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HIDETOSHI MASAI

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ON COMMENSURABILITY OF FIBRATIONS ON A HYPERBOLIC 3-MANIFOLD

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We discuss fibered commensurability of fibrations on hyperbolic 3-manifolds, a notion introduced by Calegari, Sun, and Wang (*Pacific J. Math.* 250:2 (2011), 287–317). We construct manifolds with nonsymmetric but commensurable fibrations on the same fibered face, and prove that if a given manifold M does not have hidden symmetries, then M does not admit nonsymmetric but commensurable fibrations.

It was also proved by Calegari et al that every hyperbolic fibered commensurability class contains a unique minimal element. Here we provide a detailed discussion on the proof of the theorem in the cusped case.

1. Introduction

In this paper, we are mainly interested in fibered hyperbolic 3-manifolds with the first Betti number greater than or equal to 2. Thurston [1986] showed that such a manifold admits infinitely many distinct fibrations (see also Section 4). It is an interesting question to investigate the relationship between such fibrations.

Calegari, Sun, and Wang defined the notion of fibered commensurability, which gives rise to an equivalence relation on fibrations. An *automorphism* on a surface is an isotopy class of self-homeomorphisms of the surface. For any fibration on a 3-manifold, we have the pair (F, ϕ) of the fiber surface F, and the monodromy automorphism ϕ. Since the monodromy is determined up to conjugacy in the mapping class group of F, we use the notation (F, ϕ) to denote the conjugacy class. Then commensurability of fibrations is defined as follows.

Definition 1.1 [Calegari et al. 2011]. A pair $(\widetilde{F}, \widetilde{\phi})$ *covers* (F, ϕ) if there is a finite cover $\pi : \widetilde{F} \to F$ and representative homeomorphisms $\widetilde{f}$ of $\widetilde{\phi}$ and f of ϕ so that $\pi \widetilde{f} = f \pi$ as maps $\widetilde{F} \to F$.

Definition 1.2 [Calegari et al. 2011]. Two pairs (F_1, ϕ_1) and (F_2, ϕ_2) are *commensurable* if there is a surface $\widetilde{F}$, automorphisms $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$, and nonzero integers k_1

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and k_2, so that $(\widetilde{F}, \widetilde{\phi}_i)$ covers (F_i, ϕ_i) for $i = 1, 2$ and if $\widetilde{\phi}_1^{k_1} = \widetilde{\phi}_2^{k_2}$ as automorphisms of $\widetilde{F}$.

For the remainder of the paper, we consider fibrations on hyperbolic 3-manifolds. In this case, the monodromy of each fibration is always *pseudo-Anosov* (see Definition 2.5 for the definition). The *normalized entropy* of a conjugacy class (F, ϕ) is defined as $\chi(F) \log \lambda(\phi)$, where $\chi(F)$ is the Euler characteristic of F and $\lambda(\phi)$ is the dilatation of ϕ. In Section 2, we observe that the normalized entropies of commensurable fibrations on the same hyperbolic 3-manifold agree. Then we offer an example of a manifold such that two of its fibrations are commensurable if and only if they share the same normalized entropy. We also give an example of a manifold with two noncommensurable fibrations of the same normalized entropy.

In this paper, we study commensurable fibrations on a hyperbolic 3-manifold in the context of a fibered face. A *fibered face* is a face of the Thurston norm ball whose rational points correspond to fibrations of the 3-manifold and a *fibered cone* is a cone over a fibered face (see Section 3 for details). Two fibrations on M are said to be *symmetric* if there exists a self-homeomorphism $\varphi : M \to M$ that maps one to the other. In [Calegari et al. 2011, Remark 3.9], Calegari, Sun, and Wang asked if there is an example of two fibrations on the same closed manifold, which are commensurable but have fibers distinguished by their genera. The following theorem provides such a construction in the cusped case. In this theorem fibers are distinguished by their Euler characteristics (see Section 4 for a proof).

Theorem 1.3. *There are hyperbolic* 3*-manifolds with nonsymmetric but commensurable fibrations whose corresponding elements in* $H^1(M; \mathbb{Z})$ *are in the same fibered cone.*

On the other hand, if M has no hidden symmetries, then such fibrations do not exist. Here, a (finite-volume) hyperbolic 3-manifold $M = \mathbb{H}^3 / \Gamma$ is said to have *hidden symmetries* if $[C^+(\Gamma) : N^+(\Gamma)] > 1$, where $C^+(\Gamma)$ and $N^+(\Gamma)$ are the commensurator and normalizer of Γ; see Section 4 for details.

Theorem 1.4. *Suppose that M is a hyperbolic* 3*-manifold that does not have hidden symmetries. Then*, *any pair of fibrations of M is either symmetric or noncommensurable*, *but not both.*

Theorems 1.3 and 1.4 are motivated by the fact that up to isotopy, there are only finitely many commensurable fibrations on a hyperbolic 3-manifold. This fact is a corollary of the following:

Theorem 1.5 (see also Theorem 3.1 of [Calegari et al. 2011]). *Every commensurability class of hyperbolic fibered pairs contains a unique* (*orbifold*) *minimal element.*

Here the notion of a fibered pair is a generalization of the notion of a pair (F, ϕ), see Section 2 for details. The proof in [Calegari et al. 2011] works for the closed case. In Section 2 we extend it to the case where the manifolds have boundary (Theorem 2.6). Further, as a corollary of this extension, we show examples of manifolds such that every fibration is the minimal element in its commensurability class (Corollary 2.8).

Commensurability classes are defined using the transitive hull of the relation in Definition 1.2. In Section 2 we also discuss the transitivity of commensurability. We show that if the automorphisms are pseudo-Anosov (that is to say, in the hyperbolic case), then commensurability is transitive.

