MULTIPLICATIVE DIRAC STRUCTURES

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We introduce multiplicative Dirac structures on Lie groupoids, providing a unified framework to study both multiplicative Poisson bivectors (Poisson groupoids) and multiplicative closed 2-forms such as symplectic groupoids. We prove that for every source simply connected Lie groupoid G with Lie algebroid AG, there exists a one-to-one correspondence between multiplicative Dirac structures on G and Dirac structures on AG that are compatible with both the linear and algebroid structures of AG. We explain in what sense this extends the integration of Lie bialgebroids to Poisson groupoids and the integration of Dirac manifolds. We explain the connection between multiplicative Dirac structures and higher geometric structures such as $\mathscr{LA}$-groupoids and $\mathscr{CA}$-groupoids.

1. Introduction

Dirac structures were introduced by Courant and Weinstein [1988] as a common generalization of Poisson bivectors, closed 2-forms and regular foliations. A *Dirac structure* on a smooth manifold M consists of a vector subbundle $L \subseteq \mathbb{T}M := TM \oplus T^*M$ that is maximal isotropic with respect to the nondegenerate symmetric pairing on $\mathbb{T}M$,

$$\langle (X, \alpha), (Y, \beta) \rangle = \alpha(Y) + \beta(X),$$

and that satisfies the integrability condition

$$[\![\Gamma(L), \Gamma(L)]\!] \subseteq \Gamma(L),$$

with respect to the Courant bracket $[\![\,\cdot\,,\cdot\,]\!] : \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$,

$$[\![(X,\alpha),(Y,\beta)]\!] = ([X,Y], \mathscr{L}_X\beta - i_Y d\alpha).$$

The integrability in the sense of Courant unifies different integrability conditions, including closed 2-forms, Poisson bivectors and regular foliations (see [Courant 1990b; Courant and Weinstein 1988]). More precisely, a 2-form ω on a smooth manifold M induces a bundle map $\omega^\sharp : TM \to T^*M, X \mapsto \omega(X,\cdot)$ whose graph $L_\omega = \{(X, \omega^\sharp(X) \mid X \in TM)\}$ is a Lagrangian subbundle of $\mathbb{T}M$. In this case, the Courant integrability of L_ω is equivalent to ω being a closed 2-form. Similarly, any bivector π on M defines a bundle map $\pi^\sharp : T^*M \to TM, \alpha \mapsto \pi(\alpha,\cdot)$ whose graph $L_\pi = \{(\pi^\sharp(\alpha), \alpha)\}$ is a Lagrangian subbundle of $\mathbb{T}M$. One checks that L_π satisfies the Courant integrability condition if and only if π is a Poisson bivector. Also, if $F \subseteq TM$ is a regular subbundle we denote by $F^\circ \subseteq T^*M$ its annihilator. Then the Lagrangian subbundle $F \oplus F^\circ \subseteq \mathbb{T}M$ is integrable in the sense of Courant if and only if $F \subseteq TM$ is involutive with respect to the Lie bracket of vector fields.

The main objective of this paper is to study Dirac structures defined on Lie groupoids, satisfying a suitable compatibility condition with the groupoid multiplication. Our study is motivated by a variety of geometrical structures compatible with group or groupoid structures, including:

(i) *Poisson–Lie groups*: A Poisson–Lie group is a Lie group G with a Poisson structure π that is compatible in the sense that the multiplication map $m : G \times G \to G$ is a Poisson map. Equivalently, the Poisson bivector π is *multiplicative*, that is,

$$\pi_{gh} = (l_g)_*\pi_h + (r_h)_*\pi_g,$$

for every $g, h \in G$. Here l_g and r_h denote the left and right multiplication by g and h, respectively. Poisson–Lie groups arise as semiclassical limit of quantum groups, and they are infinitesimally described by *Lie bialgebras*. See for example, [Drinfel'd 1983].

(ii) *Symplectic groupoids*: A symplectic groupoid is a Lie groupoid G with a symplectic structure ω that is compatible with the groupoid multiplication in the sense that the graph

$$\mathrm{Graph}(m) \subseteq G \times G \times \overline{G}$$

is a Lagrangian submanifold with respect to the symplectic structure $\omega \oplus \omega \ominus \omega$. This compatibility condition is equivalent to saying that ω is *multiplicative*, that is,

$$m^*\omega = pr_1^*\omega + pr_2^*\omega,$$

where $pr_1, pr_2 : G_{(2)} \to G$ are the canonical projections and $G_{(2)} \subseteq G \times G$ is the set of composable groupoid pairs. Symplectic groupoids arise in the context of quantization of Poisson manifolds [Weinstein 1987; Weinstein and Xu 1991],

connecting Poisson geometry to noncommutative geometry. In [Cattaneo and Felder 2001], symplectic groupoids appeared as phase spaces of certain sigma models. The infinitesimal description of symplectic groupoids is given by *Poisson structures*, see for example, [Weinstein 1987; Coste et al. 1987].

(iii) *Poisson groupoids*: These objects were introduced by A. Weinstein [1988] unifying Poisson–Lie groups and symplectic groupoids. A Poisson groupoid is a Lie groupoid G equipped with a Poisson structure π that is compatible with the groupoid multiplication in the sense that

$$\text{Graph}(m) \subseteq G \times G \times \overline{G}$$

is a coisotropic submanifold. These structures are related to the geometry of the classical dynamical Yang–Baxter equation, see for instance [Etingof and Varchenko 1998]. At the infinitesimal level, Poisson groupoids are described by *Lie bialgebroids* [Mackenzie and Xu 1994].

(iv) *Presymplectic groupoids*: Lie groupoids equipped with a multiplicative closed 2-form were studied in [Bursztyn et al. 2004]. A presymplectic groupoid is a Lie groupoid G with a multiplicative closed 2-form ω satisfying suitable nondegeneracy conditions. These objects arise in connection with equivariant cohomology and generalized moment maps [Bursztyn and Crainic 2009]. The infinitesimal description of presymplectic groupoids is given by *Dirac structures*, extending the infinitesimal description of symplectic groupoids. More generally, Lie groupoids endowed with arbitrary multiplicative closed 2-forms are infinitesimally described by bundle maps $\sigma : AG \to T^*M$ called *IM-2-forms*. Here AG denotes the Lie algebroid of G, and T^*M is the cotangent bundle of the base of G.

The first goal of this work is to find a suitable definition of multiplicative Dirac structure that includes both multiplicative Poisson bivectors and multiplicative closed 2-forms, and hence encompasses all examples above. This is obtained by observing that given a Lie groupoid G over M with Lie algebroid AG, the tangent bundle TG and the cotangent bundle T^*G inherit natural Lie groupoid structures over TM and A^*G, respectively. One observes that a bivector π_G is multiplicative if and only if the bundle map $\pi_G^\sharp : T^*G \to TG$ is a groupoid morphism [Mackenzie and Xu 1994]. Similarly, a 2-form ω_G is multiplicative if and only if the bundle map $\omega_G^\sharp : TG \to T^*G$ is a morphism of Lie groupoids. It turns out that the direct sum vector bundle $TG \oplus T^*G$ is a Lie groupoid over $TM \oplus A^*G$, and graphs of both multiplicative Poisson bivectors and multiplicative closed 2-forms define Lie subgroupoids of $TG \oplus T^*G$. We say that a Dirac structure L_G on a Lie groupoid G is *multiplicative* if $L_G \subseteq TG \oplus T^*G$ is a Lie subgroupoid. A Lie groupoid G equipped with a multiplicative Dirac structure is referred to as a *Dirac groupoid*.

Our main purpose is to describe multiplicative Dirac structures infinitesimally, that is, in terms of Lie algebroid data. This work can be considered as a first step toward such a description. The main result of the present work says that for every source simply connected Lie groupoid G with Lie algebroid AG, multiplicative Dirac structures on G correspond to Dirac structures on AG suitably compatible with both the linear and Lie algebroid structures on AG. In the particular case of multiplicative Poisson bivectors and multiplicative 2-forms, we explain how this is equivalent to the known infinitesimal descriptions carried out in [Mackenzie and Xu 2000] and [Bursztyn et al. 2004], respectively. Along the way, we develop techniques that can treat all multiplicative structures above in a unified manner, often simplifying existing results and proofs.

The present paper is organized as follows. In Section 2 we recall the definition of tangent and cotangent structures including tangent and cotangent groupoids and their algebroids, that is, tangent and cotangent algebroids. We also give an intrinsic construction of the tangent lift of a Dirac structure, providing an alternative proof of the results shown in [Courant 1990a]. In Section 3 we define the main objects of our study, multiplicative Dirac structures. We discuss a variety of examples arising in nature, including foliated groupoids, Dirac Lie groups, tangent lifts of multiplicative Dirac structures, symmetries of multiplicative Dirac structures (for example, reduction of Poisson groupoids), B-field transformations of multiplicative Dirac structures and generalized complex groupoids. In Section 4 we introduce the notion of Dirac algebroid and also several examples are discussed, including foliated algebroids, Dirac Lie algebras, tangent lifts of Dirac algebroids, symmetries of Dirac algebroids (for example, reduction of Lie bialgebroids), B-field transformations of Dirac algebroids and generalized complex algebroids. In Section 5 we explain how the multiplicativity of a Dirac structure is reflected at the Lie algebroid level, proving the main result of this work, which says that if G is a source simply connected Lie groupoid with Lie algebroid AG, then there is a one-to-one correspondence between Dirac groupoid structures on G and Dirac algebroid structures on AG. Along the way, we explain the relation between multiplicative Dirac structures and higher structures such as $\mathscr{CA}$-groupoids and $\mathscr{LA}$-groupoids. We also relate the examples of Section 3 with the examples of Section 4, in the spirit of the correspondence established by the main result of the paper. In Section 6, we discuss conclusions and work in progress.

1A. *Notation and conventions.* For a Lie groupoid G over M we denote by $s, t : G \to M$ the source and target maps, respectively. The multiplication map is denoted by $m : G_{(2)} \to G$, where $G_{(2)} = \{(g, h) \in G \times G \mid s(g) = t(h)\}$ is the set of composable pairs. The Lie algebroid of G is defined by $AG := \text{Ker}(Ts)|_M$, with Lie bracket given by identifying sections of AG with right-invariant vector

fields on G and anchor map $\rho_{AG} := Tt|_{AG} : AG \to TM$. Given a Lie groupoid morphism $\Psi : G_1 \to G_2$, the corresponding Lie algebroid morphism is denoted by $A(\Psi) : AG_1 \to AG_2$. Arbitrary Lie algebroids are denoted by $A \to M$ with Lie bracket $[\,\cdot\,,\,\cdot\,]_A$ and anchor map ρ_A. Also, given a smooth manifold M, the tangent bundle is denoted by $p_M : TM \to M$ and the cotangent bundle is denoted by $c_M : T^*M \to M$.

2. Tangent and cotangent structures

2A. *Tangent and cotangent groupoids.* Let G be a Lie groupoid over M with Lie algebroid AG. The tangent bundle TG has a natural Lie groupoid structure over TM. This structure is obtained by applying the tangent functor to each of the structure maps defining G (source, target, multiplication, inversion and identity section). We refer to TG with this groupoid structure over TM as the *tangent groupoid* of G. The set of composable pairs of TG is $(TG)_{(2)} = T(G_{(2)})$, and for $(g, h) \in G_{(2)}$ and a tangent groupoid pair $(X_g, Y_h) \in (TG)_{(2)}$ the multiplication map on TG is

$$X_g \bullet Y_h := Tm(X_g, Y_h).$$

Now consider the cotangent bundle T^*G. It was shown in [Coste et al. 1987] that T^*G is a Lie groupoid over A^*G. The source and target maps are defined by

$$\tilde{s}(\alpha_g)u = \alpha_g\big(Tl_g(u - Tt(u))\big) \quad \text{and} \quad \tilde{t}(\beta_g)v = \beta_g(Tr_g(v)),$$
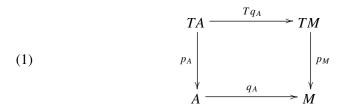
where $\alpha_g \in T_g^*G$, $u \in A_{s(g)}G$ and $\beta_g \in T_g^*G$, $v \in A_{t(g)}G$. The multiplication on T^*G is defined by

$$(\alpha_g \circ \beta_h)(X_g \bullet Y_h) = \alpha_g(X_g) + \beta_h(Y_h)$$

for $(X_g, Y_h) \in T_{(g,h)}G_{(2)}$.

We refer to T^*G with the groupoid structure over A^*G as the *cotangent groupoid* of G.

2B. *Tangent and cotangent algebroids.* Let $q_A : A \to M$ be a *vector bundle* over M. The tangent bundle TA has a natural structure of a *double vector bundle* [Pradines 1974], given by the diagram below.
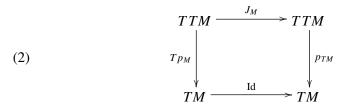
(1)

Assume now that $q_A : A \to M$ has a Lie algebroid structure with anchor map $\rho_A : A \to TM$ and Lie bracket $[\,\cdot\,,\cdot\,]$ on $\Gamma_M(A)$.

As explained in [Mackenzie 2005], there is a canonical Lie algebroid structure on the vector bundle $Tq_A : TA \to TM$. Recall that there exists a *canonical involution* $J_M : TTM \to TTM$, which is a morphism of double vector bundles. In a local coordinates system $(x^i, \dot{x}^i, \delta x^i, \delta \dot{x}^i)$ on TTM this map is given by

$$J_M((x^i, \dot{x}^i, \delta x^i, \delta \dot{x}^i)) = (x^i, \delta x^i, \dot{x}^i, \delta \dot{x}^i).$$

Notice that the map J_M yields a vector bundle isomorphism as below.

