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**RATE OF ATTRACTION FOR A SEMILINEAR WAVE
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AND CRITICAL NONLINEARITIES**

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We study the rate of convergence of global attractors and eigenvalues of the family of dissipative semilinear wave equations with variable coefficients $u_{tt} + \Lambda_\epsilon u + \Lambda_\epsilon^\delta u_t = f(u)$, where Λ_ϵ is the elliptic operator $-\operatorname{div}(a_\epsilon(x)\nabla)$ with $\epsilon \in [0, 1]$ and sufficiently smooth coefficients a_ϵ , and where $\delta \in (\frac{1}{2}, 1)$ and the nonlinearity f is a continuously differentiable function satisfying suitable growth conditions. We show that the rate of convergence, as $\epsilon \rightarrow 0^+$, of the global attractors of these problems, as well as of their eigenvalues, is proportional to the distance of the coefficients $\|a_\epsilon - a_0\|_{L^\infty(\Omega)}$.

1. Introduction and main result

In many theoretical and applied problems, it is important to understand what happens when the solutions varies parameters in the model, and wave equations with variable coefficients arise naturally in mathematical modeling of inhomogeneous media (for example, functionally graded materials or materials with damage induced inhomogeneity) in solid mechanics, electromagnetism, fluid flows through porous media (for example, modeling traveling waves in a inhomogeneous gas; see [Egorov and Shubin 1988; Suggs 2009]), and other areas of physics and engineering.

Nonlinear wave equations arise in quantum mechanics, whereas variants of the form

$$u_{tt} - \operatorname{div}(a\nabla u) + g(u, u_t) = 0$$

appear in the study of vibrating systems with or without damping, and with or without forcing terms.

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In this work, $\epsilon \in [0, 1]$, and we consider the following problem associated with a semilinear dissipative wave equation with variable coefficients:

$$(1-1) \quad \begin{cases} u_{tt} + \Lambda_\epsilon u + \Lambda_\epsilon^\delta u_t = f(u), & t > 0, x \in \Omega, \\ u(0, x) = u_0(x), u_t(0, x) = v_0(x), & x \in \Omega, \\ u(t, x) = 0, & t \geq 0, x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with boundary $\partial\Omega$ sufficiently regular, $\Lambda_\epsilon = -\operatorname{div}(a_\epsilon(x)\nabla)$, and a_ϵ is a real function defined in Ω satisfying

$$(1-2) \quad 0 < m_0 \leq a_\epsilon(x) \leq M_0 \quad \text{for all } x \in \Omega.$$

Moreover, the functions $a_\epsilon \in L^\infty(\Omega)$ converge uniformly to $a_0 \in L^\infty(\Omega)$, as $\epsilon \rightarrow 0^+$. Also, we will assume that a_ϵ is smooth for all $\epsilon \in [0, 1]$. For the system (1-1), let us consider $\delta \in (\frac{1}{2}, 1)$.

The operators $\Lambda_\epsilon^\delta := (\Lambda_\epsilon^{-\delta})^{-1}$ denote the fractional power operators associated with Λ_ϵ . Provided that Λ_ϵ with domain $D(\Lambda_\epsilon) = H^2(\Omega) \cap H_0^1(\Omega)$ is a sectorial operator with $\operatorname{Re} \sigma(\Lambda_\epsilon) > 0$, for any $\alpha \in (0, 1)$, it follows by Theorem 1.4.2 in [Henry 1981] that

$$(1-3) \quad \Lambda_\epsilon^{-\alpha} = \frac{\sin \pi \alpha}{\pi} \int_0^\infty \lambda^{-\alpha} (\lambda I + \Lambda_\epsilon)^{-1} d\lambda.$$

On the nonlinearity $f : \mathbb{R} \rightarrow \mathbb{R}$, which is continuously differentiable and bounded, we will give conditions under which the problem (1-7) is globally well posed in $H_0^1(\Omega) \times L^2(\Omega)$ and it has global attractors, in the terminology of [Hale 1988] (following closely Theorem 1.1 and Theorem 1.2 in [Carvalho and Cholewa 2002a]; see also [Carvalho and Cholewa 2002b]): if $\rho \leq (N+2)/(N-2)$, there exists a constant $C > 0$, independent of ϵ , such that

$$(1-4) \quad |f(s_1) - f(s_2)| \leq C|s_1 - s_2|(|s_1|^{\rho-1} + |s_2|^{\rho-1} + 1)$$

and

$$(1-5) \quad \limsup_{|s| \rightarrow +\infty} \frac{f(s)}{s} \leq \mu_{0,1},$$

with $\mu_{0,1}$ being the first eigenvalue of the Λ_0 in Ω .

In the rest of this paper, we will use C to denote a generic positive constant which may change from line to line (unless otherwise stated).

Since the wave equation does not have dissipative character, we have added a ‘‘damping’’ characterized by the term $\Lambda_\epsilon^\delta u_t^\epsilon$ with $\delta \in (\frac{1}{2}, 1)$. This additional term turns problem (1-1) into a sectorial structure (see [Chen and Triggiani 1989]), however, this gives us an extra difficulty, because it is necessary to perform an analysis of the rate of convergence of fractional derivatives. Although the sectorial

structure for (1-1) is preserved when the dissipative term presents the optimal power $\delta = \frac{1}{2}$ (see [Chen and Triggiani 1989]), the convergence (with rate) of attractors is an open problem for this case.

Related to this issue, in [Arrieta et al. 2013] the authors proved that the difference $\|a_\epsilon - a_0\|_{L^\infty(\Omega)}$ can be used to show the rate of convergence of attractors in the context of the heat equation. Nonlinear absorption problems with variable coefficients have been considered by many authors; see [Wu and Li 2011; Suggs 2009] and the references therein. For damped wave equations, several authors have studied existence of global attractors; see [Babin and Vishik 1989; Bruschi et al. 2006; Carvalho and Cholewa 2002a; 2002b; Cholewa and Dlotko 2006; Hale 1988; Webb 1980] and the references therein. We can still cite [Bruschi et al. 2006], where the convergence of attractors was shown, but without explicit rate.

In this work, we will investigate the relationship between the convergence of functions $a_\epsilon \in L^\infty(\Omega)$, which converge uniformly to $a_0 \in L^\infty(\Omega)$, as $\epsilon \rightarrow 0^+$, and the proximity between the perturbed and limit attractors, as well as the convergence of the eigenvalues of the operators associated with the problems in (1-1). The difference $\|a_\epsilon - a_0\|_{L^\infty(\Omega)}$ will be our measure.

To better explain the results in the paper, we introduce some terminology. Let us consider the Hilbert spaces $Y = Y^0 := L^2(\Omega)$, $Y^{1/2} := H_0^1(\Omega)$, $Y^1 := D(\Lambda_\epsilon) := \{u \in H_0^1(\Omega) : \Lambda_\epsilon u \in L^2(\Omega)\}$ and the Hilbert energy space $X = X^0 = Y^{1/2} \times Y$ equipped with the inner product

$$\left\langle \begin{bmatrix} \phi \\ \varphi \end{bmatrix}, \begin{bmatrix} \bar{\phi} \\ \bar{\varphi} \end{bmatrix} \right\rangle_X := \int_\Omega a_\epsilon(x) \nabla \phi \nabla \bar{\phi} \, dx + \int_\Omega \varphi \bar{\varphi} \, dx.$$

We define the operator $A_\epsilon : D(A_\epsilon) \subset X \rightarrow X$ by

$$A_\epsilon \begin{bmatrix} \phi \\ \varphi \end{bmatrix} = \begin{bmatrix} 0 & -I \\ \Lambda_\epsilon & \Lambda_\epsilon^\delta \end{bmatrix} \begin{bmatrix} \phi \\ \varphi \end{bmatrix} := \begin{bmatrix} -\varphi \\ \Lambda_\epsilon^\delta (\Lambda_\epsilon^{1-\delta} \phi + \varphi) \end{bmatrix}$$

and

$$D(A_\epsilon) = \left\{ \begin{bmatrix} \phi \\ \varphi \end{bmatrix} \in Y^{(3/2)-\delta} \times Y^{1/2}; \Lambda_\epsilon^{1-\delta} \phi + \varphi \in Y^\delta \right\} =: X^1,$$

with Y^δ denoting the domain of the fractional power operators associated with Λ_ϵ , that is, $Y^\delta := D(\Lambda_\epsilon^\delta)$. Let us consider Y^δ endowed with the graph norm $\|x\|_{Y^\delta} = \|\Lambda_\epsilon^\delta x\|_Y$. Notice that

$$(1-6) \quad A_\epsilon \begin{bmatrix} \phi \\ \varphi \end{bmatrix} = \begin{bmatrix} -\varphi \\ \Lambda_\epsilon \phi + \Lambda_\epsilon^\delta \varphi \end{bmatrix}, \quad \begin{bmatrix} \phi \\ \varphi \end{bmatrix} \in Y^1 \times Y^\delta,$$

where $Y^1 \times Y^\delta$ is a dense subset of $D(A_\epsilon)$.

Notice that the operator A_ϵ with domain $Y^1 \times Y^\delta$ is not a closed operator, unless $\delta = \frac{1}{2}$; see [Chen and Triggiani 1989].

Problem (1-1) can be written as

$$(1-7) \quad \begin{cases} w_t + A_\epsilon w = F(w), & t > 0, \\ w(0) = w_0 \in X \end{cases}$$

with $w = [u \ u_t]^T$ and the nonlinear map $F : X \rightarrow Y \times Y$ defined by

$$F \begin{bmatrix} \phi \\ \varphi \end{bmatrix} := \begin{bmatrix} 0 \\ f^\epsilon(\phi) \end{bmatrix},$$

where $f^\epsilon : H_0^1(\Omega) \rightarrow H^{-1/2}(\Omega)$ is the Nemytskiĭ operator associated with f .

We will show that these equations define on the space X a nonlinear semigroup $\{T_\epsilon(t) : t \geq 0\}$ having global attractors \mathcal{A}_ϵ , $\epsilon \in [0, 1]$, and that the rate of convergence of the attractors in the sense of the symmetric Hausdorff distance is given by the order of $\|a_\epsilon - a_0\|_{L^\infty(\Omega)}^\theta$ with $\theta \in (0, \frac{1}{2})$.

It is worth noting that the dependence of regular attractors on parameters is a very well-studied and well-understood topic nowadays, especially for the case when the perturbation is also regular (like in our case). Basically, all the necessary technique to handle such perturbations can be found already in the monograph of Babin and Vishik [1989]. However, the problem considered has some interesting peculiarities in a sense unusual for the attractor theory, namely, the presence of the fractional powers of the elliptic operator Λ_ϵ as well as the necessity to control the dependence of these powers on the parameter ϵ .

The main purpose of this paper is to give a proof of the following result.

Theorem 1.1. *Let $\{T_\epsilon(t) : t \geq 0\}$ be the gradient nonlinear semigroup associated with (1-7) and let \mathcal{A}_ϵ in X be its global attractor, $\epsilon \in [0, 1]$. Then there are constants $C > 0$ and $\wp \in (0, \frac{1}{2})$, independent of ϵ , such that*

$$\text{dist}(\mathcal{A}_\epsilon, \mathcal{A}_0) + \text{dist}(\mathcal{A}_0, \mathcal{A}_\epsilon) \leq C \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^\wp,$$

where

$$\text{dist}(A, B) := \sup_{x \in A} \inf_{y \in B} \|x - y\|_X, \quad A, B \subset X$$

is the Hausdorff semidistance between A and B in X .