2. Preliminaries

In this section, we recall the definitions and basic facts about commensurability of fibrations. Most of the contents in this section are discussed in [Calegari et al. 2011]. In this paper, unless otherwise stated, by a *surface* and a *hyperbolic 3-manifold*, we mean a compact connected orientable 2-manifold possibly with boundary and of negative Euler characteristic, and a connected, orientable, complete hyperbolic 3-manifold of finite volume respectively.

Fibered pairs. Given a homeomorphism $f : F \to F$, the *mapping torus* of f is the 3-manifold

$$M = F \times [0, 1] / \big((f(x), 0) \sim (x, 1)\big).$$

Mapping tori of conjugate automorphisms are homeomorphic, so if ϕ is a conjugacy class of homeomorphisms we obtain a homeomorphism class of mapping tori, which we denote by $[F, \phi]$. We call F the *fiber* and ϕ the *monodromy* of $[F, \phi]$.

We will focus on fibrations of a fixed hyperbolic 3-manifold M. Each fibration on M over the circle determines an element of $H^1(M; \mathbb{Z})$, and if $\omega \in H^1(M; \mathbb{Z})$ corresponds to a fibration, then there is an associated pair (F, ϕ), in the sense that $[F, \phi]$ is homeomorphic to M. This correspondence of ω and (F, ϕ) is well defined up to the conjugation of (F, ϕ).

Later in this section (page 317), we discuss Theorem 3.1 of [Calegari et al. 2011] for the case of fibered manifolds with boundary. To state the theorem it is convenient to define a *fibered pair* which is a generalization of a pair of type (F, ϕ). We also enlarge our attention to orbifolds. An n-orbifold is a space that is locally modeled on a quotient of an open ball in $\mathbb{R}^n$ by a finite group. See [Walsh 2011] and Chapter 13 of [Thurston 1979] for more details.

Definition 2.1 [Calegari et al. 2011]. A *fibered pair* is a pair $(M, \mathcal{F})$, where M is a compact 3-manifold with boundary a union of tori and Klein bottles, and $\mathcal{F}$ is a foliation by compact surfaces. More generally, an orbifold fibered pair is a pair $(O, \mathcal{G})$, where O is a compact 3-orbifold, and $\mathcal{G}$ is a foliation of O by compact 2-orbifolds.

Definition 2.2 [Calegari et al. 2011]. A fibered pair $(\widetilde{M}, \widetilde{\mathcal{F}})$ *covers* $(M, \mathcal{F})$ if there is a finite covering of manifolds $\pi : \widetilde{M} \to M$ such that $\pi^{-1}(\mathcal{F})$ is isotopic to $\widetilde{\mathcal{F}}$. Two fibered pairs $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ are (*fibered*) *commensurable* if there is a third fibered pair $(\widetilde{M}, \widetilde{\mathcal{F}})$ that covers both.

For a given pair (F, ϕ), the mapping torus $[F, \phi]$ has a foliation $\mathcal{F}$ by surface leaves, which are homeomorphic to F and hence there is a corresponding fibered pair $([F, \phi], \mathcal{F})$.

Unlike the case of commensurability in Definition 1.2, it is easy to see that commensurability of fibered pairs is transitive. Suppose $(M_i, \mathcal{F}_i)$ and $(M_{i+1}, \mathcal{F}_{i+1})$ are commensurable for $i = 1, 2$ and $(\widetilde{M}_{12}, \widetilde{\mathcal{F}}_{12})$ (resp. $(\widetilde{M}_{23}, \widetilde{\mathcal{F}}_{23})$) is a common covering pair of $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ (resp. $(M_2, \mathcal{F}_2)$ and $(M_3, \mathcal{F}_3)$). Then there is a covering $p : \widetilde{N} \to M_2$ that corresponds to $p^{12}_* \pi_1(\widetilde{M}_{12}) \cap p^{23}_* \pi_1(\widetilde{M}_{23}) < \pi_1(M_2)$, where $p^{12} : \widetilde{M}_{12} \to M_2$ and $p^{23} : \widetilde{M}_{23} \to M_2$ are the covering maps. Then $(\widetilde{N}, p^{-1}(\mathcal{F}_2))$ covers both $(M_1, \mathcal{F}_1)$ and $(M_3, \mathcal{F}_3)$. Thus we see that fibered commensurability is a transitive relation.

We define another equivalence relation on fibered pairs so that the covering relation will be a partial order.

Definition 2.3 [Calegari et al. 2011]. We say that two fibered pairs $(M, \mathcal{F})$ and $(N, \mathcal{G})$ are *covering equivalent* if each covers the other. We call a covering equivalence class *minimal* if no representative covers any element of another covering equivalence class.

Remark 2.4 (see also Remark 2.9 of [Calegari et al. 2011]). Each covering equivalence class of the fibered pair associated to (F, ϕ) contains exactly one fibered pair unless ϕ is periodic. Therefore, when we consider pseudo-Anosov automorphisms, by abusing notation, we use the word "element" for each covering equivalent class.

Pseudo-Anosov automorphisms. The automorphisms on a compact surface are classified into three types: periodic, reducible, and pseudo-Anosov [Thurston 1988; Casson and Bleiler 1988]. By a result of Thurston, the (interior of the) mapping torus $[F, \phi]$ admits a hyperbolic metric of finite volume if and only if the automorphism ϕ is pseudo-Anosov (see [Thurston 1988], and compare [Otal 1996]).

