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SINGULARITIES AND LIOUVILLE THEOREMS FOR SOME SPECIAL CONFORMAL HESSIAN EQUATIONS

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We develop some new techniques to get an integral estimate for some special conformal Hessian equations, and hence the classification of their singularities. This complete results of González. By this method we were able to deduce the Liouville theorem for these special conformal Hessian equations, which were understood by Yanyan Li via the method of moving planes.

1. Introduction

Consider the conformal k-Hessian equation

$$\sigma_k(A^g) = u^\alpha \quad \text{in } \Omega, \tag{1-1}$$

where Ω is the whole space $\mathbb{R}^n$ or the punctured unit ball $B \setminus \{0\} \subset \mathbb{R}^n$ and $g = u^{-2}\,dx^2$, $u > 0$, is a locally conformally flat metric. The matrix A^g is given by $A^g = g^{-1}\widetilde{A}^g$, where $\widetilde{A}^g$ is the $(0, 2)$ Schouten tensor

$$\widetilde{A}^g_{ij} = \frac{1}{n-2}\Big(\mathrm{Ric}_{ij} - \frac{R}{2(n-1)} g_{ij}\Big),$$

where Ric and R denote the Ricci tensor and the scalar curvature of g, respectively. In this metric, the $(1, 1)$ Schouten tensor becomes

$$A^g = u(D^2u) - \tfrac{1}{2}|Du|^2 I. \tag{1-2}$$

These σ_k are k-Hessians of A^g. More precisely, they are defined as the k-th elementary symmetric polynomial functions of the eigenvalues $\lambda_1, \dots, \lambda_n$ of the symmetric matrix A^g:

$$\sigma_k(A^g) := \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}.$$

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According to Caffarelli, Nirenberg, and Spruck [Caffarelli et al. 1985], we say u is *k-admissible* with respect to $\sigma_k(A^g)$ if $u \in \Gamma^k$, where Γ^k is defined by

$$\Gamma^k = \{u \in C^2(\Omega) : \sigma_s(A^g) > 0, s = 1, 2, \dots, k\}.$$

Equation (1-1) is raised in conformal geometry and has been studied extensively. For the critical case $\alpha = 0$ of (1-1), the isolated singularities at the origin were completely understood by Caffarelli, Gidas, and Spruck for $k = 1$ [Caffarelli et al. 1989] and by Han, Li, and Teixeira for $k > 1$ [Han et al. 2010], where they employed the method of moving planes; while for the subcritical case $\alpha \in (0, k)$, the isolated singularities were classified by Gidas and Spruck for $k = 1$ [1981] and by González for $1 < k < (n-1)/2$ [2006a]. The local behavior of singularities of the conformal Hessian problems was also studied by Chang, Gursky, and Yang [Chang et al. 2003], González [2006b], and Gursky and Viacolvsky [2006].

In this paper, we bring the results of [González 2006a] to completion. The main arguments in [Gidas and Spruck 1981] and [González 2006a] are some techniques of integration by parts which were due originally to Obata [1962]. Compared with the semilinear case $k = 1$, for $k > 1$, the problems are fully nonlinear and more complicated. The "almost" divergent structure for $\sigma_k(A^g)$ explored by González [2005] allows one to carry out integration by parts for the fully nonlinear cases. We develop the arguments in [Gidas and Spruck 1981] and [González 2006a] to deal with the special case $n = 2k + 1$. Note that the special case $k = 1, n = 3$ was treated separately in [Gidas and Spruck 1981]. Of course, our main idea is to use the "almost" divergent structure for $\sigma_k(A^g)$.

Our main result reads as follows.

Theorem 1.1. *Let $\alpha \in (0, k)$, $n = 2k + 1$ and $u > 0$ be a k-admissible solution of*

$$\sigma_k(A^g) = u^\alpha \quad \text{in } B \setminus \{0\} \tag{1-3}$$

with $u^{-1} \in C^3(B \setminus \{0\})$. Then there exists a constant C such that

$$u^{-1} \le \frac{C}{|x|^{2k/(2k-\alpha)}} \quad \text{near } x = 0.$$

Furthermore, if u^{-1} is not bounded near the origin, we also get

$$u^{-1} \ge \frac{1/C}{|x|^{2k/(2k-\alpha)}} \quad \text{near } x = 0.$$

González [2006a] proved the above results for $n > 2k + 1$. The main ingredient in González's proof is the following integral estimate.

