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**A STUDY OF REAL HYPERSURFACES
WITH RICCI OPERATORS
IN 2-DIMENSIONAL COMPLEX SPACE FORMS**

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A STUDY OF REAL HYPERSURFACES WITH RICCI OPERATORS IN 2-DIMENSIONAL COMPLEX SPACE FORMS

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We prove that a real hypersurface M in complex projective space $P_2(\mathbb{C})$ or complex hyperbolic space $H_2(\mathbb{C})$, whose Ricci operator is η -parallel and commutes with the structure tensor on the holomorphic distribution, is a Hopf hypersurface. We also give a characterization of this hypersurface.

1. Introduction

A complex n -dimensional Kählerian manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. As is well known, a complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n(\mathbb{C})$, a complex Euclidean space \mathbb{C}^n or a complex hyperbolic space $H_n(\mathbb{C})$, according to $c > 0$, $c = 0$ or $c < 0$.

In this paper we consider a real hypersurface M in a complex space form $M_2(c)$, $c \neq 0$. Then M has an almost contact metric structure (ϕ, g, ξ, η) induced from the Kähler metric and complex structure J on $M_n(c)$. The structure vector field ξ is said to be *principal* if $A\xi = \alpha\xi$ is satisfied, where A is the shape operator of M and $\alpha = \eta(A\xi)$. In this case, it is known that α is locally constant [Ki and Suh 1990] and that M is called a *Hopf hypersurface*.

Takagi [1973] classified homogeneous real hypersurfaces in $P_n(\mathbb{C})$ into six model spaces A_1, A_2, B, C, D and E of Hopf hypersurfaces with constant principal curvatures. Berndt [1989] classified all homogeneous Hopf hypersurfaces in $H_n(\mathbb{C})$ as four model spaces, which are said to be A_0, A_1, A_2 and B . A real hypersurface M of type A_1 or A_2 in $P_n(\mathbb{C})$ or type A_0, A_1 or A_2 in $H_n(\mathbb{C})$ is said to be *of type A* for simplicity.

As a typical characterization of real hypersurfaces of type A , the following is due to Okumura [1975] for $c > 0$, and Montiel and Romero [1986] for $c < 0$.

Theorem A [Montiel and Romero 1986; Okumura 1975]. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. It satisfies $A\phi - \phi A = 0$ on M if and only if M is locally congruent to one of the model spaces of type A .*

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The Ricci operator of M will be denoted by S , and the shape operator or the second fundamental tensor field of M by A . The holomorphic distribution T_0 of a real hypersurface M in $M_n(c)$ is defined by

$$(1-1) \quad T_0(p) = \{X \in T_p(M) \mid g(X, \xi)_p = 0\},$$

where $T_p(M)$ is the tangent space of M at $p \in M$. The Ricci operator S is said to be η -parallel if

$$(1-2) \quad g((\nabla_X S)Y, Z) = 0$$

for any vector fields X, Y and Z in T_0 .

Theorem B [Kimura and Maeda 1989; Suh 1990]. *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. Then the Ricci operator of M is η -parallel and the structure vector field ξ is a principal if and only if M is locally congruent to one of the model spaces of type A or type B.*

I.-B. Kim, K. H. Kim and one of the present authors [Kim et al. 2006; 2007] studied real hypersurfaces with certain conditions related to the Ricci operator and the structure tensor field ϕ in $M_n(c)$. As for the Ricci operator and structure tensor field ϕ , one of the present authors proved the following.

Theorem C [Sohn 2007]. *Let M be a real hypersurface with η -parallel Ricci operator in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$. If M satisfies*

$$(1-3) \quad g((S\phi - \phi S)X, Y) = 0$$

for any X and Y in T_0 , then M is locally congruent to one of the model spaces of type A or type B.

The purpose of this paper is to complete the results of [Sohn 2007] and characterize real hypersurfaces with η -parallel Ricci operator such that the Ricci operator and structure tensor field commute in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 2$. Namely, we prove:

Theorem. *A real hypersurface in a complex space form $M_2(c)$, $c \neq 0$ satisfies (1-2) and (1-3) if and only if it is pseudo-Einstein.*

The pseudo-Einstein hypersurfaces are classified by Kim and Ryan [2008] and Ivey and Ryan [2009] and are described in detail in these papers. In view of their results, we can state the following.

