THE BRIN–THOMPSON GROUPS *sV* ARE OF TYPE F$_\infty$

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We prove that the Brin–Thompson groups sV, also called higher-dimensional Thompson's groups, are of type F_∞ for all $s \in \mathbb{N}$. This result was previously shown for $s \le 3$, by considering the action of sV on a naturally associated space. Our key step is to replace this space by a subspace sX that is easier to analyze.

Recall that a group is of *type* F_∞ if it admits a classifying space with finitely many cells in each dimension. Well-known examples of groups of type F_∞ include Thompson's groups F, T, and V. Some generalizations of V were introduced by Brin [2004; 2005] and shown to be simple. We denote these groups sV, for $s \in \mathbb{N}$, with $1V = V$. These groups are usually termed higher-dimensional Thompson's groups or Brin–Thompson groups. All of the groups sV are known to be finitely presented [Hennig and Matucci 2012], and Kochloukova, Martínez-Pérez, and Nucinkis [Kochloukova et al. 2013] showed that $2V$ and $3V$ are of type F_∞. We prove that this result extends to all dimensions.

Main Theorem. *The Brin–Thompson group sV is of type F_∞ for all s.*

Fix some s. There is a natural poset $\mathcal{P}_1$ associated to sV. The realization $|\mathcal{P}_1|$ of this poset is contractible and the action of sV is proper but not cocompact. To prove the Main Theorem it suffices to produce a cocompact filtration of $|\mathcal{P}_1|$ whose connectivity tends to infinity. The tool to study relative connectivity is discrete Morse theory. This was carried out for $s = 2, 3$ in [Kochloukova et al. 2013]. However, for larger s this space quickly becomes cumbersome.

We therefore consider a subspace sX of $|\mathcal{P}_1|$, which we call the *Stein space* for sV. As before, the Stein space is contractible and the action is not cocompact. The advantage of the Stein space is that the Morse theory becomes easier to handle.

In Section 1 we recall the definition of sV. The Stein space sX is defined in Section 2 and some basic properties are verified. In Section 3 we analyze the connectivity of the subspaces in the filtration and deduce the Main Theorem.

1. The Brin–Thompson groups

The elements of the Brin–Thompson group sV can be described as dyadic self-maps of s-dimensional cubes. We will first give a brief intuition for these maps, and then delve into some formalism.

To get an intuition for the elements of sV for arbitrary s, recall first that elements of Thompson's group $V = 1V$ can be thought of as left-continuous, piecewise linear maps from the unit interval $[0, 1]$ to itself, where the slope of any linear piece is a positive dyadic rational. An equivalent description of such an element is obtained as follows: First divide the unit interval representing the domain into two halves and iterate this procedure by further subdividing some of the resulting pieces. Then similarly cut up the unit interval representing the codomain into the same number of pieces as the domain, and finally identify the pieces of the domain and codomain via a permutation. Note that the intervals identified in the last step will usually have different lengths. For more details see [Cannon et al. 1996].

To describe elements of sV, we no longer think of the unit interval but the unit s-cube $[0, 1]^s$. The unit s-cube can be halved by dyadic hyperplanes in s different directions, as can any iterated piece obtained this way. As with V, an element of sV can be described as a sequence of halvings of the domain and codomain and an identification of the resulting pieces by a permutation. Again the identification will affinely deform the individual pieces. Alternatively we can describe an element by a dyadic map from the s-cube to itself. A sequence of halvings of the s-cube will be modeled by "dyadic coverings". To get an intuition, the reader might want to look at Figure 1 (the map f_1 represents an element of $2V$). It may also be helpful to read Section 1 of [Burillo and Cleary 2010], which additionally details the *paired trees* model for elements of sV.

1A. *Dyadic maps and the group sV.* We now describe more formally the notions needed to define the group sV, and also a certain poset $\mathcal{P}_1$, which will then be used to define the space sX for our main argument.

A real number is called *dyadic* if it is of the form $k/2^l$ for some $k \in \mathbb{Z}$ and $l \in \mathbb{N}_0$. We denote by I the subspace of $[0, 1]$ of nondyadic numbers. By a *dyadic interval* we mean a set of the form $[k/2^l, (k+1)/2^l] \cap I$ with $k, l \in \mathbb{N}_0$, and the *length* of the dyadic interval is defined to be $1/2^l$. A bijection $A \to B$ between dyadic intervals is called a *simple dyadic map* if it is affine of positive slope. Note that this slope will necessarily be a power of two.

In general we consider the unit s-cube I^s (or rather, the set of nondyadic points in the unit s-cube), which is the s-fold product of I. A *brick* is a subset C of I^s that is a product of s dyadic intervals, called the *edges* of C, and the *volume* of C is the product of the lengths of its edges. Note that the volume of a brick is always a power of two. A *dyadic covering* is a finite set of bricks that disjointly cover I^s.

Note that by our definition the set I does not contain any dyadic numbers.

