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THE SHORT TIME ASYMPTOTICS OF NASH ENTROPY

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Let (M^n, g) be a complete Riemannian manifold with $\text{Rc} \geq -Kg$, $H(x, y, t)$ be the heat kernel on M^n , and $H = (4\pi t)^{-n/2}e^{-f}$. Nash entropy is defined as $N(H, t) = \int_{M^n} (fH) d\mu(x) - n/2$. We study the asymptotic behavior of $N(H, t)$ and $\partial N(H, t)/\partial t$ as $t \rightarrow 0^+$ and get the asymptotic formulas at $t = 0$. In the appendix, we get a Hamilton-type upper bound for the Laplacian of the positive solution to the heat equation on such manifolds, which is itself interesting.

1. Introduction

On a complete manifold (M^n, g) with $\text{Rc} \geq -Kg$, where $K > 0$ is a constant, for fixed $y \in M^n$, it is well known that the heat kernel $H(x, y, t)$ on (M^n, g) is unique. We assume $H = (4\pi t)^{-n/2}e^{-f}$. As in [Ni 2004b], Nash entropy is defined as follows.

Definition 1.1.
$$N(H, t) = \int_{M^n} (fH) d\mu(x) - \frac{n}{2}.$$

Nash entropy is closely related to ${}^\circ W$ -entropy for the linear heat equation, and the large time asymptotics of this entropy reflects the volume growth rate of the manifold; see [Ni 2004a; 2004b; 2010].

In this paper, we study the asymptotic behavior of $N(H, t)$ and $\partial N(H, t)/\partial t$ as $t \rightarrow 0^+$, and solve Problem 23.36 of [Chow et al. 2010]. More precisely, we prove:

Theorem 1.2. *Let (M^n, g) be a complete Riemannian manifold with $\text{Rc} \geq -Kg$, where $K > 0$ is a constant. Then*

$$(1-1) \quad N(H, t) = -\frac{1}{2}R(y) \cdot t + O(t^{3/2})$$

and

$$(1-2) \quad \frac{\partial}{\partial t}[N(H, t)] = -\frac{1}{2}R(y) + o(1),$$

where $\limsup_{t \rightarrow 0} O(t^{3/2})t^{-3/2}$ is bounded, $\lim_{t \rightarrow 0} o(1) = 0$, and t is small enough.

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One motivation to study the short time asymptotics of Nash entropy is Li–Yau–Perelman type estimates for the heat equation on manifolds with Ricci curvature bounded from below. Motivated by Perelman’s differential Harnack estimate for Ricci flow on a closed manifold (M^n, g) with $Rc \geq 0$, Ni [2004a] proved the following Li–Yau–Perelman type estimate for the heat equation when $t > 0$:

$$(1-3) \quad 2\Delta f(x, y, t) - |\nabla f(x, y, t)|^2 + \frac{f(x, y, t) - n}{t} \leq 0,$$

where $H(x, y, t) = (4\pi t)^{-n/2} e^{-f}$ is the heat kernel. In fact, (1-3) is also true for the heat kernel on a complete manifold (M^n, g) with $Rc \geq 0$; see [Chow et al. 2008].

Perelman made the following claim.

Claim 1.3 [Perelman 2002, Remark 9.6]. *If (M^n, g) is a compact Riemannian manifold, $g_{ij}(x, t)$ evolves according to $(g_{ij})_t = A_{ij}(t)$ and $g_{ij}(x, 0) = g_{ij}(x)$, $t \in (-T, 0]$. Define $\square = \partial/(\partial t) - \Delta$ and its conjugate $\square^* = -\partial/(\partial t) - \Delta - \frac{1}{2}A$ (where $A = g^{ij} A_{ij}$). Consider the fundamental solution $u = (-4\pi t)^{-n/2} e^{-f}$ for \square^* , starting as a δ -function at some point $(p, 0)$. Then, for general A_{ij} , the function $(\square \bar{f} + \bar{f}/t)(q, t)$, where $\bar{f} = f - \int_{M^n} f u$, is of order $O(1)$ for (q, t) near $(p, 0)$.*

We focus on the special case where the evolving metrics are the static metric. From Theorem 1.2, it is easy to show that Perelman’s claim in the static metric case is equivalent to the following claim on compact manifolds:

$$(1-4) \quad 2\Delta f(x, y, t) - |\nabla f(x, y, t)|^2 + \frac{f(x, y, t) - n}{t} = -R(y) + O(t + d^2(x, y)).$$

If (1-4) is true, it is an improvement of (1-3) when $t + d^2(x, y)$ is small enough and $R(y) > 0$. But, using the explicit formula (cf. [Grigor’yan 2009, Section 9.2])

$$H = (4\pi t)^{-3/2} \frac{d}{\sinh d} \exp\left(-\frac{d^2}{4t} - t\right)$$

for the heat kernel on a hyperbolic manifold \mathbb{H}^3 , it is easy to check that (1-4) is not true generally. Hence Claim 1.3 is not generally true for the static metric case on complete manifolds.

As observed in [Ni 2004b], the integrand of $\partial N(H, t)/\partial t$ is simply the expression in Li and Yau’s gradient estimate for the heat kernel multiplied with the heat kernel, which is $-(\Delta \ln H + n/(2t))H$. Because so far there is no sharp Li–Yau-type gradient estimate for the heat kernel or solutions to the heat equation on complete manifolds with Ricci curvature bounded from below by a negative constant, we hope that (1-2) will be helpful in better understanding this estimate.

On the other hand, in the case where (M^n, g) is a compact Riemannian manifold, the short time behavior of the logarithm of the heat kernel has been studied by many

probabilists. Although the heat kernel $H(x, y, t)$ has an infinite sequence expansion at $t = 0$, generally there is no such expansion of $\ln H$ at $t = 0$, and the singularity of $\ln H$ at $t = 0$ can have many complicated situations. However, Varadhan [1967] proved

$$(1-5) \quad \lim_{t \rightarrow 0} t \ln H(x, y, t) = -\frac{d^2(x, y)}{4}.$$

Moreover, using stochastic processes methods, Malliavin and Stroock proved [1996] that the above equation is preserved while taking the first and second spatial derivatives on a domain outside of the cut locus. Using analytic methods, (1-5) was proved for complete Riemannian manifolds by Cheng, Li, and Yau [Cheng et al. 1981]. We hope that Theorem 1.2 will be useful in studying the short time behavior of the logarithm of the heat kernel on complete manifolds by analytic methods.

