A STUDY OF REAL HYPERSURFACES WITH RICCI OPERATORS IN 2-DIMENSIONAL COMPLEX SPACE FORMS

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We prove that a real hypersurface $M$ in complex projective space $\mathbb{P}_2(\mathbb{C})$ or complex hyperbolic space $\mathbb{H}_2(\mathbb{C})$, whose Ricci operator is $\eta$-parallel and commutes with the structure tensor on the holomorphic distribution, is a Hopf hypersurface. We also give a characterization of this hypersurface.

1. Introduction

A complex $n$-dimensional Kählerian manifold of constant holomorphic sectional curvature $c$ is called a complex space form, which is denoted by $M_n(c)$. As is well known, a complete and simply connected complex space form is complex analytically isometric to a complex projective space $\mathbb{P}_n(\mathbb{C})$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $\mathbb{H}_n(\mathbb{C})$, according to $c > 0$, $c = 0$ or $c < 0$.

In this paper we consider a real hypersurface $M$ in a complex space form $M_2(c)$, $c \neq 0$. Then $M$ has an almost contact metric structure $(\phi, g, \xi, \eta)$ induced from the Kähler metric and complex structure $J$ on $M_n(c)$. The structure vector field $\xi$ is said to be principal if $A\xi = \alpha\xi$ is satisfied, where $A$ is the shape operator of $M$ and $\alpha = \eta(A\xi)$. In this case, it is known that $\alpha$ is locally constant [Ki and Suh 1990] and that $M$ is called a Hopf hypersurface.

Takagi [1973] classified homogeneous real hypersurfaces in $\mathbb{P}_n(\mathbb{C})$ into six model spaces $A_1$, $A_2$, $B$, $C$, $D$ and $E$ of Hopf hypersurfaces with constant principal curvatures. Berndt [1989] classified all homogeneous Hopf hypersurfaces in $\mathbb{H}_n(\mathbb{C})$ as four model spaces, which are said to be $A_0$, $A_1$, $A_2$ and $B$. A real hypersurface $M$ of type $A_1$ or $A_2$ in $\mathbb{P}_n(\mathbb{C})$ or type $A_0$, $A_1$ or $A_2$ in $\mathbb{H}_n(\mathbb{C})$ is said to be of type $A$ for simplicity.

As a typical characterization of real hypersurfaces of type $A$, the following is due to Okumura [1975] for $c > 0$, and Montiel and Romero [1986] for $c < 0$.

**Theorem A** [Montiel and Romero 1986; Okumura 1975]. *Let $M$ be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. It satisfies $A\phi - \phi A = 0$ on $M$ if and only if $M$ is locally congruent to one of the model spaces of type $A$.*

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The Ricci operator of $M$ will be denoted by $S$, and the shape operator or the second fundamental tensor field of $M$ by $A$. The holomorphic distribution $T_0$ of a real hypersurface $M$ in $M_n(c)$ is defined by

\begin{equation}
T_0(p) = \{ X \in T_p(M) \mid g(X, \xi)_p = 0 \},
\end{equation}

where $T_p(M)$ is the tangent space of $M$ at $p \in M$. The Ricci operator $S$ is said to be $\eta$-parallel if

\begin{equation}
g((\nabla_X S)Y, Z) = 0
\end{equation}

for any vector fields $X, Y$ and $Z$ in $T_0$.

**Theorem B** [Kimura and Maeda 1989; Suh 1990]. Let $M$ be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. Then the Ricci operator of $M$ is $\eta$-parallel and the structure vector field $\xi$ is a principal if and only if $M$ is locally congruent to one of the model spaces of type A or type B.

I.-B. Kim, K. H. Kim and one of the present authors [Kim et al. 2006; 2007] studied real hypersurfaces with certain conditions related to the Ricci operator and the structure tensor field $\phi$ in $M_n(c)$. As for the Ricci operator and structure tensor field $\phi$, one of the present authors proved the following.

**Theorem C** [Sohn 2007]. Let $M$ be a real hypersurface with $\eta$-parallel Ricci operator in a complex space form $M_n(c)$, $c \neq 0, n \geq 3$. If $M$ satisfies

\begin{equation}
g((S\phi - \phi S)X, Y) = 0
\end{equation}

for any $X$ and $Y$ in $T_0$, then $M$ is locally congruent to one of the model spaces of type A or type B.

The purpose of this paper is to complete the results of [Sohn 2007] and characterize real hypersurfaces with $\eta$-parallel Ricci operator such that the Ricci operator and structure tensor field $\phi$ commute in a complex space form $M_n(c)$, $c \neq 0, n \geq 2$. Namely, we prove:

**Theorem.** A real hypersurface in a complex space form $M_2(c)$, $c \neq 0$ satisfies (1-2) and (1-3) if and only if it is pseudo-Einstein.