Definition 2.5. A homeomorphism $f : F \to F$ is a *pseudo-Anosov homeomorphism* if there is a pair of transverse measured singular foliations $(\mathcal{F}^s, \mu^s)$ and $(\mathcal{F}^u, \mu^u)$ on F and a positive real number λ so that $f(\mathcal{F}^u) = \mathcal{F}^u$, $f(\mu^u) = \lambda\mu^u$ and $f(\mathcal{F}^s) = \mathcal{F}^s$, $f(\mu^s) = (1/\lambda)\mu^s$. We call $(\mathcal{F}^s, \mu^s)$ and $(\mathcal{F}^u, \mu^u)$ the stable and unstable measured singular foliations associated to f.

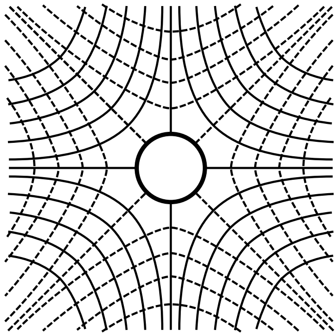

Figure 1. A shape of a singularity of degree 4 at the boundary.

See Figure 1 for a shape of the singularities of $(\mathscr{F}^s, \mu^s)$ and $(\mathscr{F}^u, \mu^u)$. An automorphism ϕ is said to be *pseudo-Anosov* if it has a pseudo-Anosov homeomorphism as a representative. We call the positive real number λ the *dilatation* of pseudo-Anosov automorphism ϕ and denote it by $\lambda(\phi)$.

In some cases, it is convenient to consider the restriction of automorphisms on the interior $\mathrm{Int}(F)$ of F. By considering $\phi|_{\mathrm{Int}(F)}$, we get a pseudo-Anosov automorphism on $\mathrm{Int}(F)$ and by abusing the notation we also denote it by ϕ. Note that $\mathrm{Int}(F)$ can be regarded as a surface with finitely many punctures, each corresponding to a boundary component of F. Then the singularities of $\mathscr{F}^s$ and $\mathscr{F}^u$ lie on $\mathrm{Int}(F)$ or the punctures. We denote the set of points and punctures that correspond to the singular points of associated singular foliations by $\mathrm{Sing}(\phi)$.

Uniqueness of the minimal element. In this subsection, we give a detailed discussion of Theorem 1.5 for the case where manifolds have boundary. By passing to a finite covering we may assume $\mathscr{F}$ to be co-orientable and hence M fibers over the circle; that is, M is the mapping torus $[F, \phi]$ of some surface F and pseudo-Anosov map ϕ. Since we are dealing with commensurability classes, it suffices to discuss the case where the foliations are co-orientable. The proof in [Calegari et al. 2011] assumes that all singular points of the singular foliations associated to ϕ lie on the interior of F. We prove this result for the case where some of the singular points lie on the boundary. This corresponds to the case where $\mathrm{Sing}(\phi)$ contains some punctures, by restricting the automorphism on the interior $\mathrm{Int}(F)$ of F.

Theorem 2.6 (see also [Calegari et al. 2011]). *Let $(M, \mathscr{F})$ be a hyperbolic co-orientable fibered pair and let (F, ϕ) be the pair associated to $(M, \mathscr{F})$. Then the commensurability class of $(M, \mathscr{F})$ contains a unique minimal (orbifold) element. Moreover, if $\mathrm{Int}(F) \cap \mathrm{Sing}(\phi) = \varnothing$, then the minimal element is a manifold.*

Proof. First, we recall the argument in [Calegari et al. 2011], since we will need it here. The stable and unstable singular foliations $\mathscr{F}^s$ and $\mathscr{F}^u$ associated to ϕ

determine a unique singular Sol metric on $\mathrm{Int}(M)$. Pulling back this metric to the universal cover $\pi : \widetilde{M} \to \mathrm{Int}(M)$, $\widetilde{M}$ becomes a simply connected singular Sol manifold. Each fiber of $\widetilde{M}$ is a singular Euclidean plane. Let Λ be the full isometry group of the singular Sol metric. By appealing to the local Sol metric of $\widetilde{M}$, it can be verified that each element of Λ preserves the foliation by the singular Euclidean planes. Since $\pi_1(M) < \Lambda$, we have the covering $(\widetilde{M}, \widetilde{\mathscr{F}})/\pi_1(M) \to (\widetilde{M}, \widetilde{\mathscr{F}})/\Lambda$. We see that for any pair $(M', \mathscr{F}')$ commensurable with (M, F) the group $\pi_1(M')$ embeds into Λ and hence $(M', \mathscr{F}')$ covers $(\widetilde{M}, \widetilde{\mathscr{F}})/\Lambda$. Thus the theorem will be proved if we establish the following claim.