The strategy to prove (1-1) is to use the infinite sequence expansion $H_N(x, y, t)$ of $H(x, y, t)$ at $t = 0$, although generally $\ln H_N$ does not converge to $\ln H$ near $t = 0$ uniformly. In the integral sense of Definition 1.1, we show there is a uniform convergence in Lemma 3.1 by using an improved estimate of $H - H_N$ obtained in Theorem 2.2. The rest of the calculation of the integral of H_N is standard, but, for completeness, we give the details.

To prove (1-2), because the manifold M^n can be noncompact, we need to be more careful when switching the order of differentiation and integration. A detailed proof of the validity of the switch is given in the beginning of Section 4. We need an upper bound of H_t/H to verify the above switch. This type of bound is known for closed manifolds [Hamilton 1993], and in [Chow et al. 2008] (see also [Ni 2006]) the proof is sketched for complete manifolds with $Rc \geq 0$ using a strategy similar to that in [Kotschwar 2007]. A detailed proof of this Hamilton-type upper bound for complete manifolds with $Rc \geq -Kg$ is included in the appendix for completeness.

The paper is organized as follows. In Section 2, we state some preliminary results about the heat kernel and get some improved estimates of $H - H_N$. In Section 3, we prove (1-1). In Section 4, using (1-1) and results in the appendix, we prove (1-2). In the appendix, we prove Hamilton-type upper bound of H_t/H on complete manifolds with Ricci curvature bounded from below.

2. Preliminaries

We first define some notations and functions. In the rest of the paper, we fix $y \in M^n$ and define

$$\Omega_y = \{x \in M^n : d(x, y) < \text{inj}_g(y)\},$$

where $\text{inj}_g(y)$ denotes the injectivity radius of the metric g at y . Define

$$B(\rho) = \{x : d(x, y) \leq \rho\} \quad \text{and} \quad B_z(\rho) = \{x : d(x, z) \leq \rho\}.$$

Hence $B(\rho) = B_y(\rho)$. $V(B_z(\rho))$ is used to denote the volume of $B_z(\rho)$ and $V_{-K}(\rho)$ is the volume of the geodesic ball of radius ρ in the constant $(-K/(n-1))$ sectional curvature space form.

Fix $r \in (0, \frac{1}{4} \text{inj}_g(y))$ and let $N_0 = n/2 + 3$. Define

$$E = (4\pi t)^{-n/2} \exp\left(-\frac{d^2(x, y)}{4t}\right) \quad \text{and} \quad \tilde{E} = (4\pi t)^{-n/2} \exp\left(-\frac{d^2(x, y)}{5t}\right).$$

When the meaning is clear from context, we sometimes simplify notation by denoting $B(r/2)$ by B and $d(x, y)$ by d .

Assume $\eta : [0, \infty) \rightarrow [0, 1]$ is a C^∞ cut-off function with

$$(2-1) \quad \eta(s) = \begin{cases} 1 & \text{if } s \leq r, \\ 0 & \text{if } s \geq 2r. \end{cases}$$

The following theorem collects several known results about the heat kernel on complete manifolds; see, for example, [Chow et al. 2010; Garofalo and Lanconelli 1989; Li 2012].

Theorem 2.1. *(M^n, g) is a complete Riemannian manifold with $\text{Rc} \geq -Kg$, where $K > 0$ is a constant. Then there exists a unique positive fundamental solution $H(x, y, t)$ to the heat equation, which is called the heat kernel. Moreover,*

$$H(x, y, t) \in C^\infty(M^n \times M^n \times (0, \infty))$$

is symmetric in x and y , and

(i)

$$(2-2) \quad \int_{M^n} H(x, y, t) d\mu(x) \equiv 1;$$

(ii)

$$(2-3) \quad H(x, y, t) = P_{N_0}(x, y, t) + F_{N_0}(x, y, t)$$

$$(2-4) \quad P_{N_0}(x, y, t) = \eta(d(x, y))H_{N_0}(x, y, t),$$

$$(2-5) \quad H_{N_0}(x, y, t) = (4\pi t)^{-n/2} \exp\left(-\frac{d^2(x, y)}{4t}\right) \cdot \sum_{k=0}^{N_0} \varphi_k(x, y)t^k,$$

where $\varphi_k(x, y) \in C^\infty(\Omega_y)$ and $k = 0, 1, \dots, N_0$. Also, H_{N_0} satisfies

$$(2-6) \quad \left(\Delta - \frac{\partial}{\partial t}\right)H_{N_0}(x, y, t) = E \Delta \varphi_{N_0} t^{N_0};$$

(iii) let $\{x^k\}_{k=1}^n$ be exponential normal coordinates centered at $y \in M^n$. Then φ_0 and φ_1 have the asymptotic expansion

$$(2-7) \quad \varphi_0(x, y) = 1 + \frac{1}{12} R_{pq}(y)x^p x^q + O(d^3(x, y)),$$

$$(2-8) \quad \varphi_1(x, y) = \frac{R(y)}{6} + O(d(x, y)).$$

We prove an estimate for F_{N_0} . This estimate is an improvement of the usual estimate of F_{N_0} , which only gives $t^{N_0+1-n/2}$ bound. The improved estimate (2-9) is the key to the proof of Lemma 3.1.

Theorem 2.2. For $F_{N_0}(x, y, t)$ in Theorem 2.1, we have the following estimates:

$$(2-9) \quad |F_{N_0}(x, y, t)| \leq Ct^4 \exp\left(-\frac{d^2(x, y)}{5t}\right),$$

$$(2-10) \quad \left| \frac{\partial}{\partial t} F_{N_0}(x, y, t) \right| \leq Ct^2 \exp\left(-\frac{d^2(x, y)}{5t}\right),$$

where t is small enough and C is a positive constant independent of x, t .

Remark 2.3. (2-9) was proved in [Garofalo and Lanconelli 1989] for uniformly parabolic operators. Our proof of (2-9) and (2-10) is motivated by an argument in [Li 2012] and is different from the proof in [Garofalo and Lanconelli 1989].