We observe that the Hausdorff semidistance between A and B , $\text{dist}(A, B)$, examines how the set A is contained in the set B . For example, if $\text{dist}(A, B) = 0$ then A is contained in the closure of the set B .

The rest of this paper is organized as follows. In Section 2 we show that the linear semigroups of contractions associated to the problems (1-7) are analytic and compact, and that their nonlinear semigroups have global attractors \mathcal{A}_ϵ in X . In Section 3 we see that the distance between the semigroups are proportional to a power of the distance between coefficients $a_\epsilon \in L^\infty(\Omega)$. We study the convergence of the operators A_ϵ^{-1} to A_0^{-1} . We also make a spectral analysis and we prove that the

convergence of the eigenvalues of the operators associated to (1-7) is proportional to measure $\|a_\epsilon - a_0\|_{L^\infty(\Omega)}$. In Section 4 we analyze the convergence of equilibria. In Section 5 we study some important properties of the Nemytskiĭ operators F . We also study the convergence of the operators $A_\epsilon - F'(w_\epsilon)$, as w_ϵ converges to w_0 in X . In Section 6 we analyze the rate of convergence of equilibria. In Section 7 we study the rate of convergence and attraction of local unstable manifolds of an equilibrium. Finally, in Section 8 we prove the main result of this paper.

2. Functional setting and background results

Our main goal in this section is to prove the well-posedness of problem (1-7) in X and to ensure that the nonlinear semigroup generated by (1-7) has global attractor with uniform bounds in X . Our approach is inspired by a similar idea from [Carvalho and Cholewa 2002a].

Under the assumption above, it is well known that the operator Λ_ϵ is a positive, self-adjoint operator with domain $D(\Lambda_\epsilon) = Y^1$. Let us denote by $\{e^{-\Lambda_\epsilon t} : t \geq 0\}$ the analytic linear semigroup generated by $-\Lambda_\epsilon$ on Y , for all $\epsilon \in [0, 1]$.

According to [Henry 1981], we still have

$$(2-1) \quad \|(\lambda I + \Lambda_\epsilon)^{-1}\|_{\mathcal{L}(Y)} \leq C \max\{1, |\lambda|^{-1}\}$$

for some $C > 0$ independent of ϵ .

Since Λ_ϵ is a sectorial operator with $\|e^{-\Lambda_\epsilon t}\|_{\mathcal{L}(Y)} \leq C$, C independent of ϵ , as a consequence of the moment inequality (see Theorem 1.4.4 in [Henry 1981]), there exists a constant $C > 0$ such that

$$(2-2) \quad \|\Lambda_\epsilon^\alpha x\|_Y \leq C \|\Lambda_\epsilon x\|_Y^\alpha \|x\|_Y^{1-\alpha}, \quad x \in Y^1,$$

with $0 \leq \alpha \leq 1$. The constant C can be chosen uniform with respect to ϵ and α .

In this way, since all operators are selfadjoint, we have that $\sigma(\Lambda_\epsilon) \subset (-\infty, \alpha]$ for some $\alpha < 0$ and, in particular, the set $\Sigma_\phi = \{\lambda \in \mathbb{C} : |\arg \lambda| \leq \phi\}$, $\phi \in (\pi/2, \pi)$, is contained in the resolvent sets of Λ_ϵ , for all $\epsilon \in [0, 1]$. Consequently,

$$(2-3) \quad \|\Lambda_\epsilon(\lambda I + \Lambda_\epsilon)^{-1}\|_{\mathcal{L}(Y^{1/2})} \leq C, \quad \lambda \in \Sigma_\phi,$$

for some $C \geq 1$ independent of ϵ .

We will show that (1-7) is defined on the phase space X , an analytic semigroup.

Proposition 2.1. *Let $\epsilon \in [0, 1]$. The following conditions hold:*

- (i) *The operator A_ϵ is closed.*
- (ii) *A_ϵ is a maximal accretive operator, or equivalently, $-A_\epsilon$ is maximal dissipative.*
- (iii) *$0 \in \rho(A_\epsilon)$ and A_ϵ has compact resolvent for each $\epsilon \in [0, 1]$.*

- (iv) *The semigroup linearly generated by $-A_\epsilon$ on X , $\{e^{-A_\epsilon t} : t \geq 0\}$, is a C^0 semigroup of contractions on X .*
- (v) *A_ϵ is a sectorial operator in X with $\text{Re } \sigma(A_\epsilon) > 0$. The semigroup of contractions $\{e^{-A_\epsilon t} : t \geq 0\}$ is analytic and compact.*

Proof. Note that (i) is immediate from the closedness of Λ_ϵ and Λ_ϵ^δ . For (ii) notice that, given $[\phi \ \varphi]^T \in X^1$, we have

$$\begin{aligned}
 (2-4) \quad & \left\langle A_\epsilon \begin{bmatrix} \phi \\ \varphi \end{bmatrix}, \begin{bmatrix} \phi \\ \varphi \end{bmatrix} \right\rangle_X \\
 &= -\langle \varphi, \phi \rangle_{Y^{1/2}} + \langle \Lambda_\epsilon \phi + \Lambda_\epsilon^\delta \varphi, \varphi \rangle_Y \\
 &= -\langle \Lambda_\epsilon^{1/2} \varphi, \Lambda_\epsilon^{1/2} \phi \rangle_Y + \langle \Lambda_\epsilon^\delta (\Lambda_\epsilon^{1-\delta} \phi + \varphi), \varphi \rangle_Y \\
 &= -\langle \Lambda_\epsilon^{1/2} \varphi, \Lambda_\epsilon^{1/2} \phi \rangle_Y + \overline{\langle \Lambda_\epsilon^{1/2} \varphi, \Lambda_\epsilon^{1/2} \phi \rangle_Y} + \langle \Lambda_\epsilon^{\delta/2} \varphi, \Lambda_\epsilon^{\delta/2} \varphi \rangle_Y,
 \end{aligned}$$

and hence

$$\text{Re} \left\langle A_\epsilon \begin{bmatrix} \phi \\ \varphi \end{bmatrix}, \begin{bmatrix} \phi \\ \varphi \end{bmatrix} \right\rangle_X = \langle \Lambda_\epsilon^{\delta/2} \varphi, \Lambda_\epsilon^{\delta/2} \varphi \rangle_Y \geq 0, \quad \begin{bmatrix} \phi \\ \varphi \end{bmatrix} \in X^1,$$

which proves accretivity of A_ϵ .

Furthermore, for each $[\bar{\phi} \ \bar{\varphi}]^T \in X$, the linear equation

$$(2-5) \quad (I + A_\epsilon) \begin{bmatrix} \phi \\ \varphi \end{bmatrix} = \begin{bmatrix} \bar{\phi} \\ \bar{\varphi} \end{bmatrix}$$

is equivalent to the system

$$\begin{cases} \phi - \varphi = \bar{\phi}, \\ \Lambda_\epsilon \phi + \varphi + \Lambda_\epsilon^\delta \varphi = \bar{\varphi}, \end{cases}$$

or to the equation

$$(2-6) \quad \Lambda_\epsilon \phi + \Lambda_\epsilon^\delta \phi + \phi = \bar{\varphi} + \bar{\phi} + \Lambda_\epsilon^\delta \bar{\phi}.$$

By elliptic theory, it follows that there exists a unique function $\phi \in Y^{1/2}$ with $\Lambda_\epsilon \phi \in Y$ satisfying (2-6) and, therefore, for each $\epsilon \in [0, 1]$, there exists a unique $[\phi \ \varphi]^T \in X^1$ solving (2-5).

Concerning $0 \in \rho(A_\epsilon)$, we recall that there exists a bounded inverse operator $A_\epsilon^{-1} : X \rightarrow X$ given by

$$A_\epsilon^{-1} = \begin{bmatrix} \Lambda_\epsilon^{-(1-\delta)} & \Lambda_\epsilon^{-1} \\ -I & 0 \end{bmatrix}, \quad \epsilon \in [0, 1],$$

where $\Lambda_\epsilon^{-\alpha}$ are bounded inverse operators of Λ_ϵ^α . Thus, the resolvent operator A_ϵ^{-1} is compact, because it takes bounded subsets of X into bounded subsets of X^1 , which is compactly embedded in X . This shows (iii).

The property (iv) that $-A_\epsilon, \epsilon \in [0, 1]$, generates a C^0 semigroup of contractions on X follows from the Lummer–Phillips theorem (see [Pazy 1983]) and the observations concerning powers of maximal accretive operators (see [Kato 1976]).

Part (v) follows as a consequence of Theorem 1.1 in [Chen and Triggiani 1989]. Finally, compactness of $\{e^{-A_\epsilon t} : t \geq 0\}, \epsilon \in [0, 1]$, is then a consequence of compactness of the resolvent operators of A_ϵ , and the proof is complete. \square

Let us denote by Γ the boundary of Σ_ϕ . The following statements are valid:

$$(2-7) \quad e^{-A_\epsilon t} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda I + A_\epsilon)^{-1} d\lambda,$$

$$(2-8) \quad \|e^{-A_\epsilon t}\|_{\mathcal{L}(X^1, X)} \leq C t^{-1/2} e^{-\omega t}, \quad t > 0,$$

for some $C > 0$ independent of ϵ .

Also, we have

$$(2-9) \quad \|(\lambda I + A_\epsilon)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C}{1 + |\lambda|}, \quad \lambda \in \Sigma_\phi,$$

$$(2-10) \quad \|(\lambda I + A_\epsilon)^{-1}\|_{\mathcal{L}(X^1)} \leq \frac{C}{1 + |\lambda|}, \quad \lambda \in \Sigma_\phi,$$

where $C = C(\phi) > 0$ independent of ϵ .

Under the assumptions (1-4), problem (1-7) is locally well posed in X ; see Theorem 1 in [Carvalho and Cholewa 2002b]. Moreover, under standard dissipative conditions like (1-5), we have the following result.

Theorem 2.2. *Assume (1-4) and (1-5) hold. The nonlinear semigroup $\{T_\epsilon(t) : t \geq 0\}$ associated with (1-7) is well defined in X and has a global attractor \mathcal{A}_ϵ in X . Furthermore,*

$$\sup_{\epsilon \in [0, 1]} \sup_{w \in \mathcal{A}_\epsilon} \|w\|_X < \infty.$$

Proof. Problem (1-7) is globally well posed in X due to Theorem 1.1 in [Carvalho and Cholewa 2002a], namely, for any $w_0^\epsilon \in X$, there exists a unique

$$w^\epsilon(\cdot, w_0^\epsilon) \in C([0, \infty), X) \cap C^1((0, \infty), X)$$

with $w^\epsilon(t, w_0^\epsilon) \in D(A_\epsilon)$, for all $t > 0$, which satisfies (1-7) and

$$w(t, w_0^\epsilon) = e^{-A_\epsilon t} w_0^\epsilon + \int_0^t e^{-A_\epsilon(t-s)} f(w(s, w_0^\epsilon)) ds, \quad t \geq 0.$$

Thus $T_\epsilon(t)w_0^\epsilon = u^\epsilon(t, w_0^\epsilon), t \geq 0$. To simplify the notation we will denote the solution $w^0(t, w_0^0)$ by $w(t, w_0)$.

