1} \cup Z_n$ is a Z-set in K_n. This easily follows from the fact that the maps $K_n \ni f \mapsto (1 - 1/k)f + 1/k(\omega + n - \frac{1}{2}) \in K_n$ send K_n into $K_n \setminus (Z_{n-1} \cup Z_n)$ and converge uniformly (as $k \to \infty$) to the identity map of K_n. Since $Q \times \{n-1, n\}$ is a Z-set in $Q \times [n-1, n]$, Theorem 7.3 provides us the existence of a homeomorphism $H_n : K_n \times Q \to Q \times [n-1, n]$ which extends both h_{n-1} and h_n. We claim that the union $H : E(X) \times Q \to Q \times [0, \infty)$ of all H_n ($n \geqslant 1$) is the homeomorphism we are searching for. It is clear that H is a well-defined bijection. Finally, notice that the interiors (in $E(X)$) of the sets $\bigcup_{j=1}^{n} K_j$ ($n \geqslant 1$) cover X and hence H is indeed a homeomorphism. $\square$

Proof of Theorem 1.1(c). Let G be a locally compact Polish group. By a theorem of Struble [1974] (see also [Abels et al. 2011]), there exists a proper left-invariant compatible metric d on G. Let $E(G)$ be the space of all Katětov maps on (G, d) endowed with the pointwise convergence topology. By Lemma 7.5, $L := E(G) \times Q$ is homeomorphic to $Q \setminus \{\text{point}\}$. So it suffices to show that there is a proper compatible metric ϱ on L such that $\operatorname{Iso}(L, \varrho)$ is isomorphic to G. For any $g \in G$ and $(f, q) \in L$ let $g.(f, q) = (f_g, q) \in L$, where $f_g(x) = f(g^{-1}x)$ (since d is left-invariant, $f_g \in E(G)$ for each $f \in E(G)$). As in the proof of point (b) of the theorem, we see that the function $G \times L \ni (g, x) \mapsto g.x \in L$ is a continuous action of G on L. It is also clear that each G-orbit (that is, each of the sets $G.x$ with $x \in L$) has empty interior. Similarly, as in point (b), we show that there is $\omega \in L$ such that G acts freely at ω (for example, $\omega = (u, q)$ with arbitrary $q \in Q$ and $u(x) = d(x, e)$, where e is the neutral element of G). So, by virtue of Theorem 7.1, it remains to check that the action is proper. To this end, take any compact set W in L. Then there is $r > 0$ such that $W \subset \{f \in E(G) \mid f(e) \leqslant r\} \times Q$. Note that the set $\{g \in G \mid g.W \cap W \neq \varnothing\}$ is closed and contained in $D \times Q$, where

$$D = \{g \in G \mid \exists f \in E(G) \text{ s.t. } f(e) \leqslant r \wedge f(g^{-1}) \leqslant r\}$$

and therefore it is enough to show that D has compact closure in G. But if $g \in D$, and $f \in E(G)$ is such that $f(e) \leqslant r$ and $f(g^{-1}) \leqslant r$ then $d(g, e) = d(e, g^{-1}) \leqslant f(e) + f(g^{-1}) \leqslant 2r$. This yields that $D \subset \bar{B}_G(e, 2r)$ and noting that d is proper finishes the proof. □

Remark 7.6. Van Dantzig and van der Waerden [1928] proved that the isometry group of a connected locally compact metric space (X, d) (possibly with nonproper or incomplete metric) is locally compact and acts properly on X. It follows from [Niemiec 2012] that there exists then a proper compatible metric ϱ on X such that $\mathrm{Iso}(X, d) = \mathrm{Iso}(X, \varrho)$. In particular,

$$\{\mathrm{Iso}(Q \setminus \{\text{point}\}, d) \mid d \text{ is a compatible metric}\}$$
$$= \{\mathrm{Iso}(Q \setminus \{\text{point}\}, \varrho) \mid \varrho \text{ is a proper compatible metric}\}$$

and hence if we omit the word *proper* in Theorem 1.1(c), we will obtain an equivalent statement.

As we mentioned in the introductory part, each (locally) compact finite-dimensional Polish group is isomorphic to the isometry group of a (proper locally) compact finite-dimensional metric space. Taking this, and Corollary 4.8, into account, the following question may be interesting:

Problem 7.7. Is every finite-dimensional metrizable (resp. finite-dimensional Polish) group isomorphic to the isometry group of a finite-dimensional (resp. finite-dimensional separable complete) metric space?

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