Pacific Journal of Mathematics

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Volume 266 No. 2 December 2013

PACIFIC JOURNAL OF MATHEMATICS
Vol. 266, No. 2, 2013
dx.doi.org/10.2140/pjm.2013.266.367

ON THE FINITE GENERATION OF A FAMILY OF EXT MODULES

TONY J. PUTHENPURAKAL

Dedicated to Professor L. L. Avramov on the occasion of his sixtieth birthday.

Let $(A, \mathfrak{m})$ be a local complete intersection ring. Let M, N be finitely generated A-modules and let I be an ideal in A. We show that

$$\bigcup_{n\geq 0}\bigcup_{i\geq 0} \operatorname{Ass}\operatorname{Ext}^i_A(M, I^n N)$$

is a finite set. We also show that there exist i_0, n_0 such that for all $i \geq i_0$ and $n \geq n_0$ we have

$$\operatorname{Ass}\operatorname{Ext}^{2i}_A(M, I^n N) = \operatorname{Ass}\operatorname{Ext}^{2i_0}_A(M, I^{n_0} N),$$
$$\operatorname{Ass}\operatorname{Ext}^{2i+1}_A(M, I^n N) = \operatorname{Ass}\operatorname{Ext}^{2i_0+1}_A(M, I^{n_0} N).$$

We prove analogous results for complete intersection rings which arise in algebraic geometry. We also prove that the complexity, $\operatorname{cx}(M, I^n N)$, is constant for all $n \gg 0$.

1. Introduction

Let A be a Noetherian ring. Let I be an ideal in A and let M be a finitely generated A-module. M. Brodmann [1979] proved that the set $\operatorname{Ass}_A M/I^n M$ is independent of n for all large n. This result is usually deduced by proving that $\operatorname{Ass}_A I^n M/I^{n+1} M$ is independent of n for all large n.

We state some generalizations of Brodmann's result. Fix $i \geq 0$. L. Melkersson and P. Schenzel [1993, Theorem 1] showed that

$$\operatorname{Ass}_A \operatorname{Tor}^A_i(M, I^n/I^{n+1}) \quad \text{and} \quad \operatorname{Ass}_A \operatorname{Tor}^A_i(M, A/I^n)$$

are independent of n for all large n. By the same argument,

$$\operatorname{Ass}_A \operatorname{Ext}^i_A(M, I^n/I^{n+1})$$

The work for this paper was done while the author was visiting University of Kentucky by a fellowship from Department of Science and Technology, India. The author is deeply grateful to DST for its financial support and University of Kentucky for its hospitality.
MSC2010: primary 13D07, 13H10; secondary 13A15, 13A02.
Keywords: local complete intersection, asymptotic associate primes, cohomological operators.

and, by [Katz and West 2004, 3.5],

$$\operatorname{Ass}_A \operatorname{Ext}^i_A(M, A/I^n)$$

are similarly independent of n. An example of A. Singh [2000] shows that

$$\operatorname{Ass}_A \varinjlim \operatorname{Ext}^i_A(A/I^n, M) \text{ need not be finite.}$$

So in this example

$$\bigcup_{n\geq 1} \operatorname{Ass}_A \operatorname{Ext}^i_A(A/I^n, M) \text{ is not even finite.}$$

I state some questions in this area that motivated me. They were raised respectively by W. Vasconcelos [1998, 3.5] and Melkersson and Schenzel [1993, page 936].

$$\text{Is the set } \bigcup_{i\geq 0} \operatorname{Ass}_A \operatorname{Ext}^i_A(M, A) \text{ finite?} \tag{1}$$

$$\text{Is the set } \bigcup_{i\geq 0}\bigcup_{n\geq 0} \operatorname{Ass}_A \operatorname{Tor}^A_i(M, A/I^n) \text{ finite?} \tag{2}$$

The motivation for the main result of this paper came from (1). I do not believe that the question has a positive answer in this generality, but I am unable to give a counterexample. Note that if A is a Gorenstein local ring then Vasconcelos's question has, trivially, a positive answer. If we change the question a little then we may ask: If M, D are two finitely generated A-modules,

$$\text{is the set } \bigcup_{i\geq 0} \operatorname{Ass}_A \operatorname{Ext}^i_A(M, D) \text{ finite?}$$

This is not known for Gorenstein rings in general. However, if $A = Q/(\boldsymbol{f})$, where $\boldsymbol{f} = f_1, \dots, f_c$ is a regular sequence, and if $\operatorname{projdim}_Q M$ is finite, then the above question has a positive answer. This can be seen by using the theory of cohomology operators over such rings. This turns $\bigoplus_{i\geq 0} \operatorname{Ext}^i_A(M, D)$ into a finitely generated module over $A[t_1, \dots, t_c]$, where t_i has degree 2 for each i.

Using Melkerson and Schenzel's question as a guidepost, I was interested to solve the the following questions: Let $(A, \mathfrak{m})$ be a local complete intersection of codimension c.

$$\text{Is the set } \bigcup_{i\geq 0}\bigcup_{j\geq 0} \operatorname{Ass}_A \operatorname{Ext}^i_A(M, D/I^jD) \text{ finite?} \tag{a}$$

$$\text{Is the set } \bigcup_{i\geq 0}\bigcup_{j\geq 0} \operatorname{Ass}_A \operatorname{Ext}^i_A(M, I^jD) \text{ finite?} \tag{b}$$

In Theorem 5.1 I prove that (b) holds. I have been unable to verify whether (a) holds.

Let $\mathcal{R}(I) = \bigoplus_{n\geq 0} I^n t^n$ be the Rees algebra of I. The main result in this paper concerns finite generation of a family of Ext modules:

Theorem 1.1. *Let Q be a Noetherian ring with finite Krull dimension and let $\boldsymbol{f} = f_1, \ldots f_c$ be a regular sequence in Q. Set $A = Q/(\boldsymbol{f})$. Let M be a finitely generated A-module with* $\operatorname{projdim}_Q M$ *finite. Let I be an ideal in A and let $N = \bigoplus_{n\geq 0} N_n$ be a finitely generated $\mathcal{R}(I)$-module. Then*

$$\mathcal{E}(N) = \bigoplus_{i\geq 0} \bigoplus_{n\geq 0} \operatorname{Ext}^i_A(M, N_n)$$

is a finitely generated *bigraded $\mathcal{S} = \mathcal{R}(I)[t_1, \ldots, t_c]$-module.*

Remark 1.2. See Section 2.3 for a description of $\mathcal{E}(N)$ as a $\mathcal{S} = \mathcal{R}(I)[t_1, \ldots, t_c]$-module.

An easy consequence of this result is that (b) holds (by taking $N = \bigoplus_{n\geq 0} I^n D$); see Theorem 5.2. A complete, local complete intersection ring is a quotient of a regular local ring mod a regular sequence. So in this case (b) holds from Theorem 5.2. The proof of (b) for local complete intersections in general is a little technical; see Theorem 5.1. We also prove (b) for complete intersection rings which arise in algebraic geometry; see Section 6.

We next discuss a surprising consequence of Theorem 1.1. Let $(A, \mathfrak{m})$ be a local complete intersection of codimension c. Let M, N be two finitely generated A-modules. Let $\mu(X)$ denote the number of minimal generators of a finitely generated A-module X. Define

$$\operatorname{cx}_A(M, N) = \inf\left\{ b \in \mathbb{N} \;\middle|\; \varlimsup_{n\to\infty} \frac{\mu(\operatorname{Ext}^n_A(M, N))}{n^{b-1}} < \infty \right\}.$$

In Section 7 we prove (see Theorem 7.1) that

$$\operatorname{cx}_A(M, I^j N) \quad \text{is constant for all } j \gg 0. \tag{†}$$

We now describe in brief the contents of this paper. In Section 2 we give a module structure to $\mathcal{E}(N)$ over $\mathcal{S}$ (as in Theorem 1.1). We also discuss a few preliminaries. The local case of Theorem 1.1 is proved in Section 3 while the global case is proved in Section 4. In Section 5 we prove our results on asymptotic primes in the case of local complete intersections. In Section 6 we prove our result on asymptotic primes in complete intersection rings which arise in algebraic geometry. In Section 7 we prove (†).

2. Module structure

Let Q be a Noetherian ring and let $\boldsymbol{f} = f_1, \ldots f_c$ be a regular sequence in Q. Set $A = Q/(\boldsymbol{f})$. Let M be a finitely generated A-module with $\operatorname{projdim}_Q M$ finite.

We will not change M throughout our discussion. Let I be an ideal in A. Let $\mathcal{R}(I) = \bigoplus_{n\geq 0} I^n X^n$ be the *Rees algebra* of I. We consider $\mathcal{R}(I)$ as a subring of the polynomial ring $A[X]$. Let $N = \bigoplus_{n\geq 0} N_n$ be a finitely generated $\mathcal{R}(I)$-module.

Set

$$\mathscr{E}(N) = \bigoplus_{i\geq 0} \bigoplus_{n\geq 0} \operatorname{Ext}^i_A(M, N_n).$$

In this section we show $\mathscr{E}(N)$ is a bigraded $\mathscr{S} = \mathcal{R}(I)[t_1, \dots, t_c]$-module. The grading on $\mathscr{S}$ is as follows: we set $\deg t_j = (0, 2)$ for $j = 1, \dots, c$, and for $a \in I^s$ we set $\deg aX^s = (s, 0)$. We also discuss two preliminary results that we will need later in this paper.