Proof. (Δ). We first prove (2-9). From the definition of $P_{N_0}(x, y, t)$, it is easy to see that $\lim_{t \rightarrow 0} P_{N_0}(x, y, t) = \delta_y(x)$. In particular,

$$(2-11) \quad \begin{aligned} F_{N_0}(x, y, t) &= H(x, y, t) - P_{N_0}(x, y, t) \\ &= - \int_0^t \frac{\partial}{\partial s} \int_{M^n} H(x, z, t-s) P_{N_0}(z, y, s) d\mu(z) ds \\ &= - \int_0^t \int_{M^n} \left(\frac{\partial}{\partial s} - \Delta_z \right) P_{N_0}(z, y, s) \cdot H(x, z, t-s) d\mu(z) ds, \end{aligned}$$

where Δ_z is the Laplacian with respect to the variable z .

From (2-6) and the definition of η , when $z \in B(r)$,

$$(2-12) \quad \left| \left(\frac{\partial}{\partial s} - \Delta_z \right) P_{N_0}(z, y, s) \right| \leq C_1 s^3 \exp\left(-\frac{d^2(z, y)}{4s}\right),$$

and when $z \in B(2r) \setminus B(r)$,

$$(2-13) \quad \left| \left(\frac{\partial}{\partial s} - \Delta_z \right) P_{N_0}(z, y, s) \right| \leq C_2 s^{-n/2-1} \exp\left(-\frac{d^2(z, y)}{4s}\right).$$

Hence

$$\begin{aligned}
 (2-14) \quad & |F_{N_0}(x, y, t)| \\
 & \leq C_1 \int_0^t s^3 \int_{B(r)} H(x, z, t-s) \exp\left(-\frac{d^2(z, y)}{4s}\right) d\mu(z) ds \\
 & \quad + C_2 \int_0^t s^{-n/2-1} \int_{B(2r) \setminus B(r)} H(x, z, t-s) \exp\left(-\frac{d^2(z, y)}{4s}\right) d\mu(z) ds \\
 & \leq (a) + (b)
 \end{aligned}$$

We can find $0 < t_1 \leq 1$ and $k_0 > 0$ such that if $s \in (0, t_1)$,

$$V(B_p(\sqrt{s})) \geq k_0 s^{n/2} \quad \text{for any } p \in B_y(3r).$$

In the rest of the proof, assume $t \in (0, t_1]$. We have two cases.

Case I. If $x \in B_y(3r)$ and $z \in B_y(2r)$, then, from [Li and Yau 1986] and the above volume lower bound,

$$\begin{aligned}
 (2-15) \quad & H(x, z, t-s) \\
 & \leq C V^{-1/2}(B_x(\sqrt{t-s})) V^{-1/2}(B_z(\sqrt{t-s})) \cdot \exp\left[CK(t-s) - \frac{6d^2(z, x)}{25(t-s)}\right] \\
 & \leq C(K, k_0, n)(t-s)^{-n/2} \exp\left(-\frac{6d^2(z, x)}{25(t-s)}\right)
 \end{aligned}$$

Case II. If $x \notin B_y(3r)$ and $z \in B_y(2r)$, using (2-15), $d(x, z) \geq r$, and the volume comparison theorem,

$$\begin{aligned}
 (2-16) \quad & H(x, z, t-s) \leq C V^{-1}(B_z(\sqrt{t-s})) \cdot \left[\frac{V_{-K}(\sqrt{t-s} + d(x, z))}{V_{-K}(\sqrt{t-s})}\right]^{1/2} \\
 & \quad \cdot \exp\left[CK(t-s) - \frac{6d^2(z, x)}{25(t-s)}\right] \\
 & \leq C(K, k_0, n, r) \exp\left(-\frac{23d^2(z, x)}{100(t-s)}\right)
 \end{aligned}$$

Note that, in Case I, $\text{inj}_g(x)$ has a uniform lower bound. Hence it is easy to get

$$(2-17) \quad \int_{B_y(r)} s^{-n/2} \exp\left(-\frac{d^2(z, x)}{100s}\right) d\mu(z) \leq C$$

for any $s \in (0, t_1]$.

Now, using (2-15), (2-16), (2-17) and the classical inequality

$$\frac{d^2(x, z)}{t-s} + \frac{d^2(y, z)}{s} \geq \frac{d^2(x, y)}{t},$$

we can get

$$\int_{B_y(r)} H(x, z, t - s) \exp\left(-\frac{d^2(z, y)}{4s}\right) d\mu(z) \leq C \exp\left(-\frac{23d^2(x, y)}{100t}\right).$$

Hence

$$(2-18) \quad (a) \leq Ct^4 \exp\left(-\frac{23d^2(x, y)}{100t}\right).$$

Similarly,

$$\int_{B_y(2r) \setminus B_y(r)} H(x, z, t - s) \exp\left(-\frac{d^2(z, y)}{4s}\right) d\mu(z) \leq C \exp\left(-\frac{3r^2}{100s}\right) \exp\left(-\frac{d^2(x, y)}{5t}\right).$$

Hence

$$(2-19) \quad (b) \leq C_2 \left[\int_0^t s^{-\frac{n}{2}-1} \exp\left(-\frac{3r^2}{100s}\right) ds \right] \exp\left(-\frac{d^2(x, y)}{5t}\right) \leq Ct^4 \exp\left(-\frac{d^2(x, y)}{5t}\right).$$

By (2-18) and (2-19), (2-9) is proved.

