Pacific Journal of Mathematics

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DONG HO LIM, WOON HA SOHN AND HYUNJUNG SONG

Volume 266 No. 2 December 2013

PACIFIC JOURNAL OF MATHEMATICS
Vol. 266, No. 2, 2013
dx.doi.org/10.2140/pjm.2013.266.305

A STUDY OF REAL HYPERSURFACES WITH RICCI OPERATORS IN 2-DIMENSIONAL COMPLEX SPACE FORMS

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We prove that a real hypersurface M in complex projective space $P_2(\mathbb{C})$ or complex hyperbolic space $H_2(\mathbb{C})$, whose Ricci operator is η-parallel and commutes with the structure tensor on the holomorphic distribution, is a Hopf hypersurface. We also give a characterization of this hypersurface.

1. Introduction

A complex n-dimensional Kählerian manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. As is well known, a complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n(\mathbb{C})$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n(\mathbb{C})$, according to $c > 0$, $c = 0$ or $c < 0$.

In this paper we consider a real hypersurface M in a complex space form $M_2(c)$, $c \neq 0$. Then M has an almost contact metric structure (ϕ, g, ξ, η) induced from the Kähler metric and complex structure J on $M_n(c)$. The structure vector field ξ is said to be *principal* if $A\xi = \alpha\xi$ is satisfied, where A is the shape operator of M and $\alpha = \eta(A\xi)$. In this case, it is known that α is locally constant [Ki and Suh 1990] and that M is called a *Hopf hypersurface*.

Takagi [1973] classified homogeneous real hypersurfaces in $P_n(\mathbb{C})$ into six model spaces A_1, A_2, B, C, D and E of Hopf hypersurfaces with constant principal curvatures. Berndt [1989] classified all homogeneous Hopf hypersurfaces in $H_n(\mathbb{C})$ as four model spaces, which are said to be A_0, A_1, A_2 and B. A real hypersurface M of type A_1 or A_2 in $P_n(\mathbb{C})$ or type A_0, A_1 or A_2 in $H_n(\mathbb{C})$ is said to be *of type A* for simplicity.

As a typical characterization of real hypersurfaces of type A, the following is due to Okumura [1975] for $c > 0$, and Montiel and Romero [1986] for $c < 0$.

Theorem A [Montiel and Romero 1986; Okumura 1975]. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. It satisfies $A\phi - \phi A = 0$ on M if and only if M is locally congruent to one of the model spaces of type A.*

MSC2010: primary 53C40; secondary 53C15.
Keywords: real hypersurface, η-parallel Ricci operator, Hopf hypersurface.

The Ricci operator of M will be denoted by S, and the shape operator or the second fundamental tensor field of M by A. The holomorphic distribution T_0 of a real hypersurface M in $M_n(c)$ is defined by

$$T_0(p) = \{X \in T_p(M) \mid g(X, \xi)_p = 0\}, \tag{1-1}$$

where $T_p(M)$ is the tangent space of M at $p \in M$. The Ricci operator S is said to be η-parallel if

$$g((\nabla_X S)Y, Z) = 0 \tag{1-2}$$

for any vector fields X, Y and Z in T_0.

Theorem B [Kimura and Maeda 1989; Suh 1990]. *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. Then the Ricci operator of M is η-parallel and the structure vector field ξ is a principal if and only if M is locally congruent to one of the model spaces of type A or type B.*

I.-B. Kim, K. H. Kim and one of the present authors [Kim et al. 2006; 2007] studied real hypersurfaces with certain conditions related to the Ricci operator and the structure tensor field ϕ in $M_n(c)$. As for the Ricci operator and structure tensor field ϕ, one of the present authors proved the following.

Theorem C [Sohn 2007]. *Let M be a real hypersurface with η-parallel Ricci operator in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$. If M satisfies*

$$g((S\phi - \phi S)X, Y) = 0 \tag{1-3}$$

for any X and Y in T_0, then M is locally congruent to one of the model spaces of type A or type B.

The purpose of this paper is to complete the results of [Sohn 2007] and characterize real hypersurfaces with η-parallel Ricci operator such that the Ricci operator and structure tensor field commute in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 2$. Namely, we prove:

Theorem. *A real hypersurface in a complex space form $M_2(c)$, $c \neq 0$ satisfies (1-2) and (1-3) if and only if it is pseudo-Einstein.*

The pseudo-Einstein hypersurfaces are classified by Kim and Ryan [2008] and Ivey and Ryan [2009] and are described in detail in these papers. In view of their results, we can state the following.