2.1. Let $\mathbb{F} : \cdots F_n \to \cdots \to F_1 \to F_0 \to 0$ be a free resolution of M as an A-module.

Let $t_1, \dots t_c : \mathbb{F}(+2) \to \mathbb{F}$ be the *Eisenbud operators*; see [Eisenbud 1980, Section 1]. Then:

(1) The t_i are uniquely determined up to homotopy.

(2) Any two of them commute up to homotopy.

Let $T = A[t_1, \dots, t_c]$ be a polynomial ring over A with variables $t_1, \dots, t_c$ of degree 2. Let D be an A-module. The operators t_j give well-defined maps

$$t_j : \operatorname{Ext}^i_A(M, D) \to \operatorname{Ext}^{i+2}_R(M, D) \quad \text{for } 1 \leq j \leq c \text{ and all } i,$$

which turn $\operatorname{Ext}^*_A(M, D) = \bigoplus_{i\geq 0} \operatorname{Ext}^i_A(M, D)$ into a module over T. Furthermore, these structures depend only on $\boldsymbol{f}$, are natural in both module arguments and commute with the connecting maps induced by short exact sequences.

2.2. Gulliksen [1974, 3.1] proved that if $\operatorname{projdim}_Q M$ is finite then $\operatorname{Ext}^*_A(M, D)$ is a finitely generated T-module. If A is local and $D = k$, the residue field of A, Avramov [1989, 3.10] proved a converse; that is, if $\operatorname{Ext}^*_A(M, k)$ is a finitely generated T-module then $\operatorname{projdim}_Q M$ is finite. For a more general result, see [Avramov et al. 1997, 4.2].

2.3. Let $N = \bigoplus_{n\geq 0} N_n$ be a finitely generated module over $\mathcal{R}(I)$. Let $a \in I^s$. Consider $u = aX^s \in \mathcal{R}(I)_s$. The map

$$N_n \xrightarrow{u} N_{n+s}$$

yields a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}(\mathbb{F}, N_n) \arrow[r, "t_j"'] \arrow[d, "u"] & \operatorname{Hom}(\mathbb{F}, N_n)(+2) \arrow[d, "u"] \\
\operatorname{Hom}(\mathbb{F}, N_{n+s}) \arrow[r, "t_j"'] & \operatorname{Hom}(\mathbb{F}, N_{n+s})(+2).
\end{tikzcd}$$

Taking homology gives that $\mathscr{E}(N) = \bigoplus_{i\geq 0}\bigoplus_{n\geq 0} \operatorname{Ext}^i_A(M, N_n)$ is a bigraded $\mathscr{S}$-module, where $\mathscr{S} = \mathscr{R}(I)[t_1, \dots, t_c]$.

Remark 2.4. (1) For each i, the $\mathscr{R}(I)$-module $\bigoplus_{n\geq 0} \operatorname{Ext}^i_A(M, N_n)$ is finitely generated.

(2) For each n, the $A[t_1, \dots, t_c]$-module $\bigoplus_{i\geq 0} \operatorname{Ext}^i_A(M, N_n)$ is finitely generated.

2.5. ***Notation.*** (1) Let $N = \bigoplus_{n\geq 0} N_n$ be a graded $\mathscr{R}(I)$-module. Fix $j \geq 0$. Set

$$N_{\geq j} = \bigoplus_{n\geq j} N_n.$$

$\mathscr{E}(N_{\geq j})$ is naturally isomorphic to the submodule

$$\mathscr{E}(N)_{\geq j} = \bigoplus_{i\geq 0}\bigoplus_{n\geq j} \mathscr{E}(N)_{ij}$$

of $\mathscr{E}(N)$.

(2) If $A \to A'$ is a ring extension and if D is an A-module then set $D' = D \otimes_A A'$. Notice that if D is a finitely generated A-module then D' is a finitely generated A'-module.

(3) Set $\mathscr{S}' = \mathscr{S} \otimes_A A'$. Notice that $\mathscr{S}'$ is a finitely generated bigraded A'-algebra. Let $U = \bigoplus_{i\geq 0}\bigoplus_{n\geq 0} U_{i,n}$ be a graded $\mathscr{S}$-module. Then

$$U' = U \otimes_A A' = \bigoplus_{i\geq 0}\bigoplus_{n\geq 0} U'_{i,n}$$

is a graded $\mathscr{S}'$-module.

We state two lemmas that will help us in proving Theorem 1.1.

Lemma 2.6. *If $\mathscr{E}(N_{\geq j})$ is a finitely generated $\mathscr{S}$-module then $\mathscr{E}(N)$ is a finitely generated $\mathscr{S}$-module.*

Proof. Set $D = \mathscr{E}(N)/\mathscr{E}(N_{\geq j})$. We have the following exact sequence of $\mathscr{S}$-modules

$$0 \to \mathscr{E}(N_{\geq j}) \to \mathscr{E}(N) \to D \to 0.$$

Using Gulliksen's result it follows that D is a finitely generated $T = A[t_1, \dots, t_c]$-module. Since T is a subring of $\mathscr{S}$, we get that D is a finitely generated $\mathscr{S}$-module. Thus if $\mathscr{E}(N_{\geq j})$ is a finitely generated $\mathscr{S}$-module then $\mathscr{E}(N)$ is a finitely generated $\mathscr{S}$-module □

Lemma 2.7. (Keep the notation of 2.5(3).) *Let $A \to A'$ be a faithfully flat extension of rings and let $U = \bigoplus_{i\geq 0}\bigoplus_{n\geq 0} U_{i,n}$ be a graded $\mathscr{S}$-module. If U' is a finitely generated $\mathscr{S}'$-module then U is a finitely generated $\mathscr{S}$-module.*

Proof. The set

$$\mathcal{D} = \{u_{in} \otimes 1 \mid u_{in} \in U_{in}, \text{ where } i, n \geq 0\}$$

generates U' as a $\mathcal{S}'$-module. As U' is a finitely generated $\mathcal{S}'$-module, we can choose a finite subset $\mathcal{C}$ of $\mathcal{D}$ which generates U' as a $\mathcal{S}'$-module. Let

$$V = \langle u \mid u \otimes 1 \in \mathcal{C} \rangle.$$

Then V is a finitely generated submodule of U. Notice that $U' = V'$. Thus $(U/V) \otimes_A A' = 0$. Since A' is a faithfully flat A-algebra we get $U = V$. So U is a finitely generated $\mathcal{S}$-module. □

3. The local case

In this section we prove Theorem 1.1 when $(Q, \mathfrak{n})$ is local. Let $\mathfrak{m}$ be the maximal ideal of A. Set $k = A/\mathfrak{m}$. Let I be an ideal in A. Let

$$F(I) = \mathcal{R}(I) \otimes_A k = \bigoplus_{n \geq 0} I^n/\mathfrak{m}I^n$$

be the *fiber cone* of I.

3.1. Assume $N = \bigoplus_{n \geq 0} N_n$ is a finitely generated $\mathcal{R}(I)$-module. Notice that

$$F(N) = N \otimes_A k = \bigoplus_{n \geq 0} N_n/\mathfrak{m}N_n$$

is a finitely generated $F(I)$-module. Define

$$\operatorname{spread}(N) := \dim_{F(I)} N/\mathfrak{m}N.$$

Proof of Theorem 1.1 in the local case.

Case 1: *The residue field $k = A/\mathfrak{m}$ is infinite.* We induct on $\operatorname{spread}(N)$. First assume $\operatorname{spread}(N) = 0$. This implies that $N_n/\mathfrak{m}N_n = 0$ for all $n \gg 0$. By Nakayama's lemma, $N_n = 0$ for all $n \gg 0$; say $N_n = 0$ for all $n \geq j$. Then $\mathcal{E}(N_{\geq j}) = 0$ and it is obviously a finitely generated $\mathcal{S}$-module. By Lemma 2.6 we get that $\mathcal{E}(N)$ is a finitely generated $\mathcal{S}$-module.

When $\operatorname{spread}(N) > 0$ then there exists $u = xt \in \mathcal{R}(I)_1$ which is $(N \oplus F(N))$-filter-regular, that is, there exists j such that

$$(0\colon {}_N u)_n = 0 \quad \text{and} \quad (0\colon {}_{F(N)} u)_n = 0 \quad \text{for all } n \geq j.$$

Set $N_{\geq j} = \bigoplus_{n \geq j} N_n$ and $U = N_{\geq j}/uN_{\geq j}$. We have an exact sequence of $\mathcal{R}(I)$-modules

$$0 \to N_{\geq j}(-1) \xrightarrow{u} N_{\geq j} \to U \to 0.$$

For each $n \geq j$ the functor $\operatorname{Hom}_A(M, -)$ induces the long exact sequence of A-modules

$$\begin{array}{llll} 0 \to \operatorname{Hom}_A(M, N_n) & \xrightarrow{u} \operatorname{Hom}_A(M, N_{n+1}) & \to \operatorname{Hom}_A(M, U_{n+1}) & \\ \to \operatorname{Ext}^1_A(M, N_n) & \xrightarrow{u} \operatorname{Ext}^1_A(M, N_{n+1}) & \to \operatorname{Ext}^1_A(M, U_{n+1}) & \\ \to \quad \cdots & \xrightarrow{u} \quad \cdots & \to \quad \cdots & \\ \to \operatorname{Ext}^i_A(M, N_n) & \xrightarrow{u} \operatorname{Ext}^i_A(M, N_{n+1}) & \to \operatorname{Ext}^i_A(M, U_{n+1}) & \\ \to \quad \cdots & \xrightarrow{u} \quad \cdots & \to \quad \cdots & . \end{array}$$

Using the naturality of Eisenbud operators we have the following exact sequence of $\mathcal{S}$-modules

$$\mathcal{E}(N_{\geq} j)(-1, 0) \xrightarrow{(u,0)} \mathcal{E}(N_{\geq} j) \to \mathcal{E}(U).$$

By construction,

$$\operatorname{spread}(U) = \operatorname{spread}(N_{\geq j}) - 1 = \operatorname{spread}(N) - 1.$$

By the induction hypothesis, $\mathcal{E}(U)$ is a finitely generated $\mathcal{S}$-module. Therefore by Lemma 3.2 we get $\mathcal{E}(N_{\geq j})$ is a finitely generated $\mathcal{S}$-module. Using Lemma 2.6 we get that $\mathcal{E}(N)$ is a finitely generated $\mathcal{S}$-module.