(2)

$$
\begin{array}{ccc}
TTM & \xrightarrow{\ \ J_M\ \ } & TTM \\
\Big\downarrow{\scriptstyle Tp_M} & & \Big\downarrow{\scriptstyle p_{TM}} \\
TM & \xrightarrow{\ \ \text{Id}\ \ } & TM
\end{array}
$$

Now we can apply the tangent functor to the anchor map $\rho_A : A \to TM$, yielding a bundle map $T\rho_A : TA \to TTM$, where TTM is equipped with bundle projection $Tp_M : TTM \to TM$. Therefore, composing $T\rho_A$ with the canonical involution J_M we obtain the bundle map $\rho_{TA} : TA \to TTM$, defined by

$$\rho_{TA} := J_M \circ T\rho_A,$$

which is a vector bundle morphism from $TA \to TTM$, where the target bundle is the one corresponding to the usual bundle projection $p_{TM} : TTM \to TM$. The map $\rho_{TA} : TA \to TTM$, as above, defines the tangent anchor map. In order to define the tangent Lie bracket, we observe that every section $u \in \Gamma_M(A)$ induces two types of sections of $Tq_A : TA \to TM$. The first type corresponds to the *linear section* $Tu : TM \to TA$, which is given by applying the tangent functor to the section $u : M \to A$. The second type of section is the *core* section $\hat{u} : TM \to TA$, which is defined by

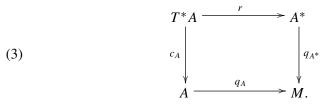$$\hat{u}(X) = T(0^A)(X) +_A \overline{u(p_M(X))},$$

where $0^A : M \to A$ denotes the zero section, and $\overline{u(p_M(X))} = d/dt(tu(p_M(X)))|_{t=0}$. As observed in [Mackenzie and Xu 1994], sections of the form Tu and $\hat{u}$ generate the module of sections $\Gamma_{TM}(TA)$. Therefore, the tangent Lie bracket is determined by

$$[Tu, Tv] = T[u, v], \quad [Tu, \hat{v}] = \widehat{[u, v]}, \quad [\hat{u}, \hat{v}] = 0,$$

and we extend to other sections by requiring the Leibniz rule with respect to the tangent anchor ρ_{TA}. This defines the natural Lie algebroid structure on $Tq_A : TA \to TM$.

Example 2.1. Assume that $A = \mathfrak{g}$ is a Lie *algebra*, that is, a Lie algebroid over a point. In this case, the tangent Lie *algebra* structure on $T\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}$ is given by the semidirect product Lie algebra determined by the adjoint representation of $\mathfrak{g}$ on itself.

Following [Mackenzie 2005], the cotangent bundle of a Lie algebroid inherits a Lie algebroid structure. For that, let us explain the vector bundle structure $T^*A \to A^*$. If (x^i, u^a) are local coordinates on A, we induce a local coordinates system $(x^i, u^a, p_i, \lambda_a)$ on T^*A, where (p_i) determines a cotangent element in T_x^*M and $(\lambda_a) \in A_x^*$ is a cotangent element with respect to the tangent direction to the fibers of A. Now the bundle projection $r : T^*A \to A^*$ is described locally by $r(x^i, u^a, p_i, \lambda_a) = (x^i, \lambda_a)$. These vector bundle structures define a commutative diagram

(3)
$$
\begin{array}{ccc}
T^*A & \xrightarrow{\;\;r\;\;} & A^* \\
{\scriptstyle c_A}\downarrow & & \downarrow{\scriptstyle q_{A^*}} \\
A & \xrightarrow{\;\;q_A\;\;} & M.
\end{array}
$$

This endows T^*A with a double vector bundle structure. Suppose that $q_A : A \to M$ carries a Lie algebroid structure. Then we can consider the dual bundle A^* endowed with the linear Poisson structure induced by A. The cotangent bundle $T^*A^* \to A^*$ has the Lie algebroid structure determined by the linear Poisson bivector on A^*. There exists a Legendre type map $R : T^*A^* \to T^*A$ that is an antisymplectomorphism with respect to the canonical symplectic structures, and it is locally defined by $R(x^i, \xi_a, p_i, u^a) = (x^i, u^a, -p_i, \xi_a)$. For an intrinsic definition see [Mackenzie and Xu 1994; Tulczyjew 1977].

Definition 2.2. The *cotangent algebroid* of A is the vector bundle $T^*A \to A^*$ equipped with the unique Lie algebroid structure that makes the Legendre type transform $R : T^*(A^*) \to T^*A$ into an isomorphism of Lie algebroids.

Example 2.3. Suppose that $A = \mathfrak{g}$ is a Lie *algebra*, that is, a Lie algebroid over a point. Then, the cotangent algebroid $T^*\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ is given by the transformation Lie algebroid with respect to the coadjoint representation of $\mathfrak{g}$ on its dual vector space $\mathfrak{g}^*$.

Finally, recall also that the *Tulczyjew map* $\Theta_M : TT^*M \to T^*TM$ is the isomorphism which, in a local coordinates system $(x^i, p_i, \dot{x}^i, \dot{p}_i)$, is given by

$$\Theta_M(x^i, p_i, \dot{x}^i, \dot{p}_i) = (x^i, \dot{x}^i, \dot{p}_i, p_i).$$

See [Mackenzie and Xu 1994; Tulczyjew 1977] for an intrinsic definition. Consider now a Lie groupoid G over M with Lie algebroid $AG = \ker Ts|_M$. There

exists a natural injective bundle map

(4) $$i_{AG} : AG \to TG.$$

The canonical involution $J_G : TTG \to TTG$ restricts to an isomorphism of Lie algebroids $j_G : T(AG) \to A(TG)$. More precisely, there exists a commutative diagram

(5)
$$
\begin{array}{ccc}
T(AG) & \xrightarrow{\ j_G\ } & A(TG) \\
\downarrow{\scriptstyle T(i_{AG})} & & \downarrow{\scriptstyle i_{A(TG)}} \\
TTG & \xrightarrow{\ J_G\ } & TTG
\end{array}
$$

In particular, the Lie algebroid $A(TG)$ of the tangent groupoid is canonically isomorphic to the tangent Lie algebroid $T(AG)$ of AG. Similarly, the Lie algebroid of the cotangent groupoid T^*G is isomorphic to the cotangent Lie algebroid $T^*(AG)$. For that, notice that the natural pairing $T^*G \oplus TG \to \mathbb{R}$ defines a groupoid morphism, and the application of the Lie functor yields a symmetric pairing $\langle\!\langle \, \cdot \, , \cdot \, \rangle\!\rangle : A(T^*G) \oplus A(TG) \to \mathbb{R}$, which is nondegenerate. See for example, [Mackenzie and Xu 1994; 2000]. In particular, we obtain an isomorphism $K_G : A(T^*G) \to A(TG)^*$, where the target dual is with respect to the fibration $A(TG) \xrightarrow{A(p_G)} AG$. Now we define a Lie algebroid isomorphism

(6) $$j'_G : A(T^*G) \to T^*(AG),$$

determined by the composition $j'_G = j_G^* \circ K_G$, where $j_G^* : A(TG)^* \to T^*(AG)$ is the bundle map dual to the isomorphism $j_G : T(AG) \to A(TG)$. As $j_G : T(AG) \to A(TG)$ is a suitable restriction of the canonical involution $J_G : TTG \to TTG$, the isomorphism j'_G is related to the Tulczyjew map $\Theta_G : TT^*G \to T^*TG$, via

$$j'_G = (Ti_{AG})^* \circ \Theta_G \circ i_{A(T^*G)}.$$

2C. *Tangent lift of a Dirac structure.* The tangent lift of Dirac structures was originally studied by T. Courant [1990a], who described tangent Dirac structures locally. I. Vaisman [2005] gives an intrinsic construction of tangent Dirac structures, where the tangent lift of a Dirac structure is described via the sheaf of local sections defining a Dirac subbundle of $TTM \oplus T^*TM$. Here, we provide an alternative description of the tangent lift of a Dirac structure relied on the tangent lift of Lie algebroid structures described in the previous section.

In order to fix our notation, we begin by summarizing some of the main properties of tangent lifts of vector fields and differential forms, see [Grabowski and Urbański 1997; Yano and Ishihara 1973]. Let $f \in C^\infty(M)$ be a smooth function. Then we

have a pair of smooth functions on TM defined by

$$f^v = f \circ p_M, \quad f^T = df.$$

We refer to f^v and f^T as the *vertical* and *tangent* lifts of f. One can see easily that the algebra of functions $C^\infty(TM)$ is generated by functions of the form f^v and f^T. Now, given a vector field X on M we define the *vertical* lift of X as the vector field X^v on TM that acts on vertical and tangent lifts of functions as

$$X^v(f^v) = 0, \quad X^v(f^T) = (Xf)^v.$$

The *tangent* lift of X is the vector field X^T on TM that acts on vertical and tangent lifts of functions in the following manner:

$$X^T(f^v) = (Xf)^v, \quad X^T(f^T) = (Xf)^T.$$

It is easy to see that vertical and tangent lifts of vector fields generate the space of all vector fields on TM. Now let us consider a 1-form α on a smooth manifold M. We define the *vertical* lift of α as the 1-form α^v on TM, which is determined by its value at vertical and tangent lifts of vector fields,

$$\alpha^v(X^v) = 0, \quad \alpha^v(X^T) = (\alpha(X))^v.$$

The *tangent* lift of α is the 1-form α^T on TM defined by

$$\alpha^T(X^v) = (\alpha(X))^v, \quad \alpha^T(X^T) = (\alpha(X))^T.$$

It is important to emphasize that vertical and tangent lifts of vector fields (resp. of 1-forms) are sections of the usual vector bundle structure $T(TM) \xrightarrow{p_{TM}} TM$ (resp. sections of $T^*(TM) \xrightarrow{c_{TM}} TM$), and they do not define sections of the tangent prolongation vector bundle $T(TM) \xrightarrow{Tp_M} TM$ (resp. of the tangent prolongation $T(T^*M) \xrightarrow{Tc_M} TM$). However, there exists a canonical relation between vector fields (resp. 1-forms) on TM and sections of the tangent prolongation vector bundle $T(TM) \to TM$ (resp. $T(T^*M) \to TM$). Given a vector field X and a 1-form α on M, we consider the linear sections $TX, T\alpha$ and the core sections $\hat{X}, \hat{\alpha}$ of the corresponding tangent prolongation vector bundles. It follows from the definition that

(7) $$J_M(TX) = X^T, \quad J_M(\hat{X}) = X^v.$$

(8) $$\Theta_M(T\alpha) = \alpha^T, \quad \Theta_M(\hat{\alpha}) = \alpha^v.$$

It turns out that many geometric properties of the direct sum vector bundle $T(TM) \oplus T^*(TM)$ can be understood in terms of tangent geometric properties of $T(TM) \oplus T(T^*M)$, using the canonical identification

$$J_M \oplus \Theta_M : T(TM) \oplus T(T^*M) \to T(TM) \oplus T^*(TM).$$

Now consider a Dirac structure L_M on M. Equivalently, we may think of L_M as a Lie algebroid over M with Lie bracket given by the Courant bracket on sections of L_M, and the anchor map ρ_M is the natural projection from $L_M \subseteq TM \oplus T^*M$ onto TM. According to a construction of K. Mackenzie and P. Xu [1994], we can consider the tangent prolongation Lie algebroid $TL_M \to TM$, with anchor map

$$\rho_{TM} = J_M \circ T\rho_M,$$

and Lie bracket defined by

$$[\hat{a}_1, \hat{a}_2]_{TL_M} = 0, \quad [Ta_1, \hat{a}_2]_{TL_M} = \widehat{[a_1, a_2]}, \quad [Ta_1, Ta_2]_{TL_M} = T[a_1, a_2],$$

where a_1, a_2 are sections of $L_M \to M$. We denote by L_{TM} the image of TL_M under the natural bundle map $J_M \oplus \Theta_M : TTM \oplus TT^*M \to TTM \oplus T^*TM$.

Proposition 2.4. *The subbundle $L_{TM} \subseteq TTM \oplus T^*TM$ is isotropic with respect to the nondegenerate symmetric pairing $\langle \cdot, \cdot \rangle_{TM}$ defined on $TTM \oplus T^*TM$.*

Proof. Let $(X, \alpha), (Y, \beta)$ be sections of L_M. Then, the tangent lifts (X^T, α^T) and (Y^T, β^T) define sections of L_{TM}. Notice that

$$\langle (X^T, \alpha^T), (Y^T, \beta^T) \rangle = (\beta(X))^T + (\alpha(Y))^T = (\langle (X, \alpha), (Y, \beta) \rangle)^T = 0.$$

This implies that L_{TM} is isotropic. $\qquad \square$

The tangent Lie algebroid $TL_M \to TM$ induces a unique Lie algebroid structure on $L_{TM} \to TM$ characterized by the property that $J_M \oplus \Theta_M : TL_M \to L_{TM}$ is a Lie algebroid isomorphism. The space of sections $\Gamma(L_{TM})$ is generated by sections of the form $a^T := (J_M \oplus \Theta_M)(Ta)$ and $a^v := (J_M \oplus \Theta_M)\hat{a}$, where a is a section of $L_M \to M$. In particular the induced Lie bracket on sections of L_{TM} is completely determined by identities

$$[a_1^v, a_2^v] = 0, \quad [a_1^T, a_2^v] = [\![a_1, a_2]\!]^v, \quad [a_1^T, a_2^T] = [\![a_1, a_2]\!]^T,$$

and the Leibniz rule with respect to the induced anchor map $pr_{TTM} : L_{TM} \to TTM$.