Proposition 1.2. *Let $\alpha \in (0, k)$, $n > 2k+1$ and $u > 0$ be a k-admissible solution of* (1-3). *Let $r > 0$ small and $M > 0$ be such that*

$$\{r < |x| < Mr\} \subset B \setminus \{0\}.$$

Then

$$\int_{r<|x|<Mr} u^{\alpha((k+1)/k)-\delta}\, dx \le C r^{n-(\delta-\alpha(k+1)/k)/(1-\alpha/2k)}, \tag{1-4}$$

where the constant $\delta < n+1$ is close enough to $n+1$ and $C > 0$ depends on M and δ but not on r.

So, to prove Theorem 1.1, we need a similar integral estimate as (1-4). In fact, in this paper, we prove the integral estimate as follows.

Proposition 1.3. *Let $\alpha \in (0, k)$, $n = 2k+1$, and $u > 0$ be a k-admissible solution of* (1-3). *Let $r > 0$ small and $M > 0$ be such that*

$$\{r < |x| < Mr\} \subset B \setminus \{0\}.$$

Then

$$\int_{r<|x|<Mr} u^{\alpha(k+1)/k-n-1}\, dx \le \frac{C}{r}, \tag{1-5}$$

where the constant $C > 0$ depends on M but not on r.

By this estimate, the rest of the proof of Theorem 1.1 can be done as in [González 2006a], and we omit it in this paper.

Meanwhile, by the method shown in this paper, we are able to get the entire Liouville theorem for this special case of conformal Hessian equations. Precisely, we have the following.

Theorem 1.4. *For $\alpha \in [0, +\infty)$ and $n = 2k+1$, consider the problem*

$$\sigma_k(A^g) = u^\alpha \quad \text{in } \mathbb{R}^n. \tag{1-6}$$

(i) *If $\alpha > 0$,* (1-6) *has no positive k-admissible solution.*

(ii) *If $\alpha = 0$, any positive k-admissible solution of* (1-6) *must be a quadratic polynomial*

$$u = a + b|x - x_0|^2 \tag{1-7}$$

for some fixed $x_0 \in \mathbb{R}^n$ and positive constants a, b.

Li and Li [2005] classified all the solutions of (1-6) for $\alpha \in [0, +\infty)$ via the method of moving planes. But our proof of Theorem 1.4 is quite different from that in [Li and Li 2005], and similar to that in [Chang et al. 2003], where they treated the case $k = 2$.

The paper is organized as follows. In Section 2, we collect some known algebraic properties of σ_k. In Section 3, we deduce some preparation decomposition results. The proofs of Proposition 1.3 and Theorem 1.4 are given in Section 4.

2. Algebraic properties of σ_k

Throughout the paper the summation convention for repeated indices is used.

For a general $n \times n$ symmetric matrix A, consider its eigenvalues $\lambda_1, \dots, \lambda_n$ and the elementary symmetric polynomial functions

$$\sigma_k = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}. \tag{2-1}$$

For $k = 1, \dots, n$, denote the Newton tensor by

$$T^k = \sigma_k I - \sigma_{k-1} A + \cdots + (-1)^k A^k = \sigma_k I - T^{k-1} A, \tag{2-2}$$

and the traceless Newton tensor by

$$L^k = \frac{n-k}{n} \sigma_k I - T^k. \tag{2-3}$$

Here we take $\sigma_0 = 1$ and $T^0_{ij} = \delta_{ij}$.

Propositions 2.1 and 2.2 are well known (see [González 2006a] and references therein) and we omit their proofs.

Proposition 2.1. *For A and T^k and L^k as above and with the constant $C > 0$ depending only on n and s, the following hold*:

(a) $(n-k)\sigma_k = \operatorname{trace}(T^k)$.

(b) $(k+1)\sigma_{k+1} = \operatorname{trace}(AT^k)$.