For a natural number m, denote by $I^s(m)$ the disjoint union

$$I^s(m) := B_1 \sqcup \cdots \sqcup B_m,$$

where each B_i is a copy of I^s. Note that I^s is the same as $I^s(1)$. We call B_i the *i-th block* of $I^s(m)$. A covering $\mathcal{U}$ of $I^s(m)$ is called *dyadic* if it is a disjoint union $\mathcal{U} = \mathcal{U}_1 \sqcup \cdots \sqcup \mathcal{U}_m$, where $\mathcal{U}_i$ is a dyadic covering of the block B_i. We denote by $\mathcal{T}_m := \{B_1, \ldots, B_m\}$ the *trivial* dyadic covering of $I^s(m)$, in which the bricks are just the blocks themselves.

Observation 1.1. The set of dyadic coverings of $I^s(m)$ is a lattice with respect to the refinement relation.

Proof. Existence of joins (that is, coarsest common refinements) as well as existence of a unique minimum (namely, $\mathcal{T}_m$) are clear. The statement now follows from standard order theory. $\square$

Let $\mathcal{U}$ and $\mathcal{V}$ be dyadic coverings of $I^s(m)$ and $I^s(n)$, respectively, and let $f : I^s(m) \to I^s(n)$ be a map. We say that the pair of dyadic coverings $(\mathcal{U}, \mathcal{V})$ is *compatible* with f if for every $C \in \mathcal{U}$, $f|_C$ is a product of simple dyadic maps and $f(C) \in \mathcal{V}$. Less formally, this means that every brick in the domain maps in an affine way to a brick in the codomain. If such a pair of dyadic coverings exists, then we say that f is a *dyadic map*. It is easy to see that composition of two dyadic maps is again a dyadic map, that every dyadic map is invertible, and that the inverse of a dyadic map is dyadic.

There is a combinatorial description of dyadic maps. If $f : I^s(m) \to I^s(n)$ is a dyadic map and $(\mathcal{U}_1, \mathcal{U}_2)$ is a compatible covering, then f induces a bijection of dyadic coverings $\mathcal{U}_1 \to \mathcal{U}_2$. Conversely, every bijection of dyadic coverings $\mathcal{U}_1 \to \mathcal{U}_2$ induces a dyadic map $I^s(m) \to I^s(n)$.

Note that two bijections $\mathcal{U}_1 \to \mathcal{V}_1$ and $\mathcal{U}_2 \to \mathcal{V}_2$ induce the same map $I^s(m) \to I^s(n)$ if and only if there are common refinements $\mathcal{U}$ and $\mathcal{V}$ such that the induced bijections $\mathcal{U} \to \mathcal{V}$ coincide.

Definition 1.2. The Brin–Thompson group sV is the group of all dyadic self maps of I^s with the multiplication given by composition, $gh := g \circ h$.

1B. *The poset $\mathcal{P}_1$.* In order to define the poset $\mathcal{P}_1$ on which sV acts we need some more notation.

Denote by $\widetilde{\mathcal{P}}_{m,n}$ the set of all dyadic maps $f : I^s(m) \to I^s(n)$, so for example $\widetilde{\mathcal{P}}_{1,1} = sV$. Let $\widetilde{\mathcal{P}}$ be the union of the $\widetilde{\mathcal{P}}_{m,n}$, where m and n range over all positive integers. Also denote by $\widetilde{\mathcal{P}}_m$ the subset of $\widetilde{\mathcal{P}}$ where the domain of the maps consists of m blocks.
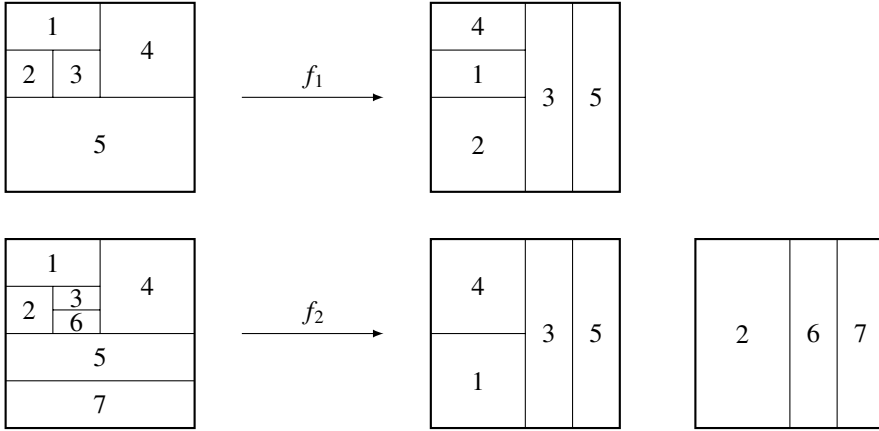
Figure 1. An example of a dyadic map $f_1: I^2(1) \to I^2(1)$ and a dyadic map $f_2: I^2(1) \to I^2(2)$, obtained from f_1 by splitting along a horizontal line. The map f_2 is equivalent in $\mathscr{P}_1$ to the one where the blocks on the right are interchanged.