Proposition 2.5. *The induced Lie bracket on sections $\Gamma(L_{TM})$ is a restriction of the Courant bracket $[\![\cdot, \cdot]\!]_{TM}$ on sections of $TTM \oplus T^*TM$.*

Proof. Due to the identities (7) and (8), we only need to check that the Courant bracket on sections of L_{TM}, naturally induced by $J_M \oplus \Theta_M$, satisfies the bracket identities that determine the induced Lie bracket on $\Gamma(L_{TM})$. One observes that vertical and tangent lifts are compatible with Lie derivatives in the sense that

(1) $\mathscr{L}_{X^v}\alpha^v = 0$,

(2) $\mathscr{L}_{X^T}\alpha^v = (\mathscr{L}_X\alpha)^v$,

(3) $\mathscr{L}_{X^T}\alpha^T = (\mathscr{L}_X\alpha)^T$,

and we conclude that

(1) $[\![X^v \oplus \alpha^v, Y^v \oplus \beta^v]\!] = 0$,

(2) $[\![X^T \oplus \alpha^T, Y^v \oplus \beta^v]\!] = [X, Y]^v \oplus (\mathscr{L}_X \beta - i_Y d\alpha)^v$,

(3) $[\![X^T \oplus \alpha^T, Y^T \oplus \beta^T]\!] = [X, Y]^T \oplus (\mathscr{L}_X \beta - i_Y d\alpha)^T$.

Thus the Lie bracket on $\Gamma_{TM}(L_{TM})$ induced by the tangent Lie bracket on $\Gamma_{TM}(TL_M)$ coincides with the Courant bracket. $\qquad\square$

We have shown the following:

Proposition 2.6. *Let M be a smooth manifold. There exists a natural map*

$$\mathrm{Dir}(M) \to \mathrm{Dir}(TM)$$
$$L_M \mapsto L_{TM},$$

where $L_{TM} := (J_M \oplus \Theta_M)(TL_M)$.

The Dirac structure $L_{TM} \in \mathrm{Dir}(TM)$ given by the proposition above is referred to as the *tangent Dirac structure* induced by $L_M \in \mathrm{Dir}(M)$. It is straightforward to check that this construction unifies the tangent lift of both closed 2-forms and Poisson bivectors. Additionally, the presymplectic foliation of L_{TM} corresponds to taking the tangent bundle of each leaf endowed with the tangent lift of the leafwise presymplectic forms defined by L_M. See also [Boumaiza and Zaalani 2009] for a general construction of tangent lifts of Dirac structures on arbitrary Courant algebroids.

3. Multiplicative Dirac structures

This section introduces the main objects of study of this work, that is, Lie groupoids equipped with Dirac structures compatible with the groupoid multiplication, including both multiplicative Poisson and closed 2-forms as particular cases.

3A. *Definition and main examples.* Let G be a Lie groupoid over M, with Lie algebroid AG. Consider the direct sum Lie groupoid $\mathbb{T}G = TG \oplus T^*G$ with base manifold $TM \oplus A^*G$.

Definition 3.1. Let G be a Lie groupoid over M. A Dirac structure L_G on G is said to be *multiplicative* if $L_G \subseteq TG \oplus T^*G$ is a subgroupoid over some subbundle $E \subseteq TM \oplus A^*G$.

We refer to a pair (G, L_G), made up of a Lie groupoid G and a multiplicative Dirac structure L_G on G, as a *Dirac groupoid*. We use the notation $\mathrm{Dir}_{\mathrm{mult}}(G)$ to indicate the set consisting of all multiplicative Dirac structures on G.

It follows from the multiplicativity of L_G that $E \subseteq TM \oplus A^*G$ is a vector subbundle. In particular, a multiplicative Dirac structure L_G on a Lie groupoid G defines a $\mathscr{VB}$-subgroupoid $L_G \subseteq \mathbb{T}G$.

Example 3.2. Let ω_G be a closed multiplicative 2-form on a Lie groupoid G. The multiplicativity property of ω_G is equivalent to saying that the bundle map $\omega_G^\sharp : TG \to T^*G$ is a morphism of Lie groupoids. Hence, the corresponding Dirac structure $L_{\omega_G} = \mathrm{Graph}(\omega_G) \subseteq \mathbb{T}G$ is a multiplicative Dirac structure. In this case we have a groupoid $L_{\omega_G} \rightrightarrows E$, where $E \subseteq TM \oplus A^*G$ is the subbundle given by the graph of the bundle map $-\sigma^t$ determined by the *IM-2-form* (see [Bursztyn et al. 2004]) σ associated to ω_G.

Example 3.3. Let (G, π_G) be a Poisson groupoid. The multiplicativity of π_G is equivalent to saying that $\pi_G^\sharp : T^*G \to TG$ is a morphism of Lie groupoids. Therefore, the associated Dirac structure $L_{\pi_G} = \mathrm{Graph}(\pi_G) \subseteq \mathbb{T}G$ defines a multiplicative Dirac structure. In this case we have a groupoid $L_{\pi_G} \rightrightarrows E$, where $E \subseteq TM \oplus A^*G$ is the subbundle given by the graph of dual anchor map $\rho_{A^*G} : A^*G \to TM$.

The examples discussed previously show that Dirac groupoids lead to a natural generalization of Poisson groupoids and presymplectic groupoids. Our main aim is to describe Dirac groupoids infinitesimally, establishing in particular, a connection between such an infinitesimal description and Lie bialgebroids and IM-2-forms.

3B. *More examples of multiplicative Dirac structures.* In addition to multiplicative closed 2-forms and multiplicative Poisson bivectors, there are several interesting multiplicative Dirac structures, which will be discussed throughout this subsection.

3B1. *Foliated groupoids.* A regular distribution $F_G \subseteq TG$ is called *multiplicative* if it defines a Lie subgroupoid of the tangent groupoid TG. A *foliated groupoid* is a pair (G, F_G), where G is a Lie groupoid and F_G is an involutive multiplicative regular distribution. In this case, the Dirac structure $F_G \oplus F_G^\circ \subseteq \mathbb{T}G$ is easily seen to be a multiplicative Dirac structure on G. The foliation tangent to an involutive multiplicative distribution is called a *multiplicative foliation*. Multiplicative foliations that are simultaneously transversal to the s-fibration and to the t-fibration were studied in [Tang 2006], providing interesting examples of noncommutative Poisson algebras. Also, multiplicative foliations arise in the context of geometric quantization of symplectic groupoids, namely, as polarizations compatible with a symplectic groupoid structure (see [Hawkins 2008]). In addition, the notion of multiplicative foliation has appeared in connection with exterior differential systems. For more details see [Crainic et al. 2012] and the references therein.

3B2. *Dirac Lie groups.* Dirac Lie groups, that is, Lie groups equipped with multiplicative Dirac structures, were first studied by the author in [Ortiz 2008], providing a generalization of Poisson Lie groups within the category of Lie groups. In that work, it is shown that, modulo regularity issues, Dirac Lie groups are given by the pull-back (in the sense of Dirac structures) of Poisson Lie groups via a surjective submersion which is also a Lie group morphism. Notice that whenever a Lie

groupoid G over M is equipped with a multiplicative Dirac structure, then for every $x \in M$, the isotropy Lie group $G_x := s^{-1}(x) \cap t^{-1}(x)$ inherits a Dirac structure L_{G_x} making the pair (G_x, L_{G_x}) into a Dirac Lie group.

We emphasize that different notions of Dirac Lie groups exist in the literature. For instance, Li-Bland and Meinrenken [2011] have proposed a notion of multiplicativity that includes interesting examples of *twisted* Dirac structures on Lie groups such as the Cartan–Dirac structure on a compact Lie group.

3B3. *Tangent lift of a multiplicative Dirac structure.* In [Grabowski and Urbański 1995] it was proved that whenever a Lie group G carries a multiplicative Poisson bivector π_G, then the tangent Lie group TG equipped with the tangent Poisson structure π_{TG} becomes a Poisson Lie group. It is easy to extend the multiplicative Poisson case to abstract multiplicative Dirac structures on Lie groupoids. Assume that G is a Lie groupoid over M and consider the tangent groupoid TG over TM explained in Section 2A. Then, the tangent Dirac structure $L_{TG} \subseteq TTG \oplus T^*TG$ induced by a multiplicative Dirac structure $L_G \subseteq TG \oplus T^*G$ is also a multiplicative Dirac structure. Indeed, first observe that the bundle map $J_G : TTG \to TTG$ is a groupoid isomorphism over $J_M : TTM \to TTM$. Similarly, the bundle map $\Theta_G : TT^*G \to T^*TG$ is a groupoid isomorphism over the canonical identification $I : T(A^*G) \to (T(AG))^*$. Since L_G is a Lie subgroupoid of $TG \oplus T^*G$, then the tangent functor yields a Lie subgroupoid TL_G of $TTG \oplus TT^*G$. Due to the fact that L_{TG} is the image of TL_G via the groupoid isomorphism $J_G \oplus \Theta_G$, we see that L_{TG} inherits a natural structure of Lie subgroupoid of $TTG \oplus T^*TG$. Hence we conclude that L_{TG} defines a multiplicative Dirac structure on TG.

3B4. *Symmetries of multiplicative Dirac structures.* Let L_G be a multiplicative Dirac structure on a Lie groupoid $G \rightrightarrows M$, and let H be a Lie group acting on G by groupoid automorphisms. Assume that the H-action is free and proper and that the H-orbits coincide with the characteristic leaves of L_G. In this case the quotient space G/H inherits the structure of a Lie groupoid over M/H. Moreover, since G/H is the space of characteristic leaves of L_G, we conclude that there exists a Poisson structure π_{red} on G/H, making the quotient map $G \to G/H$ into both a backward and forward Dirac map. This fact together with the multiplicativity of L_G imply that π_{red} is a multiplicative Poisson bivector. In other words, the quotient space G/H is a Poisson groupoid. In the case where L_G is the graph of a multiplicative Poisson bivector and the action is Hamiltonian in the sense of Fernandes and Iglesias [≥ 2013], this recovers some of the results about reduction of Poisson groupoids carried out by those authors.

3B5. *Multiplicative B-field transformations.* Let $L \subseteq \mathbb{TM}$ be a Lagrangian subbundle. Given a 2-form $B \in \Omega^2(M)$ one can construct the Lagrangian subbundle

$\tau_B(L) \subseteq \mathbb{T}M$ defined by

$$\tau_B(L) = \{X \oplus \alpha + i_X B \mid X \oplus \alpha \in L\}.$$

A straightforward computation shows that $\tau_B(L)$ defines a Dirac structure on M if and only if B is a closed 2-form. See for instance [Bursztyn 2005; Gualtieri 2003]. In this case, we say that the Dirac structure $\tau_B(L)$ is obtained out of L by a *B-field transformation*.

Assume now that L_G is a multiplicative Dirac structure on a Lie groupoid G. Given a multiplicative closed 2-form B_G on G, one can consider the bundle map $\tau_{B_G} : \mathbb{T}G \to \mathbb{T}G$, $X \oplus \alpha \mapsto X \oplus \alpha + i_X(B_G)$. It follows from the multiplicativity of B_G that τ_{B_G} is a Lie groupoid isomorphism. As a result, the Dirac structure $\tau_{B_G}(L_G)$ on G is multiplicative. Our interest in B-field transformations of multiplicative Dirac structures is motivated by the work carried out in [Bursztyn 2005; Bursztyn and Radko 2003], where the authors study the connection between certain B-field transformations of symplectic and Poisson groupoids and the notion of Morita equivalence of Poisson manifolds.

3B6. *Generalized complex groupoids.* Generalized complex structures were introduced in [Hitchin 2003] and further developed in [Gualtieri 2003]. Given a smooth manifold M, one can consider the complexified vector bundle $\mathbb{T}_{\mathbb{C}}M := \mathbb{T}M \otimes \mathbb{C}$ endowed with the complex Courant bracket and the complex pairing $\langle \cdot, \cdot \rangle$. A generalized complex structure on M is a complex Dirac structure $L \subseteq \mathbb{T}_{\mathbb{C}}M$ such that $L \cap \bar{L} = \{0\}$, where $\bar{L}$ denotes the conjugate of L. Complexified versions of multiplicative Dirac structures give rise to generalized complex groupoids. More concretely, let G be a Lie groupoid equipped with a generalized complex structure L_G. We say that (G, L_G) is a *generalized complex groupoid* if $L_G \subseteq \mathbb{T}_{\mathbb{C}}G$ is a Lie subgroupoid. Generalized complex groupoids were introduced in [Jotz et al. 2012] under the name of Glanon groupoids. Structures such as symplectic groupoids and holomorphic Poisson groupoids are special instances of generalized complex groupoids.

4. Dirac algebroids

In this section we study Lie algebroids equipped with Dirac structures compatible with both the linear and Lie algebroid structure.

4A. *Definition and main examples.* Let $A \to M$ be a vector bundle. A Poisson bivector π_A on A is *linear* if the map $\pi_A^{\sharp} : T^*A \to TA$ is a morphism of double vector bundles. Similarly, a 2-form ω_A on A is *linear* if the map $\omega_A^{\sharp} : TA \to T^*A$ is a morphism of double vector bundles. The bundle map $\omega_A^{\sharp}$ in this case covers a bundle morphism $\lambda : TM \to A^*$. As shown in [Konieczna and Urbański 1999], a linear 2-form ω_A on a vector bundle $A \to M$ is closed if and only if $\omega_A = -(\lambda^t)^* \omega_{can}$,

where ω_{can} is the canonical symplectic form on T^*M and $\lambda^t : A \to T^*M$ is a fiberwise dual map of $\lambda : TM \to A^*$. The definition below includes both linear Poisson bivectors and linear closed 2-forms as special instances.

Definition 4.1. A Dirac structure L_A on A is called *linear* if $L_A \subseteq \mathbb{T}A$ is a double vector subbundle of $\mathbb{T}A$.