(c) *If $\sigma_1, \dots, \sigma_k > 0$, then T^s is positive definite for $s = 1, \dots, k-1$, and hence $\| T_{ij}{}^s \| \le C\sigma_s$.*

(d) *If $\sigma_1, \dots, \sigma_k > 0$, then $\sigma_s \le C(\sigma_1)^s$ for $s = 1, \dots, k$.*

(e) *If $\sigma_1, \dots, \sigma_k > 0$, then $L_{ij}{}^s L_{ij}{}^1 \ge 0$ for $s = 1, \dots, k$ with equality if and only if $L^1 = 0$.*

Proposition 2.2. *For $A = A^g$, the Schouten tensor as in* (1-2), *and T^k and L^k defined as in* (2-2) *and* (2-3), *we have the following divergence formulas*:

(a) $\nabla^g_j T_{ij}{}^k = 0$,

(b) $\partial_j T_{ij}{}^k = -(n-k)\sigma_k u_i u^{-1} + n T_{ij}{}^k u_j u^{-1}$,

(c) $k\sigma_k = u \partial_j (u_i T_{ij}{}^{k-1}) - n T_{ij}{}^{k-1} u_i u_j + \dfrac{n-k+1}{2} \sigma_{k-1} |Du|^2$,

(d) $\partial_j L_{ij}{}^k = \dfrac{n-k}{n} \partial_i \sigma_k + n L_{ij}{}^k u_j u^{-1}$, *where* ∇_j^g *is the* j*-th covariant derivative with respect to the metric* $g = u^{-2}\,dx^2$ *and* $\partial_i = \partial/\partial x_i$ *is the usual derivative.*

3. Some decomposition results

Let $u > 0$ be in Γ^k. In the rest of the paper, we write $\sigma_s(A^g)$ simply as σ_s.

Let η be a smooth cut-off function supported in the ball B_{4r} satisfying

$$|D^m \eta| \lesssim \frac{1}{r^m}.$$

We use $\lesssim$, $\backsimeq$, etc. to drop some positive constants independent of r and u, and D^m means the usual m-th order multiple derivative.

Let δ, θ be constants which will be chosen later. For $s = 1, \dots, k$, set

$$b_s = -\frac{(n+\delta)k + (2k+\delta)s}{s!2^s}(n+\delta+1)\cdots(n+\delta+s-1)$$

and

$$B_s = \int \sigma_{k-s}|Du|^{2s}u^\delta \eta^\theta\,dx,$$

$$M_s = \int {T_{ij}}^{k-s} u_i u_j |Du|^{2(s-1)} u^\delta \eta^\theta\,dx,$$

$$E_s = \int {T_{ij}}^{k-s} u_i \eta_j |Du|^{2(s-1)} u^{\delta+1} \eta^{\theta-1}\,dx.$$

Throughout the paper, for convenience, we drop the domain in integrations; one can assume that all integrations are over a suitable domain such as supp η without confusion.

For computational convenience, we give the following recursion formula.

Lemma 3.1. *For* $s = 1, \dots, k-1$,

$$m_s M_s = m_{s+1} M_{s+1} + \frac{k+s}{2s} m_s B_s - \frac{n-k+s+1}{2(n+\delta+s+1)} m_{s+1} B_{s+1} + c_{s+1} E_{s+1}, \tag{3-1}$$

where

$$m_i = \frac{2i(n+\delta+i)}{(n+\delta)k + (2k+\delta)i} b_i$$

and

$$c_i = \theta \frac{m_i}{n+\delta+i}$$

for $i = 1, \dots, k$.

Proof. Using the above notation, by (2-2), Proposition 2.2(c), and integration by parts, we get

$$
\begin{aligned}
\text{(3-2)}\quad & m_s M_s \\
&= m_s \int T_{ij}{}^{k-s} u_i u_j |Du|^{2(s-1)} u^\delta \eta^\theta \, dx \\
&= m_s \int (\sigma_{k-s}\delta_{ij} - T_{il}{}^{k-s-1}(u u_{lj} - \tfrac{1}{2}|Du|^2\delta_{lj})) u_i u_j |Du|^{2(s-1)} u^\delta \eta^\theta \, dx \\
&= m_s B_s + \frac{m_s}{2} M_{s+1} - \frac{m_s}{2s} \int u_i T_{il}{}^{k-s-1} \partial_l (|Du|^{2s}) u^{\delta+1} \eta^\theta \, dx \\
&= m_s B_s + \frac{m_s}{2} M_{s+1} + \frac{m_s}{2s} \int \partial_l (u_i T_{il}{}^{k-s-1}) |Du|^{2s} u^{\delta+1} \eta^\theta \, dx \\
&\qquad + \frac{m_s}{2s}(\delta+1) M_{s+1} + \theta \frac{m_s}{2s} E_{s+1} \\
&= m_s B_s + \frac{m_s}{2} M_{s+1} + \frac{m_s}{2s} \int \Big[(k-s)\sigma_{k-s} + n T_{ij}{}^{k-s-1} u_i u_j \\
&\qquad - \frac{n-k+s+1}{2} \sigma_{k-s-1} |Du|^2 \Big] |Du|^{2s} u^\delta \eta^\theta \, dx \\
&\qquad + \frac{m_s}{2s}(\delta+1) M_{s+1} + \theta \frac{m_s}{2s} E_{s+1} \\
&= m_{s+1} M_{s+1} + \frac{k+s}{2s} m_s B_s - \frac{n-k+s+1}{2(n+\delta+s+1)} m_{s+1} B_{s+1} + c_{s+1} E_{s+1}. \quad \square
\end{aligned}
$$