The existence of global attractors \mathcal{A}_ϵ in X for semigroups $\{T_\epsilon(t) : t \geq 0\}$ and uniform bounds are also established in Theorem 1.2 in [Carvalho and Cholewa 2002a]. \square

3. Resolvent convergence

In this section we will show the convergence of the resolvent operators A_ϵ^{-1} to A_0^{-1} , as $\epsilon \rightarrow 0^+$, and we will establish that the rate of this convergence is $\|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}$.

We recall the convergence of the resolvent operators Λ_ϵ^{-1} to Λ_0^{-1} , as $\epsilon \rightarrow 0^+$, in terms of the difference $\|a_\epsilon - a_0\|_{L^\infty(\Omega)}$. This was proved in [Arrieta et al. 2013], however, for the sake of completeness, we will sketch a proof.

Lemma 3.1. *For $h \in Y$ and $\epsilon \in [0, 1]$, let us consider $u^\epsilon \in Y^1$ a solution of the problem*

$$(3-1) \quad \begin{cases} -\operatorname{div}(a_\epsilon(x)\nabla u) = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then there is a constant $C > 0$, independent of ϵ , such that

$$(3-2) \quad \|u^\epsilon\|_{Y^{1/2}} \leq C \|h\|_Y$$

and

$$(3-3) \quad \|u^\epsilon - u\|_{Y^{1/2}} \leq C \|h\|_Y \|a_\epsilon - a_0\|_\infty.$$

Proof. The estimate (3-2) follows from uniform boundedness of a_ϵ and Poincaré's inequality.

The solution of problem (3-1) can be obtained by a minimization procedure. That is, if we define

$$\lambda_\epsilon := \min_{u \in Y^{1/2}} \left\{ \frac{1}{2} \int_\Omega a_\epsilon |\nabla u|^2 dx - \int_\Omega hu dx \right\},$$

then λ_ϵ is attained at u^ϵ . Therefore,

$$(3-4) \quad \begin{aligned} \lambda_\epsilon &= \frac{1}{2} \int_\Omega a_\epsilon |\nabla u^\epsilon|^2 dx - \int_\Omega hu^\epsilon dx \\ &= \frac{1}{2} \int_\Omega a_\epsilon |\nabla u^\epsilon - \nabla u^0 + \nabla u^0|^2 dx - \int_\Omega h(u^\epsilon - u^0 + u^0) dx, \end{aligned}$$

and, evaluating this expression, using that u^ϵ solves Lemma 3.1, we easily obtain

$$(3-5) \quad \lambda_\epsilon = \lambda_0 - \frac{1}{2} \int_\Omega a_\epsilon(x) |\nabla u^\epsilon - \nabla u^0|^2 dx + \frac{1}{2} \int_\Omega (a_\epsilon(x) - a_0(x)) |\nabla u^0|^2 dx,$$

which implies

$$(3-6) \quad \lambda_\epsilon - \lambda_0 \leq \frac{1}{2} \int_\Omega (a_\epsilon(x) - a_0(x)) |\nabla u^0|^2 dx.$$

On the other hand

$$\begin{aligned}\lambda_0 &:= \min_{u \in Y^{1/2}} \left\{ \frac{1}{2} \int_{\Omega} a_0(x) |\nabla u|^2 dx - \int_{\Omega} hu dx \right\} \\ &\leq \frac{1}{2} \int_{\Omega} a_0(x) |\nabla u^\epsilon|^2 dx - \int_{\Omega} hu^\epsilon dx \\ &= \lambda_\epsilon + \frac{1}{2} \int_{\Omega} (a_0(x) - a_\epsilon(x)) |\nabla u^\epsilon|^2 dx.\end{aligned}$$

With this, we obtain

$$\lambda_\epsilon - \lambda_0 \geq -\frac{1}{2} \int_{\Omega} (a_\epsilon(x) - a_0(x)) |\nabla u^\epsilon|^2 dx,$$

which combined with (3-2) and (3-6) gives us

$$(3-7) \quad |\lambda_\epsilon - \lambda_0| \leq \|a_\epsilon - a_0\|_{L^\infty(\Omega)} \sup_{\epsilon \in [0,1]} \|u^\epsilon\|_{Y^{1/2}}^2 \leq C \|h\|_Y^2 \|a_\epsilon - a_0\|_{L^\infty(\Omega)}.$$

Finally, the estimate (3-3) is obtained by combining (3-5) and (3-7). \square

Corollary 3.2. *The operators $\Lambda_\epsilon^{-1} : Y \rightarrow Y^{1/2}$ are uniformly bounded and converge uniformly to $\Lambda_0^{-1} : Y \rightarrow Y^{1/2}$, as $\epsilon \rightarrow 0^+$. Furthermore, there exists a positive constant $C > 0$, independent of ϵ , such that*

$$(3-8) \quad \|\Lambda_\epsilon^{-1}\|_{\mathcal{L}(Y, Y^{1/2})} \leq C$$

and

$$(3-9) \quad \|\Lambda_\epsilon^{-1} - \Lambda_0^{-1}\|_{\mathcal{L}(Y, Y^{1/2})} \leq C \|a_\epsilon - a_0\|_{L^\infty(\Omega)}.$$

The uniform convergence of the operators Λ_ϵ^{-1} (see Corollary 3.2 in [Arrieta et al. 2013]) implies the convergence of their spectrum. As a matter of fact, the following result holds.

Proposition 3.3 [Carvalho and Piskarev 2006; Kato 1976]. *The following statements hold:*

- (i) *If $\mu_0 \in \sigma(-\Lambda_0)$, there exists a sequence $\epsilon_n \rightarrow 0^+$ and $\{\mu_n\}$, with $\mu_n \in \sigma(-\Lambda_{\epsilon_n})$, $n \in \mathbb{N}$, such that $\mu_n \rightarrow \mu_0$, as $n \rightarrow \infty$;*
- (ii) *If for some sequences $\epsilon_n \rightarrow 0^+$ and $\mu_n \rightarrow \mu_0$, as $n \rightarrow \infty$, with $\mu_n \in \sigma(-\Lambda_{\epsilon_n})$, $n \in \mathbb{N}$, then $\mu_0 \in \sigma(-\Lambda_0)$.*

Moreover, from Lemma 3.4 in [Arrieta et al. 2013], there exists $C > 0$, independent of ϵ , such that

$$(3-10) \quad \|(\lambda I + \Lambda_\epsilon)^{-1} - (\lambda I + \Lambda_0)^{-1}\|_{\mathcal{L}(Y, Y^{1/2})} \leq C \|a_\epsilon - a_0\|_{L^\infty(\Omega)},$$

for each $\lambda \in \Sigma_\phi$.

Proposition 3.4. *The operators $\Lambda_\epsilon^{-(1-\delta)} : Y^{1/2} \rightarrow Y^{1/2}$ are uniformly bounded and converge uniformly to $\Lambda_0^{-(1-\delta)} : Y^{1/2} \rightarrow Y^{1/2}$, as $\epsilon \rightarrow 0$. Furthermore, there exists a positive constant $C > 0$, independent of ϵ , such that*

$$(3-11) \quad \|\Lambda_\epsilon^{-(1-\delta)}\|_{\mathcal{L}(Y^{1/2})} \leq C$$

and

$$(3-12) \quad \|\Lambda_\epsilon^{-(1-\delta)} - \Lambda_0^{-(1-\delta)}\|_{\mathcal{L}(Y^{1/2})} \leq C \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}.$$

Proof. Notice that, using (2-2), we get

$$\|\Lambda_\epsilon^{-(1-\delta)} u\|_{Y^{1/2}} = \|\Lambda_\epsilon^{\delta-1/2} u\|_Y \leq C \|\Lambda_\epsilon^{-1} u\|_{Y^{1/2}}^{1-(\delta-1/2)} \|u\|_{Y^{1/2}}^{\delta-1/2},$$

where $C > 0$ is uniform with respect to ϵ and δ . Thus (3-11) follows by (3-8).

Before we prove (3-12), let us observe that (2-2) and (3-11) imply

$$(3-13) \quad \begin{aligned} & \|(\Lambda_\epsilon^{-(1-\delta)} - \Lambda_0^{-(1-\delta)})h\|_{Y^{1/2}} \\ &= \|\Lambda_\epsilon^{1/2}(\Lambda_\epsilon^{-(1-\delta)} - \Lambda_0^{-(1-\delta)})h\|_Y \\ &\leq C \|\Lambda_\epsilon(\Lambda_\epsilon^{-(1-\delta)} - \Lambda_0^{-(1-\delta)})h\|_Y^{1/2} \|(\Lambda_\epsilon^{-(1-\delta)} - \Lambda_0^{-(1-\delta)})h\|_Y^{1/2} \\ &\leq C \|\Lambda_\epsilon(\Lambda_\epsilon^{-(1-\delta)} - \Lambda_0^{-(1-\delta)})h\|_Y^{1/2} (\|\Lambda_\epsilon^{-(1-\delta)}h\|_Y^{1/2} + \|\Lambda_0^{-(1-\delta)}h\|_Y^{1/2}) \\ &\leq C \|\Lambda_\epsilon(\Lambda_\epsilon^{-(1-\delta)} - \Lambda_0^{-(1-\delta)})h\|_Y^{1/2}, \end{aligned}$$

for some $C > 0$ independent of ϵ , and for any $h \in Y$.

To prove (3-12), it follows by (3-13) that it is sufficient to obtain an estimate for the norm $\|\Lambda_\epsilon(\Lambda_\epsilon^{-(1-\delta)} - \Lambda_0^{-(1-\delta)})\|_{\mathcal{L}(Y)}$. In fact, it follows by (1-3) that

$$(3-14) \quad \Lambda_\epsilon^{-\alpha} - \Lambda_0^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{-\alpha} [(\lambda I + \Lambda_\epsilon)^{-1} - (\lambda I + \Lambda_0)^{-1}] d\lambda.$$

Using (3-14) (with $\alpha = 1 - \delta$), we can deduce

$$(3-15) \quad \begin{aligned} & \|\Lambda_\epsilon(\Lambda_\epsilon^{-(1-\delta)} - \Lambda_0^{-(1-\delta)})\|_{\mathcal{L}(Y)} \\ &\leq \int_0^\infty \|\lambda^{-(1-\delta)} \Lambda_\epsilon [(\lambda I + \Lambda_\epsilon)^{-1} - (\lambda I + \Lambda_0)^{-1}]\|_{\mathcal{L}(Y)} d|\lambda|. \end{aligned}$$

Notice that the resolvent identity

$$(3-16) \quad \begin{aligned} & (\lambda I + \Lambda_\epsilon)^{-1} - (\lambda I + \Lambda_0)^{-1} \\ &= (\lambda I + \Lambda_\epsilon)^{-1} [I - (\lambda I + \Lambda_\epsilon)(\lambda I + \Lambda_0)^{-1}] \\ &= (\lambda I + \Lambda_\epsilon)^{-1} [(\lambda I + \Lambda_0) - (\lambda I + \Lambda_\epsilon)](\lambda I + \Lambda_0)^{-1} \\ &= (\lambda I + \Lambda_\epsilon)^{-1} [\Lambda_0 - \Lambda_\epsilon](\lambda I + \Lambda_0)^{-1} \\ &= \Lambda_\epsilon(\lambda I + \Lambda_\epsilon)^{-1} [\Lambda_\epsilon^{-1} - \Lambda_0^{-1}] \Lambda_0(\lambda I + \Lambda_0)^{-1} \end{aligned}$$

holds, and, by sectoriality of Λ_ϵ , we have

$$(3-17) \quad \|\Lambda_\epsilon(\lambda I + \Lambda_\epsilon)^{-1}\|_{\mathcal{L}(Y)} \leq C, \quad \text{for all } \epsilon \in [0, 1],$$

where $C > 0$ is independent of ϵ .