(Θ). The strategy to prove (2-10) is similar.

$$\begin{aligned} \frac{\partial}{\partial t} F_{N_0}(x, y, t) &= \frac{\partial}{\partial t} \left[- \int_0^t \int_{M^n} \left(\frac{\partial}{\partial s} - \Delta_z \right) P_{N_0}(z, y, s) \cdot H(x, z, t - s) d\mu(z) ds \right] \\ &= - \int_0^t \int_{M^n} \left(\frac{\partial}{\partial s} - \Delta_z \right) P_{N_0}(z, y, s) \cdot \left(\frac{\partial}{\partial t} H(x, z, t - s) \right) d\mu(z) ds \\ &\quad + \left(\Delta_x - \frac{\partial}{\partial t} \right) P_{N_0}(x, y, t) \\ &= (\gamma) + (\tau) \end{aligned}$$

From (2-12), (2-13), and $P_{N_0}(x, y, t) = 0$, when $x \notin B(2r)$,

$$(2-20) \quad (\tau) \leq Ct^4 \exp\left(-\frac{d^2(x, y)}{5t}\right).$$

Now we estimate (γ) .

$$\begin{aligned} (\gamma) &= - \int_0^t \int_{M^n} \left(\frac{\partial}{\partial s} - \Delta_z \right) P_{N_0}(z, y, s) \cdot (\Delta_z H(x, z, t - s)) d\mu(z) ds \\ &= - \int_0^t \int_{M^n} \left[\Delta_z \left(\frac{\partial}{\partial s} - \Delta_z \right) P_{N_0}(z, y, s) \right] \cdot H(x, z, t - s) d\mu(z) ds. \end{aligned}$$

Similarly as with (2-12) and (2-13), from (2-6), when $z \in B(r)$,

$$(2-21) \quad \left| \Delta_z \left(\frac{\partial}{\partial s} - \Delta_z \right) P_{N_0}(z, y, s) \right| \leq C_3 s \exp \left(-\frac{d^2(z, y)}{4s} \right),$$

and when $z \in B(2r) \setminus B(r)$,

$$(2-22) \quad \left| \Delta_z \left(\frac{\partial}{\partial s} - \Delta_z \right) P_{N_0}(z, y, s) \right| \leq C_4 s^{-n/2-3} \exp \left(-\frac{d^2(z, y)}{4s} \right).$$

Following a similar argument as in the proof of (2-9), using (2-21) and (2-22) instead of (2-12) and (2-13),

$$(2-23) \quad (\gamma) \leq C t^2 \exp \left(-\frac{d^2(x, y)}{5t} \right).$$

From (2-20) and (2-23),

$$\left| \frac{\partial}{\partial t} F_{N_0}(x, y, t) \right| \leq (\gamma) + (\tau) \leq C t^2 \exp \left(-\frac{d^2(x, y)}{5t} \right). \quad \square$$

3. The short time asymptotics of $N(H, t)$

From (2-5) and (2-7) in Theorem 2.1, there exists $0 < t_0 \leq 1$ such that

$$(3-1) \quad \frac{1}{2} \leq (4\pi t)^{n/2} \exp \left(\frac{d^2(x, y)}{4t} \right) H_{N_0}(x, y, t) \leq 2$$

holds when $x \in B(r/2)$ and $0 < t \leq t_0$. In Sections 3 and 4, we assume that $t \in (0, t_0]$ and (M^n, g) and H are from Theorem 2.1.

Lemma 3.1.

$$(3-2) \quad \int_{B(r/2)} \left[\ln \frac{H(x, y, t)}{H_{N_0}(x, y, t)} \right] \cdot H(x, y, t) d\mu(x) = O(t^2).$$

Proof. Assume $x \in B(r/2)$, $t \leq t_0$. Then $P_{N_0}(x, y, t) = H_{N_0}(x, y, t)$. Hence

$$F_{N_0}(x, y, t) = H(x, y, t) - H_{N_0}(x, y, t).$$

From (2-9),

$$(3-3) \quad |F_{N_0}(x, y, t)| \leq C t^{N_0+1-n/2} \exp \left(-\frac{d^2(x, y)}{5t} \right).$$

If $F_{N_0}(x, y, t) > 0$,

$$\begin{aligned} \left| \ln \frac{H}{H_{N_0}} \cdot H \right|(x, y, t) &= \ln \left(1 + \frac{F_{N_0}}{H_{N_0}} \right) \cdot H \leq \frac{F_{N_0}}{H_{N_0}} \cdot H \\ &\leq C t^{N_0+1} \exp \left(\frac{d^2(x, y)}{20t} \right) \cdot H(x, y, t). \end{aligned}$$

If $F_{N_0}(x, y, t) \leq 0$, $H(x, y, t) \leq H_{N_0}(x, y, t)$ and

$$\begin{aligned} \left| \ln \frac{H}{H_{N_0}} \cdot H \right| (x, y, t) &= |\ln H(x, y, t) - \ln H_{N_0}(x, y, t)| \cdot H(x, y, t) \\ &= \left| \frac{1}{\xi} [H(x, y, t) - H_{N_0}(x, y, t)] \right| \cdot H(x, y, t), \end{aligned}$$

where $H(x, y, t) \leq \xi \leq H_{N_0}(x, y, t)$. Hence

$$\left| \ln \frac{H}{H_{N_0}} \cdot H \right| (x, y, t) \leq \left| \frac{F_{N_0}}{H} \right| \cdot H = F_{N_0} \leq C t^{N_0+1-n/2} \exp\left(-\frac{d^2(x, y)}{5t}\right).$$

By the above,

$$(3-4) \quad \left| \ln \frac{H}{H_{N_0}} \cdot H \right| (x, y, t) \leq C t^4 \left[t^{n/2} \exp\left(\frac{d^2(x, y)}{20t}\right) \cdot H + \exp\left(-\frac{d^2(x, y)}{5t}\right) \right].$$

From (3-1) and (3-3),

$$(3-5) \quad \begin{aligned} H(x, y, t) &\leq |H_{N_0}| + |F_{N_0}| \\ &\leq 2(4\pi t)^{-n/2} \exp\left(-\frac{d^2(x, y)}{4t}\right) + C t^4 \cdot \exp\left(-\frac{d^2(x, y)}{5t}\right). \end{aligned}$$

By (3-4) and (3-5),

$$(3-6) \quad \left| \ln \frac{H}{H_{N_0}} \cdot H \right| (x, y, t) \leq C t^4.$$

Hence

$$\int_{B(r/2)} \left[\ln \frac{H(x, y, t)}{H_{N_0}(x, y, t)} \right] \cdot H(x, y, t) d\mu(x) = O(t^2). \quad \square$$

Proof of (1-1).

$$-\int_{M^n} f H d\mu = \int_{M^n \setminus B(r/2)} (-f H) d\mu + \int_{B(r/2)} (-f H) d\mu = \text{(I)} + \text{(II)}.$$

First we estimate (I). From [Li and Yau 1986], we have

$$H(x, y, t) \leq C V^{-1/2}(B_x(\sqrt{t})) V^{-1/2}(B_y(\sqrt{t})) \cdot \exp\left[CKt - \frac{d^2(x, y)}{5t}\right].$$