Now we have the decomposition for the integral for σ_k.

Proposition 3.2.

$$
\int k\sigma_k u^\delta \eta^\theta \, dx = \sum_{s=1}^{k} b_s B_s + \sum_{s=1}^{k} c_s E_s. \tag{3-3}
$$

Proof. By Proposition 2.2(c) and integration by parts we get

$$
\begin{aligned}
\text{(3-4)}\quad & \int k\sigma_k u^\delta \eta^\theta \, dx \\
&= \int \Big[u \partial_j (u_i T_{ij}{}^{k-1}) - n T_{ij}{}^{k-1} u_i u_j + \frac{n-k+1}{2} \sigma_{k-1} |Du|^2 \Big] u^\delta \eta^\theta \, dx \\
&= \frac{n-k+1}{2} \int \sigma_{k-1} |Du|^2 u^\delta \eta^\theta \, dx - n \int T_{ij}{}^{k-1} u_i u_j u^\delta \eta^\theta \, dx \\
&\qquad - \int T_{ij}{}^{k-1} u_i \partial_j (u^{\delta+1} \eta^\theta) \, dx \\
&= \frac{n-k+1}{2} \int \sigma_{k-1} |Du|^2 u^\delta \eta^\theta \, dx - \theta \int T_{ij}{}^{k-1} u_i \eta_j u^{\delta+1} \eta^{\theta-1} \, dx \\
&\qquad - (n+\delta+1) \int T_{ij}{}^{k-1} u_i u_j u^\delta \eta^\theta \, dx \\
&= \frac{n-k+1}{2} B_1 + C_1 E_1 + m_1 M_1.
\end{aligned}
$$

Using the recursion formula (3-1) in (3-4) step by step, we deduce (3-3). $\square$

For the traceless Newton tensor L^k, we also have the following decomposition.

Proposition 3.3.

$$\int L_{ij}{}^k L_{ij}{}^1 u^\delta \eta^\theta\, dx \tag{3-5}$$

$$\begin{aligned}
&= -\frac{n-k}{n}\int \partial_i(\sigma_k) u_i u^{\delta+1}\eta^\theta\, dx - (n+1+\delta)\int L_{ij}{}^k u_i u_j u^\delta \eta^\theta\, dx \\
&\qquad + \frac{n-k}{n(n+2+\delta)}\int \partial_i(\sigma_k)\partial_i(\eta^\theta) u^{\delta+2}\, dx - \frac{k}{n(n+2+\delta)}\int \sigma_k \Delta(\eta^\theta) u^{\delta+2}\, dx \\
&\quad - \frac{1}{2(n+2+\delta)}\int T_{ij}{}^{k-1}\partial_{ij}(\eta^\theta)|Du|^2 u^{\delta+2}\, dx + \frac{n-k+1}{n+2+\delta}\int \sigma_{k-1} u_i u_j \partial_{ij}(\eta^\theta) u^{\delta+2}\, dx \\
&\quad - \frac{n+3+\delta}{n+2+\delta}\int T_{il}{}^{k-1} u_l u_j \partial_{ij}(\eta^\theta) u^{\delta+2}\, dx - \frac{1}{n+2+\delta}\int T_{il}{}^{k-1} u_j \partial_{ijl}(\eta^\theta) u^{\delta+3}\, dx.
\end{aligned}$$