Substituting (3-16) into (3-15), we get

$$(3-18) \quad \begin{aligned} & \|\Lambda_\epsilon(\Lambda_\epsilon^{-(1-\delta)} - \Lambda_0^{-(1-\delta)})\|_{\mathcal{L}(Y)} \\ & \leq \int_0^\infty \|\lambda^{-(1-\delta)} \Lambda_\epsilon \Lambda_\epsilon(\lambda I + \Lambda_\epsilon)^{-1} [\Lambda_\epsilon^{-1} - \Lambda_0^{-1}] \Lambda_0(\lambda I + \Lambda_0)^{-1}\|_{\mathcal{L}(Y)} d|\lambda| \\ & \leq \int_0^\infty |\lambda|^{-(1-\delta)} \|\Lambda_\epsilon^\gamma(\lambda I + \Lambda_\epsilon)^{-1} \Lambda_\epsilon^{1+(1-\gamma)} [\Lambda_\epsilon^{-1} - \Lambda_0^{-1}] \Lambda_0(\lambda I + \Lambda_0)^{-1}\|_{\mathcal{L}(Y)} d|\lambda|, \end{aligned}$$

where $\gamma \in (1, 2)$ is a constant to be chosen.

Since $Y^{1/2}$ is continuously embedded in $Y^{1+(1-\gamma)}$, by estimates (3-9) and (3-17), we can deduce from (3-18) that

$$(3-19) \quad \begin{aligned} & \|\Lambda_\epsilon(\Lambda_\epsilon^{-(1-\delta)} - \Lambda_0^{-(1-\delta)})\|_{\mathcal{L}(Y)} \\ & \leq C \|a_\epsilon - a_0\|_{L^\infty(\Omega)} \int_0^\infty |\lambda|^{-(1-\delta)} \|\Lambda_\epsilon^{\gamma-r} \Lambda_\epsilon^r(\lambda I + \Lambda_\epsilon)^{-1}\|_{\mathcal{L}(Y)} d|\lambda|, \end{aligned}$$

where $r \in (\gamma - 1, 1)$ is a constant to be chosen.

From (2-2) and the fact that Y^1 is continuously embedded in Y^r , it follows by (3-19) that

$$(3-20) \quad \begin{aligned} & \|\Lambda_\epsilon(\Lambda_\epsilon^{-(1-\delta)} - \Lambda_0^{-(1-\delta)})\|_{\mathcal{L}(Y)} \\ & \leq C \|a_\epsilon - a_0\|_{L^\infty(\Omega)} \int_0^\infty |\lambda|^{-(1-\delta)} \|\Lambda_\epsilon \Lambda_\epsilon^r(\lambda I + \Lambda_\epsilon)^{-1}\|_{\mathcal{L}(Y)}^{\gamma-r} \\ & \quad \times \|\Lambda_\epsilon^r(\lambda I + \Lambda_\epsilon)^{-1}\|_{\mathcal{L}(Y)}^{1-(\gamma-r)} d|\lambda| \\ & \leq C \|a_\epsilon - a_0\|_{L^\infty(\Omega)} \int_0^\infty |\lambda|^{-(1-\delta)} \|\Lambda_\epsilon(\lambda I + \Lambda_\epsilon)^{-1}\|_{\mathcal{L}(Y)}^{\gamma-r} \|(\lambda I + \Lambda_\epsilon)^{-1}\|_{\mathcal{L}(Y)}^{1-(\gamma-r)} d|\lambda|. \end{aligned}$$

Using (2-1) and (3-17), we get by (3-20) that

$$(3-21) \quad \begin{aligned} & \|\Lambda_\epsilon(\Lambda_\epsilon^{-(1-\delta)} - \Lambda_0^{-(1-\delta)})\|_{\mathcal{L}(Y)} \\ & \leq C \|a_\epsilon - a_0\|_{L^\infty(\Omega)} \int_0^\infty |\lambda|^{-(1-\delta)} \|(\lambda I + \Lambda_\epsilon)^{-1}\|_{\mathcal{L}(Y)}^{1-(\gamma-r)} d|\lambda| \\ & \leq C \|a_\epsilon - a_0\|_{L^\infty(\Omega)} \int_0^\infty |\lambda|^{-(1-\delta)} (\max\{1, |\lambda|^{-1}\})^{1-(\gamma-r)} d|\lambda| \\ & \leq C \|a_\epsilon - a_0\|_{L^\infty(\Omega)} \left(\int_0^1 |\lambda|^{\delta-1} d|\lambda| + \int_1^\infty |\lambda|^{-2+\delta+\gamma-r} d|\lambda| \right). \end{aligned}$$

Taking γ and r sufficiently close to 1 such that $\delta + \gamma - r < 1$, we can conclude by (3-21) the existence of a positive constant C , independent of ϵ , such that

$$(3-22) \quad \|\Lambda_\epsilon(\Lambda_\epsilon^{-(1-\delta)} - \Lambda_0^{-(1-\delta)})\|_{\mathcal{L}(Y)} \leq C \|a_\epsilon - a_0\|_{L^\infty(\Omega)}.$$

Finally, combining (3-13) and (3-22) we obtain the desired estimate (3-12). \square

Before we prove analogous result of the last proposition with A_ϵ instead of Λ_ϵ , we present the following general version of the moment inequality (see [Sobolevskii 1961]):

$$(3-23) \quad \|A_\epsilon w\|_X \leq C \|w\|_X^{1-\beta/\alpha} \|A_\epsilon^\alpha w\|_X^{\beta/\alpha}, \quad w \in D(A_\epsilon^\alpha),$$

where the constant $C > 0$ is independent of ϵ .

Proposition 3.5. *The operators $A_\epsilon^{-1} : X \rightarrow X^1$ are uniformly bounded and converge in the uniform topology to $A_0^{-1} : X \rightarrow X^1$, as $\epsilon \rightarrow 0^+$. Furthermore, there exists a positive constant $C > 0$, independent of ϵ , such that*

$$(3-24) \quad \|A_\epsilon^{-1}\|_{\mathcal{L}(X, X^1)} \leq C$$

and

$$(3-25) \quad \|A_\epsilon^{-1} - A_0^{-1}\|_{\mathcal{L}(X, X^1)} \leq C \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}.$$

Proof. For $g, h \in L^2(\Omega)$ and $\epsilon \in [0, 1]$, let $[\phi_\epsilon \ \varphi_\epsilon]^T$ be the solution of the problem

$$A_\epsilon \begin{bmatrix} \phi \\ \varphi \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix}, \quad t > 0.$$

By (1-6), there exists $C > 0$, independent ϵ , such that

$$\left\| \begin{bmatrix} \phi_\epsilon \\ \varphi_\epsilon \end{bmatrix} - \begin{bmatrix} \phi_0 \\ \varphi_0 \end{bmatrix} \right\|_{X^1} \leq C \left\| \begin{bmatrix} \phi_\epsilon \\ \varphi_\epsilon \end{bmatrix} - \begin{bmatrix} \phi_0 \\ \varphi_0 \end{bmatrix} \right\|_{Y^1 \times Y^\delta},$$

and by (3-23), we get

$$\begin{aligned} \left\| \begin{bmatrix} \phi_\epsilon \\ \varphi_\epsilon \end{bmatrix} - \begin{bmatrix} \phi_0 \\ \varphi_0 \end{bmatrix} \right\|_{Y^1 \times Y^\delta} &= \|\Lambda_\epsilon^{-(1-\delta)} g - \Lambda_0^{-(1-\delta)} g\|_{Y^1} + \|\Lambda_\epsilon^{-1} h - \Lambda_0^{-1} h\|_{Y^\delta} \\ &\leq C \|\Lambda_\epsilon^{1/2}(\Lambda_\epsilon^{-(1-\delta)} g - \Lambda_0^{-(1-\delta)} g)\|_Y + \|\Lambda_\epsilon^{-(1-\delta)} h - \Lambda_0^{-(1-\delta)} h\|_Y. \end{aligned}$$

Thus, by Corollary 3.2, we conclude that

$$\begin{aligned} \left\| \begin{bmatrix} \phi_\epsilon \\ \varphi_\epsilon \end{bmatrix} - \begin{bmatrix} \phi_0 \\ \varphi_0 \end{bmatrix} \right\|_{X^1} &\leq C (\|g\|_{Y^{1/2}} + \|h\|_Y) \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2} \\ &= C \left\| \begin{bmatrix} g \\ h \end{bmatrix} \right\|_X \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}, \end{aligned}$$

where $C > 0$ is independent of ϵ . □

The next result ensures convergence of the spectrum of operators $-A_\epsilon$.

Proposition 3.6 [Carvalho and Piskarev 2006; Kato 1976]. *The following statements hold:*

- (i) *If $\mu_0 \in \sigma(-A_0)$, there exists a sequence $\epsilon_n \rightarrow 0^+$ and $\{\mu_n\}$, with $\mu_n \in \sigma(-A_{\epsilon_n})$, $n \in \mathbb{N}$ such that $\mu_n \rightarrow \mu_0$ as $n \rightarrow \infty$.*
- (ii) *If for some sequences $\epsilon_n \rightarrow 0^+$ and $\mu_n \rightarrow \mu_0$ as $n \rightarrow \infty$, with $\mu_n \in \sigma(-A_{\epsilon_n})$, $n \in \mathbb{N}$, then $\mu_0 \in \sigma(-A_0)$.*

Now let us establish the result which treats convergence for resolvent operators.

Proposition 3.7. *For each $\phi \in (\pi/2, \pi)$, there exists a constant $C = C(\phi) > 0$ such that*

$$\sup_{\lambda \in \Sigma_\phi} \|(\lambda I + A_\epsilon)^{-1} - (\lambda I + A_0)^{-1}\|_{\mathcal{L}(X, X^1)} \leq C \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}.$$

Proof. We can see that

$$\begin{aligned} (3-26) \quad (\lambda I + A_\epsilon)^{-1} - (\lambda I + A_0)^{-1} &= (\lambda I + A_\epsilon)^{-1} A_\epsilon [A_0^{-1} - A_\epsilon^{-1}] A_0 (\lambda I + A_0)^{-1} \\ &= A_\epsilon (\lambda I + A_\epsilon)^{-1} [A_0^{-1} - A_\epsilon^{-1}] A_0 (\lambda I + A_0)^{-1}. \end{aligned}$$

Notice that, for $\lambda \in \Sigma_\phi \subset \rho(-A_\epsilon)$, we have

$$A_\epsilon (\lambda I + A_\epsilon)^{-1} = [(\lambda I + A_\epsilon) A_\epsilon^{-1}]^{-1} = [\lambda A_\epsilon^{-1} - I]^{-1},$$

and, therefore,

$$(3-27) \quad \|A_\epsilon (\lambda I + A_\epsilon)^{-1}\|_{\mathcal{L}(X^1)} \leq C, \quad \lambda \in \Sigma_\phi,$$

for some $C > 0$ independent of ϵ .