Corollary. *Let M be a real hypersurface with an η -parallel Ricci operator in a complex space form $M_2(c)$, $c \neq 0$. If M satisfies (1-3) then M is locally congruent to either a Hopf hypersurface with $A\xi = 0$ or one of the model spaces of type A.*

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M_2(c)$, and N be a unit normal vector field of M . By $\tilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini–Study metric tensor \tilde{g} of $M_2(c)$. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N \quad \text{and} \quad \tilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M , where g denotes the Riemannian metric tensor of M induced from \tilde{g} , and A is the shape operator of M in $M_2(c)$.

For any vector field X on M we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi,$$

where J is the almost complex structure of $M_2(c)$. Then we see that M induces an almost contact metric structure (ϕ, g, ξ, η) , that is,

$$\begin{aligned} \phi^2 X &= -X + \eta(X)\xi, & \phi\xi &= 0, & \eta(\xi) &= 1, \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), & \eta(X) &= g(X, \xi) \end{aligned}$$

for any vector fields X and Y on M . Since the almost complex structure J is parallel, we can verify from the Gauss formula that

$$(2-1) \quad \nabla_X \xi = \phi AX.$$

Since the ambient manifold is of constant holomorphic sectional curvature c , we have the Gauss equation

$$\begin{aligned} (2-2) \quad R(X, Y)Z &= \frac{c}{4} (g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z) \\ &\quad + g(AY, Z)AX - g(AX, Z)AY \end{aligned}$$

for any vector fields X, Y and Z on M , where R denotes the Riemannian curvature tensor of M .

From (1-3) the Ricci operator S of M is expressed by

$$(2-3) \quad SX = \frac{c}{4} ((2n + 1)X - 3\eta(X)\xi) + mAX - A^2X,$$

where $m = \text{trace } A$ is the mean curvature of M , and the covariant derivative of (2-3) is given by

$$\begin{aligned} (\nabla_X S)Y &= -\frac{3c}{4} (g(\phi AX, Y)\xi + \eta(Y)\phi AX) \\ &\quad + (Xm)AY + m(\nabla_X A)Y - (\nabla_X A)AY - A(\nabla_X A)Y. \end{aligned}$$

Let U be a unit vector field on M with the same direction of the vector field $-\phi\nabla_\xi\xi$, and let β be the length of the vector field $-\phi\nabla_\xi\xi$ if it does not vanish. It is not possible to define U without specifying that $\beta \neq 0$. Then it is easily seen from (2-1) that

$$(2-4) \quad A\xi = \alpha\xi + \beta U,$$

where $\alpha = \eta(A\xi)$. We notice here that U is orthogonal to ξ .

We put

$$\Omega = \{p \in M \mid \beta(p) \neq 0\}.$$

Then Ω is an open subset of M .

3. η -parallel Ricci operators

In this section, we assume that Ω is not empty. Then there are scalar fields γ , ε and δ and a unit vector field U and ϕU orthogonal to ξ such that

$$(3-1) \quad AU = \beta\xi + \gamma U + \varepsilon\phi U, \quad A\phi U = \varepsilon U + \delta\phi U$$

and

$$(3-2) \quad m = \text{trace } A = \alpha + \gamma + \delta$$

in $M_2(c)$.

We shall prove the following lemmas.

Lemma 3.1. *Let M be a real hypersurface in a complex space form $M_2(c)$, $c \neq 0$. If M satisfies (1-3), then we have $AU = \beta\xi + \gamma U$, $A\phi U = \delta\phi U$ and $\beta^2 = \alpha(\gamma - \delta)$.*

Proof. If we put $X = \xi$ into (2-3), we have

$$(3-3) \quad S\xi = \left(\frac{c}{2} + \alpha\gamma + \alpha\delta - \beta^2\right)\xi + \beta\delta U - \beta\varepsilon\phi U.$$

Putting $X = U$ into (2-3) and taking account of (3-1) yields

$$(3-4) \quad SU = \beta\delta\xi + \left(\frac{5c}{4} + \alpha\gamma + \gamma\delta - \beta^2 - \varepsilon^2\right)U + \alpha\varepsilon\phi U.$$

Putting $X = \phi U$ into (2-3) and using (3-1), we obtain

$$(3-5) \quad S\phi U = -\beta\varepsilon\xi + \alpha\varepsilon U + \left(\frac{5c}{4} + \alpha\delta + \gamma\delta - \varepsilon^2\right)\phi U.$$