A linear Dirac structure $L_A \subseteq \mathbb{T}A$ is not only a vector bundle over A, but also a vector bundle over a subbundle $E \subseteq TM \oplus A^*$. It follows directly from the definition that graphs of linear Poisson bivector and linear closed 2-forms define linear Dirac structures. Linear Dirac structures arise also in connection with Lagrangian and Hamiltonian mechanics, see for example, [Grabowska and Grabowski 2011].

Assume now that $A \to M$ carries also a Lie algebroid structure. Consider the direct sum Lie algebroid $\mathbb{T}A = TA \oplus T^*A$, whose base manifold is $TM \oplus A^*$.

Definition 4.2. A Dirac structure L_A on A is called *morphic* if L_A is a linear Dirac structure that is also a Lie subalgebroid of $\mathbb{T}A$.

We denote by $\mathrm{Dir}_{\mathrm{morph}}(A)$ the space of morphic Dirac structures on the Lie algebroid A.

A pair (A, L_A), where A is a Lie algebroid endowed with a morphic Dirac structure L_A, will be referred to as a *Dirac algebroid*.

Example 4.3. Let π_A be a linear Poisson bivector on a Lie algebroid $A \to M$. Then, the Dirac structure given by the graph of π_A is morphic if and only if $\pi_A^\sharp : T^*A \to TA$ is a Lie algebroid morphism. As shown in [Mackenzie and Xu 1994], this is equivalent to the pair (A, A^*) being a Lie bialgebroid.

Example 4.4. Let ω_A be a linear closed 2-form on a Lie algebroid $A \to M$, that is, $\omega_A = -\sigma^*\omega_{can}$, for some bundle map $\sigma : A \to T^*M$. The Dirac structure defined by the graph of ω_A is morphic if and only if $\omega_A^\sharp : TA \to T^*A$ is a Lie algebroid morphism. Equivalently, as shown in [Bursztyn et al. 2009a], the bundle map $\sigma : A \to T^*M$ is an IM-2-form on A. The notion of IM-2-form was introduced in [Bursztyn et al. 2004] motivated by the problem of the integration of Dirac structures. See also [Arias Abad and Crainic 2011], where IM-2-forms arise in connection with the Weil algebra and the Van Est isomorphism.

4B. *More examples of Dirac algebroids.* In addition to both morphic Poisson structures and morphic closed 2-forms, there are more examples of morphic Dirac structures, which we proceed to explain below.

4B1. *Foliated algebroids.* Let A be a Lie algebroid and $F_A \subseteq TA$ an involutive subbundle which is also a Lie subalgebroid of $TA \to TM$. In this case we say that (A, F_A) is a *foliated algebroid*. One can easily check that the Dirac structure $F_A \oplus F_A^\circ \subseteq \mathbb{T}A$ is a morphic Dirac structure. Foliated algebroids were studied

in [Hawkins 2008] as a way to promote the notion of polarization in geometric quantization to the category of Lie algebroids. Also, a detailed discussion about foliated algebroids can be found in [Jotz and Ortiz 2012].

4B2. *Dirac Lie algebras.* Let $\mathfrak{g}$ be a Lie algebra. In this case, morphic Dirac structures are Lie subalgebroids of $T\mathfrak{g} \oplus T^*\mathfrak{g} \to \mathfrak{g}^*$. It follows from [Ortiz 2008] that Dirac Lie algebras are suitable pull-backs of Lie bialgebras.

4B3. *Tangent lifts of Dirac algebroids.* Let (A, L_A) be a Dirac algebroid. Consider the tangent Dirac structure L_{TA} on TA. By definition, the tangent Dirac structure is given by $L_{TA} := (J_A \oplus \Theta_A)(TL_A)$, where $TL_A \to TM$ is the tangent algebroid associated to the Dirac structure L_A viewed as a Lie algebroid over A. Since the bundle map $J_A \oplus \Theta_A : TTA \oplus TT^*A \to TTA \oplus T^*TA$ is a Lie algebroid isomorphism, we conclude that $L_{TA} \subseteq \mathbb{T}TA$ is a Lie subalgebroid. Therefore, the pair (TA, L_{TA}) is a Dirac algebroid.

4B4. *Symmetries of Dirac algebroids.* Let (A, L_A) be a Dirac algebroid. Consider a Lie group H acting on A by Lie algebroid automorphisms. Assume that the action is free and proper and that the H-orbits coincide with the characteristic leaves of L_A. One can check that the orbit space A/H inherits a Lie algebroid structure over M/H, making the quotient map $A \to A/H$ into a Lie algebroid morphism. Since the H-orbits are exactly the characteristic leaves of L_A, one concludes that A/H is equipped with a unique Poisson bivector π_{red} determined by the fact that $A \to A/H$ is a forward and backward Dirac map. Since L_A is morphic, we conclude that π_{red} is a morphic Poisson structure on A/H. In particular, due to [Mackenzie and Xu 1994], the pair $(A/H, (A/H)^*)$ is a Lie bialgebroid. In the special case where L_A is the graph of a morphic Poisson structure on A and the action is Hamiltonian in the sense of [Fernandes and Iglesias $\geq$ 2013], this recovers the reduction of Lie bialgebroids carried out in [Fernandes and Iglesias $\geq$ 2013].

4B5. *Morphic B-field transformations.* Let (A, L_A) be a Dirac algebroid. Associated to a morphic closed 2-form B_A on A is the Lie algebroid automorphism $\tau_{B_A} : \mathbb{T}A \to \mathbb{T}A, (X, \alpha) \mapsto (X, \alpha + i_X B_A)$. The Dirac structure $\tau_{B_A}(L_A) \subseteq \mathbb{T}A$ obtained out of L_A by applying the B-field transformation τ_{B_A} is morphic. Therefore, the pair (A, τ_{B_A}) is a Dirac algebroid. In particular, B-field transformations of morphic Poisson structures (that is, Lie bialgebroid structures on (A, A^*)) by morphic closed 2-forms are always morphic Dirac structures. If the B-field transformation is admissible, that is, the resulting Dirac structure is the graph of a Poisson bivector, such a bivector is necessarily morphic as well. In particular, we get a new bialgebroid structure on (A, A^*) referred to as a *gauge transformation* of the Lie bialgebroid (A, A^*). Gauge transformations of Lie bialgebroids were introduced in [Bursztyn 2005] motivated by the study of gauge transformations of Poisson groupoids and Morita equivalence of Poisson manifolds.

4B6. *Generalized complex algebroids.* Let $A \to M$ be a Lie algebroid. Consider the complexified Lie algebroid $\mathbb{T}_{\mathbb{C}}A = (TA \oplus T^*A) \otimes \mathbb{C}$ whose base manifold is $(TM \oplus A^*) \otimes \mathbb{C}$. A generalized complex structure L_A on A is *morphic* if $L_A \subseteq \mathbb{T}_{\mathbb{C}}A$ is a Lie subalgebroid. In this case, we say that (A, L_A) is a *generalized complex algebroid*. The notion of generalized complex algebroid was introduced in [Jotz et al. 2012] under the name of Glanon algebroids. Generalized complex algebroids include holomorphic Poisson structures and holomorphic Lie bialgebroids as particular cases.

5. Infinitesimal description of multiplicative Dirac structures

This section is the main part of the present work. Here we show that Dirac algebroids correspond to the infinitesimal counterpart of Dirac groupoids.

5A. *The canonical $\mathscr{CA}$-groupoid.* The main idea for studying multiplicative Dirac structures infinitesimally is based on the following observation. Given a Lie groupoid G over M, the canonical geometric objects associated to $\mathbb{T}G$ that are used to define Dirac structures (symmetric pairing and Courant bracket) are suitably compatible with the groupoid structure of $\mathbb{T}G$. This compatibility makes $\mathbb{T}G$ into a $\mathscr{CA}$-groupoid. The notion of $\mathscr{CA}$-groupoid was suggested by Mehta [2009] and further studied by Li-Bland and Ševera [2011]. More precisely, let $\langle \cdot, \cdot \rangle_G$ be the nondegenerate symmetric pairing on the direct sum Lie groupoid $\mathbb{T}G$.

Proposition 5.1. *The canonical pairing defines a morphism of Lie groupoids*

$$\langle \cdot, \cdot \rangle_G : \mathbb{T}G \oplus \mathbb{T}G \to \mathbb{R},$$

where $\mathbb{R}$ is equipped with the usual abelian group structure.

Proof. Since $\mathbb{R}$ is a groupoid over a point, we only need to check the compatibility of $\langle \cdot, \cdot \rangle_G$ with the corresponding groupoid multiplications. For that, consider elements $(X_g \oplus \alpha_g), (Y_g \oplus \beta_g) \in \mathbb{T}_g G$ and $(X'_h \oplus \alpha'_h), (Y'_h \oplus \beta'_h) \in \mathbb{T}_h G$. Then by definition of the groupoid structure on $\mathbb{T}G \oplus \mathbb{T}G$, we have

$$\big((X_g \oplus \alpha_g) \oplus (Y_g \oplus \beta_g)\big) * \big((X'_h \oplus \alpha'_h) \oplus (Y'_h \oplus \beta'_h)\big)$$
$$= (X_g \bullet X'_h \oplus \alpha_g \circ \alpha'_h) \oplus (Y_g \bullet Y'_h \oplus \beta_g \circ \beta'_h),$$

therefore one gets

$$\big\langle (X_g \bullet X'_h \oplus \alpha_g \circ \alpha'_h), (Y_g \bullet Y'_h \oplus \beta_g \circ \beta'_h) \big\rangle_G$$
$$= (\alpha_g \circ \alpha'_h)(Y_g \bullet Y'_h) + (\beta_g \circ \beta'_h)(X_g \bullet X'_h)$$
$$= \alpha_g(Y_g) + \alpha'_h(Y'_h) + \beta_g(X_g) + \beta'_h(X'_h)$$
$$= \langle (X_g \oplus \alpha_g), (Y_g, \beta_g) \rangle_G + \langle (X'_h \oplus \alpha'_h), (Y'_h \oplus \beta'_h) \rangle_G. \qquad \square$$

In order to explain the relation between the Courant bracket and the Lie groupoid structure on the direct sum vector bundle $\mathbb{T}G = TG \oplus T^*G$, we consider the direct product Courant algebroid $\mathbb{T}G \times \mathbb{T}G \to G \times G$. Every section $a^{(2)}$ of $\mathbb{T}G \times \mathbb{T}G$ can be written as

$$a^{(2)} = a_1 \circ pr_1 \oplus a_2 \circ pr_2,$$

where a_1, a_2 are sections of $\mathbb{T}G$, and pr_1, $pr_2 : \mathbb{T}G \times \mathbb{T}G \to \mathbb{T}G$ denote the natural projections. The direct product bracket on sections of $\mathbb{T}G \times \mathbb{T}G$ is defined as usual;

$$[a^{(2)}, \bar{a}^{(2)}] = [\![a_1, \bar{a}_1]\!] \circ pr_1 \oplus [\![a_2, \bar{a}_2]\!] \circ pr_2,$$

and the anchor map $\rho_{(\mathbb{T}G)_{(2)}} : \mathbb{T}G \times \mathbb{T}G \to TG \times TG$ is given by the canonical componentwise projection.

Proposition 5.2. *Let* $m_{\mathbb{T}} : (\mathbb{T}G)_{(2)} \to \mathbb{T}G$ *denote the groupoid multiplication of* $\mathbb{T}G = TG \oplus T^*G$. *If* $a, b, a_i, b_i \in \Gamma(\mathbb{T}G)$, $i = 1, 2$ *are sections such that*

$$m_{\mathbb{T}} \circ (a_1, a_2) = a \circ m_G, \quad m_{\mathbb{T}} \circ (b_1, b_2) = b \circ m_G,$$

then the following identities hold:

(i) $Tm_G\big(\rho_{(\mathbb{T}G)_{(2)}}(X_g^1 \oplus \alpha_g^1, X_h^2 \oplus \alpha_h^2)\big) = X_g^1 \bullet X_h^2$;

(ii) $m_{\mathbb{T}} \circ ([\![a_1, b_1]\!], [\![a_2, b_2]\!]) = [\![a, b]\!] \circ m_G$.