If $x \in M^n \setminus B(r/2)$ and t is small enough, using the volume comparison theorem,

$$(3-7) \quad \begin{aligned} H(x, y, t) &\leq C V^{-1}(B_y(\sqrt{t})) \cdot \frac{V_{-K}(\sqrt{t} + d)}{V_{-K}(\sqrt{t})} \cdot \exp\left[CKt - \frac{d}{5t}\right] \\ &\leq C t^2 \exp\left(-\frac{d^2}{6t}\right), \end{aligned}$$

where C depends on n, K, r , and the metric g near y . Choose t small enough such that $H \leq Ct^2 \leq e^{-1}$. Then, by the monotonicity of $h(x) = \ln x \cdot x$ on $(0, e^{-1}]$,

$$|\ln H(x, y, t) \cdot H(x, y, t)| \leq \left| \ln \left[Ct^2 \exp \left(-\frac{d^2}{6t} \right) \right] \cdot \left[Ct^2 \exp \left(-\frac{d^2}{6t} \right) \right] \right|.$$

Hence

$$\begin{aligned} (3-8) \quad |(\text{I})| &= \left| \int_{M^n \setminus B(r/2)} \left[\ln H + \frac{n}{2} \ln(4\pi t) \right] \cdot H \, d\mu(x) \right| \\ &\leq \int_{M^n \setminus B(r/2)} \left| \ln \left[Ct^2 \exp \left(-\frac{d^2}{6t} \right) \right] \cdot \left[Ct^2 \exp \left(-\frac{d^2}{6t} \right) \right] \right| d\mu(x) \\ &\quad + \frac{n}{2} \int_{M^n \setminus B(r/2)} \left| \ln(4\pi t) \cdot \left[Ct^2 \exp \left(-\frac{d^2}{6t} \right) \right] \right| d\mu(x) \\ &\leq O(t^{3/2}). \end{aligned}$$

In the last inequality, we used $Rc \geq -Kg$ and the volume comparison theorem.

$$\begin{aligned} |(\text{II})| &= \int_{B(r/2)} \left[\ln H + \frac{n}{2} \ln(4\pi t) \right] \cdot H \, d\mu \\ &= \int_{B(r/2)} \ln \frac{H}{H_{N_0}} \cdot H \, d\mu(x) + \int_{B(r/2)} \left[\ln H_{N_0} + \frac{n}{2} \ln(4\pi t) \right] \cdot H \, d\mu(x) \\ &= (\text{III}) + (\text{IV}). \end{aligned}$$

By Lemma 3.1, $(\text{III}) = O(t^2)$. From Lemma 3.2, which follows,

$$(\text{IV}) = -\frac{n}{2} + \frac{1}{2}R(y) \cdot t + O(t^{3/2}). \quad \square$$

Lemma 3.2.

$$(3-9) \quad \int_{B(r/2)} \left[\ln H_{N_0} + \frac{n}{2} \ln(4\pi t) \right] \cdot H \, d\mu(x) = -\frac{n}{2} + \frac{1}{2}R(y) \cdot t + O(t^{3/2}).$$

Proof. Set $(\text{I}) := \int_{B(r/2)} [\ln H_{N_0} + (n/2) \ln(4\pi t)] \cdot H \, d\mu(x)$. From Theorem 2.1,

$$\ln H_{N_0} = -\frac{n}{2} \ln(4\pi t) - \frac{d^2(x, y)}{4t} + \ln \left(\sum_{k=0}^{N_0} \varphi_k t^k \right)$$

and

$$\ln \left(\sum_{k=0}^{N_0} \varphi_k t^k \right) = \ln \varphi_0 + \frac{\varphi_1}{\varphi_0} \cdot t + O(t^2).$$

Hence

$$(\text{I}) = \int_{B(r/2)} \left[-\frac{d^2(x, y)}{4t} + \ln \varphi_0 + \frac{\varphi_1}{\varphi_0} \cdot t + O(t^2) \right] \cdot H \, d\mu(x).$$

Now using Theorem 2.1(iii),

$$(3-10) \quad \begin{aligned} \text{(I)} &= \int_{B(r/2)} \left[-\frac{d^2}{4t} + \frac{1}{12} R_{pq}(y) x^p x^q + O(d^3) + \left(\frac{R(y)}{6} + O(d) \right) t \right] \\ &\quad \cdot H d\mu(x) + O(t^2) \\ &= \text{(II)} + \text{(III)} + \text{(IV)} + \text{(V)} + \text{(VI)} + O(t^2), \end{aligned}$$

where

$$\begin{aligned} \text{(II)} &= \int_{B(r/2)} \left(-\frac{d^2(x, y)}{4t} \right) \cdot H d\mu(x), \\ \text{(III)} &= \frac{1}{12} \int_{B(r/2)} \left(R_{pq}(y) x^p x^q \right) \cdot H d\mu(x), \\ \text{(IV)} &= C \int_{B(r/2)} d^3(x, y) \cdot H(x, y, t) d\mu(x), \\ \text{(V)} &= \frac{R(y)}{6} t \cdot \int_{B(r/2)} H(x, y, t) d\mu(x), \\ \text{(VI)} &= Ct \cdot \int_{B(r/2)} d(x, y) \cdot H(x, y, t) d\mu(x). \end{aligned}$$

From (3-7),

$$\int_{B(r/2)} H = \int_{M^n} H - \int_{M^n \setminus B(r/2)} H = 1 + O(t^2).$$

Hence

$$\text{(V)} = \frac{1}{6} R(y) \cdot t + O(t^2).$$

Using (3-5) and the fact that

$$\int_{R^n} O(|x|^k) (4\pi t)^{-n/2} \exp\left(-\frac{|x|^2}{4t}\right) dx = O(t^{k/2}),$$

where k is any nonnegative integer, we can get $\text{(IV)} = O(t^{3/2})$ and $\text{(VI)} = O(t^{3/2})$. Similarly,

$$\text{(III)} = \frac{1}{6} R(y) \cdot t + O(t^2)$$

Finally, from Lemma 3.3, which follows,

$$\text{(II)} = -\frac{n}{2} + \frac{1}{6} R(y) \cdot t + O(t^{3/2}). \quad \square$$

Lemma 3.3.