There is a natural action of sV on $\widetilde{\mathscr{P}}_1$ given by precomposition: $f^g := f \circ g$ for $g \in sV$ and $f \in \widetilde{\mathscr{P}}_1$. For each positive n there is also an action of the symmetric group S_n on $\widetilde{\mathscr{P}}_{m,n}$ by permuting the blocks of the codomain. We denote the quotient $\widetilde{\mathscr{P}}_{m,n}/S_n$ by $\mathscr{P}_{m,n}$. In other words, an element of $\mathscr{P}_{m,n}$ is obtained from $\widetilde{\mathscr{P}}_{m,n}$ by forgetting the order of the blocks in the codomain. We set

$$\mathscr{P} := \bigcup_{n,m \geq 1} \mathscr{P}_{m,n} \quad \text{and} \quad \mathscr{P}_1 := \bigcup_{n \geq 1} \mathscr{P}_{1,n}.$$

Note that $\widetilde{\mathscr{P}}_{1,n}$ is an sV-invariant subset of $\widetilde{\mathscr{P}}_1$, and the action of sV on $\widetilde{\mathscr{P}}_{1,n}$ commutes with the action of the symmetric group S_n, so we get an action of sV on $\mathscr{P}_{1,n}$ for every n. In particular the sV-action on $\widetilde{\mathscr{P}}_1$ induces an action of sV on $\mathscr{P}_1$.

Definition 1.3. The function $t: \mathscr{P} \to \mathbb{N}$ assigns to each $x \in \mathscr{P}$ the number of blocks in the codomain of x, that is, if $x \in \mathscr{P}_{m,n}$ for some m, then $t(x) = n$.

Next we define a poset structure on $\mathscr{P}$ using the notion of "splitting". A dyadic map $z: I^s(m) \to I^s(n)$ is called a *splitting* (*along* $\mathscr{U}$) if z is compatible with a pair of dyadic coverings of the form $(\mathscr{U}, \mathscr{T}_n)$. The splitting z is called *nontrivial* if $n > m$. Colloquially then, as the name suggests, a nontrivial splitting is given by splitting up some cubes (and then not sticking any resulting cubes together). The inverse of a splitting (along $\mathscr{U}$) is called a *merging* (along $\mathscr{U}$).

We define an order $\leq$ on $\widetilde{\mathscr{P}}$ by saying that $x < y$ if there exists a nontrivial splitting z such that $y = z \circ x$, that is, if y is obtained from x by nontrivial splitting.

We also denote the induced order on $\mathcal{P}$ by $\leq$. In particular, $\mathcal{P}_1$ is ordered by $\leq$. See Figure 1 for an example of dyadic maps and splitting.

The poset $\mathcal{P}_1$ is filtered by the t-sublevel sets

$$\mathcal{P}_1^{\leq n} = \bigcup_{1 \leq k \leq n} \mathcal{P}_{1,k}.$$

We make the following easy observations:

Observation 1.4. The poset $\widetilde{\mathcal{P}}_1$ is directed (that is, any two elements have a common upper bound). Therefore, $|\widetilde{\mathcal{P}}_1|$ and $|\mathcal{P}_1|$ are contractible.

Observation 1.5. The action of sV on $\widetilde{\mathcal{P}}_1$ is free. Thus, for any vertex x in $|\mathcal{P}_1|$, the stabilizer $\mathrm{Stab}_{sV}(x)$ is finite. Hence all cell stabilizers are finite and of type F_∞.

Observation 1.6. The action of sV on $\mathcal{P}_1^{\leq 1}$ is transitive, and for each $n \geq 1$ the sublevel set $|\mathcal{P}_1^{\leq n}|$ is locally finite. Hence $|\mathcal{P}_1^{\leq n}|$ is finite modulo sV.

These observations suggest that the filtration $(|\mathcal{P}_1^{\leq n}|)_n$ of $|\mathcal{P}_1|$ could be used to show that sV is of type F_∞, using Brown's criterion.

Brown's criterion [Brown 1987, Corollary 3.3]. *Let G be a group and X a contractible G-CW-complex such that the stabilizer of every cell is of type F_∞. Let $\{X_j\}_{j \geq 1}$ be a filtration of X such that each X_j is finite* mod G. *Suppose that the connectivity of the pair (X_{j+1}, X_j) tends to ∞ as j tends to ∞. Then G is of type F_∞.*

It would suffice now to show that the connectivity of the pair $(|\mathcal{P}_1^{\leq n+1}|, |\mathcal{P}_1^{\leq n}|)$ tends to ∞ as n tends to ∞. This was proved for the cases $s = 2, 3$ [Kochloukova et al. 2013]. However, it becomes increasingly difficult to verify for higher s. The main difference of our approach here is that we consider a certain subcomplex sX of $|\mathcal{P}_1|$. Analyzing the relative connectivity in sX turns out to be substantially easier than in $|\mathcal{P}_1|$.