Claim 2.7. *Λ is discrete with respect to the compact open topology.*

For the proof of this claim, the condition $\mathrm{Sing}(\phi) \subset \mathrm{Int}(F)$ is assumed in [Calegari et al. 2011]. We prove this claim without the assumption. Note that if $\mathrm{Sing}(\phi) \not\subset \mathrm{Int}(F)$, the singular Sol metric is not necessarily complete. Let $\Lambda' < \Lambda$ be the subgroup consisting of isometries that preserve each fiber of $\widetilde{M}$ setwise. We first prove that the subgroup Λ' is discrete. Let S be a fiber of $\widetilde{M}$ and $\bar{S}$ be its completion with respect to the singular Euclidean metric. We will extend $p = \pi|_S : S \to \mathrm{Int}(F)$ to a local isometry $\bar{p} : \bar{S} \to \mathrm{Int}(F) \cup \mathrm{Sing}(\phi)$. Let $\{x_i\}$ be a Cauchy sequence in S. Then $\{p(x_i)\}$ is a Cauchy sequence in $\mathrm{Int}(F)$ and it converges to either an interior point of F or a point in $\mathrm{Sing}(\phi)$. Since $\bar{S}$ consists of the equivalence classes of Cauchy sequences in S, we can define $\bar{p} : [(x_i)] \mapsto \lim p(x_i)$. Since p is a local isometry, $\bar{p}$ is well defined and a local isometry. Therefore we get $E := \bar{S} \setminus S = \bar{p}^{-1}(\mathrm{Sing}(\phi))$ for the natural extension $\bar{p}$ of p. Any isometry $\varphi : S \to S$ extends to an isometry $\bar{\varphi} : \bar{S} \to \bar{S}$ and by construction we get $\bar{\varphi}(E) = E$. Suppose there is a sequence $\{\bar{\varphi}_i\}$ of isometries such that $\bar{\varphi}_i \to \mathrm{id}$. Since the distances between two distinct points in E are bounded from below by a positive constant, for large enough i, $\bar{\varphi}_i$ must fix E pointwise. Suppose that $\bar{\varphi} : \bar{S} \to \bar{S}$ is an isometry which preserves E pointwise. Since $\bar{S}$ is a singular Euclidean plane, we may find two points e_1, e_2 in E which can be joined by a unique geodesic γ. By appealing to the distance from e_1 and e_2, it follows that $\bar{\varphi}$ preserves γ pointwise. Note that every isometry on $\bar{S}$ leaves the set of leaves of $p^{-1}(\mathscr{F}^s)$ and $p^{-1}(\mathscr{F}^u)$ invariant. This implies that every leaf that intersects with γ is preserved by $\bar{\varphi}$. Let l be one of such leaves. Since $\bar{\varphi}$ is a local isometry of Sol metric, it locally acts as a translation on $\bar{S}$. Therefore $\bar{\varphi}$ fixes l pointwise. Since each leaf of $\mathscr{F}^s$ or $\mathscr{F}^u$ is dense in $\mathrm{Int}(F)$, the orbit of l under the action of the deck transformation group associated to p is also dense in $\bar{S}$. Hence $\bar{\varphi}$ is identity on a dense subset of $\bar{S}$ and since it is an isometry, we get $\bar{\varphi} = \mathrm{id}$. Therefore for large enough i, we get $\bar{\varphi}_i = \mathrm{id}$. This proves the discreteness of Λ'.

The discreteness of the dynamical direction of Λ follows from exactly the same argument in [Calegari et al. 2011]. We include the proof for completeness. Note that each isometry $\varphi \in \Lambda$ extends to the metric completion $\bar{M}$ of $\widetilde{M}$. We

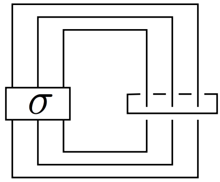

Figure 2. The fibered link associated to a braid $\sigma \in B_3$.

may parametrize each fiber by real numbers t in such a way that for any two fixed flow lines $a(t), b(t) \in E(t) \subset \bar{S}(t)$, the distance between $a(t)$ and $b(t)$ is $\sqrt{e^{2t}x^2 + e^{-2t}y^2}$ for some fixed x and y when $|t|$ is small enough. For small $|t|$ the distance between any two points in $E(t)$ are bounded from below by a constant which does not depend on t. Therefore since $\sqrt{e^{2t}x^2 + e^{-2t}y^2}$ is not a locally constant function, an isometry $\varphi \in \Lambda$ close enough to the identity must fix each fiber of the foliation by the singular Euclidean planes. Thus we see that Λ is discrete.

Since isometries may fix only singular points, if $\mathrm{Int}(F) \cap \mathrm{Sing}(\phi) = \varnothing$, then Λ has no fixed point in $\widetilde{M}$ and the last assertion holds. $\square$

Corollary 2.8. *All the fibrations of $M_1 = S^3 \setminus 6_2^2$ and the magic 3-manifold M_2 are minimal elements.*

Proof. M_1 (resp. M_2) is homeomorphic to the complement of the fibered link associated to $\sigma_1\sigma_2^{-1} \in B_3$ (resp. $\sigma_1\sigma_2^{-1}\sigma_1 \in B_3$), where B_3 is the braid group on 3 strands (see Figure 2). It is well known that for every pseudo-Anosov element of B_3, all singularities are on the punctures. Therefore it suffices to prove that M_1 and M_2 are minimal *manifolds* (not orbifolds) with respect to usual covering relation. M_1 has volume $4V_0$, where $V_0 \approx 1.01\ldots$ is the volume of the ideal regular tetrahedron (see for example [Gehring et al. 1998]). By [Cao and Meyerhoff 2001], M_1 can only cover the figure-eight knot complements or its sister (m004 or m003 in SnapPea notation). However, SnapPy [Culler et al. 2013] can enumerate all double covers of m003 and m004 and none of them are homeomorphic to M_1. Similarly, the magic 3-manifold M_2 has volume $\approx 5.33\ldots$ and if it covers a manifold with degree 2, then its volume is $\approx 2.66\ldots$, which is less than the volume of the ideal regular octahedron ($\approx 3.66\ldots$). By [Agol 2010], such a manifold has only one cusp and cannot be doubly covered by M_2, which has 3 cusps. Moreover, since $\mathrm{Vol}(M_2)/3 \approx 1.77\cdots < 2V_0$, again by [Cao and Meyerhoff 2001], M_2 cannot cover any manifold with degree greater than 2. Now the result follows from the last assertion of Theorem 2.6. $\square$

Remark 2.9. For a fixed surface, there exists a pseudo-Anosov automorphism with the smallest dilatation [Ivanov 1988]. It is interesting to compute the smallest dilatation for a given surface. Hironaka [2010] and Kin and Takasawa [2011] computed dilatations of the monodromy of each fiber of $S^3 \setminus 6_2^2$ and the magic 3-manifold respectively. It turns out that many small dilatation pseudo-Anosov automorphisms appear as the monodromies of fibrations of those manifolds. Corollary 2.8 shows that all such fibrations are minimal and hence their monodromies can be candidates for the smallest dilatation pseudo-Anosov maps.