By (3-25)–(3-27), we have the existence of a constant $C > 0$ (independent of ϵ and of $\lambda \in \Sigma_\phi$) such that

$$\|(\lambda I + A_\epsilon)^{-1} - (\lambda I + A_0)^{-1}\|_{\mathcal{L}(X, X^1)} \leq C \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}. \quad \square$$

To finish this section, we will make a spectral analysis, where we will give a characterization, as well as a rate of convergence, as $\epsilon \rightarrow 0$, for the eigenvalues associated with the operators A_ϵ .

Let γ be a closed, rectifiable, simple and oriented counterclockwise curve in $\rho(-A_0)$ around $\mu_0 \in \sigma(A_0)$ which has index 1 relative to γ . From part (ii) of Proposition 3.6, it is easy to see that there is an $\epsilon_\gamma > 0$ such that the trace of γ is in $\rho(A_\epsilon)$, for $\epsilon \in [0, \epsilon_\gamma]$. We define the spectral projection in X

$$Q_\epsilon(\mu_0) = \frac{1}{2\pi i} \int_\gamma (\lambda I + A_\epsilon)^{-1} d\lambda,$$

and, for $\mu \in \mathbb{C}$ such that $(1/(2\pi i)) \int_{\gamma} (\lambda - \mu)^{-1} d\lambda = 1$, we define the generalized eigenspace associated with μ , $W(\mu, -A_{\epsilon}) = Q_{\epsilon}(\mu_0)(X)$, $\epsilon \in [0, \epsilon_{\gamma}]$. Furthermore, $Q_{\epsilon}(\mu_0)$ is compact and $\dim W(\mu, -A_{\epsilon}) = \text{rank}(Q_{\epsilon}(\mu_0)) < \infty$.

Related to the rate of convergence, the following result holds.

Proposition 3.8. *The family of operators $Q_{\epsilon}(\mu_0) : X \rightarrow X$ converges uniformly to $Q_0(\mu_0) : X \rightarrow X$, as $\epsilon \rightarrow 0^+$. Moreover,*

$$(3-28) \quad \|Q_{\epsilon}(\mu_0) - Q_0(\mu_0)\|_{\mathcal{L}(X)} \leq C \|a_{\epsilon} - a_0\|_{L^{\infty}(\Omega)}^{1/2},$$

and

$$(3-29) \quad \|A_{\epsilon}Q_{\epsilon}(\mu_0) - A_0Q_0(\mu_0)\|_{\mathcal{L}(X)} \leq C \|a_{\epsilon} - a_0\|_{L^{\infty}(\Omega)}^{1/2},$$

where $C > 0$ independent of ϵ .

Proof. Since

$$Q_{\epsilon}(\mu_0) - Q_0(\mu_0) = \frac{1}{2\pi i} \int_{\gamma} [(\lambda I + A_{\epsilon})^{-1} - (\lambda I + A_0)^{-1}] d\lambda,$$

we can use [Proposition 3.7](#) to guarantee the estimate (3-28).

To prove (3-29), it is sufficient to use (3-26) and (3-28). \square

Remark 3.9. If μ_0 is an isolated eigenvalue for A_0 , we may define $Q_{\epsilon}(\mu_0)$ as above and it follows from [Proposition 3.6](#) that there exists μ_{ϵ} , which is an eigenvalue of A_{ϵ} such that $\mu_{\epsilon} \rightarrow \mu_0$, as $\epsilon \rightarrow 0^+$. Hence $Q_{\epsilon}(\mu_0) = Q_{\epsilon}(\mu_{\epsilon})$. We still have from [Proposition 3.8](#) that

$$\|Q_{\epsilon}(\mu_{\epsilon})Q_0(\mu_0) - Q_0(\mu_0)\|_{\mathcal{L}(X)} \leq C \|a_{\epsilon} - a_0\|_{L^{\infty}(\Omega)}^{1/2}$$

and that $Q_{\epsilon}(\mu_{\epsilon})Q_0(\mu_0)$ is an isomorphism between $R(Q_0(\mu_0))$ and $R(Q_{\epsilon}(\mu_{\epsilon}))$.

The next result deals with the characterization and rate of convergence of the eigenvalues associated to operators A_{ϵ} .

Theorem 3.10. *For each $\epsilon \in [0, 1]$, the eigenvalues of the operator A_{ϵ} are given by*

$$(3-30) \quad \lambda_{\epsilon,n}^{\pm} = \frac{\mu_{\epsilon,n}^{\delta}}{2} \pm i \frac{\sqrt{4\mu_{\epsilon,n} - \mu_{\epsilon,n}^{2\delta}}}{2}, \quad n \in \mathbb{N},$$

where $\mu_{\epsilon,n}$, $n \in \mathbb{N}$, denotes the eigenvalues of the operator Λ_{ϵ} . Furthermore, if $\text{Ker}(\lambda_{0,n}^{\pm} I - A_0) = R(Q_0(\lambda_{0,n}^{\pm}))$, then

$$|\lambda_{\epsilon,n}^{\pm} - \lambda_{0,n}^{\pm}| \leq C_n \|a_{\epsilon} - a_0\|_{L^{\infty}(\Omega)}^{1/2}, \quad n \in \mathbb{N},$$

for some constant $C_n > 0$, independent of ϵ .

Proof. To study the spectral problem for the operator A_ϵ , we consider the equation

$$(3-31) \quad A_\epsilon \begin{bmatrix} \phi \\ \varphi \end{bmatrix} = \lambda \begin{bmatrix} \phi \\ \varphi \end{bmatrix},$$

that is,

$$(3-32) \quad \lambda^2 \phi - \lambda \Lambda_\epsilon^\delta \phi + \Lambda_\epsilon \phi = 0,$$

whose solutions are the eigenvectors $\{\phi_{\epsilon,n}\}$ of Λ_ϵ :

$$(3-33) \quad \lambda^2 \phi_{\epsilon,n} - \mu_{\epsilon,n}^\delta \lambda \phi_{\epsilon,n} + \mu_{\epsilon,n} \phi_{\epsilon,n} = 0.$$

In this way, the corresponding eigenvalues $\{\lambda_{\epsilon,n}^\pm\}$ of A_ϵ are the solutions of the equation

$$\lambda^2 - \mu_{\epsilon,n}^\delta \lambda + \mu_{\epsilon,n} = 0$$

and they are given by (3-30).

Moreover, by the above remark and [Proposition 3.8](#), we have that, for each $\epsilon > 0$, there exists $[\phi \ \varphi]^T \in R(Q_0)$, $\|[\phi \ \varphi]^T\|_{X^1} = 1$, such that $Q_\epsilon [\phi \ \varphi]^T$ is an eigenvector of A_ϵ associated to λ_ϵ and

$$(3-34) \quad \begin{aligned} & |\lambda_{\epsilon,n}^\pm - \lambda_{0,n}^\pm| \\ & \leq \left\| \lambda_{\epsilon,n}^\pm Q_0 \begin{bmatrix} \phi \\ \varphi \end{bmatrix} - \lambda_{\epsilon,n}^\pm Q_\epsilon \begin{bmatrix} \phi \\ \varphi \end{bmatrix} \right\|_{X^1} + \left\| \lambda_{\epsilon,n}^\pm Q_\epsilon \begin{bmatrix} \phi \\ \varphi \end{bmatrix} - \lambda_{0,n}^\pm Q_0 \begin{bmatrix} \phi \\ \varphi \end{bmatrix} \right\|_{X^1} \end{aligned}$$

$$(3-35) \quad \leq C \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2},$$

and the proof is completed. \square

4. Rate of convergence of resolvents of linearized operators

In this section we will study the rate of convergence of the resolvents of operators which corresponds to linearizations of (1-7) around equilibria.

It is known that the Nemytskiĭ map $f^e(u) := f(u)$, $u \in Y^{1/2}$, is Fréchet continuously differentiable. Moreover, if $\{u_\epsilon\}$ converges to u_0 in $Y^{1/2}$ and $0 \notin \sigma(\Lambda_0 - (f^e)'(u_0))$, then $((f^e)'(u_\epsilon))\Lambda_\epsilon^{-1}$ converges to $((f^e)'(u_0))\Lambda_0^{-1}$ in the uniform operator topology of $\mathcal{L}(Y)$; see, for instance, [\[Arrieta et al. 2013\]](#). Hence the Nemytskiĭ map F is Fréchet continuously differentiable. Moreover, if $u_\epsilon \rightarrow u_0$ in X and $0 \notin \sigma(A_0 - F'(u_0))$, then

$$(4-1) \quad (F'(u_\epsilon))A_\epsilon^{-1} \rightarrow (F'(u_0))A_0^{-1} \quad \text{in } \mathcal{L}(X).$$

Lemma 4.1. *We assume $u_\epsilon \rightarrow u_0$ in X and $0 \notin \sigma(A_0 - F'(u_0))$. Then there exists $\epsilon_0 > 0$ such that the net of operators*

$$\{A_\epsilon^{1/2}(A_\epsilon - F'(u_\epsilon))^{-1} : \epsilon \in [0, 1]\}$$

is uniformly bounded in $\mathcal{L}(X)$ and

$$\|A_\epsilon^{1/2}(A_\epsilon - F'(u_\epsilon))^{-1} - A_0^{1/2}(A_0 - F'(u_0))^{-1}\|_{\mathcal{L}(X)} \leq C \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2},$$

where $C > 0$ is independent of ϵ .

Proof. The proof follows from the identity

$$A_\epsilon^{1/2}(A_\epsilon - F'(u_\epsilon))^{-1} = A_\epsilon^{-1/2}(I - F'(u_\epsilon)A_\epsilon^{-1})^{-1}$$

and by (4-1). □

5. Rate of convergence of the linear and nonlinear semigroups

Since the operators A_ϵ , $\epsilon \in [0, 1]$, are self-adjoint and A_ϵ^{-1} converges uniformly to A_0^{-1} as $\epsilon \rightarrow 0^+$, for each $\alpha < \lambda_1^0$ (λ_1^0 the first eigenvalue of A_0), there exists $C > 0$, independent of $\epsilon \in [0, 1]$, such that

$$(5-1) \quad \|e^{-A_\epsilon t}\|_{\mathcal{L}(X)} \leq C e^{-\alpha t} t^{-1/2}, \quad t > 0, \epsilon \in [0, 1].$$

Theorem 5.1. *If $\theta \in (0, \frac{1}{2}]$ and $\alpha < \lambda_1^0$, there exists $C > 0$, independent of ϵ , such that*

$$(5-2) \quad \|e^{-A_\epsilon t} - e^{-A_0 t}\|_{\mathcal{L}(X)} \leq C e^{-\alpha t} \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{2\theta} t^{-(1/2+\theta)}$$

for all $t > 0$ and $\epsilon \in [0, 1]$.

Proof. Considering the linear semigroup

$$e^{-A_\epsilon t} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda I + A_\epsilon)^{-1} d\lambda, \quad \epsilon \in [0, 1],$$

where Γ is the boundary of sector $\Sigma_{-\omega, \phi} = \{\lambda \in \mathbb{C} : |\arg(\lambda + \omega)| \leq \phi\}$ with $\pi/2 < \phi < \pi$, oriented in such a way that the imaginary part of λ increases as λ runs in Γ .

