A STUDY OF REAL HYPERSURFACES WITH RICCI OPERATORS IN 2-DIMENSIONAL COMPLEX SPACE FORMS

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We prove that a real hypersurface $M$ in complex projective space $P_2(\mathbb{C})$ or complex hyperbolic space $H_2(\mathbb{C})$, whose Ricci operator is $\eta$-parallel and commutes with the structure tensor on the holomorphic distribution, is a Hopf hypersurface. We also give a characterization of this hypersurface.

1. Introduction

A complex $n$-dimensional Kählerian manifold of constant holomorphic sectional curvature $c$ is called a complex space form, which is denoted by $M_n(c)$. As is well known, a complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n(\mathbb{C})$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n(\mathbb{C})$, according to $c > 0$, $c = 0$ or $c < 0$.

In this paper we consider a real hypersurface $M$ in a complex space form $M_2(c)$, $c \neq 0$. Then $M$ has an almost contact metric structure $(\phi, g, \xi, \eta)$ induced from the Kähler metric and complex structure $J$ on $M_n(c)$. The structure vector field $\xi$ is said to be principal if $A\xi = \alpha \xi$ is satisfied, where $A$ is the shape operator of $M$ and $\alpha = \eta(A\xi)$. In this case, it is known that $\alpha$ is locally constant [Ki and Suh 1990] and that $M$ is called a Hopf hypersurface.

Takagi [1973] classified homogeneous real hypersurfaces in $P_n(\mathbb{C})$ into six model spaces $A_1$, $A_2$, $B$, $C$, $D$ and $E$ of Hopf hypersurfaces with constant principal curvatures. Berndt [1989] classified all homogeneous Hopf hypersurfaces in $H_n(\mathbb{C})$ as four model spaces, which are said to be $A_0$, $A_1$, $A_2$ and $B$. A real hypersurface $M$ of type $A_1$ or $A_2$ in $P_n(\mathbb{C})$ or type $A_0$, $A_1$ or $A_2$ in $H_n(\mathbb{C})$ is said to be of type $A$ for simplicity.

As a typical characterization of real hypersurfaces of type $A$, the following is due to Okumura [1975] for $c > 0$, and Montiel and Romero [1986] for $c < 0$.

**Theorem A** [Montiel and Romero 1986; Okumura 1975]. *Let $M$ be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. It satisfies $A\phi - \phi A = 0$ on $M$ if and only if $M$ is locally congruent to one of the model spaces of type $A$.***

**MSC2010:** primary 53C40; secondary 53C15.

**Keywords:** real hypersurface, $\eta$-parallel Ricci operator, Hopf hypersurface.
The Ricci operator of $M$ will be denoted by $S$, and the shape operator or the second fundamental tensor field of $M$ by $A$. The holomorphic distribution $T_0$ of a real hypersurface $M$ in $M_n(c)$ is defined by

\begin{equation}
T_0(p) = \{ X \in T_p(M) \mid g(X, \xi)_p = 0 \},
\end{equation}

where $T_p(M)$ is the tangent space of $M$ at $p \in M$. The Ricci operator $S$ is said to be $\eta$-parallel if

\begin{equation}
g((\nabla_X S)Y, Z) = 0
\end{equation}

for any vector fields $X$, $Y$ and $Z$ in $T_0$.

**Theorem B** [Kimura and Maeda 1989; Suh 1990]. *Let $M$ be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. Then the Ricci operator of $M$ is $\eta$-parallel and the structure vector field $\xi$ is a principal if and only if $M$ is locally congruent to one of the model spaces of type A or type B.*

I.-B. Kim, K. H. Kim and one of the present authors [Kim et al. 2006; 2007] studied real hypersurfaces with certain conditions related to the Ricci operator and the structure tensor field $\phi$ in $M_n(c)$. As for the Ricci operator and structure tensor field $\phi$, one of the present authors proved the following.

**Theorem C** [Sohn 2007]. *Let $M$ be a real hypersurface with $\eta$-parallel Ricci operator in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$. If $M$ satisfies

\begin{equation}
g((S\phi - \phi S)X, Y) = 0
\end{equation}

for any $X$ and $Y$ in $T_0$, then $M$ is locally congruent to one of the model spaces of type A or type B.*

The purpose of this paper is to complete the results of [Sohn 2007] and characterize real hypersurfaces with $\eta$-parallel Ricci operator such that the Ricci operator and structure tensor field commute in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 2$. Namely, we prove:

**Theorem.** *A real hypersurface in a complex space form $M_2(c)$, $c \neq 0$ satisfies (1-2) and (1-3) if and only if it is pseudo-Einstein.*

The pseudo-Einstein hypersurfaces are classified by Kim and Ryan [2008] and Ivey and Ryan [2009] and are described in detail in these papers. In view of their results, we can state the following.