Corollary. *Let M be a real hypersurface with an η-parallel Ricci operator in a complex space form $M_2(c)$, $c \neq 0$. If M satisfies (1-3) then M is locally congruent to either a Hopf hypersurface with $A\xi = 0$ or one of the model spaces of type A.*

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M_2(c)$, and N be a unit normal vector field of M. By $\widetilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini–Study metric tensor $\tilde{g}$ of $M_2(c)$. Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N \quad \text{and} \quad \widetilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M, where g denotes the Riemannian metric tensor of M induced from $\tilde{g}$, and A is the shape operator of M in $M_2(c)$.

For any vector field X on M we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi,$$

where J is the almost complex structure of $M_2(c)$. Then we see that M induces an almost contact metric structure (ϕ, g, ξ, η), that is,

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi)$$

for any vector fields X and Y on M. Since the almost complex structure J is parallel, we can verify from the Gauss formula that

$$\nabla_X \xi = \phi AX. \tag{2-1}$$

Since the ambient manifold is of constant holomorphic sectional curvature c, we have the Gauss equation

$$\begin{aligned} R(X, Y)Z &= \frac{c}{4}\big(g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\big) \\ &\qquad + g(AY, Z)AX - g(AX, Z)AY \end{aligned} \tag{2-2}$$

for any vector fields X, Y and Z on M, where R denotes the Riemannian curvature tensor of M.

From (1-3) the Ricci operator S of M is expressed by

$$SX = \frac{c}{4}\big((2n+1)X - 3\eta(X)\xi\big) + mAX - A^2 X, \tag{2-3}$$

where $m = \operatorname{trace} A$ is the mean curvature of M, and the covariant derivative of (2-3) is given by

$$\begin{aligned} (\nabla_X S)Y = -\frac{3c}{4}\big(g(\phi AX, Y)\xi + \eta(Y)\phi AX\big) \\ + (Xm)AY + m(\nabla_X A)Y - (\nabla_X A)AY - A(\nabla_X A)Y. \end{aligned}$$

Let U be a unit vector field on M with the same direction of the vector field $-\phi\nabla_\xi\xi$, and let β be the length of the vector field $-\phi\nabla_\xi\xi$ if it does not vanish. It is not possible to define U without specifying that $\beta \neq 0$. Then it is easily seen from (2-1) that

$$A\xi = \alpha\xi + \beta U, \tag{2-4}$$

where $\alpha = \eta(A\xi)$. We notice here that U is orthogonal to ξ.

We put

$$\Omega = \{p \in M \mid \beta(p) \neq 0\}.$$

Then Ω is an open subset of M.

3. η-parallel Ricci operators

In this section, we assume that Ω is not empty. Then there are scalar fields γ, ε and δ and a unit vector field U and ϕU orthogonal to ξ such that

$$AU = \beta\xi + \gamma U + \varepsilon\phi U, \quad A\phi U = \varepsilon U + \delta\phi U \tag{3-1}$$

and

$$m = \operatorname{trace} A = \alpha + \gamma + \delta \tag{3-2}$$

in $M_2(c)$.

We shall prove the following lemmas.

Lemma 3.1. *Let M be a real hypersurface in a complex space form $M_2(c)$, $c \neq 0$. If M satisfies (1-3), then we have $AU = \beta\xi + \gamma U$, $A\phi U = \delta\phi U$ and $\beta^2 = \alpha(\gamma - \delta)$.*

Proof. If we put $X = \xi$ into (2-3), we have

$$S\xi = \left(\frac{c}{2} + \alpha\gamma + \alpha\delta - \beta^2\right)\xi + \beta\delta U - \beta\varepsilon\phi U. \tag{3-3}$$

Putting $X = U$ into (2-3) and taking account of (3-1) yields

$$SU = \beta\delta\xi + \left(\frac{5c}{4} + \alpha\gamma + \gamma\delta - \beta^2 - \varepsilon^2\right) + \alpha\varepsilon\phi U. \tag{3-4}$$

Putting $X = \phi U$ into (2-3) and using (3-1), we obtain

$$S\phi U = -\beta\varepsilon\xi + \alpha\varepsilon U + \left(\frac{5c}{4} + \alpha\delta + \gamma\delta - \varepsilon^2\right)\phi U. \tag{3-5}$$

If we apply ϕ to (3-4), then we have

$$(S\phi - \phi S)U = -\beta\varepsilon\xi + 2\alpha\varepsilon U + (\alpha\delta - \alpha\gamma + \beta^2)\phi U. \tag{3-6}$$

From condition (1-3), we have, for all $X \in T_0$,

$$(S\phi - \phi S)X = -\beta g(\varepsilon U + \delta\phi U, X)\xi \tag{3-7}$$

If we substitute $X = U$ into (3-7), then we obtain

$$(S\phi - \phi S)U = -\beta\varepsilon\xi. \tag{3-8}$$

Comparing (3-6) and (3-8), we get $\varepsilon = 0$ and $\beta^2 = \alpha(\gamma - \delta)$. It follows that AU is expressed in terms of ξ and U only and $A\phi U$ is given by ϕU. □