2. The Stein space for sV

The idea of passing to what we are calling a "Stein space" was first introduced by Stein [1992], and in particular was used to obtain a new proof that F is of type F_∞. This construction generalizes nicely to deal with some more complicated versions of Thompson's groups. For example Stein spaces were used in [Bux et al. 2012] to prove that braided Thompson's groups are of type F_∞. The key idea is that the splitting establishing a relation $x \leq y$ can be obtained from "elementary splittings" that give rise to elementary relations $x \preceq x_1 \preceq \cdots \preceq x_r \preceq y$, and these small steps are much easier to understand locally. Heuristically, an elementary splitting amounts to halving an s-cube at most once in any given direction. We now describe more rigorously the construction of the Stein space.

Definition 2.1. Call a brick C *elementary* if every edge of C has length at least $\frac{1}{2}$. Call an elementary brick *very elementary* if it has volume at least $\frac{1}{2}$. A dyadic covering $\mathcal{U}$ is called (*very*) *elementary* if every brick in $\mathcal{U}$ has this property. Likewise, a splitting or merging along $\mathcal{U}$ is (*very*) *elementary* if $\mathcal{U}$ is.

For $x, y \in \mathscr{P}$, if y can be obtained from x by an elementary splitting, write $x \preceq y$; if moreover $x \neq y$ then we write $x \prec y$. If y is obtained from x by a very elementary splitting, write $x \sqsubseteq y$; if moreover $x \neq y$, then we write $x \sqsubset y$. Note that the relations $\preceq$ and $\sqsubseteq$ are not transitive. In particular, the length of a chain of very elementary splittings is bounded by the number of blocks. However, if $x_1 \leq x_2 \leq x_3$ and $x_1 \preceq x_3$ then $x_1 \preceq x_2$ and $x_2 \preceq x_3$, and analogously for $\sqsubseteq$. It is clear that the action of sV respects the relations $\leq$, $\preceq$ and $\sqsubseteq$.

Clearly $I^s(m)$ has a unique maximal elementary covering $\mathscr{E}$ by $m \cdot 2^s$ bricks all of which have volume 2^{-s}. A covering is elementary if and only if $\mathscr{E}$ is a refinement of it.

The closed interval $[x, y]$ in $\mathscr{P}_1$ is defined to be $[x, y] := \{w \in \mathscr{P}_1 \mid x \leq w \leq y\}$; the open and half-open intervals are defined analogously. Call an interval $[x, y]$ in $|\mathscr{P}_1|$ *elementary* if $x \preceq y$, and *very elementary* if $x \sqsubseteq y$. A simplex of $|\mathscr{P}_1|$ is (*very*) *elementary* if there is a (very) elementary interval that contains all of its vertices.

Definition 2.2. The *Stein space for* sV, denoted sX, is the subcomplex of $|\mathscr{P}_1|$ consisting of elementary simplices.

The following statement is the key to showing the contractibility of the Stein space:

Lemma 2.3. *Let* $x, y \in \mathscr{P}_1$ *with* $x \leq y$. *There exists a unique* $y_0 \in [x, y]$ *such that* $x \preceq y_0$ *and for any* $x \preceq w \leq y$, *we have* $w \leq y_0$. *If* $x < y$, *then* $x < y_0$.

Proof. Set $m := t(x)$ and $n := t(y)$. Let $\tilde{x}$ be a representative in $\widetilde{\mathscr{P}}_1$ for x. Let $\mathcal{U}$ be the dyadic covering of $I^s(m)$ such that y is obtained from $\tilde{x}$ by splitting along $\mathcal{U}$. Let $\mathscr{E}$ be the maximal elementary covering of $I^s(m)$. The element y_0 is obtained from $\tilde{x}$ by splitting along the finest common coarsening $\mathscr{E} \wedge \mathcal{U}$. The desired properties follow from Observation 1.1. $\qquad\square$

For $x \leq y$, call the y_0 from the lemma the *elementary core of* y *with respect to* x, and denote it $\text{core}_x(y) := y_0$. When x is understood we omit the subscript. Observe that if $y_1 \leq y_2$ then $\text{core}(y_1) \leq \text{core}(y_2)$, that is, taking elementary cores respects the poset relation. Figure 2 gives an example of an elementary core.

Lemma 2.4. *For* $x < y$ *with* $x \not\prec y$, $|(x, y)|$ *is contractible.*

The proof is essentially the same as the proof of the lemma in Section 4 of [Brown 1992].
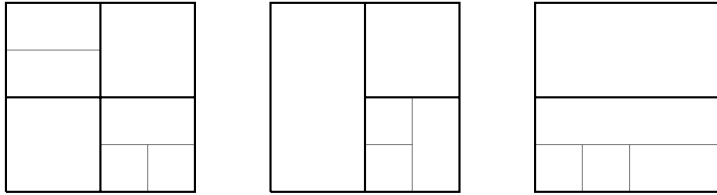
Figure 2. A nonelementary dyadic covering, for $s = 2$. The thick lines indicate the elementary core.