The estimate

$$(5-3) \quad \|e^{-A_\epsilon t} - e^{-A_0 t}\|_{\mathcal{L}(X)} \leq \|e^{-A_\epsilon t}\|_{\mathcal{L}(X)} + \|e^{-A_0 t}\|_{\mathcal{L}(X)} \leq C e^{-\alpha t} t^{-1/2}$$

follows by (5-1).

On the other hand, using Proposition 3.7, we have

$$(5-4) \quad \|e^{-A_\epsilon t} - e^{-A_0 t}\|_{\mathcal{L}(X)} \leq C e^{-\alpha t} \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2} t^{-1}.$$

Therefore, for $\theta \in (0, \frac{1}{2}]$, we obtain

$$\begin{aligned} \|e^{-A_\epsilon t} - e^{-A_0 t}\|_{\mathcal{L}(X)} &\leq C e^{-\alpha(1-2\theta)t} t^{-1/2(1-2\theta)} e^{-\alpha(2\theta)t} \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{2\theta} t^{-2\theta} \\ &\leq C e^{-\alpha t} \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{2\theta} t^{-(1/2+\theta)}. \end{aligned}$$

where $C > 0$ is independent of ϵ . □

Theorem 5.2. *Let $[u_\epsilon \ v_\epsilon]^T, [u \ v]^T \in X$, and $\theta \in (0, \frac{1}{2})$. Then there are positive constants C and L_f such that*

$$\left\| T_\epsilon(t) \begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - T_0(t) \begin{bmatrix} u \\ v \end{bmatrix} \right\|_X \leq C e^{L_f t} t^{-(1/2+\theta)} \left(\left\| \begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - \begin{bmatrix} u \\ v \end{bmatrix} \right\|_X + \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^\theta \right)$$

for all $t \geq 0$.

Proof. For $t \geq 0$ and $[u_\epsilon \ v_\epsilon]^T \in X$ we have

$$T_\epsilon(t) \begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} = e^{-A_\epsilon t} \begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} + \int_0^t e^{-A_\epsilon(t-s)} f \left(T_\epsilon(s) \begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} \right) ds, \quad \epsilon \in [0, 1],$$

and therefore

$$(5-5) \quad \left\| T_\epsilon(t) \begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - T_0(t) \begin{bmatrix} u \\ v \end{bmatrix} \right\|_X \\ \leq \left\| e^{-A_\epsilon t} \begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - e^{-A_0 t} \begin{bmatrix} u \\ v \end{bmatrix} \right\|_X + \int_0^t \left\| e^{-A_\epsilon(t-s)} f \left(T_\epsilon(s) \begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} \right) - e^{-A_0(t-s)} f \left(T_0(s) \begin{bmatrix} u \\ v \end{bmatrix} \right) \right\|_X ds.$$

From (5-1) and (5-2) we get

$$(5-6) \quad \left\| e^{-A_\epsilon t} \begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - e^{-A_0 t} \begin{bmatrix} u \\ v \end{bmatrix} \right\|_X \\ \leq C t^{-(1/2+\theta)} \left\| \begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - \begin{bmatrix} u \\ v \end{bmatrix} \right\|_X + C \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{2\theta} t^{-(1/2+\theta)}.$$

We still have

$$(5-7) \quad \int_0^t \left\| e^{-A_\epsilon(t-s)} f \left(T_\epsilon(s) \begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} \right) - e^{-A_0(t-s)} f \left(T_0(s) \begin{bmatrix} u \\ v \end{bmatrix} \right) \right\|_X ds \\ \leq C L_f \int_0^t (t-s)^{-1/2} e^{-\alpha(t-s)} \left\| T_\epsilon(s) \begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - T_0(s) \begin{bmatrix} u \\ v \end{bmatrix} \right\|_X ds \\ + C \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{2\theta} \int_0^t (t-s)^{-(1/2+\theta)} e^{-\alpha(t-s)} ds.$$

Substituting (5-6) and (5-7) in (5-5), it follows that

$$\begin{aligned} & \left\| T_\epsilon(t) \begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - T_0(t) \begin{bmatrix} u \\ v \end{bmatrix} \right\|_X \\ & \leq C \left(\left\| \begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - \begin{bmatrix} u \\ v \end{bmatrix} \right\|_X + \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{2\theta} \right) t^{-(1/2+\theta)} e^{-\alpha t} \\ & \quad + CL_f \int_0^t (t-s)^{-1/2} e^{-\alpha(t-s)} \left\| T_\epsilon(s) \begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - T_0(s) \begin{bmatrix} u \\ v \end{bmatrix} \right\|_X ds. \end{aligned}$$

Thus the singular Gronwall inequality (see Lemma 7.1.1 in [Henry 1981]) guarantees the existence of a constant $L > 0$ such that

$$\begin{aligned} & \left\| T_\epsilon(t) \begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - T_0(t) \begin{bmatrix} u \\ v \end{bmatrix} \right\|_X \\ & \leq C e^{L_f t} t^{-(1/2+\theta)} \left(\left\| \begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - \begin{bmatrix} u \\ v \end{bmatrix} \right\|_X + \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^\theta \right). \quad \square \end{aligned}$$

6. Rate of convergence of the equilibria and of the linearizations

Now we will work to control the behavior of equilibria in terms of $\|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}$. First, we will give the definition of equilibrium of problem (1-7).

Definition 6.1. The equilibrium solutions of (1-7) are the functions that solve the stationary

$$(6-1) \quad A_\epsilon w^\epsilon = F(w^\epsilon), \quad \epsilon \in [0, 1].$$

For each $\epsilon \in [0, 1]$, we denote by \mathcal{E}_ϵ the set of the equilibrium solutions of (1-7). We say that an equilibrium w_*^ϵ of (1-7) is hyperbolic if the spectrum $\sigma(A_\epsilon - F'(w_*^\epsilon))$ of $A_\epsilon - F'(w_*^\epsilon)$ is disjoint from the imaginary axis.

We start by proving the upper semicontinuity of the family of equilibria.

Proposition 6.2. *The family $\{\mathcal{E}_\epsilon : \epsilon \in [0, 1]\}$ is upper semicontinuous at $\epsilon = 0$.*

Proof. Since \mathcal{E}_ϵ is contained in \mathcal{A}_ϵ , $\sup\{\|w^\epsilon\|_X : w^\epsilon \in \mathcal{E}_\epsilon, \epsilon \in [0, 1]\} < \infty$. Using the fact that $F : X \rightarrow Y \times Y$ is bounded, for each $w^\epsilon \in \mathcal{E}_\epsilon$, we have that $w^\epsilon = A_\epsilon^{-1} F(w^\epsilon)$, and the result follows from the uniform convergence of A_ϵ^{-1} to A_0^{-1} . \square

The proof of lower semicontinuity requires additional assumptions. We need to assume that the equilibrium points of (1-7) are stable under perturbation. This stability under perturbation will be given by the hyperbolicity.

Proposition 6.3. *Any hyperbolic point of \mathcal{E}_0 is isolated.*

Proof. We note that $w_* \in \mathcal{E}_0$ is a solution of (6-1) if and only if w_* is a fixed point of

$$\Psi(w) := (A_0 - F'(w_*))^{-1}(F(w) - F'(w_*)w).$$

If we show that, for some $r > 0$, $\Psi : \bar{B}_r(w_*) \rightarrow \bar{B}_r(w_*)$ is a contraction, where $\bar{B}_r(w_*) := \{w \in X : \|w - w_*\|_X \leq r\}$, then w_* is a unique element in $\bar{B}_r(w_*) \cap \mathcal{E}_0$ and, consequently, is isolated. In fact, letting $r > 0$ and $u, v \in \bar{B}_r(w_*)$, we observe by (4-1) that

$$\begin{aligned} \|\Psi(u) - \Psi(v)\|_X &\leq \|(A_0 - F'(w_*))^{-1}\|_{\mathcal{L}(X)}\|(F(u) - F(v) - F'(w_*)(u - v))\|_X \\ &\leq Cr\|u - v\|_X. \end{aligned}$$

Thus, choosing r such that $Cr < 1$, we have Ψ is a contraction. We can see that, if $v \in \bar{B}_r(w_*)$, then $\|\Psi(v) - w_*\|_X = \|\Psi(v) - \Psi(w_*)\|_X \leq C\|v - w_*\|_X < r$, for some constant $C \in [0, 1)$. Then $\Psi(\bar{B}_r(w_*)) \subset \bar{B}_r(w_*)$. This implies that Ψ has a unique fixed point in $\bar{B}_r(w_*)$ and the proof is complete. \square

Corollary 6.4. *The set \mathcal{E}_0 has at most a finite number of hyperbolic points.*

Proof. It follows directly of the compactness of \mathcal{E}_ϵ . \square

Now we are going to study the convergence properties of resolvent operators of the form $(A_\epsilon + V_\epsilon)^{-1}$, with $V_\epsilon \in \mathcal{L}(X^1, X)$. This is because we are interested in comparing the resolvent operators of the linearization around equilibrium.

The convergence of resolvents of $A_\epsilon + V_0$ follows from the convergence of resolvents of A_ϵ (see Proposition 3.5) and the lemma below, whose proof is immediate.

Lemma 6.5. *The operator $A_\epsilon + V_0, \epsilon \in [0, 1]$, satisfies the identity*

$$\begin{aligned} (6-2) \quad (A_\epsilon + V_0)^{-1} - (A_0 + V_0)^{-1} \\ = [I - (A_\epsilon + V_0)^{-1}V_0](A_\epsilon^{-1} - A_0^{-1})[I - V_0(A_0 + V_0)^{-1}]. \end{aligned}$$

Theorem 6.6. *Let us consider w_* a hyperbolic of \mathcal{E}_0 with $0 \notin \sigma(A_0 - f'(w_*))$. Then there exist $\epsilon_1 > 0$ and $r > 0$ such that problem (1-7) has exactly one equilibrium solution w_*^ϵ in $\bar{B}_r(w_*) := \{w \in X : \|w - w_*\|_X \leq r\}$ for $\epsilon \in [0, \epsilon_1]$. Furthermore, $\|w_*^\epsilon - w_*\|_X \leq C\|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}$ for some $C > 0$ independent of ϵ .*

Proof. The hyperbolicity of w_* means that $\sigma(A_\epsilon - f'(w_*))$ is disjoint from the imaginary axis. Thus, by Lemma 4.1, we can guarantee the existence of a constant $C > 0$ such that

$$\|(A_\epsilon - F'(w_*))^{-1}\|_{\mathcal{L}(X)} \leq C, \quad \epsilon \in [0, 1].$$