Case 2: *The residue field k is finite.*

In this case we do the standard trick. Let $Q' = Q[X]_{\mathfrak{n}Q[X]}$. Set $A' = A \otimes_Q Q'$. Notice that $A' = A[X]_{\mathfrak{m}A[X]}$ is a flat A-algebra with residue field $k(X)$ which is infinite. Notice that $f_1, \ldots, f_c$ is a Q'-regular sequence and $Q'/(\boldsymbol{f}) = A'$. Set $I' = IA'$ and $M' = M \otimes_Q Q' = M \otimes_A A'$. Notice that $\operatorname{projdim}_{Q'} M'$ is finite. Set $\mathcal{R}(I)' = \mathcal{R}(I')$, the Rees algebra of I'. Then $N' = N \otimes_A A'$ is a finitely generated $\mathcal{R}(I)'$-module. Also note that $\mathcal{E}(N') = \mathcal{E}(N) \otimes_A A'$.

By Case 1 we have that $\mathcal{E}(N')$ is a finitely generated $\mathcal{S}'$-module. So by Lemma 2.7 we get that $\mathcal{E}(N)$ is a finitely generated $\mathcal{S}$-module. $\square$

The next lemma is a bigraded version of Lemma 2.8(1) of [Puthenpurakal 2005].

Lemma 3.2. *Let R be a Noetherian ring (not necessarily local) and let $B = \bigoplus_{i,j \geq 0} B_{i,j}$ be a finitely generated bigraded R-algebra with $B_{0,0} = R$. Note that B need not be standard graded. Set*

$$B_y = \bigoplus_{j \geq 0} B_{(0,j)}.$$

Let $V = \bigoplus_{i,j \geq 0} V_{i,j}$ be a bigraded B-module satisfying these conditions:

(1) *For each $i \geq 0$, $V_i = \bigoplus_{j \geq 0} V_{i,j}$ is finitely generated as a B_y-module.*

(2) *There exists* $z \in B_{(r,0)}$ (*with* $r \geq 1$) *and a finitely generated bigraded* B*-module* D *such that we have an exact sequence of* B*-modules*

$$V(-r,0) \xrightarrow{z} V \xrightarrow{\psi} D.$$

Then V *is a finitely generated* B*-module.*

Proof. Step 1. We begin by reducing to the case when ψ is *surjective*. Notice that $D' = \text{image}\, \psi$ is a finitely generated bigraded B-module. If $\psi' \colon V \to D'$ is the map induced by ψ then we have an exact sequence

$$V(-r,0) \xrightarrow{z} V \xrightarrow{\psi'} D' \to 0.$$

Thus we may assume ψ is surjective.

Step 2. Choosing generators:

2.1. Choose a *finite* set W in V of homogeneous elements such that

$$\psi(W) = \{\psi(w) \mid w \in W\}$$

is a *generating set* for D.

2.2. Assume all the elements in W have x-coordinate $\leq c$.

2.3. For each $i \geq 0$, by hypothesis, V_i is a finitely generated B_y-module. So we may choose a *finite set* P_i of homogeneous elements in V_i which generates V_i as a B_y-module.

2.4. Set

$$G = W \cup \left(\bigcup_{i=0}^{c} P_i \right).$$

Clearly G is a *finite* set.

Claim. *G is a generating set for V.*

Let U be the B-submodule of V generated by G. It suffices to prove that $U_{i,j} = V_{i,j}$ for all $i, j \geq 0$. By construction we have that for $0 \leq i \leq c$

$$U_{i,j} = V_{i,j} \quad \text{for each } j \geq 0. \tag{*}$$

We give $X := \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ the lex-order $\preceq$, making it well ordered. So we can prove our result by induction on X with respect to $\preceq$.

The base case is $(0,0)$. In this case $U_{0,0} = V_{0,0}$ by (*). Let $(i,j) \in X \setminus \{(0,0)\}$ and assume that for all $(r,s) \prec (i,j)$ we have $U_{r,s} = V_{r,s}$.

Subcase 1: $i \leq c$. By (*) we have $U_{i,j} = V_{i,j}$.

Subcase 2: $i > c$. Let $p \in V_{i,j}$. By construction, there exist $w_1, \dots, w_m \in W \subseteq G$ such that

$$\psi(p) = \sum_{l=1}^{m} h_l \psi(w_l), \quad \text{where } h_l \in B.$$

We may assume that $\deg h_l w_l = (i, j)$ for each l. Set $p' = \sum_{l=1}^{m} h_l w_l \in V_{i,j}$. Then $p' \in U_{i,j}$ and $p - p' \in \ker \psi$. So

$$p - p' = z \cdot q, \quad \text{where } q \in V_{(i-r,j)}.$$

If $q = 0$ then $p = p' \in U_{i,j}$. Otherwise, note that $(i - r, j) \prec (i, j)$. So by induction hypothesis, $q \in U_{(i-r,j)}$. It follows that $p \in U_{i,j}$. Thus $V_{i,j} \subseteq U_{i,j}$. Since $U_{i,j} \subseteq V_{i,j}$, by construction it follows that $U_{i,j} = V_{i,j}$. The result follows by induction on X. □

4. The global case

We need quite a few preliminaries to prove the global case of Theorem 1.1. See Section 4.2 for the difficulty in going from the local to the global case. Note that in the local case we proved the result by inducting on spread(N). This is unavailable to us in the global situation as there are usually infinitely many maximal ideals in a global ring. Most of this section will discuss two invariants of a graded $\mathscr{R}(I)$-module $N = \bigoplus_{n \geq 0} N_n$. We will use these invariants to prove Theorem 1.1 by induction.

4.1. ***Notation and conventions.*** We take the dimension of the zero-module to be -1. We also set the degree of the zero-polynomial to be -1.

Let $\mathfrak{P} \in \operatorname{Spec} Q$. If $\mathfrak{P} \supseteq \boldsymbol{f}$ then set $\mathfrak{p} = \mathfrak{P}/\boldsymbol{f}$. If $\mathfrak{P} \not\supseteq \boldsymbol{f}$ then any A-module localized at $\mathfrak{P}$ is zero. So assume $\mathfrak{P} \supseteq \boldsymbol{f}$.

(1) $\mathscr{R}(I)_\mathfrak{p} \cong \mathscr{R}(IA_\mathfrak{p})$ and $\mathscr{S}_\mathfrak{p} \cong \mathscr{R}(I)_\mathfrak{p}[t_1, \dots, t_c]$.

(2) $M_\mathfrak{p} = M_\mathfrak{P}$ has finite projective dimension as a $Q_\mathfrak{P}$-module.

(3) $\mathscr{E}(N)_\mathfrak{p} \cong \mathscr{E}(N_\mathfrak{p})$.

4.2. ***The difficulty in going from local to global.*** For each $\mathfrak{p} \in \operatorname{Spec} A$ it follows from Section 4.1 that $\mathscr{E}(N_\mathfrak{p})$ is a finitely generated $\mathscr{S}_\mathfrak{p}$-module. Usually $\operatorname{Supp}_A \mathscr{E}(N)$ will be an infinite set. So we cannot apply the local case and conclude.

The situation when $\operatorname{Supp}_A \mathscr{E}(N)$ is a finite set will help in the base step of our induction argument to prove Theorem 1.1. So we show it separately.

Lemma 4.3. *If* $\operatorname{Supp}_A \mathscr{E}(N)$ *is a finite set then* $\mathscr{E}(N)$ *is a finitely generated* $\mathscr{S}$-*module.*

Proof. We may choose a finite subset C of $\mathscr{E}(N)$ such that its image in $\mathscr{E}(N)_\mathfrak{p}$ generates $\mathscr{E}(N)_\mathfrak{p}$ for each $\mathfrak{p} \in \operatorname{Supp}_A \mathscr{E}(N)$. Set U to be the finitely generated submodule of $\mathscr{E}(N)$ generated by C.

Set $D = \mathscr{E}(N)/U$. Notice that $D_{\mathfrak{p}} = 0$ for each $\mathfrak{p} \in \operatorname{Spec} A$. So $D = 0$. Therefore $\mathscr{E}(N) = U$ is a finitely generated $\mathscr{S}$-module. $\square$

4.4. ***First inductive device.*** Since N is a finitely generated $\mathscr{R}(I)$-module we have $\operatorname{ann}_A N_n \subseteq \operatorname{ann}_A N_{n+1}$ for all $n \gg 0$. Since A is Noetherian it follows that $\operatorname{ann}_A N_n$ is constant for all $n \gg 0$. Call this stable value $\mathfrak{L}_N$. This enables us to define the *limit dimension* of N.

$$\lim\dim N = \lim_{n\to\infty} \dim_A N_n = \dim A/\mathfrak{L}_N.$$

Since A has finite Krull dimension we get that $\lim\dim N$ is finite.