**Corollary.** *Let $M$ be a real hypersurface with an $\eta$-parallel Ricci operator in a complex space form $M_2(c)$, $c \neq 0$. If $M$ satisfies (1-3) then $M$ is locally congruent to either a Hopf hypersurface with $A\xi = 0$ or one of the model spaces of type A.*
2. Preliminaries

Let \( M \) be a real hypersurface immersed in a complex space form \( M_2(c) \), and \( N \) be a unit normal vector field of \( M \). By \( \tilde{\nabla} \) we denote the Levi-Civita connection with respect to the Fubini–Study metric tensor \( \tilde{g} \) of \( M_2(c) \). Then the Gauss and Weingarten formulas are given respectively by

\[
\tilde{\nabla}_X Y = \nabla_X Y + g(A X, Y) N \quad \text{and} \quad \tilde{\nabla}_X N = -AX
\]

for any vector fields \( X \) and \( Y \) tangent to \( M \), where \( g \) denotes the Riemannian metric tensor of \( M \) induced from \( \tilde{g} \), and \( A \) is the shape operator of \( M \) in \( M_2(c) \).

For any vector field \( X \) on \( M \) we put

\[
J X = \phi X + \eta(X) N, \quad J N = -\xi,
\]

where \( J \) is the almost complex structure of \( M_2(c) \). Then we see that \( M \) induces an almost contact metric structure \((\phi, g, \xi, \eta)\), that is,

\[
\phi^2 X = -X + \eta(X) \xi, \quad \phi \xi = 0, \quad \eta(\xi) = 1,
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X) \eta(Y), \quad \eta(X) = g(X, \xi)
\]

for any vector fields \( X \) and \( Y \) on \( M \). Since the almost complex structure \( J \) is parallel, we can verify from the Gauss formula that

(2-1) \[ \nabla_X \xi = \phi AX. \]

Since the ambient manifold is of constant holomorphic sectional curvature \( c \), we have the Gauss equation

(2-2) \[ R(X, Y)Z = \frac{c}{4} (g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z) + g(AY, Z)AX - g(AX, Z)AY \]

for any vector fields \( X, Y \) and \( Z \) on \( M \), where \( R \) denotes the Riemannian curvature tensor of \( M \).

From (1-3) the Ricci operator \( S \) of \( M \) is expressed by

(2-3) \[ SX = \frac{c}{4} ((2n + 1)X - 3\eta(X)\xi) + mAX - A^2 X, \]

where \( m = \text{trace} A \) is the mean curvature of \( M \), and the covariant derivative of (2-3) is given by

\[
(\nabla_X S)Y = -\frac{3c}{4} (g(\phi AX, Y)\xi + \eta(Y)\phi AX)
+ (Xm)AY + m(\nabla_X A)Y - (\nabla_X A)AY - A(\nabla_X A)Y.
\]
Let $U$ be a unit vector field on $M$ with the same direction of the vector field $-\phi \nabla_\xi \xi$, and let $\beta$ be the length of the vector field $-\phi \nabla_\xi \xi$ if it does not vanish. It is not possible to define $U$ without specifying that $\beta \neq 0$. Then it is easily seen from (2-1) that

\begin{equation}
(2-4) \quad A\xi = \alpha \xi + \beta U,
\end{equation}

where $\alpha = \eta(A\xi)$. We notice here that $U$ is orthogonal to $\xi$.

We put

$$\Omega = \{ p \in M \mid \beta(p) \neq 0 \}.$$ 

Then $\Omega$ is an open subset of $M$.

3. $\eta$-parallel Ricci operators

In this section, we assume that $\Omega$ is not empty. Then there are scalar fields $\gamma$, $\epsilon$ and $\delta$ and a unit vector field $U$ and $\phi U$ orthogonal to $\xi$ such that

\begin{equation}
(3-1) \quad AU = \beta \xi + \gamma U + \epsilon \phi U, \quad A\phi U = \epsilon U + \delta \phi U
\end{equation}

and

\begin{equation}
(3-2) \quad m = \text{trace } A = \alpha + \gamma + \delta
\end{equation}

in $M_2(c)$.

We shall prove the following lemmas.

**Lemma 3.1.** Let $M$ be a real hypersurface in a complex space form $M_2(c)$, $c \neq 0$. If $M$ satisfies (1-3), then we have $AU = \beta \xi + \gamma U$, $A\phi U = \delta \phi U$ and $\beta^2 = \alpha(\gamma - \delta)$.

**Proof.** If we put $X = \xi$ into (2-3), we have

\begin{equation}
(3-3) \quad S\xi = \left( \frac{c}{2} + \alpha \gamma + \alpha \delta - \beta^2 \right) \xi + \beta \delta U - \beta \epsilon \phi U.
\end{equation}

Putting $X = U$ into (2-3) and taking account of (3-1) yields

\begin{equation}
(3-4) \quad SU = \beta \delta \xi + \left( \frac{5c}{4} + \alpha \gamma + \gamma \delta - \beta^2 - \epsilon^2 \right) + \alpha \epsilon \phi U.
\end{equation}