Proof. By Proposition 2.2(d) and integration by parts we get

$$\int L_{ij}{}^k L_{ij}{}^1 u^\delta \eta^\theta\, dx \tag{3-6}$$

$$\begin{aligned}
&= \int L_{ij}{}^k u_{ij} u^{\delta+1}\eta^\theta\, dx \\
&= -\int \partial_j(L_{ij}{}^k) u_i u^{\delta+1}\eta^\theta\, dx - (\delta+1)\int L_{ij}{}^k u_i u_j u^\delta \eta^\theta\, dx - \int L_{ij}{}^k u_i \partial_j(\eta^\theta) u^{\delta+1}\, dx \\
&= -\int \Big[\frac{n-k}{n}\partial_i(\sigma_k) + n L_{ij}{}^k u_j u^{-1}\Big] u_i u^{\delta+1}\eta^\theta\, dx \\
&\qquad - (\delta+1)\int L_{ij}{}^k u_i u_j u^\delta \eta^\theta\, dx - \int L_{ij}{}^k u_i \partial_j(\eta^\theta) u^{\delta+1}\, dx \\
&= -\frac{n-k}{n}\int \partial_i(\sigma_k) u_i u^{\delta+1}\eta^\theta\, dx - (n+\delta+1)\int L_{ij}{}^k u_i u_j u^\delta \eta^\theta\, dx \\
&\qquad - \int L_{ij}{}^k u_i \partial_j(\eta^\theta) u^{\delta+1}\, dx.
\end{aligned}$$

For the last term in (3-6), integrating once again, we have

$$-\int L_{ij}{}^k u_i \partial_j(\eta^\theta) u^{\delta+1}\, dx \tag{3-7}$$

$$\begin{aligned}
&= \int \partial_i(L_{ij}{}^k)\partial_j(\eta^\theta) u^{\delta+2}\, dx + \int L_{ij}{}^k \partial_{ij}(\eta^\theta) u^{\delta+2}\, dx + (\delta+1)\int L_{ij}{}^k \partial_j(\eta^\theta) u_i u^{\delta+1}\, dx \\
&= \int \Big[\frac{n-k}{n}\partial_i(\sigma_k) + n L_{ij}{}^k u_j u^{-1}\Big]\partial_i(\eta^\theta) u^{\delta+2}\, dx \\
&\qquad + \int L_{ij}{}^k \partial_{ij}(\eta^\theta) u^{\delta+2}\, dx + (\delta+1)\int L_{ij}{}^k \partial_j(\eta^\theta) u_i u^{\delta+1}\, dx \\
&= \frac{n-k}{n}\int \partial_i(\sigma_k)\partial_i(\eta^\theta) u^{\delta+2}\, dx + \int L_{ij}{}^k \partial_{ij}(\eta^\theta) u^{\delta+2}\, dx \\
&\qquad + (n+\delta+1)\int L_{ij}{}^k \partial_j(\eta^\theta) u_i u^{\delta+1}\, dx.
\end{aligned}$$

Transposition of the term implies

$$-\int L_{ij}{}^k u_i \partial_j(\eta^\theta) u^{\delta+1}\,dx = \frac{n-k}{n(n+2+\delta)} \int \partial_i(\sigma_k)\partial_i(\eta^\theta) u^{\delta+2}\,dx + \frac{1}{n+2+\delta} \int L_{ij}{}^k \partial_{ij}(\eta^\theta) u^{\delta+2}\,dx. \tag{3-8}$$

For the last term in (3-8), we have

$$\begin{aligned}\int L_{ij}{}^k \partial_{ij}(\eta^\theta) u^{\delta+2}\,dx &= \int \Big(T_{il}{}^{k-1} A_{lj} - \frac{k}{n}\sigma_k \delta_{ij}\Big)\partial_{ij}(\eta^\theta) u^{\delta+2}\,dx \\ &= \int T_{il}{}^{k-1}(u u_{lj} - \tfrac{1}{2}|Du|^2\delta_{lj})\partial_{ij}(\eta^\theta)u^{\delta+2}\,dx - \frac{k}{n}\int \sigma_k \Delta(\eta^\theta) u^{\delta+2}\,dx \\ &= -\frac{k}{n}\int \sigma_k \Delta(\eta^\theta)u^{\delta+2}\,dx - \tfrac{1}{2}\int T_{ij}{}^{k-1}\partial_{ij}(\eta^\theta)|Du|^2 u^{\delta+2}\,dx \\ &\qquad + \int T_{il}{}^{k-1} u_{lj}\partial_{ij}(\eta^\theta) u^{\delta+3}\,dx.\end{aligned} \tag{3-9}$$