4.5. Let $\mathfrak{P}$ be a prime ideal in A. If D is a finitely generated A-module then

$$\operatorname{ann}_{A_{\mathfrak{P}}} D_{\mathfrak{P}} = (\operatorname{ann}_A D)_{\mathfrak{P}} = (\operatorname{ann}_A D)A_{\mathfrak{P}}.$$

Therefore

$$(\mathfrak{L}_N)_{\mathfrak{P}} = \mathfrak{L}_{N_{\mathfrak{P}}}.$$

4.6. Note that if $\lim\dim(N) = -1$ then $N_j = 0$, say for all $j \geq j_0$. So $\mathscr{E}(N_{\geq j_0}) = 0$. Using Lemma 2.6 it follows that $\mathscr{E}(N)$ is a finitely generated $\mathscr{S}$-module. The first nontrivial case is the following:

Proposition 4.7. *If* $\lim\dim(N) = 0$ *then* $\mathscr{E}(N)$ *is a finitely generated* $\mathscr{S}$*-module.*

Proof. This implies that $A/\mathfrak{L}_N$ is Artinian. Say $\dim N_n = 0$ for $n \geq r$. Clearly,

$$\operatorname{Supp}_A \mathscr{E}(N_{\geq r}) \subseteq \operatorname{Supp}_A A/\mathfrak{L}_N,$$

a finite set of maximal ideals in A. It follows from Lemma 4.3 that $\mathscr{E}(N_{\geq r})$ is a finitely generated $\mathscr{S}$-module. Using Lemma 2.6 we get that $\mathscr{E}(N)$ is a finitely generated $\mathscr{S}$-module. $\square$

4.8. ***Higher-degree filter-regular element.*** We do not have filter-regular elements of degree 1 in the global situation. However we can do the following:

Set $E = N/H^0_{R_+}(N)$. Assume $E \neq 0$. As $H^0_{R_+}(E) = 0$ there exists homogeneous $u \in R_+$ such that u is E-regular [Bruns and Herzog 1993, 1.5.11]. Say $\deg u = s$. Since $E_n = N_n$ for all $n \gg 0$ it follows that the map $N_i \to N_{i+s}$ induced by multiplication by u is injective for all $i \gg 0$. We will say that u is an N filter-regular element of degree s.

4.9. ***The second inductive device.*** We now discuss a global invariant of N which patches well with local ones.

4.10. ***The local invariant.*** Let $(A, \mathfrak{m})$ be local and let $W = \bigoplus_{n\geq 0} W_n$ be a finitely generated $\mathcal{R}(I)$-module. Suppose $\mathfrak{L}_W = \operatorname{ann}_A W_n$ for all $n \geq c$. Let $\mathfrak{a} \subseteq \mathfrak{L}_W$ be an ideal. Fix $j \geq 0$. Set

$$d_{\mathfrak{a}}(W, j) = \begin{cases} 0 & \text{if } j < c, \\ 0 & \text{if } j \geq c \text{ and } \dim W_j < \dim A/\mathfrak{a}, \\ e(\mathfrak{m}, W_j) & \text{otherwise.} \end{cases}$$

Note that for $j \geq c$, W_j is an $A/\mathfrak{a}$-module. Furthermore $d_{\mathfrak{a}}(W, j)$ is the modified multiplicity function on the $A/\mathfrak{a}$-module W_j.

Remark 4.11. Notice if $\dim W_j = \dim A/\mathfrak{a}$ and $j \geq c$ then

$$d_{\mathfrak{a}}(W, j) = d_{\mathfrak{L}_W}(W, j).$$

Let $\mu(D)$ denote the minimal number of generators of an A-module D.

Lemma 4.12. *The function $d_{\mathfrak{a}}(W, -)$ is of polynomial type of degree $\leq \mu(I) - 1$.*

Proof. We may assume that the residue field of A is infinite. Set $T = \mathcal{R}(I)/\mathfrak{a}\mathcal{R}(I) = \bigoplus_{n\geq 0} T_n$. Notice $T_0 = A/\mathfrak{a}$. Let $\boldsymbol{x} = x_1, \ldots, x_r$ be a minimal reduction of $\mathfrak{m}(A/\mathfrak{a})$. So $e(\mathfrak{m}, -) = e(\boldsymbol{x}, -)$ [Bruns and Herzog 1993, 4.6.5]. By a result due to Serre [Bruns and Herzog 1993, 4.7.6], we get that

$$e(\boldsymbol{x}, W_j) = \sum_{i=0}^{r} (-1)^i \ell\big(H_i(\boldsymbol{x}, W_j)\big).$$

Notice $H_i(\boldsymbol{x}, W) = \bigoplus_{j\geq c} H_i(\boldsymbol{x}, W_j)$ is a finitely generated $T/\boldsymbol{x}T$-module. Notice $(T/\boldsymbol{x}T)_0 = A/(\mathfrak{a}+\boldsymbol{x})$ is Artinian. Furthermore $(T/\boldsymbol{x}T)_1$ is a quotient of $\mathcal{R}(I)_1$ and so can be generated by $\mu(I)$ elements. Therefore the function $j \mapsto \ell\big(H_i(\boldsymbol{x}, W_j)\big)$ is of polynomial type of degree $\leq \mu(I) - 1$. The result follows. $\square$

Definition 4.13. $\theta(\mathfrak{a}, W)$ is the degree of the polynomial function $d_{\mathfrak{a}}(W, -)$.

Remark 4.14. Clearly $\theta(\mathfrak{a}, W)$ is nonnegative if and only if $\lim\dim W = \dim R/\mathfrak{a}$ and is -1 otherwise. Note that if $\dim A/\mathfrak{a} = \lim\dim W$ then $\theta(\mathfrak{a}, W) = \theta(\mathfrak{L}_W, W)$ is independent of $\mathfrak{a}$.

4.15. ***The global invariant.*** Let A be a Noetherian ring with finite Krull dimension. Let $I = (x_1, \ldots, x_s)$ be an ideal in A. Let $W = \bigoplus_{n\geq 0} W_n$ be a finitely generated $\mathcal{R}(I)$-module. We assume that $\mathfrak{L}_W = \operatorname{ann}_A W_n$ for all $n \geq c$. Let $\mathfrak{a} \subseteq \mathfrak{L}_W$ be an ideal.

Set

$$\mathscr{C}(\mathfrak{a}) = \{\mathfrak{m} \mid \mathfrak{m} \in \text{m-Spec}(A), \mathfrak{m} \supseteq \mathfrak{a} \text{ and } \dim(A/\mathfrak{a})_{\mathfrak{m}} = \dim A/\mathfrak{a}\}.$$

Let $I = (x_1, \ldots, x_s)$. If $\mathfrak{m} \in \mathscr{C}(\mathfrak{a})$ we have:

(a) $W_{\mathfrak{m}} = \bigoplus_{n\geq 0} (W_n)_{\mathfrak{m}}$.

(b) $\mathfrak{L}_{W_\mathfrak{m}} = (\mathfrak{L}_W)_\mathfrak{m}$. So $\mathfrak{a}_\mathfrak{m} \subseteq \mathfrak{L}_{W_\mathfrak{m}}$.

(c) $\theta(\mathfrak{a}_\mathfrak{m}, W_\mathfrak{m}) \leq s - 1$.

Define

$$\theta(\mathfrak{a}, W) = \max\{\theta(\mathfrak{a}_\mathfrak{m}, W_\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{C}(\mathfrak{a})\}.$$

By (c) above we get that $\theta(\mathfrak{a}, W)$ is finite and is $\leq s - 1$.

4.16. ***Properties of $\theta(\mathfrak{a}, W)$.*** We describe some properties of $\theta(\mathfrak{a}, W)$ we need for the proof of the global case of Theorem 1.1. Let $I = (x_1, \ldots, x_s)$.

(i) $\theta(\mathfrak{a}, W) \leq s - 1$. This is clear.

(ii) If $\mathfrak{L}_W \neq A$ then $\theta(\mathfrak{L}_W, W) \geq 0$. It suffices to consider the local case. Note that then $d_{\mathfrak{L}_W}(W, j) > 0$ for all $j \geq c$. It follows that $\theta(\mathfrak{L}_W, W) \geq 0$.

(iii) $\theta(\mathfrak{a}, W) = -1$ if and only if $\lim\dim W < \dim A/\mathfrak{a}$. If $\theta(\mathfrak{a}, W) = -1$ then $\theta(\mathfrak{a}_\mathfrak{m}, W_\mathfrak{m}) = -1$ for all $\mathfrak{m} \in \mathscr{C}(\mathfrak{a})$. This is equivalent to saying that $\lim\dim W_\mathfrak{m} < \dim(A/\mathfrak{a})_\mathfrak{m}$ for all $\mathfrak{m} \in \mathscr{C}(\mathfrak{a})$. By definition of $\mathscr{C}(\mathfrak{a})$ we have that

$$\dim A/\mathfrak{a} = \dim(A/\mathfrak{a})_\mathfrak{m} \quad \text{for each } \mathfrak{m} \in \mathscr{C}(\mathfrak{a}).$$

Also note that as $\mathfrak{a} \subseteq \mathfrak{L}_W$ we have

$$\lim\dim W = \max\{\lim\dim W_\mathfrak{m} \mid \mathfrak{m} \in \mathscr{C}(\mathfrak{a})\}.$$

So $\lim\dim W < \dim A/\mathfrak{a}$.