Proof. If $w \in (x, y]$, then core$(w) \in [x, y)$ because $x \not\prec y$, and core$(w) \in (x, y]$ because $x < w$. So in fact core$(w) \in (x, y)$. Also, core$(w) \le$ core(y) by the previous discussion. The inequalities $w \ge$ core$(w) \le$ core(y) provide a contraction of $|(x, y)|$, by Section 1.5 of [Quillen 1978]. □

As was done in [Brown 1992] for the Stein space of V, we can build up from sX to $|\mathscr{P}_1|$ to show that sX is contractible.

Corollary 2.5. *The Stein space sX is contractible for all s.*

Proof. By Observation 1.4, $|\mathscr{P}_1|$ is contractible. We build up from sX to $|\mathscr{P}_1|$ without changing the homotopy type.

Given a closed interval $[x, y]$, define $r([x, y]) := t(y) - t(x)$. We attach the contractible subcomplexes $|[x, y]|$ for $x \not\prec y$ to sX in increasing order of r-value. In particular, when we attach $|[x, y]|$, we attach it along $|[x, y)| \cup |(x, y]|$. But this is the suspension of $|(x, y)|$, and so is contractible by the previous lemma. We conclude that attaching $|[x, y]|$ does not change the homotopy type, and since $|\mathscr{P}_1|$ is contractible, so is sX. □

For each $n \ge 1$ let $sX^{\le n}$ be the full subcomplex of sX spanned by vertices x with $t(x) \le n$. Similarly define $sX^{<n}$, and let $sX^{=n}$ be the set of vertices x with $t(x) = n$. Note that all of these sets are invariant under the action of sV. We will show that the filtration $(sX^{\le n})_n$ of sX satisfies the assumptions of Brown's criterion.

Thanks to Observations 1.5 and 1.6 and to Corollary 2.5, the only remaining feature of the filtration $(sX^{\le n})_n$ of sX that we need to verify is that the connectivity of the pair $(sX^{\le n+1}, sX^{\le n})$ tends to ∞ as n tends to ∞. This is exactly the condition that proved difficult to verify for the filtration of $|\mathscr{P}_1|$ in [Kochloukova et al. 2013].

We will verify the relative connectivity in the next section using discrete Morse theory. The idea is to treat t as a height function on sX and inspect descending links.

3. Connectivity of the descending links and proof of the Main Theorem

We will use the following Morse-theoretic tools: Fix a vertex x in sX, say with $t(x) = n$, and call n the *height* of x. The *descending link* $\text{lk}{\downarrow}(x)$ of x is defined to be the intersection of $\text{lk}(x)$ with $sX^{<n}$. The fact that vertices with equal heights

cannot share an edge means that we can obtain $sX^{\leq n}$ from $sX^{<n}$ by "gluing in" each vertex at height n along its descending link. This is made rigorous by the Morse lemma (compare Corollary 2.6 of [Bestvina and Brady 1997]):

Lemma 3.1. *Let X be a simplicial complex and let $f : X^{(0)} \to \mathbb{Z}$ be such that $f(x) \neq f(y)$ for adjacent vertices x and y of X. If $\mathrm{lk}{\downarrow}(x)$ is $(k-1)$-connected for every vertex $x \in X^{=n}$, then the pair $(X^{\leq n}, X^{<n})$ is k-connected, that is, the inclusion $X^{<n} \hookrightarrow X^{\leq n}$ induces an isomorphism in π_j, $j < k$ and an epimorphism in π_k.*

Fix a vertex x in sX and consider $L(x) := \mathrm{lk}{\downarrow}(x)$. As a subcomplex of $|\mathcal{P}_1|$, $L(x)$ is the collection of simplices given by chains $y_k < \cdots < y_0 < x$ with $y_k \prec x$. We first consider the subcomplex $L_0(x)$ of $L(x)$ consisting of such chains with $y_k \sqsubset x$.

The complex $L_0(x)$ naturally projects onto a matching complex.

Definition 3.2. Let Γ be a graph. The *matching complex* $\mathcal{M}(\Gamma)$ of Γ is the simplicial complex with a k-simplex for every collection $\{e_0, \ldots, e_k\}$ of $k+1$ pairwise disjoint edges, with the face relation given by inclusion. If we regard every edge as consisting of two oriented edges (effectively doubling each edge), we get the *oriented matching complex* $\mathcal{M}^o(\Gamma)$.