Proof. We begin by checking (i). For that, consider a section $a^{(2)} = a_1 \circ pr_1 \oplus a_2 \circ pr_2$ of $(\mathbb{T}G)_{(2)}$, where $a_1 = X^1 \oplus \alpha^1$ and $a_2 = X^2 \oplus \alpha^2$ are sections of $\mathbb{T}G$. The multiplication on the Lie groupoid $\mathbb{T}G$ maps the section $a^{(2)}$ into

$$m_{\mathbb{T}}(a_1 \circ pr_1 \oplus a_2 \circ pr_2)(g, h) = X_g^1 \bullet X_h^2 \oplus \alpha_g^1 \circ \alpha_h^2.$$

Applying the anchor map of $\mathbb{T}G$ we obtain

$$\rho_{\mathbb{T}G}(X_g^1 \bullet X_h^2 \oplus \alpha_g^1 \circ \alpha_h^2) = X_g^1 \bullet X_h^2.$$

On the other hand, the componentwise anchor map of $(\mathbb{T}G)_{(2)}$ applied to the section $a^{(2)}$ gives rise to

$$\rho_{(\mathbb{T}G)_{(2)}}(a_1 \circ pr_1 \oplus a_2 \circ pr_2)(g, h) = (X_g^1, X_h^2),$$

which followed by the derivative of $m_G : G_{(2)} \to G$ yields

$$Tm_G\big(\rho_{(\mathbb{T}G)_{(2)}}(X_g^1 \oplus \alpha_g^1, X_h^2 \oplus \alpha_h^2)\big) = X_g^1 \bullet X_h^2,$$

as required. In order to prove identity (ii), one considers

(9) $$m_{\mathbb{T}} \circ a^{(2)} = a \circ m_G,$$

(10) $$m_{\mathbb{T}} \circ \bar{a}^{(2)} = \bar{a} \circ m_G,$$

where $a^{(2)}, \bar{a}^{(2)} \in \Gamma_{G_{(2)}}((\mathbb{T}G)_{(2)})$ and $a, \bar{a} \in \Gamma_G(\mathbb{T}G)$. More concretely, write down sections as

$$a^{(2)} = (X^1 \oplus \alpha^1) \circ pr_1 \oplus (X^2 \oplus \alpha^2) \circ pr_2,$$
$$\bar{a}^{(2)} = (\bar{X}^1 \oplus \bar{\alpha}^1) \circ pr_1 \oplus (\bar{X}^2 \oplus \bar{\alpha}^2) \circ pr_2,$$
$$a = Y \oplus \beta,$$
$$\bar{a} = \bar{Y} \oplus \bar{\beta}.$$

The identities (9), (10) then become

(11) $$X_g^1 \bullet X_h^2 \oplus \alpha_g^1 \circ \alpha_h^2 = Y_{gh} \oplus \beta_{gh},$$

(12) $$\bar{X}_g^1 \bullet \bar{X}_h^2 \oplus \bar{\alpha}_g^1 \circ \bar{\alpha}_h^2 = \bar{Y}_{gh} \oplus \bar{\beta}_{gh},$$

for any composable pair $(g, h) \in G \times G$. Now it follows directly from the definition of the direct product bracket that

$$[a^{(2)}, \bar{a}^{(2)}]$$
$$= \left([X^1, \bar{X}^1] \oplus \mathscr{L}_{X^1}\bar{\alpha}^1 - i_{\bar{X}^1}d\alpha^1\right) \circ pr_1 \oplus \left([X^2, \bar{X}^2] \oplus \mathscr{L}_{X^2}\bar{\alpha}^2 - i_{\bar{X}^2}d\alpha^2\right) \circ pr_2.$$

Then, composing with the groupoid multiplication of $\mathbb{T}G$, we have

$$m_{\mathbb{T}} \circ [a^{(2)}, \bar{a}^{(2)}]_{(g,h)}$$
$$= [X^1, \bar{X}^1]_g \bullet [X^2, \bar{X}^2]_h \oplus (\mathscr{L}_{X^1}\bar{\alpha}^1 - i_{\bar{X}^1}d\alpha^1)_g \circ (\mathscr{L}_{X^2}\bar{\alpha}^2 - i_{\bar{X}^2}d\alpha^2)_h.$$

On the other hand,

$$[\![a, \bar{a}]\!] \circ m_G(g, h) = [Y, \bar{Y}]_{gh} \oplus (\mathscr{L}_Y\bar{\beta} - i_{\bar{Y}}d\beta)_{gh},$$

and using the identities (11) and (12) one concludes that

$$[Y, \bar{Y}]_{gh} = [X^1, \bar{X}^1]_g \bullet [X^2, \bar{X}^2]_h.$$

Thus, the tangent component of $[\![a, \bar{a}]\!]_{gh}$ coincides with the tangent component of $m_{\mathbb{T}} \circ [a^{(2)}, \bar{a}^{(2)}]_{(g,h)}$. It remains to show that we also have the equality of the corresponding cotangent parts. This is equivalent to showing that

$$(\mathscr{L}_Y\bar{\beta} - \mathscr{L}_{\bar{Y}}\beta - d\langle\beta, \bar{Y}\rangle)_{gh}$$
$$= \left(\mathscr{L}_{X^1}\bar{\alpha}^1 - \mathscr{L}_{\bar{X}^1}\alpha^1 - d\langle\alpha^1, \bar{X}^1\rangle\right)_g \circ \left(\mathscr{L}_{X^2}\bar{\alpha}^2 - \mathscr{L}_{\bar{X}^2}\alpha^2 - d\langle\alpha^2, \bar{X}^2\rangle\right)_h,$$

for every composable pair $(g, h) \in G_{(2)}$. In order to prove this identity, we need to check that the left hand side (*LHS*), and the right hand side (*RHS*) above coincide at elements of the form $U_g \bullet V_h$. For that consider the 1-form on G defined by $\gamma := \mathscr{L}_Y\bar{\beta} - \mathscr{L}_{\bar{Y}}\beta - d\langle\beta, \bar{Y}\rangle$. We can look at the pull-back 1-form $m_G^*\gamma \in \Omega^1(G_{(2)})$, which at every tangent vector $(U_g, V_h) \in T_{(g,h)}G_{(2)}$ is given by

$$(m_G^*\gamma)_{(g,h)}(U_g, V_h) = \gamma_{gh}(U_g \bullet V_h) = (LHS)(U_g \bullet V_h).$$

The pull-back form $m_G^* \gamma$ involves three terms. Let us analyze the first term $m_G^*(\mathcal{L}_Y \bar{\beta})$ of this pull-back form. It follows from the relation $Y = (m_G)_*(X^1, X^2)$ that

$$m_G^*(\mathcal{L}_Y \bar{\beta}) = \mathcal{L}_{(X^1, X^2)} m_G^* \bar{\beta}.$$

Notice that (12) implies that

$$
\begin{aligned}
(m_G^* \bar{\beta})_{(g,h)}(U_g, V_H) &= \bar{\beta}_{gh}(U_g \bullet V_h) = (\bar{\alpha}_g^1 \circ \bar{\alpha}_h^2)(U_g \bullet V_h) \\
&= \bar{\alpha}_g^1(U_g) + \bar{\alpha}_h^2(V_h) = (\bar{\alpha}^1, \bar{\alpha}^2)_{(g,h)}(U_g, V_h).
\end{aligned}
$$

That is, $m_G^*(\mathcal{L}_Y \bar{\beta}) = \mathcal{L}_{X^1} \bar{\alpha}^1 \oplus \mathcal{L}_{X^2} \bar{\alpha}^2$. A similar argument can be applied to the other terms of the pull-back form $m_G^* \gamma$, yielding

$$
\begin{aligned}
(LHS)(U_g \bullet V_h) &= (m_G^* \gamma)_{(g,h)}(U_g, V_h) \\
&= (\mathcal{L}_{X^1} \bar{\alpha}^1)_g(U_g) + (\mathcal{L}_{X^2} \bar{\alpha}^2)_h(V_h) - (\mathcal{L}_{\bar{X}^1} \alpha^1)_g(U_g) \\
&\quad - (\mathcal{L}_{\bar{X}^2} \alpha^2)_h(V_h) - d\langle \alpha^1, \bar{X}^1 \rangle_g(U_g) - d\langle \alpha^2, \bar{X}^2 \rangle_h(V_h) \\
&= (RHS)(U_g \bullet V_h).
\end{aligned}
$$

Thus *RHS* and *LHS* coincide at elements of the form $U_g \bullet V_h$, and we conclude that $(m_{\mathbb{T}}, m_G)$ is bracket preserving. $\square$

Recall that, given a Courant algebroid $(\mathbb{E}, \rho, [\![\cdot, \cdot]\!])$ over smooth manifold M and a submanifold $Q \subseteq M$, a *Dirac structure supported* on Q (see [Alekseev and Xu 2011; Bursztyn et al. 2009b]) is a subbundle $K \subset \mathbb{E}|_Q$ such that $K_x \subseteq \mathbb{E}_x$ is Lagrangian for all $x \in Q$ and the following conditions are fulfilled:

(1) $\rho(K) \subseteq TQ$;

(2) whenever $a_1, a_2 \in \Gamma(\mathbb{E})$ satisfy $a_1|_Q, a_2|_Q \in \Gamma(K)$, then $[\![a_1, a_2]\!]|_Q \in \Gamma(K)$.

Dirac structures with support were used in [Bursztyn et al. 2009b] to introduce a natural notion of morphism between Courant algebroids. Let $\mathbb{E}_1, \mathbb{E}_2$ be Courant algebroids over M, N, respectively. A *Courant algebroid morphism* from $\mathbb{E}_1$ to $\mathbb{E}_2$ is a Dirac structure in $\mathbb{E}_2 \times \overline{\mathbb{E}}_1$ supported on Graph(f), where $f : M \to N$ is a smooth map. Here $\overline{\mathbb{E}}_1$ denotes the Courant algebroid structure on the vector bundle $\mathbb{E}_1$ with the same bracket on $\Gamma(\mathbb{E}_1)$, anchor map and minus the usual symmetric pairing.
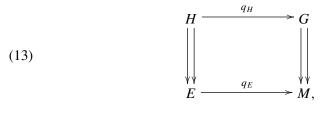
Combining Propositions 5.1 and 5.2, we obtain:

Proposition 5.3. *Let G be a Lie groupoid over M with multiplication map $m_G : G_{(2)} \to G$. Let $m_{\mathbb{T}} : (\mathbb{T}G)_{(2)} \to \mathbb{T}G$ denote the groupoid multiplication on $\mathbb{T}G$. Then* Graph($m_{\mathbb{T}}$) $\subseteq \mathbb{T}G \times \overline{\mathbb{T}G \times \mathbb{T}G}$ *is a Dirac structure supported on* Graph(m_G) $\subseteq G \times G \times G$. *That is,* Graph($m_{\mathbb{T}}$) *is a Courant algebroid morphism from $\mathbb{T}G \times \mathbb{T}G$ to $\mathbb{T}G$.*

Using the terminology of [Li-Bland and Ševera 2011], Proposition 5.3 says that $\mathbb{T}G$ with its canonical Courant algebroid structure and groupoid multiplication is an example of $\mathscr{CA}$-groupoid. See also Example 2.3.1 of [Li-bland 2012].
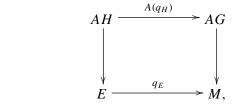
5B. *The $\mathscr{LA}$-groupoid of a multiplicative Dirac structure.*

5B1. *Review of $\mathscr{LA}$-groupoids.* An $\mathscr{LA}$-groupoid is a Lie groupoid object in the category of Lie algebroids. More precisely, an $\mathscr{LA}$-*groupoid* [Mackenzie 1992] is a square

(13)
$$
\begin{array}{ccc}
H & \xrightarrow{\ q_H\ } & G \\
\Big\Vert\Big\downarrow & & \Big\Vert\Big\downarrow \\
E & \xrightarrow{\ q_E\ } & M,
\end{array}
$$

where the single arrows denote Lie algebroids and the double arrows denote Lie groupoids. These structures are compatible in the sense that all the structure mappings (that is, source, target, unit section, inversion and multiplication) defining the Lie groupoid H are Lie algebroid morphisms over the corresponding structure mappings which define the Lie groupoid G. We also require that the anchor map $\rho_H : H \to TG$ be a groupoid morphism over the anchor map $\rho_E : E \to TM$. Here TG is endowed with the tangent groupoid structure over TM. For describing the square given by an $\mathscr{LA}$-groupoid we use the notation (H, G, E, M). It is worthwhile to explain how the groupoid multiplication defines a morphism of Lie algebroids. For that, let $m_H : H_{(2)} \subseteq H \times H \to H$ denote the groupoid multiplication of H, and similarly let $m_G : G_{(2)} \subseteq G \times G \to G$ denote the multiplication of G. The direct product vector bundle $H \times H \to G \times G$ inherits a natural Lie algebroid structure, and we have a Lie subalgebroid $H_{(2)}$ over $G_{(2)}$ which is just a pull-back algebroid, see for example, [Higgins and Mackenzie 1990] for details about the pull-back operation in the category of Lie algebroids. With respect to this Lie algebroid structure, the multiplication map m_H is required to be a Lie algebroid morphism covering m_G.

The Lie functor applied to an $\mathscr{LA}$-groupoid (13) determines a double vector bundle

(14)
$$
\begin{array}{ccc}
AH & \xrightarrow{\ A(q_H)\ } & AG \\
\downarrow & & \downarrow \\
E & \xrightarrow{\ q_E\ } & M,
\end{array}
$$

where each of the arrows define Lie algebroids. The top Lie algebroid structure

is nontrivial, and it deserves a detailed explanation. The Lie algebroid structure $AH \to AG$ was constructed in [Mackenzie 2000] as a prolongation procedure similar to the tangent prolongation of a Lie algebroid, except that we replace the tangent functor by the Lie functor.

Definition 5.4. The *prolonged anchor map* $AH \to T(AG)$ is defined by

$$\tilde{\rho} := j_G^{-1} \circ A(\rho_H),$$

where $j_G : T(AG) \to A(TG)$ is the canonical identification defined in Equation (5).

Now we study the space of sections $\Gamma_{AG}(AH)$.

Definition 5.5. A section $u \in \Gamma_G(H)$ is called a *star section* if there exists a section $u_0 \in \Gamma_M(E)$ such that

(1) $\epsilon_E \circ u_0 = u \circ \epsilon_M$,

(2) $s_H \circ u = u_0 \circ s_G$.

Notice that since every star section $u : G \to H$ preserves the units and the source fibrations, we are allowed to apply the Lie functor to u, yielding a section $A(u)$ of the vector bundle $AH \xrightarrow{A(q_H)} AG$.