Transitivity of commensurability in Definition 1.2. In this subsection, we discuss the subtle difference between fibered commensurability and commensurability in the sense of Definition 1.2. Here, two pairs of type (F, ϕ) are said to be *fibered commensurable* if associated fibered pairs are commensurable. It is easy to see that if two pairs (F_1, ϕ_1) and (F_2, ϕ_2) are fibered commensurable, they are commensurable in the sense of following definition.

Definition 2.10 [Carlson 2010]. Two pairs (F_1, ϕ_1) and (F_2, ϕ_2) are *commensurable* if there is a surface $\widetilde{F}$, an automorphism $\widetilde{\phi}$, and nonzero integers k_1 and k_2, so that $(\widetilde{F}, \widetilde{\phi})$ covers $(F_i, \phi_i^{k_i})$ for $i = 1, 2$.

In [Calegari et al. 2011], it is claimed without proof that two pairs (F_1, ϕ_1) and (F_2, ϕ_2) are fibered commensurable if and only if they are commensurable in the sense of Definition 1.2. Since a map cannot always be lifted even if a power of it can be lifted, the claim is not trivial. The claim would follow from the transitivity of commensurability in the sense of Definition 1.2, because taking powers of an automorphism is tantamount to a covering. In this subsection, we will prove that the transitivity of commensurability in Definition 1.2 is valid if the automorphisms are pseudo-Anosov.

Proposition 2.11. *Suppose that (F_i, ϕ_i) and (F_{i+1}, ϕ_{i+1}) are commensurable in the sense of Definition 1.2 for $i = 1, 2$. Suppose further that ϕ_i are pseudo-Anosov for $i = 1, 2, 3$. Then there exists a pair $(F_{123}, \widetilde{\phi}_i)$ that covers (F_i, ϕ_i) for each $i = 1, 2, 3$ such that $\widetilde{\phi}_1^{k_1} = \widetilde{\phi}_2^{k_2} = \widetilde{\phi}_3^{k_3}$ for some $k_1, k_2, k_3 \in \mathbb{Z} \setminus \{0\}$. In particular, commensurability in the sense of Definition 1.2 is transitive.*

Proof. In Theorem 2.6 we proved that each hyperbolic fibered commensurability class contains a unique minimal element. Let $M = [F_1, \phi_1]$. Recall that Λ is the group of isometries of the singular Sol metric on the universal cover $\widetilde{M}$ (see the proof of Theorem 2.6). By considering the subgroup Λ^+ that consists of isometries which preserve the orientation of $\widetilde{M}$ and the orientation of the leaf space of $\widetilde{M}$. By taking $M_{\min}^+ := \widetilde{M} / \Lambda^+$, we get a unique minimal element among all commensurable fibered pairs both orientable and co-orientable. Although there is a natural extension of this proof in the case where $\widetilde{M} / \Gamma^+$ is an orbifold, such a proof would require more terminology and could obfuscate the key ideas of the proof. Therefore, we only present the case where $\widetilde{M} / \Gamma^+$ is a manifold. In this case we get an associated pair $(F_{\min}, \phi_{\min})$ since $M_{\min}^+$ is orientable and co-orientable. Each

(F_i, ϕ_i) covers $(F_{\min}, \phi_{\min}^{l_i})$ for some $l_i \in \mathbb{Z} \setminus \{0\}$ $(i = 1, 2, 3)$. Note that $\phi_{\min}$ is not always lifted to F_i. Let $H_i < \pi_1(F_{\min})$ be a subgroup which is the image of $\pi_1(F_i)$ by the covering map for each $i = 1, 2, 3$. Further let $d = [\pi_1(F_{\min}) : H_1 \cap H_2 \cap H_3]$, and take $H_{123} := \bigcap\{H < \pi_1(F_{\min}) \mid [\pi_1(F_{\min}) : H] = d\}$. Recall that for a group G, a subgroup $H < G$ is called *characteristic* if for every isomorphism $f : G \to G$, we get $f(H) = H$. H_{123} is a characteristic subgroup and hence every homeomorphism on $F_{\min}$ lifts to the covering F_{123} that corresponds to $H_{123} < \pi_1(F_{\min})$. Since each $\phi_i : F_i \to F_i$ is a lift of $\phi_{\min}^{l_i}$, it can be lifted to $\widetilde{\phi}_i : F_{123} \to F_{123}$. Let l be the least common multiple of l_i's, then by putting $k_i = l/l_i$, we get $\widetilde{\phi}_1^{k_1} = \widetilde{\phi}_2^{k_2} = \widetilde{\phi}_3^{k_3}$ on F_{123}. □

Remark 2.12. We do not know if the transitivity or the equivalence of fibered commensurability and commensurability in the sense of Definition 1.2 holds for the case where the automorphisms are periodic or reducible.