The specific graphs that we will need are generalizations of complete graphs. For $s \in \mathbb{N}$, let sK_n be the graph with n nodes and s edges between any two distinct nodes. In particular $1K_n$ is just K_n, the complete graph on n nodes. Color the edges from 1 to s so that any two distinct nodes have precisely one edge of each color between them. For a fixed labeling 1 through n of the nodes of each sK_n, we have a projection $s\pi : sK_n \to K_n$ for each s, given by sending an edge with endpoints i and j to the unique edge of K_n with endpoints i and j. Since disjoint edges map to disjoint edges, this induces a map $\mathcal{M}(s\pi) : \mathcal{M}(sK_n) \to \mathcal{M}(K_n)$.

For any $l \in \mathbb{Z}$, define $\nu(l) := \lfloor (l-2)/3 \rfloor$.

Lemma 3.3. $\mathcal{M}(sK_n)$ is $(\nu(n)-1)$-connected, as is $\mathcal{M}^o(sK_n)$.

Proof. It is well known that $\mathcal{M}(K_n)$ is $(\nu(n)-1)$-connected; see for example [Athanasiadis 2004; Bux et al. 2012; Björner et al. 1994]. For any k-simplex σ in $\mathcal{M}(K_n)$, the fiber $\mathcal{M}(s\pi)^{-1}(\sigma)$ is the join of the fibers of the vertices of σ, so it is homotopy equivalent to a wedge of spheres of dimension k. It is clear also that links in $\mathcal{M}(K_n)$ are themselves matching complexes of complete graphs. Therefore the hypotheses of Theorem 9.1 in [Quillen 1978] are satisfied, and we conclude that $\mathcal{M}(sK_n)$ is $(\nu(n)-1)$-connected. We also have an obvious map $\mathcal{M}^o(sK_n) \twoheadrightarrow \mathcal{M}(sK_n)$ obtained by forgetting the orientation on the edges. The fibers of this map are similarly spherical of the right dimension, so again using Theorem 9.1 of [Quillen 1978] we conclude that $\mathcal{M}^o(sK_n)$ is $(\nu(n)-1)$-connected. $\square$
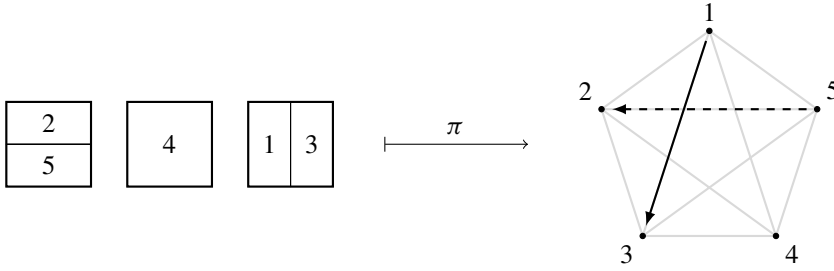
Figure 3. An example of $\pi \colon VE_n \to \mathcal{M}^o(sK_n)$ in the case $n = 5$ and $s = 2$. The solid arrow corresponds to a merge along a vertical face, and the dashed arrow corresponds to a merge along a horizontal face.

Every vertex $y \in L_0(x)$, say with $t(y) = m$, is obtained from x by applying a nontrivial very elementary merging. The merging is given by a very elementary covering $\mathcal{U}$ of m blocks whose n bricks are indexed by the blocks of x. Two such mergings define the same element y if and only if they differ by a permutation of the blocks. Consequently, denoting by VE_n the set of very elementary coverings by n labeled bricks up to permutation of the blocks, we get a one-to-one correspondence between $L_0(x)$ and VE_n. We obtain a partial order VE_n from the partial order on $\mathcal{P}_1$ via this identification.

Corollary 3.4. *VE_n, and therefore $L_0(x)$, is isomorphic to $\mathcal{M}^o(sK_n)$. Hence, both are $(v(n)-1)$-connected.*

Proof. Consider a nontrivial very elementary dyadic covering $\mathcal{U}$ of $I^s(m)$ with n bricks labeled 1 to n. Since $\mathcal{U}$ is very elementary, each block consists of at most two bricks. If it does consist of two bricks, then it defines an oriented edge in the graph sK_n as follows. The two bricks are

$$I^{k-1} \times \left(I \cap \left[0, \tfrac{1}{2}\right]\right) \times I^{s-k} \quad \text{and} \quad I^{k-1} \times \left(I \cap \left[\tfrac{1}{2}, 1\right]\right) \times I^{s-k}$$

for some $1 \le k \le s$. Say the first brick is labeled i and the second brick is labeled j. Then the edge in sK_n defined by this block points from i to j and has color k. See Figure 3 for an example.

This procedure defines an isomorphism of ordered sets $VE_n \to \mathcal{M}^o(sK_n)$. The connectivity statement now follows from Lemma 3.3. $\qquad\square$

The next step is to show that $L(x)$ is highly connected by building up from $L_0(x)$ to $L(x)$ along highly connected links. If $s = 1$, then $L_0(x) = L(x)$ so we may assume $s > 1$ in what follows.