$$(3-11) \quad -\frac{1}{4t} \int_{B(r/2)} d^2(x, y) \cdot H d\mu(x) = -\frac{n}{2} + \frac{1}{6} R(y) \cdot t + O(t^{3/2}).$$

Proof. (II) := $-(1/(4t)) \int_{B(r/2)} d^2(x, y) \cdot H d\mu(x)$. Then

$$(3-12) \quad (\text{II}) = -\frac{1}{4t} \int_{B(r/2)} d^2(x, y) \cdot (H_{N_0} + F_{N_0}) \cdot \alpha dx,$$

where dx in the integral of (3-12) is the volume element of Euclidean space \mathbb{R}^n , and

$$\alpha = \sqrt{\det(g)} = 1 - \frac{1}{6} R_{pq}(y) x^p x^q + O(d^3(x, y)).$$

Then

$$\begin{aligned} (\text{II}) &= -\frac{1}{4t} \int_{B(r/2)} d^2(x, y) \cdot (4\pi t)^{-n/2} \exp\left(-\frac{d^2(x, y)}{4t}\right) (\varphi_0 + \varphi_1 t) \cdot \alpha dx + O(t^2) \\ &= \left[-\frac{1}{4t} - \frac{1}{24} R(y)\right] \cdot \int_{B(r/2)} d^2 \cdot (4\pi t)^{-n/2} \exp\left(-\frac{d^2}{4t}\right) dx \\ &\quad + \frac{1}{48t} \int_{B(r/2)} (4\pi t)^{-n/2} (R_{pq}(y) x^p x^q) d^2 \cdot \exp\left(-\frac{d^2}{4t}\right) dx + O(t^{3/2}) \\ &= \left[-1/4t - 1/24R(y)\right] \cdot 2nt + \frac{1}{48t} I_n + O(t^{3/2}), \end{aligned}$$

where

$$I_n = \int_{\mathbb{R}^n} (4\pi t)^{-n/2} \left(\sum_{k=1}^n \lambda_k x_k^2\right) \cdot \left(\sum_{i=1}^n x_i^2\right) \exp\left(-\frac{1}{4t} \cdot \sum_{j=1}^n x_j^2\right) dx.$$

In the above we diagonalize $R_{pq}(y)$ and let $\lambda_k = R_{kk}(y)$.

We can get $I_1 = 12\lambda_1 t^2$ and the induction formula

$$I_n = I_{n-1} + 4\left(\sum_{i=1}^n \lambda_i\right) t^2 + 4(n+1)\lambda_n t^2.$$

Then it is easy to get

$$(3-13) \quad I_n = 4(n+2)\left(\sum_{i=1}^n \lambda_i\right) t^2 = 4(n+2)R(y)t^2.$$

By all the above (II) = $-n/2 + (R(y)/6)t + O(t^{3/2})$. □

4. The short time asymptotics of $\partial N(H, t)/\partial t$

To study $\partial N(H, t)/\partial t$, we must first switch the order of differentiation with integration. Because the manifold M^n can be noncompact, we need to be more careful when doing this. The following lemma justifies this switch in our case.

Lemma 4.1.

$$(4-1) \quad \frac{\partial}{\partial t} \left[\int_{M^n} H(-f) d\mu(x) \right] = \int_{M^n} [H(-f)]_t d\mu(x).$$

Proof. Define $\varphi_\rho(x) = \phi(d(x, y)/\rho)$, where ϕ is defined in the appendix, $\rho > 1$ is a constant. Fix $t > 0$ and define $G(x, t) = [H(-f)](x, y, t)$. For any $\epsilon > 0$, assume $1 > l > 0$ (if $l < 0$, a similar argument works). Then

$$\begin{aligned} & \left| \int_{M^n} \frac{G(x, t+l) - G(x, t)}{l} d\mu(x) - \int_{M^n} G_t \varphi_\rho d\mu(x) \right| \\ & \leq \int_{B(\rho)} |G_t(x, t + \xi_x l) - G_t(x, t)| d\mu(x) + 2 \int_{M^n \setminus B(\rho)} \sup_{s \in [t, t+l]} |G_t(x, s)| d\mu(x) \\ & \leq \int_{B(\rho)} \left| \frac{\partial^2}{\partial^2 t} G(x, t + \xi_x l) \right| d\mu(x) \cdot l + 2 \int_{M^n \setminus B(\rho)} \left(\sup_{s \in [t, t+l]} |G_t(x, s)| \right) d\mu(x) \\ & \leq \text{(I)} + \text{(II)}. \end{aligned}$$

We first estimate (II). From [Li and Yau 1986], for $s \in [t, t + l]$,

$$(4-2) \quad \frac{H_t}{H}(x, y, s) \geq \frac{1}{2} \left[\frac{|\nabla H|^2}{H^2} - \frac{2n}{s} - CK \right] \geq -\frac{C}{s},$$

where $C = C(K, n)$. From Corollary A.10,

$$(4-3) \quad \begin{aligned} & \frac{H_t}{H}(x, s) \\ & \leq \frac{2}{s} \left\{ n + (4 + Ks) \ln \frac{C(K, t+1)}{H(x, y, s) V^{1/2}(B_x(\sqrt{s/2})) V^{1/2}(B_y(\sqrt{s/2}))} \right\} \\ & \leq \frac{C}{s} \left(1 + |\ln H| + \left| \ln \left[V \left(B_x \left(\sqrt{\frac{s}{2}} \right) \right) \cdot V \left(B_y \left(\sqrt{\frac{s}{2}} \right) \right) \right] \right| \right). \end{aligned}$$

When $x \in M^n \setminus B(\rho)$, using the volume comparison theorem,

$$(4-4) \quad \begin{aligned} & \left| \ln \left[V \left(B_x \left(\sqrt{\frac{s}{2}} \right) \right) \cdot V \left(B_y \left(\sqrt{\frac{s}{2}} \right) \right) \right] \right| \\ & \leq 2 \left| \ln V \left(B_y \left(\sqrt{\frac{s}{2}} \right) \right) \right| + \left| \ln V_{-K} \left(\sqrt{\frac{s}{2}} \right) \right| + \left| \ln V_{-K}(s + d(x, y)) \right| \\ & \leq C(|\ln s| + s + d), \end{aligned}$$

where C is independent of ρ . From (4-2), (4-3), and (4-4),

$$(4-5) \quad \left| \frac{H_t}{H} \right| (x, s) \leq \frac{C}{s} (|\ln H| + |\ln s| + s + d)$$

when $x \in M^n \setminus B(\rho)$.