3. Thurston norm and normalized entropy

Thurston norm. Let M be a fibered hyperbolic 3-manifold. In this subsection we recall briefly the Thurston norm on $H^1(M; \mathbb{R})$ and discuss the relationship between fibered commensurability of fibrations on a fixed manifold M and the normalized entropy. For more details about the Thurston norm, see [Thurston 1986; Kapovich 2001; Kin and Takasawa 2011]. For any (possibly disconnected) compact surface $F = F_1 \sqcup F_2 \sqcup \cdots \sqcup F_n$, let $\chi_-(F)$ be the sum of the absolute values of Euler characteristics $|\chi(F_i)|$ of components with negative Euler characteristics. For a given $\omega \in H^1(M; \mathbb{Z}) \subset H^1(M; \mathbb{R})$, we define $\|\omega\|$ to be

$$\min\big\{\chi_-(F) \mid F \text{ is an embedded orientable surface } (F, \partial F) \subset (M, \partial M), \text{ and } [F] \in H_2(M, \partial M; \mathbb{Z}) \text{ is the Poincaré dual of } \omega \in H^1(M; \mathbb{Z})\big\}.$$

If F realizes the minimum, we call F a minimal representative of ω. We can extend this norm to $H^1(M; \mathbb{Q})$ by $\|\omega\| = \|r\omega\|/r$. It turns out that $\|\cdot\|$ extends continuously to $H^1(M; \mathbb{R})$. Further, this $\|\cdot\|$ turns out to be seminorm on $H^1(M; \mathbb{R})$ and the unit ball $U = \{\omega \in H^1(M; \mathbb{R}) \mid \|\omega\| \leq 1\}$ is a compact convex polygon [Thurston 1986]. The seminorm $\|\cdot\|$ is called the Thurston norm on $H^1(M; \mathbb{R})$. We need some more terminologies to explain the relationship between $\|\cdot\|$ and fibrations on M. We denote

- the cone over a top-dimensional face Δ of the unit ball U by C_Δ,
- the set of integral classes on $\mathrm{Int}(C_\Delta)$ by $\mathrm{Int}(C_\Delta(\mathbb{Z}))$, and
- the set of rational classes on a top-dimensional face Δ by $\Delta(\mathbb{Q})$.

Theorem 3.1 [Thurston 1986]. *Let M be a fibered hyperbolic 3-manifold and F the fiber. Then there is a top-dimensional face Δ of U such that*

- *the dual of* $[F] \in H_2(M, \partial M; \mathbb{Z})$ *belongs to* $\mathrm{Int}(C_\Delta(\mathbb{Z}))$, *and*
- *for every primitive class* ω *in* $\mathrm{Int}(C_\Delta(\mathbb{Z}))$, *a minimal representative of* ω *is the fiber of a fibration on* M.

We call the face Δ in Theorem 3.1 a *fibered face* and the cone over a fibered face a *fibered cone*.

As a corollary, we see that if the first Betti number $b_1(M) > 1$ and M is fibered, then M has infinitely many distinct fibrations. We will discuss fibered commensurability of fibrations of a hyperbolic fibered 3-manifold.

Normalized entropy. The normalized entropy is shared by commensurable fibrations on a fixed hyperbolic 3-manifold.

Proposition 3.2. *Suppose that* $[F_1, \phi_1] = [F_2, \phi_2]$ *and their interior admit hyperbolic metrics. If* (F_1, ϕ_1) *is commensurable to* (F_2, ϕ_2), *then*

$$\chi(F_1) \log \lambda(\phi_1) = \chi(F_2) \log \lambda(\phi_2).$$

Proof. There are pairs $(\widetilde{F}, \widetilde{\phi_i})$ that cover (F_i, ϕ_i) and $k_i \in \mathbb{Z} \setminus \{0\}$ for $i = 1, 2$ such that $\widetilde{\phi}_1^{k_1} = \widetilde{\phi}_2^{k_2}$. Then the mapping torus $[\widetilde{F}, \widetilde{\phi}_i^{k_i}]$ covers $[F_i, \phi_i]$ and the degree of this cover is $k_i \chi(\widetilde{F})/\chi(F_i)$. Since $[F_1, \phi_1] = [F_2, \phi_2]$, we get $k_1/\chi(F_1) = k_2/\chi(F_2)$. Since $\lambda(\phi) = \lambda(\widetilde{\phi})$,

$$\chi(\widetilde{F}) \log \lambda(\widetilde{\phi}_i^{k_i}) = \frac{\chi(\widetilde{F})}{\chi(F_1)} \chi(F_1) k_1 \log \lambda(\phi_1) = \frac{\chi(\widetilde{F})}{\chi(F_2)} \chi(F_2) k_2 \log \lambda(\phi_2).$$

Putting them all together, we get $\chi(F_1) \log \lambda(\phi_1) = \chi(F_2) \log \lambda(\phi_2)$. □

Each primitive integral class in $C_\Delta(\mathbb{Z})$ corresponds to a rational class in $\mathrm{Int}(\Delta)$. The normalized entropy defines a function $\mathrm{ent} : \Delta(\mathbb{Q}) \to \mathbb{R}$. In [Fried 1982], the function $1/\mathrm{ent}$ is shown to be concave and therefore it extends to $\mathrm{Int}(\Delta)$. Moreover:

Theorem 3.3 [McMullen 2000]. *The function* $1/\mathrm{ent} : \mathrm{Int}(\Delta) \to \mathbb{R}$ *is strictly concave.*

In Example 3.12 of [Calegari et al. 2011], it is remarked that some fibrations on $S^3 \setminus 6_2^2$ are not commensurable. In Corollary 2.8, it is proved that all fibrations on $S^3 \setminus 6_2^2$ are minimal elements and since each minimal element is unique, we see that two fibrations of $S^3 \setminus 6_2^2$ are either symmetric or noncommensurable. Here, we give an alternative proof of this fact in terms of the normalized entropy. In [Hironaka 2010; McMullen 2000], the unit ball of the Thurston norm on $H^1(S^3 \setminus 6_2^2)$ is computed to be a square. Further, the symmetries of the square all come from the symmetries of the manifold (see Example 4.5 for more details about the symmetries of $S^3 \setminus 6_2^2$). Therefore the function $1/\mathrm{ent}$ is invariant under the action of the symmetries of the unit ball. Since $1/\mathrm{ent}$ is *strictly* concave, this proves that any two fibrations that correspond to distinct elements in $H^1(M; \mathbb{Z})$ are either symmetric or noncommensurable. In other words, the normalized entropy determines the commensurability class of a fibration on $S^3 \setminus 6_2^2$ up to symmetry.