We start by giving a combinatorial description of $L(x)$ similar to the one given for $L_0(x)$ before. Every vertex in $L(x)$ is obtained from x via a nontrivial elementary merging. We can therefore replace "very elementary" by "elementary" in the

discussion of VE_n above. We get that the poset E_n of elementary mergings of n labeled bricks up to permutation of blocks is isomorphic to $L(x)$.

We now describe the Morse function that determines in which order we build up from $L_0(x)$ to $L(x)$. For any $\mathcal{U} \in E_n$, the volume of any brick in $\mathcal{U}$ is at least $1/2^s$. For each $0 \le i \le s$ define c_i to be the number of bricks in $\mathcal{U}$ with volume $1/2^i$. Then define c to be the lexicographically ordered function $c = (c_s, c_{s-1}, \ldots, c_3, c_2)$. Note that we *do not* include c_1 or c_0 in this tuple; this will be crucial to our arguments. Denote by b the number of blocks of $\mathcal{U}$. The *height* h of $\mathcal{U}$ is now defined to be $h = (c, b)$, ordered lexicographically.

Observation 3.5. Let $\mathcal{X}$ and $\mathcal{Y}$ be vertices in E_n with $\mathcal{X} < \mathcal{Y}$. Then $c(\mathcal{X}) \ge c(\mathcal{Y})$ and $b(\mathcal{X}) < b(\mathcal{Y})$, so in particular $h(\mathcal{X}) < h(\mathcal{Y})$ if and only if $c(\mathcal{X}) = c(\mathcal{Y})$, and $h(\mathcal{X}) > h(\mathcal{Y})$ if and only if $c(\mathcal{X}) > c(\mathcal{Y})$.

Fix a vertex $\mathcal{U}$ in $E_n \setminus VE_n$. The descending link of $\mathcal{U}$ with respect to h will be denoted $\mathrm{lk}\!\downarrow_h(\mathcal{U})$. There are two types of vertices $\mathcal{V}$ in $\mathrm{lk}\!\downarrow_h(\mathcal{U})$. First, we could have $\mathcal{U} > \mathcal{V}$ and $h(\mathcal{U}) > h(\mathcal{V})$, which by the above observation implies that $c(\mathcal{U}) = c(\mathcal{V})$. The full subcomplex of $\mathrm{lk}\!\downarrow_h(\mathcal{U})$ spanned by such vertices will be called the (*descending*) *down-link*. Second, we could have $\mathcal{U} < \mathcal{V}$ and $h(\mathcal{U}) > h(\mathcal{V})$, which implies that $c(\mathcal{U}) > c(\mathcal{V})$. The full subcomplex of $\mathrm{lk}\!\downarrow_h(\mathcal{U})$ spanned by these vertices will be called the (*descending*) *up-link*.

Observation 3.6. Vertices $\mathcal{V}$ in the down-link and $\mathcal{W}$ in the up-link automatically satisfy $\mathcal{V} < \mathcal{W}$. Therefore $\mathrm{lk}\!\downarrow_h(\mathcal{U})$ is a join of the down-link and the up-link.

This allows us to study the up-link and the down-link separately.

Lemma 3.7. *If $\mathcal{U}$ has a block with precisely two bricks, then the up-link of $\mathcal{U}$ is contractible, and hence so is $\mathrm{lk}\!\downarrow_h(\mathcal{U})$.*

Proof. Let B be a block in $\mathcal{U}$ with two bricks. Note that splitting only B does not yield a vertex with lower height. For a vertex $\mathcal{V}$ of the up-link we define a vertex $\mathcal{V}_0$ as follows (see Figure 4): Since $\mathcal{V}$ is in the up-link, it is obtained from $\mathcal{U}$ by splitting. Let $\mathcal{V}_0$ be the covering obtained from $\mathcal{U}$ by doing all the same splittings as for $\mathcal{V}$, except that B is not split (whether or not it was split for $\mathcal{V}$). Then $\mathcal{V}_0 > \mathcal{U}$, since $\mathcal{V}$ was obtained by splitting more than just B, as observed above. It is also clear that $c(\mathcal{V}_0) < c(\mathcal{U})$, and so $\mathcal{V}_0$ is again in the up-link of $\mathcal{U}$. Now let $\mathcal{Z}_B$ be the maximal elementary splitting of $\mathcal{U}$ that does not split B. Then for all $\mathcal{V}$ in the up-link, we have $\mathcal{V}_0 \le \mathcal{Z}_B$. Hence we have the inequalities $\mathcal{V} \ge \mathcal{V}_0 \le \mathcal{Z}_B$, which provide a contraction of the up-link of $\mathcal{U}$, by Section 1.5 of [Quillen 1978]. $\square$

For $l \in \mathbb{Z}$, define $\eta(l) := \lfloor (l-2)/2^s \rfloor$. Note that, for a fixed s, as $n \to \infty$, $\eta(n)$ increases monotonically to ∞.