It follows from (2-3) and (3-1) that

$$S\xi = \left(\frac{c}{2} + 2\alpha\delta\right)\xi + \beta\delta U, \tag{3-9}$$

$$SU = \beta\delta\xi + \left(\frac{5c}{4} + \gamma\delta + \alpha\delta\right)U, \tag{3-10}$$

$$S\phi U = \left(\frac{5c}{4} + \gamma\delta + \alpha\delta\right)\phi U. \tag{3-11}$$

Lemma 3.2. *Under the assumptions of Lemma 3.1, if M has the η-parallel Ricci operator S, then we have $AU = \beta\xi + \gamma U$, $A\phi U = 0$ and $\beta^2 = \alpha\gamma$.*

Proof. Differentiating (3-10) covariantly along vector field X in T_0, we obtain

$$(\nabla_X S)U = \left(\left(\frac{5c}{4} + \gamma\delta + \alpha\delta\right)I - S\right)\nabla_X U + \beta\delta\phi AX + X(\beta\delta)\xi + X\left(\frac{5c}{4} + \gamma\delta + \alpha\delta\right)U.$$

Taking the inner product of this equation with U and ϕU and making use of (3-9)–(3-11) and Lemma 3.1, we obtain

$$(\alpha + \gamma)\nabla\delta + \delta(\nabla\gamma + \nabla\alpha) = 2\beta\delta^2\phi U \tag{3-12}$$

and

$$\delta\gamma = 0.$$

If we differentiate this along the vector field X in T_0, then (3-12) is reduced to

$$\alpha\nabla\delta + \delta\nabla\alpha = 2\beta\delta^2\phi U. \tag{3-13}$$

Differentiating (3-11) covariantly along vector field X in T_0, we obtain

$$(\nabla_X S)\phi U = \left(\left(\frac{5c}{4} + \gamma\delta + \alpha\delta\right)I - S\right)\nabla_X \phi U + \left(X\left(\frac{5c}{4} + \gamma\delta + \alpha\delta\right)\right)\phi U. \tag{3-14}$$

If we take the inner product of (3-14) with ϕU and use (3-9)–(3-11), then we have

$$\alpha\nabla\delta + \delta\nabla\alpha = 0. \tag{3-15}$$

Comparing (3-13) and (3-15), we obtain $\delta = 0$ and $\beta^2 = \alpha\gamma$ from Lemma 3.1. From this and Lemma 3.1 we conclude that AU is expressed in terms of ξ and U only and $A\phi U = 0$. □

4. Proof of the main theorem

Assume that M satisfies (1-2) and (1-3). We first show that M is Hopf. If the open set Ω is not empty, then Lemma 3.2 yields $\delta = 0$. Thus the Ricci operator, as expressed in (3-9)–(3-11), has the property that ξ, U and ϕU are eigenvectors and that U and ϕU have the same eigenvalue. That is, M is pseudo-Einstein with

$$SX = \frac{5c}{4}X - \frac{3c}{4}g(X,\xi)\xi.$$

This contradicts a result from [Kim and Ryan 2008]. Thus we conclude that any hypersurface satisfying (1-2) and (1-3) must be Hopf.

Since M is Hopf, condition (1-3) yields $\alpha(\gamma - \delta) = 0$ and that the criteria for Proposition 2.21 in [Kim and Ryan 2008] are satisfied. Thus M is pseudo-Einstein.

Conversely, if M is pseudo-Einstein, observe that (1-2) and (1-3) must be satisfied. □

Acknowledgments

The authors would like to express their sincere gratitude to the referee who gave them valuable suggestions and comments.

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Received September 20, 2012. Revised May 6, 2013.

DONG HO LIM
DEPARTMENT OF MATHEMATICS
HANKUK UNIVERSITY OF FOREIGN STUDIES
SEOUL 130-791
SOUTH KOREA
dhlnys@hufs.ac.kr

WOON HA SOHN
DEPARTMENT OF MATHEMATICS
CATHOLIC UNIVERSITY OF DAEGU
DAEGU 712-702
SOUTH KOREA
kumogawa@cu.ac.kr

HYUNJUNG SONG
DEPARTMENT OF MATHEMATICS
HANKUK UNIVERSITY OF FOREIGN STUDIES
SEOUL 130-791
SOUTH KOREA
hsong@hufs.ac.kr

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PACIFIC JOURNAL OF MATHEMATICS

Volume 266 No. 2 December 2013