On the other hand, in [Kin et al. 2012, §2], it is observed that for the magic 3-manifold N there are rational points on a fibered face which share the same normalized entropy but which are not symmetric to each other. However, again by Corollary 2.8, we also see that any two distinct fibrations of N are either symmetric or noncommensurable. Hence for the magic 3-manifold, the commensurability classes of fibrations are not determined by the normalized entropies. We do not know for what kind of hyperbolic 3-manifolds the commensurability classes of fibrations on the same hyperbolic 3-manifold are determined by the normalized entropy up to symmetry.

4. Commensurability of fibrations on a hyperbolic 3-manifold

In this section we prove Theorems 1.4 and 1.3.

Manifolds without hidden symmetries. We start with some definitions. A *Kleinian group* is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$. Two Kleinian groups Γ_1 and Γ_2 are said to be *commensurable* if $\Gamma_1 \cap \Gamma_2$ is a finite-index subgroup of both Γ_1 and Γ_2. Let Γ be a Kleinian group. The *commensurator* $C^+(\Gamma)$ of Γ is

$$C^+(\Gamma) = \{h \in \mathrm{PSL}(2, \mathbb{C}) \mid \Gamma \text{ and } h\Gamma h^{-1} \text{ are commensurable}\},$$

and the *normalizer* $N^+(\Gamma)$ is

$$N^+(\Gamma) = \{h \in \mathrm{PSL}(2, \mathbb{C}) \mid \Gamma = h\Gamma h^{-1}\}.$$

Note that $N^+(\Gamma) < C^+(\Gamma)$.

Let M be a hyperbolic 3-manifold and $\rho : \pi_1(M) \to \Gamma < \mathrm{PSL}(2, \mathbb{C})$ a holonomy representation of $\pi_1(M)$. By the Mostow–Prasad rigidity theorem, any self-homeomorphism $\varphi : M \to M$ corresponds to a conjugation of Γ. Therefore we get $N(\Gamma)/\Gamma \cong \mathrm{Isom}(M)$, where $\mathrm{Isom}(M)$ is the group of self-homeomorphisms of M. If $C^+(\Gamma) \setminus N^+(\Gamma) \neq \varnothing$, each nontrivial element $h \in C^+(\Gamma) \setminus N^+(\Gamma)$ is said to be a *hidden symmetry*. Then M is said to have no hidden symmetries if Γ has no hidden symmetries. Note that by the Mostow–Prasad rigidity theorem, the holonomy representations of $\pi_1(M)$ are related by a conjugation. Hence the definition does not depend on the choice of a holonomy representation.

Proof of Theorem 1.4. Let $(M, \mathcal{F}_1)$ and $(M, \mathcal{F}_2)$ be commensurable fibered pairs that correspond to two distinct fibrations on M. By Theorem 2.6 we have a unique minimal element $(N, \mathcal{G})$ in the commensurability class. Let $\rho : \pi_1(N) \to \mathrm{PSL}(2, \mathbb{C})$ be a holonomy representation and $\Gamma := \rho(\pi_1(N))$. Since $(M, \mathcal{F}_1)$ and $(M, \mathcal{F}_2)$ cover $(N, \mathcal{G})$, there are two corresponding coverings $p_1, p_2 : M \to N$.

Let $\Gamma_i = \rho p_{i*}(\pi_1(M))$ for $i = 1, 2$. By the Mostow–Prasad rigidity theorem, there is $h \in \mathrm{PSL}(2, \mathbb{C})$ such that $h\Gamma_1 h^{-1} = \Gamma_2$. Further, since $\Gamma_2 < \Gamma \cap h\Gamma h^{-1}$, $h \in C^+(\Gamma) = C^+(\Gamma_1) = N^+(\Gamma_1)$. The last equality holds since M has no hidden symmetries. It follows that $\Gamma_1 = \Gamma_2$ and hence there exists a homeomorphism $\varphi : M \to M$ such that $p_1\varphi = p_2$. Therefore ω_1 and ω_2 are symmetric. □

Remark 4.1. Hyperbolic 3-manifolds with hidden symmetries are "rare" among all nonarithmetic hyperbolic 3-manifolds (see for example, [Goodman et al. 2008]). Hence we may expect that "most" hyperbolic 3-manifolds have no hidden symmetries and therefore have no nonsymmetric but commensurable fibration.

Remark 4.2. As mentioned above, there are no nonsymmetric but commensurable fibrations on $S^3 \setminus 6_2^2$ and the magic 3-manifold. However, $S^3 \setminus 6_2^2$ and the magic 3-manifold are arithmetic and by a result of Margulis [1991], they have lots of hidden symmetries. Therefore even though a manifold has hidden symmetries, it might not have any nonsymmetric but commensurable fibrations.

Nonsymmetric and commensurable fibrations. We now prove Theorem 1.3 by constructing examples of manifolds that have nonsymmetric but commensurable fibrations.