For the last term in (3-9), by Proposition 2.2(b), we compute

$$\begin{aligned}\int T_{il}{}^{k-1} u_{lj}\partial_{ij}(\eta^\theta)u^{\delta+3}\,dx &= -\int \partial_l(T_{il}{}^{k-1}) u_j \partial_{ij}(\eta^\theta) u^{\delta+3}\,dx - \int T_{il}{}^{k-1} u_j \partial_{ijl}(\eta^\theta) u^{\delta+3}\,dx \\ &\qquad - (\delta+3)\int T_{il}{}^{k-1} u_j u_l \partial_{ij}(\eta^\theta) u^{\delta+2}\,dx \\ &= -\int [-(n-k+1)\sigma_{k-1}u_i u^{-1} + n T_{il}{}^{k-1} u_l u^{-1}] u_j \partial_{ij}(\eta^\theta) u^{\delta+3}\,dx \\ &\qquad - \int T_{il}{}^{k-1} u_j \partial_{ijl}(\eta^\theta)u^{\delta+3}\,dx - (\delta+3)\int T_{il}{}^{k-1} u_j u_l \partial_{ij}(\eta^\theta)u^{\delta+2}\,dx \\ &= (n-k+1)\int \sigma_{k-1} u_i u_j \partial_{ij}(\eta^\theta) u^{\delta+2}\,dx - \int T_{il}{}^{k-1} u_j \partial_{ijl}(\eta^\theta) u^{\delta+3}\,dx \\ &\qquad - (n+\delta+3)\int T_{il}{}^{k-1} u_j u_l \partial_{ij}(\eta^\theta) u^{\delta+2}\,dx.\end{aligned} \tag{3-10}$$

Inserting this into (3-9), we get

$$\begin{aligned}\int L_{ij}{}^k \partial_{ij}(\eta^\theta) u^{\delta+2}\,dx &= -\frac{k}{n}\int \sigma_k \Delta(\eta^\theta) u^{\delta+2}\,dx - \tfrac{1}{2}\int T_{ij}{}^{k-1}\partial_{ij}(\eta^\theta)|Du|^2 u^{\delta+2}\,dx \\ &\qquad + (n-k+1)\int \sigma_{k-1} u_i u_j \partial_{ij}(\eta^\theta) u^{\delta+2}\,dx \\ &\qquad - (n+3+\delta)\int T_{il}{}^{k-1} u_j u_l \partial_{ij}(\eta^\theta) u^{\delta+2}\,dx - \int T_{il}{}^{k-1} u_j \partial_{ijl}(\eta^\theta) u^{\delta+3}\,dx.\end{aligned} \tag{3-11}$$

Substituting this into (3-8) and then (3-6), we get (3-5) as desired. □

To end this section, we give the estimate on the "error" terms "E_s" in (3-3).

Lemma 3.4.

$$|E_s| \lesssim \varepsilon \sum_{m=s}^{k} B_m + \frac{1}{r^{2k}} \int u^{\delta+2k} \eta^{\theta-2k}\, dx. \tag{3-12}$$

Proof. First, by $|D\eta| \lesssim 1/r$ and Proposition 2.1(c), we have

$$|E_s| \lesssim \frac{1}{r} \int \sigma_{k-s} |Du|^{2s-1} u^{\delta+1} \eta^{\theta-1}\, dx.$$

Using Young's inequality with exponent pair $(2s/(2s-1), 2s)$ and $\varepsilon > 0$ small, the last inequality turns into

$$|E_s| \lesssim \varepsilon \int \sigma_{k-s} |Du|^{2s} u^{\delta} \eta^{\theta}\, dx + \frac{C(\varepsilon)}{r^{2s}} \int \sigma_{k-s} u^{\delta+2s} \eta^{\theta-2s}\, dx. \tag{3-13}$$