From (4-5), on $M^n \setminus B(\rho)$,

$$\begin{aligned}
 (4-6) \quad |(-f)H_t|(x, s) &\leq \left[|\ln H| + \frac{n}{2} |\ln(4\pi s)| \right] \cdot |H_t|(x, s) \\
 &\leq \left[|\ln H| + \frac{n}{2} |\ln(4\pi s)| \right] \cdot C|H| \cdot s^{-1} (|\ln H| + |\ln s| + s + d) \\
 &\leq \frac{C}{s} \cdot H [|\ln H|^2 + |\ln s|^2 + s^2 + d^2].
 \end{aligned}$$

From (4-5) and (4-6), if $s \in [t, t+l]$ and $x \in M^n \setminus B(\rho)$,

$$\begin{aligned}
 (4-7) \quad |G_t(x, s)| &\leq \left[|H_t| + \frac{n}{2s} |H| + |(-f)H_t| \right] (x, s) \\
 &\leq \frac{C}{s} H \cdot (|\ln H| + |\ln s| + s + d) + \frac{n}{2s} |H| + \frac{C}{s} H \cdot (|\ln H|^2 + |\ln s|^2 + s^2 + d^2) \\
 &\leq \frac{C}{s} H \cdot (|\ln H|^2 + |\ln s|^2 + s^2 + d^2),
 \end{aligned}$$

where C is independent of ρ . We can choose l smooth enough such that $(t+l) \leq 2t$. Then, using (3-7) and (4-7), on $x \in M^n \setminus B(\rho)$,

$$\begin{aligned}
 (4-8) \quad |G_t(x, s)| &\leq Cs \exp\left(-\frac{d^2}{6s}\right) \cdot \left[\left| C + 2 \ln s - \frac{d^2}{6s} \right|^2 + |\ln s|^2 + s^2 + d^2 \right] \\
 &\leq C(t+l) \exp\left(-\frac{d^2}{6(t+l)}\right) \cdot \left[t^2 + d^2 + |\ln t|^2 + \left(\frac{d^2}{t}\right)^2 \right] \\
 &\leq Ct \exp\left(-\frac{d^2}{12t}\right) \cdot \left[t^2 + |\ln t|^2 + \left(\frac{d^2}{t}\right)^2 \right].
 \end{aligned}$$

Hence, for any $\epsilon > 0$, we can find $\rho_0 > 1$ such that if $\rho \geq \rho_0$,

$$(4-9) \quad \int_{M^n \setminus B(\rho)} \left(\sup_{s \in [t, t+l]} |G_t(x, s)| \right) d\mu(x) < \frac{\epsilon}{4}$$

On the other hand, because $0 < l < 1$,

$$(4-10) \quad \int_{B(\rho)} |G_{tt}(x, t + \zeta_x l)| d\mu(x) \leq \int_{B(\rho)} \sup_{s \in [t, t+1]} |G_{tt}(x, s)| d\mu(x) \leq C(\rho).$$

Choose $l \leq \epsilon / (4C(\rho))$. From (4-9) and (4-10), if $\rho > \rho_0$,

$$(4-11) \quad \left| \int_{M^n} \frac{G(x, t+l) - G(x, t)}{l} d\mu(x) - \int_{M^n} G_t \phi_\rho d\mu(x) \right| < \epsilon.$$

It is easy to see from Lemma 4.3 and its proof that

$$\lim_{\rho \rightarrow \infty} \int_{M^n} G_t \phi_\rho$$

exists and

$$(4-12) \quad \lim_{\rho \rightarrow \infty} \int_{M^n} G_t \phi_\rho = \int_{M^n} G_t.$$

From (4-11) and (4-12), we get our conclusion. □

From results in [Cheng et al. 1981], $\lim_{t \rightarrow 0} t \ln H = -d^2/4$ and the limit is uniform for any x in $B(r)$. Hence we can assume

$$t \ln H(x, y, t) = -\frac{d^2(x, y)}{4} + \epsilon(t, x, y).$$

We sometimes simplify notation by denoting $\epsilon(t, x, y)$ by ϵ . Then

$$(4-13) \quad t(-f) = \frac{n}{2}t \ln(4\pi t) - \frac{d^2}{4} + \epsilon,$$

where $\lim_{t \rightarrow 0} \epsilon(t, x, y) = 0$, and the limit is uniform for any x in $B(r)$. Without loss of generality, we can assume that $\varphi_0(x, y) \geq 1/2$ when $x \in B(r/2)$.

Lemma 4.2.

$$(4-14) \quad \int_{B(r/2)} E(-f) d\mu(x) = -\frac{n}{2} + \frac{1}{3}R(y) \cdot t + o(t),$$

$$(4-15) \quad \int_{B(r/2)} E(-f) O(d(x, y)) d\mu(x) = o(1),$$

where $\lim_{t \rightarrow 0} o(t)/t = 0$.

Proof.

$$(4-16) \quad \begin{aligned} & \int_B E(-f) d\mu(x) \\ &= \int_B \frac{H_{N_0}}{\sum_{k=0}^{N_0} \varphi_k t^k} \cdot (-f) d\mu(x) \\ &= \int_B \left(\frac{1}{\varphi_0} - \frac{\varphi_1}{\varphi_0^2} t \right) H(-f) d\mu(x) + o(t) \\ &= \int_B \left(1 + \frac{1}{12} R_{pq}(y) x^p x^q - \frac{R(y)}{6} t \right) H(-f) + o(t) \\ &= -\frac{n}{2} + \left(\frac{1}{2} + \frac{n}{12} \right) R(y)t + \frac{1}{12} \int_B R_{pq}(y) x^p x^q \cdot H(-f) d\mu(x) + o(t). \end{aligned}$$

In the last equation, we used (1-1).