We have that w^ϵ is a solution of (6-1) if and only if it is a fixed point of the map

$$\Psi_\epsilon(w) := (A_\epsilon - F'(w_*))^{-1}(F(w) - F'(w_*)w).$$

From [Lemma 4.1](#), we get that $A_\epsilon^{1/2}(A_\epsilon - F'(w_*))^{-1}$ converges uniformly to $A_0^{1/2}(A_0 - F'(w_*))^{-1}$, which implies

$$\Psi_\epsilon(w_*) \rightarrow \Psi_0(w_*) \quad \text{in } X.$$

Now we will prove the existence of $r > 0$ and $\epsilon_1 \in [0, 1]$ such that Ψ_ϵ is a contraction of $\bar{B}_r(w_*) = \{w \in X : \|w - w_*\|_X < r\}$ into itself, uniformly in $[0, \epsilon_1]$. In fact, first we will see that Ψ_ϵ is a contraction map. For this, we take u^ϵ and v^ϵ in $\bar{B}_r(w_*)$. In this way,

$$\begin{aligned} & \|\Psi_\epsilon(u^\epsilon) - \Psi_\epsilon(v^\epsilon)\|_X \\ &= \|(A_\epsilon - F'(w_*))^{-1}[F(u^\epsilon) - F(v^\epsilon) - F'(w_*)(u^\epsilon - v^\epsilon)]\|_X \\ &\leq \|(A_\epsilon - F'(w_*))^{-1}\|_{\mathcal{L}(X, X^1)} \|F(u^\epsilon) - F(v^\epsilon) - F'(w_*)(u^\epsilon - v^\epsilon)\|_X \\ &= \|A_\epsilon^{-1}(I - F'(w_*)^{-1}A_\epsilon^{-1})\|_{\mathcal{L}(X, X^1)} \|F(u^\epsilon) - F(v^\epsilon) - F'(w_*)(u^\epsilon - v^\epsilon)\|_X, \end{aligned}$$

and, according to [Proposition 3.5](#) and (4-1), there exist $C > 0$ and $\epsilon_1 > 0$ such that

$$(6-3) \quad \|\Psi_\epsilon(u^\epsilon) - \Psi_\epsilon(v^\epsilon)\|_{X^1} \leq C\delta \|u^\epsilon - v^\epsilon\|_X, \quad \text{for all } \delta > 0 \text{ and all } \epsilon \in [0, \epsilon_1].$$

Therefore, choosing δ such that $C\delta \leq a < 1$, it follows that Ψ_ϵ is a contraction as claimed.

Let us show now that $\Psi_\epsilon(\bar{B}_r(w_*)) \subset \bar{B}_r(w_*)$. Taking $u^\epsilon \in \bar{B}_r(w_*)$, we obtain by (6-3) that

$$(6-4) \quad \begin{aligned} \|\Psi_\epsilon(u^\epsilon) - w_*\|_X &\leq \|\Psi_\epsilon(u^\epsilon) - \Psi_\epsilon(w_*)\|_X + \|\Psi_\epsilon(w_*) - w_*\|_X \\ &\leq a \|u^\epsilon - w_*\|_X + \|\Psi_\epsilon(w_*) - w_*\|_X \\ &\leq ar + \|\Psi_\epsilon(w_*) - w_*\|_X, \quad \text{for all } \epsilon \in (0, \bar{\epsilon}]. \end{aligned}$$

It follows from [Lemma 4.1](#) that there exists $\epsilon_1 > 0$ such that

$$(6-5) \quad \|\Psi_\epsilon(w_*) - w_*\|_X \leq r/2, \quad \text{for all } \epsilon \in [0, \epsilon_1].$$

Combining (6-4) and (6-5), and considering $a \leq 1/2$, we deduce that

$$\|\Psi_\epsilon(u^\epsilon) - w_*\|_X \leq r, \quad \text{for all } \epsilon \in [0, \epsilon_1],$$

and, therefore, $\Psi_\epsilon : \bar{B}_r(w_*) \rightarrow \bar{B}_r(w_*)$ is a contraction, for all $\epsilon \in [0, \epsilon_1]$. Hence, there exists a fixed point of Ψ_ϵ in $\bar{B}_r(w_*)$, which we will call w_*^ϵ .

Finally, we will find an estimate of the difference $w_*^\epsilon - w_*$ in terms of $\|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}$.

Observe that $w_*^\epsilon = \Psi_\epsilon(w_*^\epsilon)$ and $w_* = \Psi_0(w_*)$. If we denote $F'(w_*) = V_0$, we have

$$(6-6) \quad \begin{aligned} \|w_*^\epsilon - w_*\|_X &\leq \|((A_\epsilon + V_0)^{-1} - (A_0 + V_0)^{-1})[F(w_*^\epsilon) + V_0 w_*^\epsilon] \\ &\quad + (A_0 + V_0)^{-1}[F(w_*^\epsilon) - F(w_*) + V_0(w_*^\epsilon - w_*)]\|_X. \end{aligned}$$

Identity (6-2) and Proposition 3.5 give us

$$(6-7) \quad \|(A_\epsilon + V_0)^{-1} - (A_0 + V_0)^{-1}\|_{\mathcal{L}(X)} \leq C \|A_\epsilon^{-1} - A_0^{-1}\|_{\mathcal{L}(X)} \leq C \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2},$$

where the constant $C > 0$ is independent of ϵ .

On the other hand, denoting $z_*^\epsilon = F(w_*^\epsilon) - F(w_*) + V_0(w_*^\epsilon - w_*)$ and using the differentiability of the map $F : X \rightarrow Y \times Y$ (see (4-1)), we get that, for every $r > 0$, $\|z_\epsilon\|_X \leq r \|w_*^\epsilon - w_*\|_X$. Hence,

$$(6-8) \quad \|(A_0 + V_0)^{-1} z_*^\epsilon\|_X \leq r \|(A_0 + V_0)^{-1}\|_{\mathcal{L}(X)} \|w_*^\epsilon - w_*\|_X.$$

Substituting (6-7) and (6-8) in (6-5) and choosing $r > 0$ such that $r \|(A_0 + V_0)^{-1}\|_{\mathcal{L}(X)} \leq 1/2$, we obtain

$$\|w_*^\epsilon - w_*\|_X \leq C \|F(w_*^\epsilon) + V_0 w_*^\epsilon\|_X \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2} + \frac{1}{2} \|w_*^\epsilon - w_*\|_X,$$

which, combined with the fact that f and its derivative are limited, allows us to conclude

$$\|w_*^\epsilon - w_*\|_X \leq C \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}. \quad \square$$

Remark 6.7. Notice that, by assuming that elements of $\mathcal{E}_0 = \{w_*^{1,0}, \dots, w_*^{n,0}\}$ are hyperbolic, we have that the points of $\mathcal{E}_\epsilon = \{w_*^{1,\epsilon}, \dots, w_*^{n,\epsilon}\}$, with $\epsilon \in (0, 1]$, satisfy the estimate $\|w_*^{i,\epsilon} - w_*^{i,0}\|_X \leq C \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}$. We still have by (4-1) that, writing $V_\epsilon = F'(w_*^\epsilon)$ with $w_*^\epsilon \in \mathcal{E}_\epsilon$, V_ϵ converges to V_0 in the uniform topology.

Lemma 6.8. *There exists a constant $C > 0$, independent of ϵ , such that*

$$\|V_\epsilon A_\epsilon^{-1} - V_0 A_0^{-1}\|_{\mathcal{L}(X)} \leq C \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}.$$

Proof. The estimate follows by the decomposition

$$V_\epsilon A_\epsilon^{-1} - V_0 A_0^{-1} = V_\epsilon (A_\epsilon^{-1} - A_0^{-1}) + (V_\epsilon - V_0) A_0^{-1},$$

(3-25), and Theorem 6.6. □

The next result shows an analogous property found in Proposition 3.5 with $A_\epsilon + V_\epsilon$ instead of A_ϵ . This will be important in the analysis in the next section.

Proposition 6.9. *Let us consider $\bar{A}_\epsilon = A_\epsilon + V_\epsilon$ for all $\epsilon \in [0, 1]$. If $0 \in \rho(\bar{A}_0)$, then $0 \in \rho(\bar{A}_\epsilon)$, for all $\epsilon \in (0, 1]$, and the following identity holds:*

$$(6-9) \quad \bar{A}_\epsilon^{-1} - \bar{A}_0^{-1} \\ = (A_\epsilon^{-1} - A_0^{-1})(I + V_0 A_0^{-1})^{-1} - A_\epsilon^{-1}(I + V_0 A_0^{-1})^{-1}(V_\epsilon A_\epsilon^{-1} - V_0 A_0^{-1})(I + V_\epsilon A_\epsilon^{-1})^{-1}.$$

Furthermore

$$(6-10) \quad \|\bar{A}_\epsilon^{-1} - \bar{A}_0^{-1}\|_{\mathcal{L}(X)} \leq C \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2},$$

for some $C > 0$ independent of ϵ .

Proof. The first part follows from [Proposition 2.1](#). The identity (6-9) is immediate, and (6-10) follows using (6-9), [Proposition 3.5](#), and [Lemma 6.8](#). \square

The last proposition enables us to prove similar results as [Proposition 3.6](#), [Proposition 3.7](#), [Theorem 5.1](#) and [Theorem 5.2](#) for $A_\epsilon + V_\epsilon$ instead of A_ϵ .

7. Rate of convergence and attraction of local unstable manifolds

The main aim of this section is the proof of the existence unstable local manifolds as a graph of a Lipschitz function, its convergence, and exponential attraction.

For each $\epsilon \in [0, \epsilon_1]$, let us consider w_*^ϵ to be an equilibrium solution for (1-7). We assume the existence of a constant $C > 0$ such that $\|w_*^\epsilon - w_*\|_X \leq C \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}$, for all $\epsilon \in [0, \epsilon_1]$, and that $w_* := w_*^0$ is hyperbolic. To deal with a neighborhood of the equilibrium point w_*^ϵ , we rewrite the problems (1-7) as

$$(7-1) \quad z_t^\epsilon + \bar{A}_\epsilon z^\epsilon = F(u^\epsilon + w_*^\epsilon) - F(w_*^\epsilon) - F'(w_*^\epsilon)u^\epsilon,$$

where $z^\epsilon = u^\epsilon - w_*^\epsilon$ and $\bar{A}_\epsilon = A_\epsilon - F'(w_*^\epsilon)$. With this, one can look for [Proposition 3.7](#) with \bar{A}_ϵ instead of A_ϵ .