For the last term of (3-13), by Proposition 2.2(c), we deduce

$$\begin{aligned} &\frac{C(\varepsilon)}{r^{2s}} \int \sigma_{k-s} u^{\delta+2s} \eta^{\theta-2s}\, dx \\ &\backsimeq \frac{1}{r^{2s}} \int \Big[u \partial_j (u_i {T_{ij}}^{k-s-1}) - n {T_{ij}}^{k-s-1} u_i u_j \\ &\qquad\qquad + \frac{n-k+s+1}{2} \sigma_{k-s-1} |Du|^2 \Big] u^{\delta+2s} \eta^{\theta-2s}\, dx \\ &\backsimeq \frac{1}{r^{2s}} \int \sigma_{k-s-1} |Du|^2 u^{\delta+2s} \eta^{\theta-2s}\, dx - \frac{1}{r^{2s}} \int {T_{ij}}^{k-s-1} u_i u_j u^{\delta+2s} \eta^{\theta-2s}\, dx \\ &\qquad\qquad - \frac{1}{r^{2s}} \int {T_{ij}}^{k-s-1} u_i \eta_j u^{\delta+2s+1} \eta^{\theta-2s-1}\, dx \\ &\lesssim \frac{1}{r^{2s}} \int \sigma_{k-s-1} |Du|^2 u^{\delta+2s} \eta^{\theta-2s}\, dx + \frac{1}{r^{2s+1}} \int \sigma_{k-s-1} |Du| u^{\delta+2s+1} \eta^{\theta-2s-1}\, dx \\ &\lesssim \varepsilon \int \sigma_{k-s-1} |Du|^{2(s+1)} u^{\delta} \eta^{\theta}\, dx + \frac{C(\varepsilon)}{r^{2(s+1)}} \int \sigma_{k-s-1} u^{\delta+2(s+1)} \eta^{\theta-2(s+1)}\, dx, \end{aligned} \tag{3-14}$$

where we have used Young's inequality in the last step in (3-13).

Substituting (3-14) into (3-13) step by step shows (3-12). □

4. Proofs of Proposition 1.3 and Theorem 1.4

For $n = 2k+1$, if we choose $\delta = -2k = 1-n$, (3-12) implies

$$|E_s| \lesssim \varepsilon \sum_{m=s}^{k} B_m + r. \tag{4-1}$$

Moreover, by this choice of δ we see that $b_s < 0 (s = 1, 2, \dots, k)$. Hence if we take ε small enough, combining (3-3) with (4-1), we have

$$\int \sigma_k u^{1-n} \eta^\theta \, dx + \sum_{s=1}^{k} B_s \lesssim r. \tag{4-2}$$

On the other hand, if we choose $\delta = -n - 1$ in (3-5) , then

$$\begin{aligned} &\int L_{ij}{}^k L_{ij}{}^1 u^{-n-1} \eta^\theta \, dx \\ &= -\frac{n-k}{n} \int \partial_i(\sigma_k) u_i u^{-n} \eta^\theta \, dx + \frac{n-k}{n} \int \partial_i(\sigma_k) \partial_i(\eta^\theta) u^{1-n} \, dx - \frac{k}{n} \int \sigma_k \Delta(\eta^\theta) u^{1-n} \, dx \\ &\quad - \tfrac{1}{2} \int T_{ij}{}^{k-1} \partial_{ij}(\eta^\theta) |Du|^2 u^{1-n} \, dx + (n-k+1) \int \sigma^{k-1} u_i u_j \partial_{ij}(\eta^\theta) u^{1-n} \, dx \\ &\quad -2 \int T_{il}{}^{k-1} u_l u_j \partial_{ij}(\eta^\theta) u^{1-n} \, dx - \int T_{il}{}^{k-1} u_j \partial_{ijl}(\eta^\theta) u^{2-n} \, dx. \end{aligned} \tag{4-3}$$

By (1-1) and $|D^m \eta| \lesssim 1/r^m$ we deduce

$$\begin{aligned} &\int L_{ij}{}^k L_{ij}{}^1 u^{-n-1} \eta^\theta \, dx \\ &\quad \lesssim -\frac{n-k}{n} \alpha \int |Du|^2 u^{\alpha-n-1} \eta^\theta \, dx + \frac{n-k}{n} \alpha\theta \int u_i \eta_i u^{\alpha-n} \eta^{\theta-1} \, dx \\ &\qquad + \frac{1}{r^2} \int u^{\alpha+1-n} \eta^{\theta-2} \, dx + \frac{1}{r^2} \int \sigma_{k-1} |Du|^2 u^{1-n} \eta^{\theta-2} \, dx \\ &\qquad + \frac{1}{r^3} \int \sigma_{k-1} |Du| u^{2-n} \eta^{\theta-3} \, dx. \end{aligned} \tag{4-4}$$