Definition 5.6. Let (H, G, E, M) be an $\mathcal{LA}$-groupoid. The *core* of H is the vector bundle over M defined by

$$K := \epsilon_M^* \ker(s_H).$$

Every section $k \in \Gamma(K)$ induces a section $k_H \in \Gamma_G(H)$ in the following way:

$$k_H(g) := k(t_G(g))0_g^H,$$

where 0_g^H is the zero element in the fiber H_g above $g \in G$. Notice that for every section $k \in \Gamma(K)$ the induced section $k_H \in \Gamma_G(H)$ satisfies

$$k_H \circ \epsilon_M = k.$$

It was proved in [Mackenzie 2000] that the core of the double vector bundle (AH, AG, E, M) is the vector bundle $K \to M$. Notice that a core element $k \in K$ induces a Lie algebroid element $\bar{k} \in AH$. Indeed, we observe that every element in AH has the form

$$W = \frac{d}{dt}(h_t)|_{t=0},$$

where h_t is a curve in H sitting in a fixed source fiber $s_H^{-1}(e)$ with $h_0 = \epsilon_E(e)$. Thus, for every core element $k \in K$ above $x \in M$ — that is, $s_H(k) = 0_x^E$ and $q_H(k) = \epsilon_M(x)$ — there exists a natural element $\bar{k} \in AH$, defined by

$$\bar{k} := \frac{d}{dt}(tk)|_{t=0}.$$

Definition 5.7. Given a section $k \in \Gamma(K)$, the *core* section induced by k is the section $k^{\mathrm{core}} \in \Gamma_{AG}(AH)$ defined by

$$k^{\mathrm{core}}(u_x) := A(0^H)u_x + \overline{k(x)}.$$

Proposition 5.8 [Mackenzie 2000]. *The space of sections $\Gamma_{AG}(AH)$ is generated by sections of the form $A(u)$, where $u : G \to H$ is a star section, and by sections of the form k^{core}, where $k : M \to K$ is a section of the core of H.*

The Lie bracket on $\Gamma_{AG}(AH)$ is defined in terms of star sections and core sections. First we observe that whenever $u, v \in \Gamma_G(H)$ are star sections, then the Lie bracket $[u, v] \in \Gamma_G(H)$ is also a star section. Thus the Lie bracket between sections of the form $A(u)$, $A(v)$ is defined by

$$[A(u), A(v)] = A([u, v]).$$

The bracket of a pair of core sections is defined by

$$[k_1^{\mathrm{core}}, k_2^{\mathrm{core}}] = 0.$$

In order to define the bracket of a star section and a core section we notice that every star section $u : G \to H$ induces a covariant differential operator

$$D_u : \Gamma(K) \to \Gamma(K), \quad k \mapsto [u, k_H] \circ \epsilon_M.$$

Now we define $[A(u), k^{\mathrm{core}}] = (D_u(k))^{\mathrm{core}}$.

The Lie bracket of other sections of $\Gamma_{AG}(AH)$ is defined by requiring the Leibniz rule

$$[w, f w'] = f[w, w'] + (\mathscr{L}_{\tilde{\rho}(w)} f)w'.$$

The vector bundle $AH \xrightarrow{A(q_H)} AG$ endowed with the anchor map

$$\tilde{\rho} = j_G^{-1} \circ A(\rho)$$

and the Lie bracket $[\,\cdot\,, \cdot\,]$ on $\Gamma_{AG}(AH)$ becomes a Lie algebroid called the *prolonged Lie algebroid* induced by $H \to G$, see [Mackenzie 2000].

Although the following remark is not mentioned in [Mackenzie 2000], it is important to notice that Mackenzie's construction of the prolonged Lie algebroid is natural in the following sense.

Proposition 5.9. *Let (H, G, E, M) be an $\mathscr{LA}$-groupoid. Consider the canonical embeddings $i_{AH} : AH \to TH$ and $i_{AG} : AG \to TG$. Endow $TH \to TG$ with the tangent algebroid structure and $AH \to AG$ with the prolonged algebroid structure. Then i_{AH} is a Lie algebroid morphism covering i_{AG}.*

Recall that (see for example, [Mackenzie 2005]) a vector bundle map $\Psi : A \to B$, covering $\psi : M \to N$, is a *Lie algebroid morphism* if

$$\rho_B \circ \Psi = T\psi \circ \rho_A,$$

and the following compatibility with brackets holds: For sections $u, v \in \Gamma(A)$ such that $\Psi(u) = \sum_j f_j \psi^* u_j$ and $\Psi(v) = \sum_i g_i \psi^* v_i$, where $f_j, g_i \in C^\infty(M)$ and $u_j, v_i \in \Gamma(B)$, we have

$$(15) \quad \Psi([u, v]_A) = \sum_{i,j} f_j g_i \psi^* [u_j, v_i]_B + \sum_i \mathcal{L}_{\rho_A(u)} g_i \psi^* v_i - \sum_j \mathcal{L}_{\rho_A(v)} f_j \psi^* u_j.$$

Proof. The compatibility with the anchor maps reads

$$\rho_{TH} \circ i_{AH} = T i_{AG} \circ \tilde{\rho},$$

which is exactly the definition of the prolonged anchor map.

Let us check now the compatibility with the Lie brackets. For that, consider a star section $u : G \to H$. Then, there are sections $Tu : TG \to TH$ and $A(u) : AG \to AH$. Both are related by $A(u) = Tu|_{AG}$. In particular, $i_{AH} \circ A(u) = Tu \circ i_{AG}$ holds. Similarly, every section $k \in \Gamma(K)$ of the core of H induces a section of the tangent prolongation $TH \to TG$. Indeed, first consider the induced section $k_H \in \Gamma_G(H)$ and then construct the core section $\hat{k}_H \in \Gamma_{TG}(TH)$ defined in the usual way:

$$\hat{k}_H(X_g) = T(0^H) X_g + \overline{k_H(g)}.$$

For every $x \in \epsilon_M(M) \subseteq G$ one has $k_H(x) = k(x)$, and thus at any $u_x \in (AG)_x \subseteq T_x G$ we get

$$\hat{k}_H(u_x) = A(0^H) u_x + \overline{k(x)}.$$

Hence we conclude that $i_{AH} \circ k^{\mathrm{core}} = \hat{k}_H \circ i_{AG}$. Let us show that (15) holds for a pair of sections $A(u), A(v)$, where $u, v : G \to H$ are star sections. Indeed,

$$i_{AH} \circ [A(u), A(v)] = i_{AH} \circ A[u, v] = T[u, v] \circ i_{AG} = [Tu, Tv] \circ i_{AG},$$

as desired. It remains to show the bracket condition (15) for sections of the form $A(u), k^{\mathrm{core}}$, where $u : G \to H$ is a star section and $k : M \to K$ is a section of the core. On the one hand, one has that

$$i_{AH} \circ [A(u), k^{\mathrm{core}}] = i_{AH} \circ (D_u k)^{\mathrm{core}} = (\widehat{D_u k})_H \circ i_{AG}.$$

On the other hand,

$$[Tu, \hat{k}_H] \circ i_{AG} = \widehat{[u, k_H]} \circ i_{AG}.$$

Notice that to conclude that (15) holds in this case it suffices to show that $(\widehat{D_u k})_H \circ i_{AG} = \widehat{[u, k_H]} \circ i_{AG}$. Indeed, using the fact that $k = k_H \circ \epsilon_M$ for every
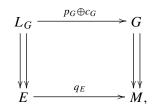
section $k : M \to K$, we conclude that if $v_x \in A_x G$, then

$$\widehat{[u, k_H]}(u_x) = T0_G^H(u_x) + \frac{d}{dt}(t[u, k_H](x))|_{t=0}$$

$$= T0_G^H(u_x) + \frac{d}{dt}(t(D_u k)_H(x))|_{t=0} = \widehat{(D_u k)}_H(v_x). \qquad \square$$

5B2. *Dirac groupoids as $\mathscr{L}\mathscr{A}$-groupoids.* Let L_G be a multiplicative Dirac structure on a Lie groupoid $G \rightrightarrows M$. This means that we have a $\mathscr{V}\mathscr{B}$-subgroupoid $L_G \rightrightarrows E$ of $\mathbb{T}G \rightrightarrows TM \oplus A^* G$, such that $L_G \subseteq \mathbb{T}G$ is also a Dirac subbundle. In particular there is a canonical Lie algebroid structure on $L_G \to G$ with anchor map $L_G \to TG$ the natural projection and Lie bracket $[\![\cdot , \cdot]\!]$ on $\Gamma_G(L_G)$. Given sections e_1, e_2 of E, there exist star sections a_1, a_2 of L_G covering e_1 and e_2, respectively. Since L_G is involutive with respect to the Courant bracket, we conclude that $[\![a_1, a_2]\!]$ is a star section of L_G covering a section e of E. We define $[e_1, e_2] := e$. A straightforward computation shows that with respect to this Lie bracket and the natural projection $E \to TM$, the vector bundle $E \to M$ becomes a Lie algebroid.

Proposition 5.10. *A multiplicative Dirac structure L_G on G gives rise to an $\mathscr{L}\mathscr{A}$-groupoid*

(16)

$$
\begin{array}{ccc}
L_G & \xrightarrow{p_G \oplus c_G} & G \\
\downdownarrows & & \downdownarrows \\
E & \xrightarrow{\quad q_E \quad} & M,
\end{array}
$$

where p_G and c_G denote the tangent projection and the cotangent projection, respectively.

Proof. Since the structure mappings defining the Lie groupoid $L_G \rightrightarrows E$ are restrictions of the structure mappings of the tangent and cotangent groupoids, a straightforward computation shows that these structure mappings are Lie algebroid morphisms over the structure mapping of G. The fact that the multiplication on L_G is a Lie algebroid morphism over the multiplication on G follows from Proposition 5.2. An argument similar to the one used in the proof of Proposition 5.2 shows that the inversion map on L_G is a Lie algebroid morphism. $\qquad \square$

5C. *The Lie algebroid of a multiplicative Dirac structure.* We let G be a Lie groupoid over M with Lie algebroid AG. Let L_G be a multiplicative Dirac structure on a Lie groupoid G. According to Proposition 5.1, the canonical pairing $\langle \cdot , \cdot \rangle_G : \mathbb{T}G \oplus \mathbb{T}G \to \mathbb{R}$ is a Lie groupoid morphism. Applying the Lie functor yields a nondegenerate symmetric pairing

$$A(\langle \cdot , \cdot \rangle_G) : (A(TG) \oplus A(T^*G)) \times_{AG} (A(TG) \oplus A(T^*G)) \to \mathbb{R}.$$

Let $\langle\,\cdot\,,\,\cdot\,\rangle_{AG}$ denote the canonical nondegenerate symmetric pairing on $\mathbb{T}(AG)$. Recall that there exist canonical isomorphisms of Lie algebroids $j_G : T(AG) \to A(TG)$ and $j'_G : A(T^*G) \to T^*(AG)$ (see (5) and (6)). Since $\langle\,\cdot\,,\,\cdot\,\rangle_{AG}$ is just a suitable restriction of $T\langle\,\cdot\,,\,\cdot\,\rangle_G$, one concludes that the canonical map

$$j_G^{-1} \oplus j'_G : A(TG) \oplus A(T^*G) \to T(AG) \oplus T^*(AG)$$

is a fiberwise isometry with respect to $A(\langle\,\cdot\,,\,\cdot\,\rangle_G)$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{AG}$. This is a useful tool for transporting Lagrangian subbundles of $TG \oplus T^*G$ to Lagrangian subbundles of $T(AG) \oplus T^*(AG)$. For instance, given a $\mathcal{VB}$-subgroupoid L_G of $TG \oplus T^*G$, we can apply the Lie functor to obtain a $\mathcal{VB}$-subalgebroid $A(L_G) \subseteq A(TG) \oplus A(T^*G)$. We mimic the construction of tangent Dirac structures, giving rise to a $\mathcal{VB}$-subalgebroid of $T(AG) \oplus T^*(AG)$ defined by

$$L_{AG} := (j_G^{-1} \oplus j'_G)(A(L_G)).$$

The following result is a straightforward consequence of the previous discussion.

Proposition 5.11. *Let $L_G \subseteq TG \oplus T^*G$ be a $\mathcal{VB}$-subgroupoid. Consider the associated $\mathcal{VB}$-subalgebroid $L_{AG} \subseteq T(AG) \oplus T^*(AG)$. Then L_G is isotropic with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_G$ if and only if L_{AG} is isotropic with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_{AG}$.*

In particular, if $L_G \subseteq TG \oplus T^*G$ is a $\mathcal{VB}$-subgroupoid with associated $\mathcal{VB}$-subalgebroid $L_{AG} \subseteq T(AG) \oplus T^*(AG)$ then L_G is an almost Dirac structure on G if and only if L_{AG} is an almost Dirac structure on AG.

Now we want to deal with integrability issues. For that, consider a multiplicative Dirac structure $L_G \subseteq \mathbb{T}G$ and let (L_G, G, E, M) be the associated $\mathcal{LA}$-groupoid. Applying the Lie functor we obtain the prolonged Lie algebroid structure on $A(L_G) \to AG$, and we use the canonical map

$$j_G^{-1} \oplus j'_G : A(TG) \oplus A(T^*G) \to T(AG) \oplus T^*(AG)$$

to define a Lie algebroid $L_{AG} = (j_G^{-1} \oplus j'_G)(A(L_G))$ over AG, characterized by the fact that $j_G^{-1} \oplus j'_G : A(L_G) \to L_{AG}$ is a Lie algebroid isomorphism. We have seen that $L_{AG} \subseteq \mathbb{T}(AG)$ is a Lagrangian subbundle with respect to the canonical pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{AG}$ on $\mathbb{T}(AG)$. We claim that the Lie bracket on $\Gamma_{AG}(L_{AG})$ induced by the prolonged Lie bracket on $\Gamma_{AG}(A(L_G))$ coincides with the Courant bracket. Indeed, since the tangent Lie algebroid $TL_G \to TG$ is isomorphic to $L_{TG} \to TG$, where the latter is equipped with the algebroid structure induced by the tangent Dirac structure $L_{TG} \subset TTG \oplus T^*TG$, and $A(L_G)$ is a Lie subalgebroid of TL_G (Proposition 5.9), then the bracket on sections of L_{AG} induced by the identification $A(L_G) = L_{AG}$ is exactly the restriction of the Courant bracket on $\Gamma(T(AG) \oplus T^*(AG))$. We have proved:

Theorem 5.12. *Given a Lie groupoid G with Lie algebroid AG, there is a canonical map*

$$\mathrm{Dir}_{\mathrm{mult}}(G) \to \mathrm{Dir}_{\mathrm{morph}}(AG), L_G \mapsto L_{AG} := (j_G^{-1} \oplus j_G')(A(L_G)).$$

That is, up to a canonical identification, the Lie algebroid of a multiplicative Dirac structure $L_G \subset \mathbb{T}G$ defines a Dirac structure L_{AG} on AG which is also a Lie subalgebroid of $\mathbb{T}(AG)$.