Conversely if $\lim\dim W < \dim A/\mathfrak{a}$ then for all $\mathfrak{m} \in \mathscr{C}(\mathfrak{a})$ we have

$$\lim\dim W_\mathfrak{m} \leq \lim\dim W < \dim A/\mathfrak{a} = \dim(A/\mathfrak{a})_\mathfrak{m}.$$

So $\theta(\mathfrak{a}_\mathfrak{m}, W_\mathfrak{m}) = -1$ for all $\mathfrak{m} \in \mathscr{C}(\mathfrak{a})$. Thus $\theta(\mathfrak{a}, W) = -1$.

(iv) If $\theta(\mathfrak{a}, W) \geq 0$ then $\theta(\mathfrak{L}_W, W) \leq \theta(\mathfrak{a}, W)$. By (iii) we get that $\lim\dim W = \dim A/\mathfrak{a}$. By hypothesis we also have $\mathfrak{a} \subseteq \mathfrak{L}_W$. Since $\dim A/\mathfrak{a} = \dim A/\mathfrak{L}_W$ it follows that $\mathscr{C}(\mathfrak{L}_W) \subseteq \mathscr{C}(\mathfrak{a})$. Using Remark 4.11 it follows that $\theta(\mathfrak{L}_W, W) \leq \theta(\mathfrak{a}, W)$.

(v) Let $u \in \mathscr{R}(I)_+$ be homogeneous of degree b. Assume u is W-filter regular and $W_n \neq 0$ for all $n \gg 0$. Set $E = W/uW$. Notice that $\mathfrak{L}_W \subseteq \mathfrak{L}_E$. Then

$$\theta(\mathfrak{L}_W, E) \leq \theta(\mathfrak{L}_W, W) - 1.$$

We have nothing to show if $\theta(\mathfrak{L}_W, E) = -1$. So assume $\theta(\mathfrak{L}_W, E) \geq 0$. Suppose $\theta(\mathfrak{L}_W, E) = \theta((\mathfrak{L}_W)_\mathfrak{p}, E_\mathfrak{p})$ for some $\mathfrak{p} \in \mathscr{C}(\mathfrak{L}_W)$. Since u is W-filter-regular, multiplication by u induces the exact sequence

$$0 \to W_{j-b} \to W_j \to E_j \to 0 \quad \text{for all } j \gg 0.$$

Localization at $\mathfrak{p}$ yields an exact sequence

$$0 \to (W_{j-b})_{\mathfrak{p}} \to (W_j)_{\mathfrak{p}} \to (E_j)_{\mathfrak{p}} \to 0 \quad \text{for all } j \gg 0.$$

Since $d_{\mathfrak{L}_{W_{\mathfrak{p}}}}(-,-)$ is an additive functor on $(A/\mathfrak{L}_W)_{\mathfrak{p}}$-modules we get that

$$\theta\big((\mathfrak{L}_W)_{\mathfrak{p}}, E_{\mathfrak{p}}\big) = \theta\big((\mathfrak{L}_W)_{\mathfrak{p}}, W_{\mathfrak{p}}\big) - 1.$$

The result follows since

$$\theta\big((\mathfrak{L}_W)_{\mathfrak{p}}, E_{\mathfrak{p}}\big) = \theta(\mathfrak{L}_W, E) \quad \text{and} \quad \theta\big((\mathfrak{L}_W)_{\mathfrak{p}}, W_{\mathfrak{p}}\big) \le \theta(\mathfrak{L}_W, W).$$

Proof of Theorem 1.1. We induct on $\operatorname{lim\,dim} N$. If $\operatorname{lim\,dim} N = -1, 0$ then the result follows from Section 4.6 and Proposition 4.7.

Assume $\operatorname{lim\,dim} N \ge 1$ and assume the result holds for all $\mathcal{R}(I)$-modules E with $\operatorname{lim\,dim} E \le \operatorname{lim\,dim} N - 1$. Let $x \in \mathcal{R}(I)_+$ be homogeneous and an N-filter-regular element. Let $\deg x = r$. Set $D = N/xD$. By Lemma 2.6 it suffices to assume the case when x is N-regular.

We now induct on $\theta(\mathfrak{L}_N, N)$. If $\theta(\mathfrak{L}_N, N) = 0$ then $\theta(\mathfrak{L}_N, D) \le -1$, by Section 4.16(v). Using Section 4.16(iii) we get that

$$\operatorname{lim\,dim} D < \dim A/\mathfrak{L}_N = \operatorname{lim\,dim} N.$$

By the induction hypothesis (on lim dim) the module $\mathscr{E}(D)$ is a finitely generated $\mathcal{S}$-module. The short exact sequence of $\mathcal{R}(I)$-modules

$$0 \to N(-r) \xrightarrow{x} N \to D \to 0$$

induces an exact sequence of $\mathcal{S}$-modules

$$\mathscr{E}(N)(-r, 0) \xrightarrow{x} \mathscr{E}(N) \to \mathscr{E}(D).$$

By Lemma 3.2 we get that $\mathscr{E}(N)$ is a finitely generated $\mathcal{S}$-module.

We assume the result if $\theta(\mathfrak{L}_N, N) \le i$ and prove it when $\theta(\mathfrak{L}_N, N) = i + 1$. Let D be as above. So $\theta(\mathfrak{L}_N, D) \le i$, by Section 4.16(v). If $\theta(\mathfrak{L}_N, D) = -1$ then the argument as above yields $\mathscr{E}(N)$ to be a finitely generated $\mathcal{S}$-module.

If $\theta(\mathfrak{L}_N, D) \ge 0$ then by Section 4.16(iv) we get that $\theta(\mathfrak{L}_D, D) \le \theta(\mathfrak{L}_N, D) \le i$. So by induction hypothesis on $\theta(-,-)$ we get that $\mathscr{E}(D)$ is a finitely generated $\mathcal{S}$-module. By an argument similar to the one above we get that $\mathscr{E}(N)$ is a finitely generated $\mathcal{S}$-module. □

5. Application I: Asymptotic associated primes — the local case

In this section we give an answer to our main motivating question.

Theorem 5.1. *Let $(A, \mathfrak{m})$ be a local complete intersection. Let M be a finitely generated A-module. Let I be an ideal in A and let $N = \bigoplus_{n\ge 0} N_n$ be a finitely* *generated $\mathcal{R}(I)$-module. Then*

$$\bigcup_{n\geq 0}\bigcup_{i\geq 0}\operatorname{Ass}_A \operatorname{Ext}^i_A(M, N_n) \textit{ is a finite set.}$$

Furthermore there exist i_0, n_0 such that for all $i \geq i_0$ and $n \geq n_0$ we have

$$\operatorname{Ass}_A \operatorname{Ext}^{2i}_A(M, N_n) = \operatorname{Ass}_A \operatorname{Ext}^{2i_0}_A(M, N_{n_0}),$$
$$\operatorname{Ass}_A \operatorname{Ext}^{2i+1}_A(M, N_n) = \operatorname{Ass}_A \operatorname{Ext}^{2i_0+1}_A(M, N_{n_0}).$$

Recall a local ring A is said to be a complete intersection if $\hat{A} = Q/(f_1, \dots, f_c)$, where $(Q, \mathfrak{n})$ is a complete regular local ring and $\boldsymbol{f}$ is a Q-regular sequence. If A is a complete intersection and a quotient of a regular local ring T then it can be shown that $A = T/(g_1, \dots, g_c)$, where $\boldsymbol{g}$ is a T-regular sequence (see [Matsumura 1980, 21.2]). In this case Theorem 5.1 holds by the following more general result:

Theorem 5.2. *Let Q be a Noetherian ring with finite Krull dimension and let $\boldsymbol{f} = f_1, \dots f_c$ be a regular sequence in Q. Set $A = Q/(\boldsymbol{f})$. Let M be a finitely generated A-module with* $\operatorname{projdim}_Q M$ *finite. Let I be an ideal in A and let $N = \bigoplus_{n\geq 0} N_n$ be a finitely generated $\mathcal{R}(I)$-module. Then*

$$\bigcup_{n\geq 0}\bigcup_{i\geq 0}\operatorname{Ass}\operatorname{Ext}^i_A(M, N_n) \textit{ is a finite set.}$$

Furthermore there exist i_0, n_0 such that for all $i \geq i_0$ and $n \geq n_0$ we have

$$\operatorname{Ass}_A \operatorname{Ext}^{2i}_A(M, N_n) = \operatorname{Ass}_A \operatorname{Ext}^{2i_0}_A(M, N_{n_0}),$$
$$\operatorname{Ass}_A \operatorname{Ext}^{2i+1}_A(M, N_n) = \operatorname{Ass}_A \operatorname{Ext}^{2i_0+1}_A(M, N_{n_0}).$$

The following example shows that two sets of stable values of associate primes can occur.

Example 5.3. Let $Q = k[\![u, x]\!]$, $A = Q/(ux)$. Let $M = Q/(u)$, $I = A$ and $N = M[t]$ (so $N_n = M$ for all n).

For $i \geq 1$ one has (see [Avramov and Buchweitz 2000, 4.3])

$$\operatorname{Ext}^{2i-1}_A(M, M) = 0 \quad \text{and} \quad \operatorname{Ext}^{2i}_A(M, M) = k.$$

5.4. We now state a special case of a result due to E. West [2004, 3.2 and 5.1].

Let $R = A[x_1, \dots, x_r; y_1, \dots y_s]$ be a bigraded A-algebra with $\deg x_i = (2, 0)$ and $\deg y_j = (0, 1)$. Let $M = \bigoplus_{i,n\geq 0} M_{(i,n)}$ be a finitely generated R-module. Then:

(1) $\bigcup_{i\geq 0}\bigcup_{n\geq 0}\operatorname{Ass}_A M_{(i,n)}$ is a finite set.