Using Young's inequality, by (4-4), we can get

$$\begin{aligned} &\int L_{ij}{}^k L_{ij}{}^1 u^{-n-1} \eta^\theta \, dx \\ &\quad \lesssim \Big(\varepsilon - \frac{n-k}{n}\Big) \alpha \int |Du|^2 u^{\alpha-n-1} \eta^\theta \, dx + \frac{1}{r^2} \int u^{\alpha+1-n} \eta^{\theta-2} \, dx \\ &\qquad + \frac{1}{r^2} \int \sigma_{k-1} |Du|^2 u^{1-n} \eta^{\theta-2} \, dx + \frac{1}{r^4} \int \sigma_{k-1} u^{3-n} \eta^{\theta-4} \, dx. \end{aligned} \tag{4-5}$$

For the last term of (4-5), using (3-14) (with $\delta = 1 - n$) step by step, we have

$$\begin{aligned} \frac{1}{r^4} \int \sigma_{k-1} u^{3-n} \eta^{\theta-4} \, dx &\lesssim \frac{1}{r^2} \Big[\sum_{s=2}^{k} B_s + \frac{1}{r^{2k}} \int u^{1-n+2k} \eta^{\theta-2-2k} \, dx \Big] \\ &\lesssim \frac{1}{r^2} \sum_{s=2}^{k} B_s + \frac{1}{r}. \end{aligned} \tag{4-6}$$

Taking ε small, inserting (4-6) into (4-5), and combining with (4-2) (replacing θ with $\theta-2$), we get

$$\int L_{ij}{}^{k} L_{ij}{}^{1} u^{-n-1}\eta^{\theta}\,dx+\alpha\int |Du|^2 u^{\alpha-n-1}\eta^{\theta}\,dx \lesssim \frac{1}{r^2}\Big[\int u^{\alpha+1-n}\eta^{\theta-2}\,dx+\sum_{s=1}^{k} B_s\Big]+\frac{1}{r} \lesssim \frac{1}{r}. \tag{4-7}$$

Now, from (4-7), we can prove Theorem 1.4 and Proposition 1.3.

Proof of Theorem 1.4. Let $\eta\equiv 1$ in B_r, $0<\eta<1$ in $B_{2r}\setminus B_r$. Taking $r\to+\infty$ in (4-7), we can get

$$\int_{\mathbb{R}^n} L_{ij}{}^{k} L_{ij}{}^{1} u^{-n-1}\,dx+\alpha\int_{\mathbb{R}^n} |Du|^2 u^{\alpha-n-1}\,dx\le 0. \tag{4-8}$$

By Proposition 2.1(e), if $\alpha>0$, (4-8) shows u must be a positive constant solution of (1-6), which is impossible; if $\alpha=0$, (4-8) shows $L^1=0$ and hence u must be the quadratic polynomial as in (1-7). □

Proof of Proposition 1.3. Let $\eta\equiv 1$ for $r\le |x|\le Mr$ and $\eta=0$ for $0<|x|<r/2$, $2Mr<|x|$. By (1-3) and Proposition 2.1(d) we have

$$\int u^{\alpha/k+\alpha-n-1}\eta^{\theta}\,dx=\int (\sigma_k)^{1/k}u^{\alpha-n-1}\eta^{\theta}\,dx\lesssim\int \sigma_1 u^{\alpha-n-1}\eta^{\theta}\,dx = -\frac{n}{2}\int |Du|^2u^{\alpha-n-1}\eta^{\theta}\,dx+\int \Delta u u^{\alpha-n}\eta^{\theta}\,dx. \tag{4-9}$$

For the last term in (4-9), integrating by parts and using Young's inequality, we deduce

$$\int\Delta u u^{\alpha-n}\eta^{\theta}\,dx=(n-\alpha)\int|Du|^2u^{\alpha-n-1}\eta^{\theta}\,dx-\theta\int u_i\eta_i u^{\alpha-n}\eta^{\theta-1}dx \lesssim (n-\alpha+\varepsilon)\int|Du|^2u^{\alpha-n-1}\eta^{\theta}\,dx+\frac{1}{r^2}\int u^{\alpha-n+1}\eta^{\theta-2}dx. \tag{4-10}$$

Inserting this into (4-9) and combining with (4-7)and (4-2), we have

$$\int u^{((k+1)/k)\alpha-n-1}\eta^{\theta}\,dx \lesssim \Big(\frac{n}{2}-\alpha+\varepsilon\Big)\int |Du|^2u^{\alpha-n-1}\eta^{\theta}\,dx+\frac{1}{r^2}\int u^{\alpha-n+1}\eta^{\theta-2}\,dx\lesssim\frac{1}{r}. \tag{4-11}$$

This implies (1-5) and hence the proof of Proposition 1.3 is completed. □

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