It is interesting to observe that since L_{AG} is the Lie algebroid of the $\mathscr{L}\mathcal{A}$-groupoid L_G, in particular L_{AG} inherits the structure of a *double Lie algebroid* [Mackenzie 2000]. Double Lie algebroids were introduced by Mackenzie [2011] as a way to understand Drinfeld's doubles of Lie bialgebroids. As a result, multiplicative Dirac structures provide interesting examples of double Lie algebroids.

5D. *Dirac groupoids vs. Dirac algebroids.* This section is concerned with the statement and proof of the main result of this work. We will prove that, whenever G is a source simply connected Lie groupoid with Lie algebroid AG, then the map in Theorem 5.12 is a bijection.

For that, recall that if M is a smooth manifold and $L \subset \mathbb{T}M$ is a Lagrangian subbundle, then there is a well-defined element $\mu_L \in \Gamma(\bigwedge^3 L^*)$ given by

(17) $\mu_L(a_1, a_2, a_3) := \langle [\![a_1, a_2]\!], a_3 \rangle.$

The element $\mu_L \in \Gamma(\bigwedge^3 L^*)$ is referred to as the *Courant 3-tensor* of L. Notice that a Lagrangian subbundle $L \subset \mathbb{T}M$ is a Dirac structure if and only if μ_L vanishes.

Proposition 5.13. *Let G be a Lie groupoid over M. Consider a Lagrangian subbundle $L_G \subset TG \oplus T^*G$, which is also a Lie subgroupoid. Then, the Courant 3-tensor of L_G is multiplicative; that is,*

$$\mu_{L_G} : \prod_{p_G \oplus c_G}^{3} L_G \to \mathbb{R}$$

is a groupoid morphism.

Proof. Let us consider composable pairs $a_g^i, \bar{a}_h^i$ in L_G with $i = 1, 2, 3$. Set $c_{gh}^i = m_{\mathbb{T}}(a_g^i, \bar{a}_h^i) \in (L_G)_{gh}$, for $i = 1, 2, 3$. Choose a section $c^i \in \Gamma(L_G)$ such that $c^i(gh) = c_{gh}^i$. Since L_G is a $\mathscr{V}\mathscr{B}$-groupoid, the multiplication on L_G is fiberwise surjective. In particular, there exist sections $a^i, \bar{a}^i \in \Gamma(L_G)$ such that $m_{\mathbb{T}} \circ (a^i, \bar{a}^i) = c^i \circ m_G$, for every $i = 1, 2, 3$. Clearly $a^i(g) = a_g^i$ and $\bar{a}^i(h) = \bar{a}_h^i$, for $i = 1, 2, 3$.

Then,

$$\mu_{L_G}\big((a_g^1, a_g^2, a_g^3) * (\bar{a}_h^1, \bar{a}_h^2, \bar{a}_h^3)\big) = \mu_{L_G}(c_{gh}^1, c_{gh}^2, c_{gh}^3) = \langle [\![c^1, c^2]\!](gh), c^3(gh) \rangle$$

$$= \big\langle m_{\mathbb{T}}([\![a^2, a^2]\!], [\![\bar{a}^1, \bar{a}^2]\!])(g, h), m_{\mathbb{T}}(a^3, \bar{a}^3)(g, h) \big\rangle$$

The last identity follows from the fact that $(m_{\mathbb{T}}, m_G)$ is a Courant morphism (see Proposition 5.2). Now we use the fact that $\langle \cdot, \cdot \rangle_G$ is a groupoid morphism to conclude that

$$\mu_{L_G}\big((a_g^1, a_g^2, a_g^3) * (\bar{a}_h^1, \bar{a}_h^2, \bar{a}_h^3)\big) = \mu_{L_G}(a_g^1, a_g^2, a_g^3) + \mu_{L_G}(\bar{a}_h^1, \bar{a}_h^2, \bar{a}_h^3).$$

This proves that the function μ_{L_G} is multiplicative. $\square$

We would like to describe explicitly the Lie algebroid morphism induced by the multiplicative tensor $\mu_{L_G} : \prod^3_{p_G \oplus c_G} L_G \to \mathbb{R}$. For that, we need the next lemma.

Lemma 5.14. *Let M be a smooth manifold. Consider a Lagrangian subbundle $L_M \subset \mathbb{T}M$. Then, for every $(\dot{a}_1, \dot{a}_2, \dot{a}_3) \in TL_M$ the following identity holds:*

$$T\mu_{L_M}(\dot{a}_1, \dot{a}_2, \dot{a}_3) = \mu_{L_{TM}}\big((J_M \oplus \Theta_M)\dot{a}_1, (J_M \oplus \Theta_M)\dot{a}_2, (J_M \oplus \Theta_M)\dot{a}_3\big),$$

where $L_{TM} \subset \mathbb{T}(TM)$ is the tangent lift of L_M.

Proof. One has $T\mu_{L_M}(Ta_1, Ta_2, Ta_3) = T(\mu_{L_M}(a_1, a_2, a_3))$ for every $a_1, a_2, a_3 \in \Gamma_M(L_M)$. On the other hand, the canonical map $J_M \oplus \Theta_M$ applied to each of the sections Ta_1, Ta_2, Ta_3 gives $a_1^T, a_2^T, a_3^T \in \Gamma_{TM}(L_{TM})$. Thus we conclude that

$$\mu_{L_{TM}}(a_1^T, a_2^T, a_3^T) = \langle [\![a_1^T, a_2^T]\!], a_3^T \rangle_{TM} = (\langle [\![a_1, a_2]\!], a_3 \rangle_M)^T,$$

which is exactly the tangent functor applied to the function $\mu_{L_M}(a_1, a_2, a_3)$. Therefore, for every triple of sections a_1, a_2, a_3 of L_M we get

(18) $$T\mu_{L_M}(Ta_1, Ta_2, Ta_3) = \mu_{L_{TM}}(a_1^T, a_2^T, a_3^T).$$

Now we notice, using local coordinates, that for every point $\dot{a} \in TL_M$ above $\dot{x} \in TM$ there exists a section $a \in \Gamma_M(L_M)$ such that $Ta(\dot{x}) = \dot{a}$, where $Ta \in \Gamma_{TM}(TL_M)$ is the section obtained by applying the tangent functor to the section a of L_M. This fact together with identity (18) prove the statement. $\square$

As a consequence we obtain a direct proof of the Courant integrability of the tangent lift of a Dirac structure L_M on M.

Corollary 5.15. *Let L_M be an almost Dirac structure on M, and consider the induced almost Dirac structure L_{TM} on TM. Then L_{TM} is Courant integrable if L_M is Courant integrable.*

Now consider a multiplicative Dirac structure L_G on G. The application of the Lie functor to the groupoid morphism μ_{L_G} of Proposition 5.13 yields a Lie algebroid morphism

$$A(\mu_{L_G}) : \prod_{A(p_G \oplus c_G)}^3 A(L_G) \to \mathbb{R}.$$

Since $A(\mu_{L_G}) = T\mu_{L_G}|_{A(L_G)}$, we conclude:

Proposition 5.16. *For the Lagrangian subbundle $L_{AG} = (j_G^{-1} \oplus j_G')A(L_G) \subseteq \mathbb{T}(AG)$, we have*

$$A(\mu_{L_G}) = \mu_{L_{AG}} \circ (j_G^{-1} \oplus j_G')^{(3)},$$

where $(j_G^{-1} \oplus j_G')^{(3)} : \prod_{A(p_G \oplus c_G)}^3 A(L_G) \to \prod_{p_{AG} \oplus c_{AG}}^3 L_{AG}$ denotes the natural extension of $(j_G^{-1} \oplus j_G')$.

Proof. This follows directly from Lemma 5.14 and the fact that j_G and j_G' are suitable restrictions of J_G and Θ_G, respectively.   □

Now we are ready to state the main theorem of this work.

Theorem 5.17. *Let G be a source simply connected Lie groupoid with Lie algebroid AG. There is a one-to-one correspondence between multiplicative Dirac structures on G and morphic Dirac structures on AG. The correspondence is given by the map in Theorem 5.12.*

Proof. Let L_G be a multiplicative Dirac structure on G. Consider the Lagrangian subbundle $L_{AG} := (j_G^{-1} \oplus j_G')(A(L_G)) \subset \mathbb{T}AG$. Since $\mu_{L_G} \equiv 0$, then Proposition 5.16 implies that $\mu_{L_{AG}} \equiv 0$. Thus, L_{AG} is a Dirac structure on AG which is clearly morphic. Notice that the integrability of L_{AG} is also a consequence of Theorem 5.12. Conversely, consider an element $L_A \in \mathrm{Dir}_{\mathrm{morph}}(AG)$; thus L_A is a linear Dirac structure on AG such that $L_A \subseteq \mathbb{T}AG$ is a $\mathscr{VB}$-subalgebroid. Since G is source simply connected, $\mathbb{T}G$ is the source simply connected Lie groupoid which integrates the Lie algebroid $\mathbb{T}AG$. As explained in [Bursztyn et al. $\geq$ 2013], the $\mathscr{VB}$-subalgebroid $L_A \subseteq \mathbb{T}A$ integrates to a source simply connected $\mathscr{VB}$-subgroupoid $L_G \subseteq \mathbb{T}G$. We will prove that L_G is a multiplicative Dirac structure on G. Since $L_{AG} \subseteq \mathbb{T}AG$ is Lagrangian with respect to the canonical symmetric pairing $\langle \cdot, \cdot \rangle_{AG}$ on $\mathbb{T}AG$, we conclude from Proposition 5.11 that L_G is Lagrangian with respect to the canonical symmetric pairing $\langle \cdot, \cdot \rangle_G$ on $\mathbb{T}G$. It remains to show that $L_G \subseteq \mathbb{T}G$ is integrable with respect to the Courant bracket. Equivalently, we have to prove that the Courant 3-tensor $\mu_{L_G} \in \Gamma(\bigwedge^3 L_G^*)$ is zero. Since $L_A \subseteq \mathbb{T}AG$ is a Dirac structure, the induced Courant 3-tensor $\mu_{L_A} \in \Gamma(\bigwedge^3 L_A^*)$ vanishes. Therefore, combining Proposition 5.16 (applied to the zero Lie algebroid morphism) with Lie's second theorem we conclude that $\mu_{L_G} \equiv 0$, as desired. This shows that L_G is a Dirac structure on G, which by definition is multiplicative.   □

Remark 5.18. Theorem 5.17 provides a direct proof of the Notice that integrability of the Lagrangian subbundle $L_{AG} \subset \mathbb{T}(AG)$ associated to a multiplicative Dirac structure $L_G \subset \mathbb{T}G$, without using the theory of $\mathscr{LA}$-groupoids. In spite of this, we believe that the fact that L_{AG} inherits the structure of a double Lie algebroid is interesting in itself. This relies on the observation that L_G is an $\mathscr{LA}$-groupoid.

5E. *Main examples revisited.* We have shown several examples of Dirac groupoids and Dirac algebroids. See Sections 3 and 4, respectively. Here we will see that both classes of examples are related by the construction explained in Section 5C. Throughout this subsection G denotes a Lie groupoid over M with Lie algebroid AG.

5E1. *Poisson groupoids and Lie bialgebroids.* Consider a multiplicative Poisson bivector π_G on G. It is well known that in this case $M \subseteq G$ is a coisotropic submanifold and, in particular, the conormal bundle $N^*M \cong A^*G$ inherits a Lie algebroid structure. The Dirac structure L_G on G defined by the graph of π_G is a multiplicative Dirac structure. The multiplicativity of this Dirac structure is equivalent to $\pi_G^\sharp : T^*G \to TG$ being a morphism of Lie groupoids, and the associated Lie algebroid morphism coincides, up to identifications, with $\pi_{AG}^\sharp : T^*(AG) \to T(AG)$, where π_{AG} denotes the linear Poisson bivector on AG dual to the Lie algebroid A^*G. One concludes that the corresponding Dirac structure L_{AG} on AG is exactly the graph of π_{AG}. Since L_{AG} is a Lie subalgebroid of $\mathbb{T}AG$, the bundle map $\pi_{AG}^\sharp : T^*(AG) \to T(AG)$ is a Lie algebroid morphism. This is equivalent to saying that (AG, A^*G) is a Lie bialgebroid. As a corollary of Theorem 5.17 we obtain:

Corollary 5.19 [Mackenzie and Xu 2000]. *Let G be a source simply connected Lie groupoid with Lie algebroid AG. There is a one-to-one correspondence between multiplicative Poisson bivectors on G and Lie bialgebroid structures on (AG, A^*G).*

5E2. *Multiplicative 2-forms and IM-2-forms.* Assume that $\omega_G \in \Omega^2(G)$ is a multiplicative closed 2-form on G. The Dirac structure L_G given by the graph of $\omega_G^\sharp : TG \to T^*G$ is multiplicative. In this case, the corresponding Dirac structure L_{AG} on AG is given by the graph of the closed 2-form $\omega_{AG} := -\sigma^*\omega_{can}$, where $\sigma : AG \to T^*M$ is defined by $\sigma(u) = i_u\omega_G|_{TM}$. Since the Dirac structure L_{AG} is a Lie subalgebroid of $\mathbb{T}(AG)$, we conclude that the bundle map $\omega_{AG}^\sharp : T(AG) \to T^*(AG)$ is a Lie algebroid morphism. As shown in [Bursztyn et al. 2009a], this is equivalent to the bundle map $\sigma : AG \to T^*M$ being an *IM-2-form* on AG; that is, for every $u, v \in \Gamma(AG)$, the following conditions hold:

- $\langle \sigma(u), \rho_{AG}(v) \rangle = -\langle \sigma(v), \rho_{AG}(u) \rangle$;
- $\sigma[u, v] = \mathscr{L}_{\rho_{AG}(u)}\sigma(v) - \mathscr{L}_{\rho_{AG}(v)}\sigma(u) + d\langle \sigma(u), \rho_{AG}(v) \rangle$.