If we apply ϕ to (3-4), then we have

$$(3-6) \quad (S\phi - \phi S)U = -\beta\varepsilon\xi + 2\alpha\varepsilon U + (\alpha\delta - \alpha\gamma + \beta^2)\phi U.$$

From condition (1-3), we have, for all $X \in T_0$,

$$(3-7) \quad (S\phi - \phi S)X = -\beta g(\varepsilon U + \delta\phi U, X)\xi$$

If we substitute $X = U$ into (3-7), then we obtain

$$(3-8) \quad (S\phi - \phi S)U = -\beta\varepsilon\xi.$$

Comparing (3-6) and (3-8), we get $\varepsilon = 0$ and $\beta^2 = \alpha(\gamma - \delta)$. It follows that AU is expressed in terms of ξ and U only and $A\phi U$ is given by ϕU . \square

It follows from (2-3) and (3-1) that

$$(3-9) \quad S\xi = \left(\frac{c}{2} + 2\alpha\delta\right)\xi + \beta\delta U,$$

$$(3-10) \quad SU = \beta\delta\xi + \left(\frac{5c}{4} + \gamma\delta + \alpha\delta\right)U,$$

$$(3-11) \quad S\phi U = \left(\frac{5c}{4} + \gamma\delta + \alpha\delta\right)\phi U.$$

Lemma 3.2. *Under the assumptions of Lemma 3.1, if M has the η -parallel Ricci operator S , then we have $AU = \beta\xi + \gamma U$, $A\phi U = 0$ and $\beta^2 = \alpha\gamma$.*

Proof. Differentiating (3-10) covariantly along vector field X in T_0 , we obtain

$$(\nabla_X S)U = \left(\left(\frac{5c}{4} + \gamma\delta + \alpha\delta\right)I - S\right)\nabla_X U + \beta\delta\phi AX + X(\beta\delta)\xi + X\left(\frac{5c}{4} + \gamma\delta + \alpha\delta\right)U.$$

Taking the inner product of this equation with U and ϕU and making use of (3-9)–(3-11) and Lemma 3.1, we obtain

$$(3-12) \quad (\alpha + \gamma)\nabla\delta + \delta(\nabla\gamma + \nabla\alpha) = 2\beta\delta^2\phi U$$

and

$$\delta\gamma = 0.$$

If we differentiate this along the vector field X in T_0 , then (3-12) is reduced to

$$(3-13) \quad \alpha\nabla\delta + \delta\nabla\alpha = 2\beta\delta^2\phi U.$$

Differentiating (3-11) covariantly along vector field X in T_0 , we obtain

$$(3-14) \quad (\nabla_X S)\phi U = \left(\left(\frac{5c}{4} + \gamma\delta + \alpha\delta\right)I - S\right)\nabla_X \phi U + \left(X\left(\frac{5c}{4} + \gamma\delta + \alpha\delta\right)\right)\phi U.$$

If we take the inner product of (3-14) with ϕU and use (3-9)–(3-11), then we have

$$(3-15) \quad \alpha\nabla\delta + \delta\nabla\alpha = 0.$$

Comparing (3-13) and (3-15), we obtain $\delta = 0$ and $\beta^2 = \alpha\gamma$ from Lemma 3.1. From this and Lemma 3.1 we conclude that AU is expressed in terms of ξ and U only and $A\phi U = 0$. \square

4. Proof of the main theorem

Assume that M satisfies (1-2) and (1-3). We first show that M is Hopf. If the open set Ω is not empty, then Lemma 3.2 yields $\delta = 0$. Thus the Ricci operator, as expressed in (3-9)–(3-11), has the property that ξ , U and ϕU are eigenvectors and that U and ϕU have the same eigenvalue. That is, M is pseudo-Einstein with

$$SX = \frac{5c}{4}X - \frac{3c}{4}g(X, \xi)\xi.$$

This contradicts a result from [Kim and Ryan 2008]. Thus we conclude that any hypersurface satisfying (1-2) and (1-3) must be Hopf.

Since M is Hopf, condition (1-3) yields $\alpha(\gamma - \delta) = 0$ and that the criteria for Proposition 2.21 in [Kim and Ryan 2008] are satisfied. Thus M is pseudo-Einstein.

Conversely, if M is pseudo-Einstein, observe that (1-2) and (1-3) must be satisfied. \square

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
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