The pseudo-Einstein hypersurfaces are classified by Kim and Ryan [2008] and Ivey and Ryan [2009] and are described in detail in these papers. In view of their results, we can state the following.

**Corollary.** Let $M$ be a real hypersurface with an $\eta$-parallel Ricci operator in a complex space form $M_2(c)$, $c \neq 0$. If $M$ satisfies (1-3) then $M$ is locally congruent to either a Hopf hypersurface with $A\xi = 0$ or one of the model spaces of type A.
2. Preliminaries

Let $M$ be a real hypersurface immersed in a complex space form $M_2(c)$, and $N$ be a unit normal vector field of $M$. By $\tilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini–Study metric tensor $\tilde{g}$ of $M_2(c)$. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(A X, Y)N \quad \text{and} \quad \tilde{\nabla}_X N = -AX$$

for any vector fields $X$ and $Y$ tangent to $M$, where $g$ denotes the Riemannian metric tensor of $M$ induced from $\tilde{g}$, and $A$ is the shape operator of $M$ in $M_2(c)$.

For any vector field $X$ on $M$ we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi,$$

where $J$ is the almost complex structure of $M_2(c)$. Then we see that $M$ induces an almost contact metric structure $(\phi, g, \xi, \eta)$, that is,

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi \xi = 0, \quad \eta(\xi) = 1,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi)$$

for any vector fields $X$ and $Y$ on $M$. Since the almost complex structure $J$ is parallel, we can verify from the Gauss formula that

(2-1) \hspace{1cm} \nabla_X \xi = \phi AX.$$

Since the ambient manifold is of constant holomorphic sectional curvature $c$, we have the Gauss equation

(2-2) \hspace{1cm} R(X, Y)Z = \frac{c}{4}(g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z)

+ g(AY, Z)AX - g(AX, Z)AY

for any vector fields $X$, $Y$ and $Z$ on $M$, where $R$ denotes the Riemannian curvature tensor of $M$.

From (1-3) the Ricci operator $S$ of $M$ is expressed by

(2-3) \hspace{1cm} SX = \frac{c}{4}((2n+1)X - 3\eta(X)\xi) + mAX - A^2X,$$

where $m = \text{trace } A$ is the mean curvature of $M$, and the covariant derivative of (2-3) is given by

$$\nabla_X S = -\frac{3c}{4}(g(\phi AX, Y)\xi + \eta(Y)\phi AX)

+ (Xm)AY + m(\nabla_X A)Y - (\nabla_X A)AY - A(\nabla_X A)Y.$$
Let $U$ be a unit vector field on $M$ with the same direction of the vector field $-\phi \nabla_\xi \xi$, and let $\beta$ be the length of the vector field $-\phi \nabla_\xi \xi$ if it does not vanish. It is not possible to define $U$ without specifying that $\beta \neq 0$. Then it is easily seen from (2-1) that

\[(2-4) \quad A\xi = \alpha\xi + \beta U,\]

where $\alpha = \eta(A\xi)$. We notice here that $U$ is orthogonal to $\xi$.

We put

\[\Omega = \{ p \in M \mid \beta(p) \neq 0 \}.\]

Then $\Omega$ is an open subset of $M$.

### 3. $\eta$-parallel Ricci operators

In this section, we assume that $\Omega$ is not empty. Then there are scalar fields $\gamma$, $\epsilon$ and $\delta$ and a unit vector field $U$ and $\phi U$ orthogonal to $\xi$ such that

\[(3-1) \quad AU = \beta\xi + \gamma U + \epsilon \phi U, \quad A\phi U = \epsilon U + \delta \phi U\]

and

\[(3-2) \quad m = \text{trace } A = \alpha + \gamma + \delta\]

in $M_2(c)$.

We shall prove the following lemmas.

**Lemma 3.1.** Let $M$ be a real hypersurface in a complex space form $M_2(c)$, $c \neq 0$. If $M$ satisfies (1-3), then we have $AU = \beta \xi + \gamma U$, $A\phi U = \delta \phi U$ and $\beta^2 = \alpha(\gamma - \delta)$.