Putting $X = \phi U$ into (2-3) and using (3-1), we obtain

\begin{equation}
(3-5) \quad S\phi U = - \beta \epsilon \xi + \alpha \epsilon U + \left( \frac{5c}{4} + \alpha \delta + \gamma \delta - \epsilon^2 \right) \phi U.
\end{equation}

If we apply $\phi$ to (3-4), then we have

\begin{equation}
(3-6) \quad (S\phi - \phi S)U = - \beta \epsilon \xi + 2\alpha \epsilon U + (\alpha \delta - \alpha \gamma + \beta^2) \phi U.
\end{equation}

From condition (1-3), we have, for all $X \in T_0$,

\begin{equation}
(3-7) \quad (S\phi - \phi S)X = - \beta g(\epsilon U + \delta \phi U, X) \xi
\end{equation}
If we substitute $X = U$ into (3-7), then we obtain

\[(3-8) \quad (S\phi - \phi S)U = -\beta \varepsilon \xi.\]

Comparing (3-6) and (3-8), we get $\varepsilon = 0$ and $\beta^2 = \alpha(\gamma - \delta)$. It follows that $AU$ is expressed in terms of $\xi$ and $U$ only and $A\phi U$ is given by $\phi U$. \qed

It follows from (2-3) and (3-1) that

\[(3-9) \quad S\xi = \left(\frac{c}{2} + 2\alpha \delta\right)\xi + \beta \delta U,\]

\[(3-10) \quad SU = \beta \delta \xi + \left(\frac{5c}{4} + \gamma \delta + \alpha \delta\right)U,\]

\[(3-11) \quad S\phi U = \left(\frac{5c}{4} + \gamma \delta + \alpha \delta\right)\phi U.\]

**Lemma 3.2.** Under the assumptions of Lemma 3.1, if $M$ has the $\eta$-parallel Ricci operator $S$, then we have $AU = \beta \xi + \gamma U$, $A\phi U = 0$ and $\beta^2 = \alpha \gamma$.

**Proof.** Differentiating (3-10) covariantly along vector field $X$ in $T_0$, we obtain

\[(\nabla_X S)U = \left(\left(\frac{5c}{4} + \gamma \delta + \alpha \delta\right)I - S\right)\nabla_X U + \beta \delta \phi AX + X(\beta \delta)\xi + X\left(\frac{5c}{4} + \gamma \delta + \alpha \delta\right)U.\]

Taking the inner product of this equation with $U$ and $\phi U$ and making use of (3-9)–(3-11) and Lemma 3.1, we obtain

\[(3-12) \quad (\alpha + \gamma)\nabla \delta + \delta(\nabla \gamma + \nabla \alpha) = 2\beta \delta^2 \phi U\]

and

\[\delta \gamma = 0.\]

If we differentiate this along the vector field $X$ in $T_0$, then (3-12) is reduced to

\[(3-13) \quad \alpha \nabla \delta + \delta \nabla \alpha = 2\beta \delta^2 \phi U.\]

Differentiating (3-11) covariantly along vector field $X$ in $T_0$, we obtain

\[(3-14) \quad (\nabla_X S)\phi U = \left(\left(\frac{5c}{4} + \gamma \delta + \alpha \delta\right)I - S\right)\nabla_X \phi U + \left(X\left(\frac{5c}{4} + \gamma \delta + \alpha \delta\right)\right)\phi U.\]

If we take the inner product of (3-14) with $\phi U$ and use (3-9)–(3-11), then we have

\[(3-15) \quad \alpha \nabla \delta + \delta \nabla \alpha = 0.\]

Comparing (3-13) and (3-15), we obtain $\delta = 0$ and $\beta^2 = \alpha \gamma$ from Lemma 3.1. From this and Lemma 3.1 we conclude that $AU$ is expressed in terms of $\xi$ and $U$ only and $A\phi U = 0$. \qed
4. Proof of the main theorem

Assume that $M$ satisfies (1-2) and (1-3). We first show that $M$ is Hopf. If the open set $\Omega$ is not empty, then Lemma 3.2 yields $\delta = 0$. Thus the Ricci operator, as expressed in (3-9)–(3-11), has the property that $\xi$, $U$ and $\phi U$ are eigenvectors and that $U$ and $\phi U$ have the same eigenvalue. That is, $M$ is pseudo-Einstein with

$$SX = \frac{5c}{4}X - \frac{3c}{4}g(X, \xi)\xi.$$ 

This contradicts a result from [Kim and Ryan 2008]. Thus we conclude that any hypersurface satisfying (1-2) and (1-3) must be Hopf.

Since $M$ is Hopf, condition (1-3) yields $\alpha(\gamma - \delta) = 0$ and that the criteria for Proposition 2.21 in [Kim and Ryan 2008] are satisfied. Thus $M$ is pseudo-Einstein.

Conversely, if $M$ is pseudo-Einstein, observe that (1-2) and (1-3) must be satisfied.

\[\square\]

Acknowledgments

The authors would like to express their sincere gratitude to the referee who gave them valuable suggestions and comments.

References


Received September 20, 2012. Revised May 6, 2013.

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Acknowledgement