Lemma 3.8. *If $\mathcal{U}$ has no block with precisely two bricks, then $\mathrm{lk}\!\downarrow_h(\mathcal{U})$ is at least $(\eta(n)-2)$-connected.*
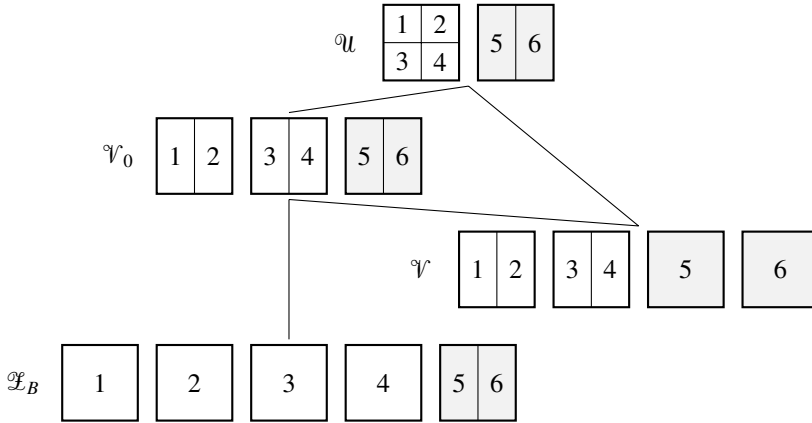
Figure 4. A step in building up from VE_6 to E_6 as described in the proof of Lemma 3.7. The block B of the covering $\mathcal{U}$ and its images under the various splittings are highlighted.

Proof. Call a block in $\mathcal{U}$ with more than two bricks *big*, and a block with only one brick *small*. Let k_b be the number of big blocks and k_s the number of small blocks. By assumption $k_b + k_s$ is the number m of blocks in $\mathcal{U}$.

The up-link of $\mathcal{U}$ is clearly at least $(k_b - 2)$-connected, since splitting a big block in any way produces a vertex with lower height, and so each big block contributes a nonempty join factor to the up-link. The down-link of $\mathcal{U}$ is isomorphic to VE_{k_s}, and therefore is $(\nu(k_s) - 1)$-connected by Corollary 3.4. This implies that $\mathrm{lk}{\downarrow_h}(\mathcal{U})$ is $(k_b + \nu(k_s) - 1)$-connected. Also, n is the number of bricks in $\mathcal{U}$, so $n \le 2^s k_b + k_s$.

Since $s > 1$, $2^s > 3$, so we have

$$k_b + \nu(k_s) - 1 \ge k_b + \left\lfloor \frac{k_s - 2}{2^s} \right\rfloor - 1 \ge k_b + \frac{k_s - 2}{2^s} - 2$$

$$= \frac{2^s k_b + k_s - 2}{2^s} - 2 \ge \frac{n - 2}{2^s} - 2 \ge \eta(n) - 2.$$

We conclude that $\mathrm{lk}{\downarrow_h}(\mathcal{U})$ is at least $(\eta(n) - 2)$-connected. $\square$

Corollary 3.9. *If $s = 1$ then E_n, and hence $L(x)$ is $(\nu(n) - 1)$-connected. If $s > 1$, then E_n, and hence $L(x)$ is $(\eta(n) - 1)$-connected.*

Proof. The $s = 1$ case is already done, since then $E_n = VE_n$. Now suppose $s > 1$. Then $\eta \le \nu$, so VE_n is at least $(\eta(n) - 1)$-connected. Also, for $\mathcal{U} \in E_n \setminus VE_n$, $\mathrm{lk}{\downarrow_h}(\mathcal{U})$ is $(\eta(n) - 2)$-connected by Lemmas 3.7 and 3.8. It follows from Lemma 3.1 that E_n is at least $(\eta(n) - 1)$-connected. $\square$

Proposition 3.10. *For each $n \ge 1$, the pair $(sX^{\le n}, sX^{<n})$ is $\eta(n)$-connected for $s > 1$, and the pair $(1X^{\le n}, 1X^{<n})$ is $\nu(n)$-connected.*

Proof. Let x be a vertex in $sX^{=n}$. By Corollary 3.9, the descending link $\mathrm{lk}{\downarrow}(x)$ of x in sX is at least $(\eta(n)-1)$-connected for $s > 1$, or $(\nu(n)-1)$-connected for $s = 1$. The result now follows from Lemma 3.1. □

We are now in a position to apply Brown's criterion.

Proof of Main Theorem. Consider the action of sV on sX. By Corollary 2.5, sX is contractible, by Observation 1.5, the stabilizer of every cell is finite, and by Observation 1.6, each $sX^{\leq n}$ is finite modulo sV. By Proposition 3.10, the connectivity of the pairs $(sX^{\leq n}, sX^{<n})$ tends to ∞ as n tends to ∞. Hence, sV is of type F_∞ by Brown's criterion. □

Acknowledgments

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