(2) There exist i_0, n_0 such that for all $i \geq i_0$ and $n \geq n_0$ we have

$$\operatorname{Ass}_A M_{(2i,n)} = \operatorname{Ass}_A M_{(2i_0,n_0)}, \quad \operatorname{Ass}_A M_{(2i+1,n)} = \operatorname{Ass}_A M_{(2i_0+1,n_0)}.$$

Proof of Theorem 5.2. The result follows from our main theorem (1.1) and 5.4. $\square$

We need the following exercise problem from [Matsumura 1980, 6.7, page 42].

Fact 5.5. *Let $f\colon A \to B$ be a ring homomorphism of Noetherian rings. Let U be a finitely generated B-module. Then*

$$\operatorname{Ass}_A U = \{\mathfrak{P} \cap A \mid \mathfrak{P} \in \operatorname{Ass}_B U\}.$$

In particular $\operatorname{Ass}_A U$ *is a finite set.*

There exist complete intersection rings which are not quotients of a regular local ring (see [Heitmann and Jorgensen 2012]). So Theorem 5.2 does not settle Theorem 5.1. To prove an analog of Theorem 5.2 for a local complete intersection we need the following result.

Lemma 5.6. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $\hat{A}$ be the completion of A with respect to $\mathfrak{m}$. Let B be a finitely generated $\hat{A}$-algebra containing $\hat{A}$. Let E be an A-module such that $E \otimes_A \hat{A}$ is a finitely generated B-module. Let D be any A-module. Then*:

(a) $\operatorname{Ass}_{\hat{A}} E \otimes_A \hat{A}$ *is a finite set.*

(b) $\operatorname{Ass}_A D = \{\mathfrak{P} \cap A \mid \mathfrak{P} \in \operatorname{Ass}_{\hat{A}}(D \otimes_A \hat{A})\}$.

(c) $\operatorname{Ass}_A E$ *is a finite set.*

To prove this result we need Theorem 23.3 from [Matsumura 1980]. Unfortunately, there is a typographical error there, so we state it here.

Theorem 5.7. *Let $\varphi\colon A \to B$ be a homomorphism of Noetherian rings, and let E be an A-module and G a B-module. Suppose that G is flat over A; then we have the following*:

(i) *If* $\mathfrak{p} \in \operatorname{Spec} A$ *and* $G/\mathfrak{p}G \neq 0$ *then*

$$^{a}\varphi(\operatorname{Ass}_B(G/\mathfrak{p}G)) = \operatorname{Ass}_A(G/\mathfrak{p}G) = \{\mathfrak{p}\}.$$

(ii) $\operatorname{Ass}_B(E \otimes_A G) = \bigcup_{\mathfrak{p} \in \operatorname{Ass}_A(E)} \operatorname{Ass}_B(G/\mathfrak{p}G).$

Remark 5.8. In [Matsumura 1980], $\operatorname{Ass}_A(E \otimes G)$ is written instead of $\operatorname{Ass}_B(E \otimes G)$. Also note that $^{a}\varphi(\mathfrak{P}) = \mathfrak{P} \cap A$ for $\mathfrak{P} \in \operatorname{Spec} B$.

Proof of Lemma 5.6. We consider the natural ring homomorphisms

$$\alpha\colon A \hookrightarrow \hat{A}, \quad \beta\colon \hat{A} \hookrightarrow B.$$

(a) We use the map β and Fact 5.5 to get our result.

(b) Set $X = \{\mathfrak{P} \cap A \mid \mathfrak{P} \in \operatorname{Ass}_{\hat{A}}(D \otimes_A \hat{A})\}$. We consider the flat map α.

Let $\mathfrak{q} \in X$. Say $\mathfrak{q} = \mathfrak{P} \cap A$, where $\mathfrak{P} \in \operatorname{Ass}_{\hat{A}} D \otimes \hat{A}$. By Theorem 5.7(ii), $\mathfrak{P} \in \operatorname{Ass}_{\hat{A}} \hat{A}/\mathfrak{p}\hat{A}$ for some $\mathfrak{p} \in \operatorname{Ass}_A D$. Notice $\hat{A}/\mathfrak{p}\hat{A} \neq 0$. By Theorem 5.7.(i) it follows that $\mathfrak{p} = \mathfrak{P} \cap A = \mathfrak{q}$. So $X \subseteq \operatorname{Ass}_A D$.

Conversely, if $\mathfrak{p} \in \operatorname{Ass}_A D$, then by Theorem 5.7(ii), $\operatorname{Ass}_{\hat{A}} \hat{A}/\mathfrak{p}\hat{A} \subseteq \operatorname{Ass}_{\hat{A}} D \otimes \hat{A}$. Notice $\hat{A}/\mathfrak{p}\hat{A} \neq 0$. Let $\mathfrak{P} \in \operatorname{Ass}_{\hat{A}} \hat{A}/\mathfrak{p}\hat{A}$. Then by Theorem 5.7(i) we have $\mathfrak{p} = \mathfrak{P} \cap A \in X$. Thus $\operatorname{Ass}_A D \subseteq X$. It follows that $\operatorname{Ass}_A D = X$.

(c) This follows from (a) and (b). □

Proof of Theorem 5.1. We consider the flat extension $\alpha\colon A \to \hat{A}$. Say $\hat{A} = Q/(\boldsymbol{f})$, where $(Q, \mathfrak{n})$ is a regular local ring and $\boldsymbol{f} = f_1, \dots, f_c \in \mathfrak{n}^2$ is a regular sequence.

(1) Consider $\mathscr{E}(N) = \bigoplus_{i \geq 0} \bigoplus_{n \geq 0} \operatorname{Ext}^i_A(M, N_n)$ as an A-module. By Theorem 1.1, $\mathscr{E}(N) \otimes \hat{A}$ is a finitely generated $B = \mathscr{R}(I\hat{A})[t_1, \dots t_c]$-algebra. By Lemma 5.6 we get that $\operatorname{Ass}_A \mathscr{E}(N)$ is a finite set. Notice that

$$\operatorname{Ass}_A \mathscr{E}(N) = \bigcup_{n \geq 0} \bigcup_{i \geq 0} \operatorname{Ass}_A \operatorname{Ext}^i_A(M, N_n).$$

(2) Set $\mathscr{E} = \mathscr{E}(N)$. By Theorem 1.1 there exist i_0 and n_0 such that for all $i \geq i_0$ and $n \geq n_0$ we have

$$\operatorname{Ass}_{\hat{A}} \mathscr{E}_{2i,n} \otimes \hat{A} = \operatorname{Ass}_{\hat{A}} \mathscr{E}_{2i_0,n_0} \otimes \hat{A}, \quad \operatorname{Ass}_{\hat{A}} \mathscr{E}_{2i+1,n} \otimes \hat{A} = \operatorname{Ass}_{\hat{A}} \mathscr{E}_{2i_0+1,n_0} \otimes \hat{A}.$$

By Lemma 5.6(b) it follows that for all $i \geq i_0$ and $n \geq n_0$ we have

$$\operatorname{Ass}_A \mathscr{E}_{2i,n} = \operatorname{Ass}_A \mathscr{E}_{2i_0,n_0}, \quad \operatorname{Ass}_A \mathscr{E}_{2i+1,n} = \operatorname{Ass}_A \mathscr{E}_{2i_0+1,n_0}.$$ □

6. Application II: Asymptotic associated primes — the geometric case

Let V be an affine or projective variety over an algebraically closed field K. Then V is said to be a local complete intersection if all of its local rings are complete intersections. Let A be the coordinate ring of V. In the affine case we have $A_{\mathfrak{p}}$ is a complete intersection for all $\mathfrak{p} \in \operatorname{Spec}(A)$. In the projective case we have $A_{(\mathfrak{p})}$ is a complete intersection for every $\mathfrak{p} \in \operatorname{Proj}(A)$. In this section we prove results analogous to Theorem 5.1 to coordinate rings of locally complete intersection varieties.

We first consider the affine case. In this case we prove the following general result. Recall a ring R is regular (a complete intersection) if $R_{\mathfrak{p}}$ is regular (a complete intersection) for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

Theorem 6.1. *Let Q be a regular ring of finite Krull dimension and let $\mathfrak{a}$ be an ideal in Q with $A = Q/\mathfrak{a}$ a complete intersection. Let M be a finitely generated A-module and let I be an ideal in A. Let $N = \bigoplus_{n \geq 0} N_n$ be a finitely generated $\mathscr{R}(I)$-module.*

Then

$$\bigcup_{n\geq 0}\bigcup_{i\geq 0} \operatorname{Ass}\operatorname{Ext}^i_A(M, N_n) \text{ is a finite set.}$$

Furthermore there exist i_0, n_0 such that for all $i \geq i_0$ and $n \geq n_0$ we have

$$\begin{aligned}\operatorname{Ass}_A \operatorname{Ext}^{2i}_A(M, N_n) &= \operatorname{Ass}_A \operatorname{Ext}^{2i_0}_A(M, N_{n_0}),\\ \operatorname{Ass}_A \operatorname{Ext}^{2i+1}_A(M, N_n) &= \operatorname{Ass}_A \operatorname{Ext}^{2i_0+1}_A(M, N_{n_0}).\end{aligned}$$

6.2. Before proving Theorem 6.1 we state the analogous result in the projective case. Let $\mathfrak{a}$ be a graded ideal in $Q = K[X_0, X_1, \dots, X_m]$, where $\deg X_i = 1$ for all i. Here K is not necessarily algebraically closed. Set $A = Q/\mathfrak{a}$. We assume $A_{(\mathfrak{p})}$ is a complete intersection for every $\mathfrak{p} \in \operatorname{Proj} A$. Recall that if U is the set of homogeneous elements in $A \setminus \mathfrak{p}$ then $A_{(\mathfrak{p})}$ is the degree-zero part of the graded ring $U^{-1}A$.