Let γ be a smooth, closed, simple, rectifiable curve in $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, oriented counterclockwise and such that the bounded connected component of $\mathbb{C} \setminus \{\gamma\}$ (here $\{\gamma\}$ denotes the trace of γ) contains $\{z \in \sigma(-\bar{A}_0) : \operatorname{Re} z > 0\}$. From part (ii) of [Proposition 3.6](#), there exists $\epsilon_1 > 0$ such that $\{\gamma\} \subset \rho(-\bar{A}_\epsilon)$ for all $\epsilon \in [0, \epsilon_1]$. We define \bar{Q}_ϵ by

$$\bar{Q}_\epsilon = \frac{1}{2\pi i} \int_\gamma (\lambda I - \bar{A}_\epsilon)^{-1} d\lambda, \quad \text{for all } \epsilon \in [0, \epsilon_1].$$

The operator \bar{A}_ϵ is self-adjoint and there exist $\beta > 0$ and $C \geq 1$ such that

$$\|e^{-\bar{A}_\epsilon t} \bar{Q}_\epsilon\|_{\mathcal{L}(X)} \leq C e^{-\beta t}, \quad \text{for all } t \geq 0 \text{ and all } \epsilon \in [0, \epsilon_1]$$

and

$$\|e^{-\bar{A}_\epsilon t} (I - \bar{Q}_\epsilon)\|_{\mathcal{L}(X)} \leq C t^{-1/2} e^{-\beta t}, \quad t > 0.$$

Using the decomposition $X = \bar{Q}_\epsilon X \oplus (I - \bar{Q}_\epsilon)X$ (the solution z^ϵ of (7-1) can be decomposed as $z^\epsilon = \bar{Q}_\epsilon z^\epsilon + (I - \bar{Q}_\epsilon)z^\epsilon$), we rewrite (7-1) as

$$(7-2) \quad \begin{cases} \frac{d}{dt}(\bar{Q}_\epsilon z^\epsilon) + \bar{A}_\epsilon \bar{Q}_\epsilon z^\epsilon = H_\epsilon(\bar{Q}_\epsilon z^\epsilon, (I - \bar{Q}_\epsilon)z^\epsilon), \\ \frac{d}{dt}[(I - \bar{Q}_\epsilon)z^\epsilon] + \bar{A}_\epsilon (I - \bar{Q}_\epsilon)z^\epsilon = G_\epsilon(\bar{Q}_\epsilon z^\epsilon, (I - \bar{Q}_\epsilon)z^\epsilon), \end{cases}$$

where

$$(7-3) \quad \begin{aligned} & H_\epsilon(\bar{Q}_\epsilon z^\epsilon, (I - \bar{Q}_\epsilon)z^\epsilon) \\ & := \bar{Q}_\epsilon [F(\bar{Q}_\epsilon z^\epsilon + (I - \bar{Q}_\epsilon)z^\epsilon + w_*^\epsilon) - F(w_*^\epsilon) - F'(w_*^\epsilon)(\bar{Q}_\epsilon z^\epsilon + (I - \bar{Q}_\epsilon)z^\epsilon)], \end{aligned}$$

and

$$(7-4) \quad G_\epsilon(\bar{Q}_\epsilon z^\epsilon, (I - \bar{Q}_\epsilon)z^\epsilon) \\ := (I - \bar{Q}_\epsilon)[F(\bar{Q}_\epsilon z^\epsilon + (I - \bar{Q}_\epsilon)z^\epsilon + w_*^\epsilon) - F(w_*^\epsilon) - F'(\bar{w}_*^\epsilon)(\bar{Q}_\epsilon z^\epsilon + (I - \bar{Q}_\epsilon)z^\epsilon)].$$

The functions H_ϵ and G_ϵ are continuously differentiable with

$$H_\epsilon(0, 0) = G_\epsilon(0, 0) = 0$$

and $H'_\epsilon(0, 0) = 0 = G'_\epsilon(0, 0) \in \mathcal{L}(X)$. For simplicity of notation, we write $\omega^\epsilon = \bar{Q}_\epsilon z^\epsilon$ and $\vartheta^\epsilon = (I - \bar{Q}_\epsilon)z^\epsilon$. Hence, given $\rho > 0$, there exist $\epsilon_1 > 0$ and $r > 0$ such that if $\|\omega^\epsilon\|_{\bar{Q}_\epsilon X} + \|\vartheta^\epsilon\|_{(I - \bar{Q}_\epsilon)X} < r$ and $\epsilon \in [0, \epsilon_1]$, then

$$(7-5) \quad \|H_\epsilon(\omega^\epsilon, \vartheta^\epsilon)\|_{\bar{Q}_\epsilon X} \leq \rho \quad \text{and} \quad \|G_\epsilon(\omega^\epsilon, \vartheta^\epsilon)\|_{(I - \bar{Q}_\epsilon)X} \leq \rho,$$

$$(7-6) \quad \|H_\epsilon(\omega^\epsilon, \vartheta^\epsilon) - H_\epsilon(\bar{\omega}^\epsilon, \bar{\vartheta}^\epsilon)\|_{\bar{Q}_\epsilon X} \leq \rho(\|\omega^\epsilon - \bar{\omega}^\epsilon\|_{\bar{Q}_\epsilon X} + \|\vartheta^\epsilon - \bar{\vartheta}^\epsilon\|_{(I - \bar{Q}_\epsilon)X})$$

and

$$(7-7) \quad \|G_\epsilon(\omega^\epsilon, \vartheta^\epsilon) - G_\epsilon(\bar{\omega}^\epsilon, \bar{\vartheta}^\epsilon)\|_{(I - \bar{Q}_\epsilon)X} \leq \rho(\|\omega^\epsilon - \bar{\omega}^\epsilon\|_{\bar{Q}_\epsilon X} + \|\vartheta^\epsilon - \bar{\vartheta}^\epsilon\|_{(I - \bar{Q}_\epsilon)X}).$$

Considering the coupled system (7-2), we can show an unstable manifold theorem using similar arguments to those in the results of Chapter 6 in [Henry 1981].

Theorem 7.1. *There exists a map $s_*^\epsilon : \bar{Q}_\epsilon X \rightarrow (I - \bar{Q}_\epsilon)X$ such that the unstable manifold of w_*^ϵ is given by*

$$W^u(w_*^\epsilon) = \{(\omega, \vartheta) \in X : \vartheta = s_*^\epsilon(\omega), \omega \in \bar{Q}_\epsilon X\}.$$

The map s_*^ϵ satisfies

$$\|s_*^\epsilon\| := \sup_{\omega \in \bar{Q}_\epsilon X} \|s_*^\epsilon(\omega)\|_X \leq C, \quad \|s_*^\epsilon(\omega) - s_*^\epsilon(\tilde{\omega})\|_X \leq \bar{C}\|\omega - \tilde{\omega}\|_{\bar{Q}_\epsilon X},$$

where $C > 0$ is a constant independent of ϵ , and for $\theta \in (0, \frac{1}{2})$ there exists a $C > 0$, independent of ϵ , such that

$$(7-8) \quad \|s_*^\epsilon - s_*^0\| \leq C \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{2\theta}.$$

Furthermore, there exists $\rho_1 > 0$, $C > 0$ (independent of ϵ), and $t_0 > 0$ such that, for any solution $(\omega^\epsilon(t), \vartheta^\epsilon(t)) \in X$, $t \in [t_0, \infty)$, of (7-2), we have

$$(7-9) \quad \|\vartheta^\epsilon(t) - s_*^\epsilon(\omega^\epsilon(t))\|_X \leq C e^{-\rho_1(t-t_0)} \|\vartheta^\epsilon(t_0) - s_*^\epsilon(\omega^\epsilon(t_0))\|_X, \quad \text{for all } t \geq t_0.$$

Proof. We consider the set

$$\Sigma_\epsilon = \{s : \bar{Q}_\epsilon X^1 \rightarrow (I - \bar{Q}_\epsilon)X : \|s\| \leq C, \|s(\omega) - s(\tilde{\omega})\|_X \leq C\|\omega - \tilde{\omega}\|_{\bar{Q}_\epsilon X}\}.$$

It is not difficult to see that $(\Sigma_\epsilon, \|\cdot\|)$ is a complete metric space.

Given $s \in \Sigma_\epsilon$ and $\eta \in \bar{Q}_\epsilon X$, we denote by $\omega^\epsilon(t) = \psi(t, \tau, \eta, s)$ the solution of

$$\begin{cases} \omega_t^\epsilon(t) + B_\epsilon \omega^\epsilon(t) = H_\epsilon(\omega^\epsilon(t), s(\omega^\epsilon(t))), & t < \tau \\ \omega^\epsilon(\tau) = \eta. \end{cases}$$

We define $\Psi_\epsilon : \Sigma_\epsilon \rightarrow \Sigma_\epsilon$ by

$$\Psi_\epsilon(s)\eta = \int_{-\infty}^{\tau} e^{-\tilde{A}_\epsilon(\tau-\xi)} G_\epsilon(\omega^\epsilon(\xi), s(\omega^\epsilon(\xi))) d\xi.$$

According to Theorem 7.1 in [Arrieta et al. 2013], we can deduce that Ψ_ϵ is a contraction. Therefore, there is a fixed point $s_*^\epsilon = \Psi(s_*^\epsilon)$ in Σ_ϵ .

Now we shall prove that the graph of s_*^ϵ $\{(\omega^\epsilon, s_*^\epsilon(\omega^\epsilon)) : \omega^\epsilon \in \bar{Q}_\epsilon X\}$ is invariant for (7-2), in the sense that initial data for (7-2) in $\{(\omega^\epsilon, s_*^\epsilon(\omega^\epsilon)) : \omega^\epsilon \in \bar{Q}_\epsilon X\}$ lead to solutions in this space. In fact, we take $(\omega_0^\epsilon, \vartheta_0^\epsilon) \in W^u(w_*^\epsilon)$ ($\vartheta_0^\epsilon = s_*^\epsilon(\omega_0^\epsilon)$). We denote by $\omega_*^\epsilon(t)$ the solution of the initial value problems

$$\begin{cases} \frac{d}{dt}(\bar{Q}_\epsilon z^\epsilon) + \bar{A}_\epsilon \bar{Q}_\epsilon z^\epsilon = H_\epsilon(\omega^\epsilon, s_*^\epsilon(\omega^\epsilon)), \\ \omega^\epsilon(0) = \omega_0^\epsilon, \end{cases}$$

where $z^\epsilon = \omega^\epsilon + \vartheta^\epsilon \in \bar{Q}_\epsilon X \oplus (I - \bar{Q}_\epsilon)X$. This defines a curve $(\omega_*^\epsilon(t), s_*^\epsilon(\omega_*^\epsilon(t))) \in W^u(w_*^\epsilon)$, $t \in \mathbb{R}$. Also, the unique solution of

$$\frac{d}{dt}[(I - \bar{Q}_\epsilon)z^\epsilon] + \bar{A}_\epsilon(I - \bar{Q}_\epsilon)z^\epsilon = G_\epsilon(\omega^\epsilon, s_*^\epsilon(\omega^\epsilon)),$$

which remains bounded as $t \rightarrow -\infty$, is

$$\vartheta_*^\epsilon(t) = (I - \bar{Q}_\epsilon)z_*^\epsilon(t) = \int_{-\infty}^t e^{\bar{A}_\epsilon(I - \bar{Q}_\epsilon)(t-\xi)} G_\epsilon(\omega_*^\epsilon(\xi), s_*^\epsilon(\omega_*^\epsilon(\xi))) d\xi = s_*^\epsilon(\omega_*^\epsilon(t)).$$

Therefore $(\omega_*^\epsilon(t), s_*^\epsilon(\omega_*^\epsilon(t)))$ is a solution of the system (7-2) through the point $(\omega_0^\epsilon, \vartheta_0^\epsilon)$, proving the invariance of the graph of s_*^ϵ .

To show (7-8), we can proceed as in the proof of Proposition 6.1 in [Arrieta et al. 2009].

Finally, the proof that the graph of s_*^ϵ is the unstable manifold that attracts exponentially, uniformly in ϵ , that is, the inequality (7-9) holds, follows by similar arguments to those in the proof of (A.8) in [Bruschi et al. 2006]. \square

Now we are able to prove our main result.

8. Proof of Theorem 1.1

The purpose of this section is to emphasize the proof of our main result. For this, we return to Theorem 1.1 to establish its proof.

Proof of Theorem 1.1. This proof follows by [Theorem 5.2](#), [Theorem 6.6](#), and [Theorem 7.1](#) jointly with [Theorems 2.1 and 2.2](#), and [Corollary 2.1](#) in Chapter 8 of [\[Babin and Vishik 1989\]](#). \square

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