As a corollary of Theorem 5.17, we get:

Corollary 5.20 [Bursztyn et al. 2004]. *Let G be a source simply connected Lie groupoid with Lie algebroid AG. There is a one-to-one correspondence between multiplicative closed 2-forms on G and IM-2-forms on AG.*

5E3. *Foliated groupoids and foliated algebroids.* Let $F_G \subseteq TG$ be a multiplicative involutive subbundle. Then, the Dirac structure $L_G = F_G \oplus F_G^\circ$ is multiplicative.

The corresponding Dirac structure L_{AG} on AG associated to L_G is given by $L_{AG} = F_{AG} \oplus F_{AG}^\circ \subset \mathbb{T}(AG)$, where $F_{AG} := j_G^{-1}(A(F_G)) \subseteq T(AG)$. Since L_{AG} is a Dirac structure which is also a Lie subalgebroid of $\mathbb{T}(AG)$, we conclude that $F_{AG} \subseteq T(AG)$ is an involutive subbundle which is also a Lie subalgebroid of $T(AG) \to TM$. We refer to such a subbundle as a *morphic foliation* on AG. As a corollary of Theorem 5.17, we obtain the next result.

Corollary 5.21 [Hawkins 2008]. *Let G be a source simply connected Lie groupoid with Lie algebroid AG. There exists a one-to-one correspondence between multiplicative foliations on G and morphic foliations on AG.*

As shown in [Hawkins 2008; Jotz and Ortiz 2012], having a morphic foliation on AG is equivalent to AG be equipped with an *IM-foliation*, that is, a triple (F_M, K, ∇) where $F_M \subseteq TM$ is an involutive subbundle, $K \subseteq AG$ is a Lie subalgebroid with $\rho_{AG}(K) \subseteq F_M$, and ∇ is an F_M-connection on AG/K satisfying the following conditions:

- ∇ is flat.

- If $u \in \Gamma(AG)$ satisfies $\nabla_{\Gamma(F_M)}(u + K) \in \Gamma(K)$, then $[u, \Gamma(K)] \subseteq \Gamma(K)$.

- If $u, v \in \Gamma(AG)$ are such that $\nabla_{\Gamma(F_M)}(u + K), \nabla_{\Gamma(F_M)}(v + K) \in \Gamma(K)$, it follows that $\nabla_{\Gamma(F_M)}([u, v] + K) \in \Gamma(K)$.

- If $u \in \Gamma(AG)$ satisfies $\nabla_{\Gamma(F_M)}(u + K) \in \Gamma(K)$, then $[\rho_{AG}(u), \Gamma(F_M)] \subseteq \Gamma(F_M)$.

The properties as above determine completely the morphic foliation F_{AG} on AG. In particular, Dirac structures of the form $L_{AG} = F_{AG} \oplus F_{AG}^\circ$ are in one-to-one correspondence with IM-foliations. Additionally, there exists a conceptually clear interpretation of IM-foliations in terms of representations up to homotopy. See [Drummond et al. 2013] for more details.

5E4. *Dirac Lie groups and Dirac Lie algebras.* Let G be a Lie *group* with Lie algebra $\mathfrak{g}$ and let $L_G \in \text{Dir}_{\text{mult}}(G)$ be a multiplicative Dirac structure. Consider the Dirac structure $L_{\mathfrak{g}}$ on $\mathfrak{g}$ associated to L_G. It was shown in [Ortiz 2008] that $\ker(L_G) := L_G \cap TG$ is a regular involutive subbundle of TG, in particular $\ker(L_{\mathfrak{g}}) = j_G^{-1}(A(\ker(L_G)))$ is an involutive subbundle of $T\mathfrak{g}$. Since $\ker(L_{\mathfrak{g}})$ is a linear foliation on $\mathfrak{g}$, that is, multiplicative with respect to the abelian group structure on $\mathfrak{g}$, then the leaf through $0 \in \mathfrak{g}$ is a vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$. The other leaves are affine subspaces of $\mathfrak{g}$ modeled on $\mathfrak{h}$. In particular, the space of characteristic leaves of $L_{\mathfrak{g}}$ coincides with the quotient space $\mathfrak{g}/\mathfrak{h}$. The fact that $L_g \subseteq \mathbb{T}\mathfrak{g}$ is a Lie subalgebroid implies that $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal. Therefore, the space of characteristic leaves $\mathfrak{g}/\mathfrak{h}$ of $L_{\mathfrak{g}}$ inherits a unique Lie algebra structure making the quotient map $\phi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ into a surjective Lie algebra morphism. Since $\mathfrak{g}/\mathfrak{h}$ is the space of characteristic

leaves of $L_\mathfrak{g}$, there is a unique Poisson structure π on $\mathfrak{g}/\mathfrak{h}$ making the quotient map $\phi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ into a forward and backward Dirac map. Since $L_\mathfrak{g}$ is a morphic Dirac structure, we conclude that π is a morphic bivector on $\mathfrak{g}/\mathfrak{h}$. In particular, the pair $(\mathfrak{g}/\mathfrak{h}, (\mathfrak{g}/\mathfrak{h})^*)$ is a Lie bialgebra. Conversely, given a Lie algebra $\mathfrak{g}$ and an ideal $\mathfrak{h} \subseteq \mathfrak{g}$ such that $(\mathfrak{g}/\mathfrak{h}, (\mathfrak{g}/\mathfrak{h})^*)$ is a Lie bialgebra, then the linear Poisson bivector π on $\mathfrak{g}/\mathfrak{h}$ is morphic. The surjective Lie algebra morphism $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ induces a Dirac structure $L_\mathfrak{g}$ on $\mathfrak{g}$ (the pull-back of π) which is morphic as well. We have proved the following result.

Proposition 5.22. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. There is a one-to-one correspondence between*

(1) *morphic Dirac structures on $\mathfrak{g}$, and*

(2) *ideals $\mathfrak{h} \subseteq \mathfrak{g}$ such that $(\mathfrak{g}/\mathfrak{h}, (\mathfrak{g}/\mathfrak{h})^*)$ is a Lie bialgebra.*

The proposition above recovers the results of [Ortiz 2008].

5E5. *Tangent lifts of Dirac structures.* Let L_G be a multiplicative Dirac structure on G. Consider the associated morphic Dirac structure L_{AG} on the Lie algebroid of G. We can lift L_G to a multiplicative Dirac structure on the tangent groupoid TG. Similarly, as explained in Section 4B3, the morphic Dirac structure L_{AG} can be lifted to a morphic Dirac structure $L_{T(AG)}$ on the tangent Lie algebroid $T(AG) \to TM$. It is straightforward to check that the morphic Dirac structure on $T(AG)$ associated to L_{TG} as in Theorem 5.17 coincides with the tangent lift $L_{T(AG)}$ of L_{AG}. That is, the tangent functor commutes with the Lie functor.

5E6. *Symmetries of Dirac groupoids.* Let L_G be a multiplicative Dirac structure on G. Consider the associated morphic Dirac structure L_{AG} on AG as in Theorem 5.17. Let H be a Lie group acting freely and properly on G by groupoid automorphisms $\Phi_h : G \to G$, $h \in H$. Applying the Lie functor to each $\Phi_h : G \to G$ yields a free and proper H-action on AG by Lie algebroid automorphisms $A(\Phi_h) : AG \to AH$, $h \in H$. Assume that the H-orbits of G coincide with the characteristic leaves of L_G. Then, the H-orbits of AG coincide with the characteristic leaves of L_{AG}. We have shown that in this situation we can endow the space of characteristic leaves G/H of L_G with a unique multiplicative Poisson bivector $\pi_{G/H}$ making the quotient map $G \to G/H$ into a forward and backward Dirac map. Similarly, the space of characteristic leaves AG/H of L_{AG} inherits a unique morphic Poisson structure $\pi_{AG/H}$ making the quotient map $AG \to AG/H$ into a forward and backward Dirac map. One can easily see that the morphic Dirac structure $L_{AG/H}$ associated to $\pi_{G/H}$ as in Section 5E1 coincides with the morphic Dirac structure on AG/H given by the graph of $\pi_{AG/H}$. As a consequence, the Lie bialgebroid of $(G/H, \pi_{G/H})$ is exactly $(AG/H, (AG/H)^*)$.

5E7. *B-field transformations.* Let L_G be a multiplicative Dirac structure on G. Assume that B_G is a multiplicative closed 2-form on G. Consider the Dirac structure L_G^B on G, obtained out of L_G by applying the B-field transformation with respect to B_G. As observed in [Bursztyn et al. 2009a], every multiplicative closed 2-form on G induces a morphic closed 2-form B_{AG} on AG. A direct computation shows that the morphic Dirac structure L_{AG}^B corresponding to L_G^B (as in Theorem 5.17) is given by the B-field transformation of L_{AG} with respect to B_{AG}, in agreement with [Ortiz 2012].

5E8. *Generalized complex groupoids.* Let $L_G \subseteq \mathbb{T}_\mathbb{C} G$ be a multiplicative generalized complex structure on G. The construction explained in Theorem 5.12 applies also to the case of multiplicative generalized complex structures. As a result, there is a morphic Dirac structure $L_{AG} \subseteq \mathbb{T}_\mathbb{C} AG$ given by $L_{AG} := (j_G^{-1} \oplus j_G')_\mathbb{C}(A(L_G))$, where $(j_G^{-1} \oplus j_G')_\mathbb{C} : A(\mathbb{T}_\mathbb{C} G) \to \mathbb{T}_\mathbb{C}(AG)$ denotes the complexification of the canonical isomorphism $(j_G^{-1} \oplus j_G') : A(\mathbb{T}G) \to \mathbb{T}(AG)$. Observe that $L_{AG} \subseteq \mathbb{T}_\mathbb{C} AG$ is in fact a generalized complex structure making the pair (AG, L_{AG}) into a generalized Lie algebroid. For that, we only need to check that $L_{AG} \cap \overline{L_{AG}} = \{0\}$. Indeed, one easily checks that the conjugation map $\overline{(\cdot)}_G : \mathbb{T}_\mathbb{C} G \to \mathbb{T}_\mathbb{C} G$ is a Lie groupoid isomorphism. Therefore, the generalized complex structure $\bar{L}_G$ on G is also multiplicative. Since $\mathbb{T}_\mathbb{C} G = L_G \oplus \bar{L}_G$, the application of the Lie functor yields a decomposition

$$(19) \qquad\qquad A(\mathbb{T}_\mathbb{C} G) = A(L_G) \oplus A(\bar{L}_G).$$

Straightforward computation shows that the Lie algebroid isomorphism $A(\overline{(\cdot)}_G) : A(\mathbb{T}_\mathbb{C} G) \to A(\mathbb{T}_\mathbb{C} G)$ satisfies

$$(j_G^{-1} \oplus j_G')_\mathbb{C} \circ A(\overline{(\cdot)}_G) = \overline{(\cdot)}_{AG},$$

where the map of the right hand side of the identity above is the conjugation map $\mathbb{T}_\mathbb{C}(AG) \to \mathbb{T}_\mathbb{C}(AG)$. Hence, applying the canonical isomorphism $(j_G^{-1} \oplus j_G')_\mathbb{C} : A(\mathbb{T}_\mathbb{C} G) \to \mathbb{T}_\mathbb{C}(AG)$ on both sides of (19), gives rise to

$$\mathbb{T}_\mathbb{C} AG = L_{AG} \oplus \bar{L}_{AG}.$$

Therefore, L_{AG} is transversal to $\bar{L}_{AG}$ and we conclude that L_{AG} is a morphic generalized complex structure. In this situation, Theorem 5.17 gives rise to the following result.

Proposition 5.23 [Jotz et al. 2012]. *Let G be a source simply connected Lie groupoid with Lie algebroid AG. There is a one-to-one correspondence between multiplicative generalized complex structures on G and morphic generalized complex structures on AG.*

6. Conclusions and final remarks

This work can be considered as the first step toward describing multiplicative Dirac structures infinitesimally. We have seen that every multiplicative Dirac structure L_G on a Lie groupoid G induces a Dirac structure L_{AG} on its Lie algebroid AG which is compatible with the algebroid structure in the sense that $L_{AG} \subseteq \mathbb{T}(AG)$ is a Lie subalgebroid. Notice that in the special situation of Poisson groupoids (resp. multiplicative closed 2-forms, multiplicative foliations) the induced Dirac structure on AG is equivalent to endowing (AG, A^*G) with a Lie bialgebroid structure (resp. IM-2-form, IM-foliation). Therefore, it would be interesting to introduce a suitable notion of *IM-Dirac structure*, providing a more explicit description of Dirac structures compatible with a Lie algebroid, unifying different infinitesimal structures such as Lie bialgebroids, IM-2-forms and IM-foliations. This study will be part of a future work.

Acknowledgements

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