Lemma 4.3. *Let M be a fibered hyperbolic 3-manifold. Suppose two primitive elements $\omega_1 \neq \pm\omega_2 \in H^1(M; \mathbb{Z})$ correspond to fibrations with the fibers and the monodromies (F_1, ϕ_1) and (F_2, ϕ_2) respectively. We suppose further $(F_1, \phi_1) = (F_2, \phi_2)$ (that is, conjugate to each other). Then, for all large enough $n \in \mathbb{N}$, there exists a degree n covering space $p_n : M_n \to M$ such that $p_n^*(\omega_1)$ and $p_n^*(\omega_2)$ correspond to commensurable but nonsymmetric fibrations.*

Proof. Note that by the universal coefficient theorem, we have

$$H^1(M; \mathbb{Z}) \cong \mathrm{Hom}(H_1(M)/\mathrm{Tor}, \mathbb{Z}),$$

where Tor is the torsion part. This isomorphism is determined by a choice of a basis of $H_1(M; \mathbb{Z})/\mathrm{Tor}$. Let $A_i = \mathrm{ab}(\pi_1(F_i))/\mathrm{Tor}$, where $\mathrm{ab} : \pi_1(M) \to H_1(M)$ is the abelianization and $\pi_1(F_i) \hookrightarrow \pi_1(M)$ is an injection induced by the fiber bundle structure of M associated to (F_i, ϕ_i) for $i = 1, 2$. The fiber bundle structure of M gives the exact sequence

$$0 \to \pi_1(F_i) \to \pi_1(M) \xrightarrow{\rho_i} \pi_1(S^1) \cong \mathbb{Z} \to 0.$$

The map ρ_i factors through the abelianization since $\pi_1(S^1) \cong \mathbb{Z}$ is abelian. Hence we get $A_i = \mathrm{Ker}(\omega_i) \cong \mathbb{Z}^{b-1}$, where b is the first Betti number of M. We consider the dynamical covering $p_n : M_n \to M$ of degree n with respect to ω_1 (that is, the covering corresponding to (F_1, ϕ_1^n)). This is the covering corresponding to the

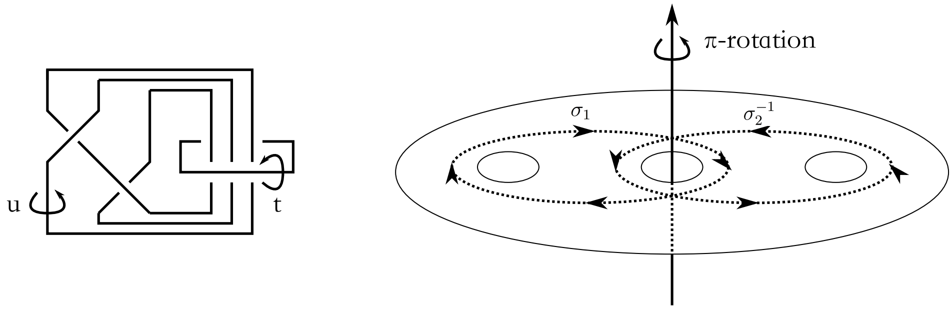

Figure 3. An involution map h on the 4-holed sphere.

surjective map

$$\pi_1(M) \xrightarrow{\text{ab}} H_1(M) \xrightarrow{\omega_1} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}.$$

For sufficiently large n, there exists $a \in A_2$ such that a maps to a nonzero element by the above surjective map. This means that each component of $p_n^{-1}(F_2)$ is not homeomorphic to F_2. $\square$

Example 4.4. The 3-manifold $S^3 \setminus 6_2^2$ and the magic manifold have symmetries that permute cusps, and therefore they do have two distinct elements in their first cohomology with homeomorphic fibers and conjugate monodromies.

Example 4.5. In this example we observe that $M := S^3 \setminus 6_2^2$ has two symmetric fibrations in the same fibered cone in $H^1(M)$. Although this fact can be checked by computing the symmetry group by SnapPy [Culler et al. 2013], we give a geometric proof. The first half of the following argument is due to Eriko Hironaka, see also [Hironaka 2010].

Let u, t be the generators of $H_1(M, \mathbb{Z})$ that correspond to the meridians of 6_2^2 (see the left picture of Figure 3). Let $U, T \in H^1(M; \mathbb{Z})$ be the dual of u, t respectively. Then U corresponds to the fibration of M with monodromy f that corresponds to $\sigma_1\sigma_2^{-1} \in B_3$. Let h be a π-rotation, which is depicted in Figure 3. We can see that $f^{-1} = \sigma_2\sigma_1^{-1} = hfh$, that is f and f^{-1} are conjugate to each other. Then we take the mirror image of 6_2^2. By isotopy and above conjugacy, we see that 6_2^2 is amphicheiral. The induced map on $H^1(M; \mathbb{Z})$ of the symmetry on M that gives amphicheirality satisfies $U \mapsto -U$ and $T \mapsto T$. This symmetry preserves the fibered face $\Delta := \{aU + bT \mid -1 < a < 1, b = 1\}$. By this symmetry, we see that fibrations on the cone C_Δ over Δ of the form $nU + mT$ and $-nU + mT$ $(n, m \in \mathbb{Z})$ are symmetric.

Proof of Theorem 1.3. Putting Lemma 4.3 and Example 4.5 together, we have a proof. $\square$

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HIDETOSHI MASAI
DEPARTMENT OF MATHEMATICAL AND COMPUTING SCIENCES
TOKYO INSTITUTE OF TECHNOLOGY
2-12-1 O-OKAYAMA, MEGURO-KU
TOKYO 152-8552
JAPAN

masai9@is.titech.ac.jp

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