Let $\mathfrak{m}$ be the unique maximal homogeneous ideal of A. If E is a graded A-module then note that all its associate primes are homogeneous prime ideals of A. Set

$${}^*\operatorname{Ass}_A(E) = \operatorname{Ass}_A(E) \setminus \{\mathfrak{m}\},$$

the relevant associate primes of E. In the projective case our main theorem is this:

Theorem 6.3. (Keep the hypotheses of Section 6.2; note that $\mathcal{R}(I)$ is a bigraded ring.) *Let M be a finitely generated graded A-module and let I be a homogeneous ideal in A. Let $N = \bigoplus_{n\geq 0} N_n$ be a finitely generated bigraded $\mathcal{R}(I)$-module* (*so each N_n is a graded A-module*)*. Then*

$$\bigcup_{n\geq 0}\bigcup_{i\geq 0} {}^*\operatorname{Ass}\operatorname{Ext}^i_A(M, N_n) \text{ is a finite set.}$$

Furthermore there exist i_0, n_0 such that for all $i \geq i_0$ and $n \geq n_0$ we have

$$\begin{aligned}{}^*\operatorname{Ass}_A \operatorname{Ext}^{2i}_A(M, N_n) &= {}^*\operatorname{Ass}_A \operatorname{Ext}^{2i_0}_A(M, N_{n_0}),\\ {}^*\operatorname{Ass}_A \operatorname{Ext}^{2i+1}_A(M, N_n) &= {}^*\operatorname{Ass}_A \operatorname{Ext}^{2i_0+1}_A(M, N_{n_0}).\end{aligned}$$

We now prove Theorems 6.1 and 6.3. We begin with the affine case. We need the following:

Lemma 6.4. *Suppose $A = Q/\mathfrak{a}$, where Q is a regular ring. Suppose for some $\mathfrak{p} \in \operatorname{Spec} A$ the ring $A_{\mathfrak{p}}$ is a complete intersection. Let $\mathfrak{q} \in \operatorname{Spec} Q$ with $\mathfrak{q}/\mathfrak{a} = \mathfrak{p}$. Then there exist $g \in Q \setminus \mathfrak{q}$ such that $\mathfrak{a}Q_g$ is generated by a Q_g-regular sequence.*

Proof. We have $A_\mathfrak{p} = Q_\mathfrak{q}/\mathfrak{a}Q_\mathfrak{q}$. Since $A_\mathfrak{p}$ is a complete intersection it follows from [Matsumura 1980, 21.2] that $\mathfrak{a}Q_\mathfrak{q}$ is generated by a regular sequence, say $f_1, \dots, f_c$. We may assume $f_i \in \mathfrak{a}$ for all i.

Set

$$E = \frac{\mathfrak{a}}{(f_1, \dots, f_c)} \quad \text{and} \quad D_i = \frac{(f_1, \dots, f_{i-1})\colon f_i}{(f_1, \dots, f_{i-1})} \quad \text{for } i = 1, \dots, c.$$

Let

$$L = E \oplus \left(\bigoplus_{i=1}^{c} D_i \right).$$

Then L is a finitely generated Q-module and $L_\mathfrak{q} = 0$. So there exists $g \in Q \setminus \mathfrak{q}$ such that $L_g = 0$. In Q_g note that $\mathfrak{a}Q_g = (f_1, \dots, f_c)Q_g$. Also as $(f_1, \dots, f_c)Q_\mathfrak{q} \neq Q_\mathfrak{q}$ we have that $(f_1, \dots, f_c)Q_g \neq Q_g$. Since $(D_i)_g = 0$ for $i = 1, \dots, c$ we get that $f_1, \dots, f_c$ is a Q_g-regular sequence. □

Proof of Theorem 6.1. Let $\mathfrak{p} \in \operatorname{Spec} A$. Then $A_\mathfrak{p}$ is a complete intersection. Let $\mathfrak{q} \in \operatorname{Spec} Q$ with $\mathfrak{q}/\mathfrak{a} = \mathfrak{p}$. Then by Lemma 6.4 there exist $g \in Q \setminus \mathfrak{q}$ such that $\mathfrak{a}Q_g$ is generated by a Q-regular sequence. Let $g_\mathfrak{p}$ be the image of g in A.

For $x \in A$ let $D(x) = \{\mathfrak{P} \in \operatorname{Spec} A \mid x \notin \mathfrak{P}\}$. Then $D(x)$ is a basic open set in $\operatorname{Spec}(A)$. Note that $\mathfrak{p} \in D(g_\mathfrak{p})$. Clearly

$$\operatorname{Spec} A = \bigcup_{\mathfrak{p} \in \operatorname{Spec} A} D(g_\mathfrak{p}).$$

As $\operatorname{Spec} A$ is quasicompact we have

$$\operatorname{Spec} A = D(g_{\mathfrak{p}_1}) \cup \dots \cup D(g_{\mathfrak{p}_m}) \quad \text{for some } m \geq 1.$$

Set $g_i = g_{\mathfrak{p}_i}$. Note that for any A-module E we have

$$\operatorname{Ass}_A E = \bigcup_{i=1}^{m} (\operatorname{Ass}_{A_{g_i}} E_{g_i}) \cap A,$$

and that $\mathscr{E}(N)_g = \mathscr{E}(N_g)$. Thus it suffices to prove the result for A_{g_i} for each i.

For each $i = 1, \dots, m$ we have that

$$A_{g_i} = \frac{Q_i \text{ a regular ring of finite Krull dimension}}{\text{regular sequence in } Q_i}.$$

As Q_i is a regular ring of finite Krull dimension we get that $\operatorname{projdim}_{Q_i} M_{g_i}$ is finite. So we can apply Theorem 5.2 to get the result. □

To prove Theorem 6.3 we need a few preliminaries. Recall that a $\mathbb{Z}$-graded ring $S = \bigoplus_{n \in \mathbb{Z}} S_n$ is said to be $*$-local if it has a unique proper maximal homogeneous ideal $\mathfrak{P}$. Note that $\mathfrak{P}$ is a prime ideal in S but not necessarily a maximal ideal in S. The functor $- \bigotimes S_\mathfrak{P}$ from the category of graded S-modules to the category of

$S_{\mathfrak{P}}$-modules is faithfully exact, by [Bruns and Herzog 1993, 1.5.15]. The following result is well known and can be easily proved using the same reference.

Lemma 6.5. *Let $S = \bigoplus_{n\in\mathbb{Z}} S_n$ be a $*$-local Cohen–Macaulay ring with unique maximal homogeneous ideal $\mathfrak{P}$. Let $\mathfrak{a}$ be a homogeneous ideal in S. If $\mathfrak{a}_{\mathfrak{P}}$ is generated by a regular sequence then $\mathfrak{a}$ is generated by a regular sequence of homogeneous elements. Furthermore if $\mathscr{C} = \{x_\alpha \mid \alpha \in \Delta\}$ is a generating set of $\mathfrak{a}$ consisting of homogeneous elements then we may choose $\boldsymbol{x} = x_1, \dots, x_c \in \mathscr{C}$ with $\mathfrak{a} = (\boldsymbol{x})$ and $\boldsymbol{x}$ is an S-regular sequence.*

To prove Theorem 6.3 we need the following analogue of Lemma 6.4.

Lemma 6.6. *Suppose $A = Q/\mathfrak{a}$, where $Q = K[X_0, \dots, X_n]$ is graded with $\deg X_i = 1$ for all i and $\mathfrak{a}$ is a homogeneous ideal in Q. Suppose for some $\mathfrak{p} \in \operatorname{Proj} A$ the ring $A_{(\mathfrak{p})}$ is a complete intersection. Let $\mathfrak{q} \in \operatorname{Proj} Q$ with $\mathfrak{q}/\mathfrak{a} = \mathfrak{p}$. Then there exist homogeneous $g \in Q \setminus \mathfrak{q}$ such that $\mathfrak{a}Q_g$ is generated by a Q_g-regular sequence.*

Proof. Set

$$U = \{h \in A \mid h \text{ homogeneous and } h \notin \mathfrak{p}\},$$
$$W = \{h \in Q \mid h \text{ homogeneous and } h \notin \mathfrak{q}\}.$$

Then $U^{-1}A = W^{-1}Q/W^{-1}\mathfrak{a}$. Also note that some $X_i \notin \mathfrak{q}$. It follows that

$$U^{-1}A \cong A_{(\mathfrak{p})}[t, t^{-1}] \quad \text{and} \quad W^{-1}Q \cong Q_{(\mathfrak{q})}[t, t^{-1}].$$

Claim. *$U^{-1}A$ is a complete intersection.*

To see this, first observe that as $Q_{\mathfrak{q}}$ is a localization of $W^{-1}Q$ we have a flat map $Q_{(\mathfrak{q})} \to Q_{\mathfrak{q}}$ of local rings. As $Q_{\mathfrak{q}}$ is regular we have that $Q_{(\mathfrak{q})}$ is regular (see [Matsumura 1980, 23.7]). Notice that $A_{(\mathfrak{p})}$ is a quotient of a regular local ring $Q_{(\mathfrak{q})}$. So by [Bruns and Herzog 1993, 2.3.6], we have that $A_{(\mathfrak{p})}[t]$ is a complete intersection. As $U^{-1}A$ is a localization of $A_{(\mathfrak{p})}[t]$, it is also a complete intersection.