**Proof.** If we put $X = \xi$ into (2-3), we have

\[(3-3) \quad S\xi = (\frac{c}{2} + \alpha \gamma + \alpha \delta - \beta^2)\xi + \beta \delta U - \beta \epsilon \phi U.\]

Putting $X = U$ into (2-3) and taking account of (3-1) yields

\[(3-4) \quad SU = \beta \delta \xi + \left(\frac{5c}{4} + \alpha \gamma + \gamma \delta - \beta^2 - \epsilon^2\right) + \alpha \epsilon \phi U.\]

Putting $X = \phi U$ into (2-3) and using (3-1), we obtain

\[(3-5) \quad S\phi U = -\beta \epsilon \xi + \alpha \epsilon U + \left(\frac{5c}{4} + \alpha \delta + \gamma \delta - \epsilon^2\right) \phi U.\]

If we apply $\phi$ to (3-4), then we have

\[(3-6) \quad (S\phi - \phi S)U = -\beta \epsilon \xi + 2\alpha \epsilon U + (\alpha \delta - \alpha \gamma + \beta^2) \phi U.\]

From condition (1-3), we have, for all $X \in T_0$,

\[(3-7) \quad (S\phi - \phi S)X = -\beta g(\epsilon U + \delta \phi U, X)\xi.\]
If we substitute $X = U$ into (3-7), then we obtain

\[(S\phi - \phi S)U = -\beta \varepsilon \xi.\]  

Comparing (3-6) and (3-8), we get $\varepsilon = 0$ and $\beta^2 = \alpha(\gamma - \delta)$. It follows that $AU$ is expressed in terms of $\xi$ and $U$ only and $A\phi U$ is given by $\phi U$. \(\square\)

It follows from (2-3) and (3-1) that

\[(S\xi)(U) = \left(\frac{c}{2} + 2\alpha \delta\right)\xi + \beta \delta U,\]
\[(SU)(U) = \beta \delta \xi + \left(\frac{5c}{4} + \gamma \delta + \alpha \delta\right)U,\]
\[(S\phi U)(U) = \left(\frac{5c}{4} + \gamma \delta + \alpha \delta\right)\phi U.\]

**Lemma 3.2.** Under the assumptions of Lemma 3.1, if $M$ has the $\eta$-parallel Ricci operator $S$, then we have $AU = \beta \xi + \gamma U$, $A\phi U = 0$ and $\beta^2 = \alpha \gamma$.

**Proof.** Differentiating (3-10) covariantly along vector field $X$ in $T_0$, we obtain

\[\nabla_X S(U) = \left(\left(\frac{5c}{4} + \gamma \delta + \alpha \delta\right)I - S\right)\nabla_X U + \beta \delta \phi AX + X(\beta \delta)\xi + X\left(\frac{5c}{4} + \gamma \delta + \alpha \delta\right)U.\]

Taking the inner product of this equation with $U$ and $\phi U$ and making use of (3-9)–(3-11) and Lemma 3.1, we obtain

\[(\alpha + \gamma)\nabla \delta + \delta(\nabla \gamma + \nabla \alpha) = 2\beta \delta^2 \phi U\]
and
\[\delta \gamma = 0.\]

If we differentiate this along the vector field $X$ in $T_0$, then (3-12) is reduced to

\[(\alpha + \gamma)\nabla \delta + \delta \nabla \alpha = 2\beta \delta^2 \phi U.\]

Differentiating (3-11) covariantly along vector field $X$ in $T_0$, we obtain

\[(\nabla_X S)(\phi U) = \left(\left(\frac{5c}{4} + \gamma \delta + \alpha \delta\right)I - S\right)\nabla_X \phi U + \left(X\left(\frac{5c}{4} + \gamma \delta + \alpha \delta\right)\right)\phi U.\]

If we take the inner product of (3-14) with $\phi U$ and use (3-9)–(3-11), then we have

\[\alpha \nabla \delta + \delta \nabla \alpha = 0.\]

Comparing (3-13) and (3-15), we obtain $\delta = 0$ and $\beta^2 = \alpha \gamma$ from Lemma 3.1. From this and Lemma 3.1 we conclude that $AU$ is expressed in terms of $\xi$ and $U$ only and $A\phi U = 0$. \(\square\)
4. Proof of the main theorem

Assume that $M$ satisfies (1-2) and (1-3). We first show that $M$ is Hopf. If the open set $\Omega$ is not empty, then Lemma 3.2 yields $\delta = 0$. Thus the Ricci operator, as expressed in (3-9)–(3-11), has the property that $\xi$, $U$ and $\phi U$ are eigenvectors and that $U$ and $\phi U$ have the same eigenvalue. That is, $M$ is pseudo-Einstein with

$$S X = \frac{5c}{4} X - \frac{3c}{4} g(X, \xi)\xi.$$  

This contradicts a result from [Kim and Ryan 2008]. Thus we conclude that any hypersurface satisfying (1-2) and (1-3) must be Hopf.

Since $M$ is Hopf, condition (1-3) yields $\alpha(\gamma - \delta) = 0$ and that the criteria for Proposition 2.21 in [Kim and Ryan 2008] are satisfied. Thus $M$ is pseudo-Einstein.

Conversely, if $M$ is pseudo-Einstein, observe that (1-2) and (1-3) must be satisfied. □

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