By Lemma 6.5 we have that $W^{-1}\mathfrak{a}$ is generated by a regular sequence $\boldsymbol{x} = x_1, \dots, x_c$ with $\boldsymbol{x} \in \mathfrak{a}$ homogeneous. Set

$$E = \frac{\mathfrak{a}}{(x_1, \dots, x_c)} \quad \text{and} \quad D_i = \frac{(x_1, \dots, x_{i-1}) : x_i}{(x_1, \dots, x_{i-1})} \quad \text{for } i = 1, \dots, c.$$

Set

$$L = E \oplus \left(\bigoplus_{i=1}^{c} D_i \right).$$

We have $W^{-1}L = 0$. Also, L is a finitely generated Q-module. So there exist $g \in W$ with $L_g = 0$. In Q_g note that $\mathfrak{a}Q_g = (x_1, \dots, x_c)Q_g$. Also, as $(x_1, \dots, x_c)W^{-1}Q \neq W^{-1}Q$ we have that $(x_1, \dots, x_c)Q_g \neq Q_g$. Since $(D_i)_g = 0$ for $i = 1, \dots, c$ we get that $x_1, \dots, x_c$ is a Q_g-regular sequence. □

The proof of Theorem 6.3 is similar to that of Theorem 6.1, so we just sketch it.

Sketch of proof of Theorem 6.3. We use Lemma 6.6 and an argument analogous to the one used Theorem 6.1 to obtain

$$\operatorname{Proj} A = {}^*D(g_1) \cup \cdots \cup {}^*D(g_r) \quad \text{for some } r \geq 1,$$

for some homogeneous $g_i \in A$ and $A_{g_i} = Q_i/\mathfrak{a}_i$, where Q_i is regular of finite Krull dimension and $\mathfrak{a}_i$ is generated by a regular sequence. Note that for x homogeneous, ${}^*D(x) = \{\mathfrak{P} \in \operatorname{Proj} A \mid x \notin \mathfrak{P}\}$.

Let E be a graded A-module. Note that

$${}^*\operatorname{Ass}_A E = \bigcup_{i=1}^{r} (\operatorname{Ass}_{A_{g_i}} E_{g_i}) \cap A.$$

The result now follows by applying Theorem 5.2 to each $A_{g_i} = Q_i/\mathfrak{a}_i$. □

7. Application III: Support varieties

Let $(A, \mathfrak{m})$ be a local complete intersection of codimension c. Let M, N be two finitely generated A-modules. Define

$$\operatorname{cx}_A(M, N) = \inf\left\{ b \in \mathbb{N} \;\middle|\; \varlimsup_{n\to\infty} \frac{\mu(\operatorname{Ext}^n_A(M, N))}{n^{b-1}} < \infty \right\}.$$

In this section we prove the following theorem:

Theorem 7.1. *Let $(A, \mathfrak{m})$ be a local complete intersection, M, N two finitely generated A-modules and let I be a proper ideal in A. Then*

$$\operatorname{cx}_A(M, I^j N) \quad \textit{is constant for all } j \gg 0.$$

7.2. ***Reduction to the case when A is complete and the residue field of A is algebraically closed.***

7.3. Suppose A' is a flat local extension of A such that $\mathfrak{m}' = \mathfrak{m}A'$ is the maximal ideal of A'. If E is an A-module then set $E' = E \otimes_A A'$. Notice that $I' \cong IA'$; we consider it as an ideal in A'. By [Avramov 1998, 7.4.3], A' is also a complete intersection. It can be easily checked that

$$\operatorname{cx}_{A'}(M', (I')^j N') = \operatorname{cx}_A(M, I^j N) \quad \text{for all } n \geq 0.$$

We now do our reduction in two steps.

By [Bourbaki 1983, Chapitre 9, appendice, corollaire du théoréme 1, p. IX.41], there exists a flat local extension $A \subseteq \widetilde{A}$ such that $\widetilde{\mathfrak{m}} = \mathfrak{m}\widetilde{A}$ is the maximal ideal of $\widetilde{A}$ and the residue field $\widetilde{k}$ of $\widetilde{A}$ is an algebraically closed extension of k. By

Section 7.3 it follows that we may assume k to be algebraically closed. We now complete A. Note that $\hat{A}$ is a flat extension of A which satisfies Section 7.3.

Thus we may assume that our local complete intersection A

(1) is complete. So $A = Q/(f_1, \dots, f_c)$, where $(Q, \mathfrak{n})$ is regular local and $f_1, \dots, f_c \in \mathfrak{n}^2$ is a regular sequence.

(2) has an algebraically closed residue field k.

Of course there exist many Q and $f_1, \dots, f_c$ of the type as indicated above. We simply fix one such representation of A.

7.4. Let U, V be two finitely generated A-modules.

Let $\operatorname{Ext}^*(U, V) = \bigoplus_{n\geq 0} \operatorname{Ext}^n_A(U, V)$ be the total ext module of U and V. We consider it as a (finitely generated) module over the ring of cohomological operators $A[t_1, \dots, t_c]$. Since $\operatorname{projdim}_Q U$ is finite $\operatorname{Ext}^*(U, V)$ is a finitely generated $A[t_1, \dots, t_c]$-module.

7.5. Let $\mathcal{C}(U, V) = \operatorname{Ext}^*(U, V) \otimes_A k$. Clearly $\mathcal{C}(U, V)$ is a finitely generated $T = k[t_1, \dots, t_c]$-module. (Here the degree of t_i is 2 for each $i = 1, \dots, c$). Set

$$\mathfrak{a}(U, V) = \operatorname{ann}_T \mathcal{C}(U, V).$$

Notice that $\mathfrak{a}(U, V)$ is a homogeneous ideal.

7.6. We now forget the grading of T and consider the affine space $\mathbb{A}^c(k)$. Let

$$\mathcal{V}(U, V) = \mathcal{V}(\mathfrak{a}(U, V)) \subseteq \mathbb{A}^c(k).$$

Since $\mathfrak{a}(U, V)$ is a graded ideal we get that $\mathcal{V}(U, V)$ is a cone.

7.7. By [Avramov and Buchweitz 2000, 2.4] we get that

$$\dim \mathcal{V}(U, V) = \operatorname{cx}_A(M, N).$$

Lemma 7.8. *If I is an ideal in A then there exists $j_0 \geq 0$ such that*

$$\mathcal{V}(U, I^j V) = \mathcal{V}(U, I^{j_0} V) \quad \textit{for all } j \geq j_0.$$

Proof of Theorem 7.1 assuming the lemma. By 7.3 we may assume that A is complete and has an algebraically closed residue field. The result now follows from 7.7 and Lemma 7.8. □

7.9. Let $N = \bigoplus_{n\geq 0} I^n V$. Set $\mathcal{E}(N) = \bigoplus_{n\geq 0} \bigoplus_{i\geq 0} \operatorname{Ext}^i_A(U, I^n V)$. Set $\mathcal{C}(N) = \mathcal{E}(N) \otimes_A k$. By Theorem 1.1, $\mathcal{E}(N)$ is a finitely generated $\mathcal{S} = \mathcal{R}(I)[t_1, \dots, t_c]$-module. It follows that $\mathcal{C}(N)$ is a finitely generated, bigraded, $G = F(I)[t_1, \dots, t_c]$-module. Recall that $F(I)$, the fiber cone of I, is a finitely generated k-algebra.

So we may as well consider $\mathcal{C}(N)$ as a bigraded $R = k[X_1, \dots, X_m, t_1, \dots t_c]$-module (of course here $X_1, \dots, X_m$ are variables). Furthermore $\deg X_l = (1, 0)$ for $l = 1, \dots m$ and $\deg t_s = (0, 2)$ for $s = 1, \dots, c$. Set $T = k[t_1, \dots, t_c]$.

7.10. ***Advantages of coarsening the grading on $\mathcal{C}(N)$.*** By forgetting the degree on the t_i we may consider $R = T[X_1, \dots, X_m]$. Notice that

$$\mathcal{C}(N) = \bigoplus_{n \geq 0} \mathcal{C}(U, I^n V)$$

as a graded R-module.

Proof of Lemma 7.8. We make the constructions as in Section 7.10. So $\mathcal{C}(N)$ is a finitely generated graded $R = T[X_1, \dots, X_m]$-module. Notice that R is $\mathbb{N}$-standard graded. So there exists j_0 such that

$$\operatorname{ann}_T \mathcal{C}(N)_j = \operatorname{ann}_T \mathcal{C}(N)_{j_0} \quad \text{for all } j \geq 0.$$

The results follows. □

Question 7.11 (With hypotheses as in Theorem 7.1).

Is $\operatorname{cx}_A(M, N/I^j N)$ *constant for all* $j \gg 0$?

Acknowledgements

I thank Professor L. L. Avramov and Professor J. Herzog for many discussions regarding this paper. I also thank the referee for many pertinent comments.

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Received September 18, 2008. Revised April 11, 2013.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 266 No. 2 December 2013

0030-8730(201312)266:2;1-3