We estimate the third term on the right side of (4-16).

$$\begin{aligned}
 (4-17) \quad (I) &:= \frac{1}{12} \int_B R_{pq}(y)x^p x^q \cdot H(-f) d\mu(x) \\
 &= \frac{1}{12} \int_B R_{pq}(y)x^p x^q \cdot H \left[\ln H_{N_0} + \frac{n}{2} \ln(4\pi t) \right] d\mu(x) \\
 &= \frac{1}{12} \int_B R_{pq}(y)x^p x^q \cdot H_{N_0} \left[-\frac{d^2}{4t} + \ln \varphi_0 \right] \cdot \alpha dx + o(t) \\
 &= -\frac{1}{48t} \int_B E \cdot d^2 \cdot R_{pq}(y)x^p x^q dx + o(t) \\
 &= -\frac{n+2}{12} R(y)t + o(t).
 \end{aligned}$$

In the last equation above, we used (3-13). From (4-16) and (4-17), we get (4-14). To prove (4-15), we follow a similar strategy.

$$\begin{aligned}
 \int_B E(-f)O(d) d\mu(x) &= \int_B \left(\frac{1}{\varphi_0} - \frac{\varphi_1}{\varphi_0^2} t \right) H(-f)O(d) d\mu(x) + o(1) \\
 &= \int_B H_{N_0} \left[\ln H_{N_0} + \frac{n}{2} \ln(4\pi t) \right] O(d) d\mu(x) + o(1) \\
 &= \int_B E \left(-\frac{d^2}{4t} + \ln \varphi_0 \right) O(d) d\mu(x) + o(1) = o(1). \quad \square
 \end{aligned}$$

Lemma 4.3.

$$\int_{M^n \setminus B} |(-f)H_t| d\mu(x) = O(t^{1/2}),$$

where $t \ll 1$ is small enough.

Proof. Similarly as with (4-6), on $M^n \setminus B$,

$$(4-18) \quad |(-f)H_t| \leq \frac{C}{t} \cdot H[|\ln H|^2 + |\ln t|^2 + t^2 + d^2].$$

Hence

$$\begin{aligned}
 \int_{M^n \setminus B} |(-f)H_t| &\leq \frac{C}{t} \int_{M^n \setminus B} H \cdot |\ln H|^2 + \frac{C}{t} \int_{M^n \setminus B} H(|\ln t|^2 + t^2 + d^2) \\
 &= (I) + (II).
 \end{aligned}$$

Similarly as in the proof of (3-8), using (3-7), the volume comparison theorem, and the monotonicity of $h(x) = x(\ln x)^2$, when $x \in (0, e^{-2}]$,

$$(I) \leq O(t^{1/2}).$$

Using (2-9), when $x \in M^n \setminus B$,

$$(4-19) \quad H \leq |\eta H_{N_0}| + |F_{N_0}| \leq C \left[t^{-n/2} \exp\left(-\frac{d^2}{4t}\right) + t^4 \cdot \exp\left(-\frac{d^2}{5t}\right) \right] = O(t^2) \tilde{E}.$$

From (4-19), it is easy to get

$$(II) \leq O(t). \quad \square$$

Proof of (1-2).

$$\begin{aligned} \frac{\partial}{\partial t} \left[\int_{M^n} H(-f) d\mu(x) \right] &= \int_{M^n} \left[H_t + \frac{n}{2t} H + (-f) H_t \right] d\mu(x) \\ &= \frac{n}{2t} + \int_{M^n \setminus B(r/2)} (-f) H_t d\mu(x) + \int_{B(\frac{r}{2})} (-f) H_t d\mu(x) \\ &= \frac{n}{2t} + (I) + (II). \end{aligned}$$

From Lemma 4.3, we have

$$(I) = O(t^{1/2}).$$

From Lemma 4.4, which follows, we get

$$(II) = -\frac{n}{2t} + \frac{1}{2} R(y) + o(1). \quad \square$$

Lemma 4.4.

$$\int_B (-f) H_t d\mu(x) = -\frac{n}{2t} + \frac{1}{2} R(y) + o(1).$$

Proof. From (2-10) and (4-13),

$$\int_B (-f) H_t d\mu(x) = \int_B (-f) \cdot (H_{N_0})_t + O(t)$$

and

$$\begin{aligned} &\int_B (-f) (H_{N_0})_t d\mu(x) \\ &= \int_B \left(\frac{d^2}{4t^2} - \frac{n}{2t} \right) H_{N_0} \cdot (-f) d\mu(x) + \int_B E\varphi_1(-f) d\mu(x) + o(1) \\ &= \frac{1}{4t^2} \int_B H_{N_0} (-f) d^2 d\mu(x) - \frac{n}{2t} \int_B H_{N_0} (-f) d\mu(x) + \int_B E\varphi_1(-f) d\mu(x) + o(1) \\ &= (I) + (II) + (III) + o(1). \end{aligned}$$

Using Lemma 4.2,

$$\begin{aligned} \text{(III)} &= \int_B E\varphi_1(-f) d\mu(x) = \frac{1}{6}R(y) \int_B E(-f) d\mu(x) + \int_B E(-f) \cdot O(d) \\ &= -\frac{n}{12}R(y) + o(1). \end{aligned}$$

From (2-9) and (1-1),

$$\begin{aligned} \text{(II)} &= -\frac{n}{2t} \int_B H_{N_0}(-f) d\mu(x) \\ &= -\frac{n}{2t} \int_B H(-f) d\mu(x) - \frac{n}{2t} \int_B O(t^{N_0+1})\tilde{E}(-f) d\mu(x) \\ &= \frac{n^2}{4t} - \frac{n}{4}R(y) + o(1). \end{aligned}$$

Similarly, by Lemma 4.5, which follows,

$$\begin{aligned} \text{(I)} &= \frac{1}{4t^2} \int_B (H + O(t^{N_0+1})\tilde{E})(-f) \cdot d^2 d\mu(x) = \frac{1}{4t^2} \int_B H(-f) \cdot d^2 d\mu(x) + o(1) \\ &= -\frac{n(n+2)}{4t} + \left(\frac{n}{3} + \frac{1}{2}\right)R(y) + o(1). \end{aligned}$$

From all the above,

$$\int_B (-f)H_t d\mu(x) = -\frac{n}{2t} + \frac{1}{2}R(y) + o(1) \quad \square$$

Lemma 4.5.

$$\frac{1}{4t^2} \int_B H(-f) \cdot d^2 d\mu(x) = -\frac{n(n+2)}{4t} + \left(\frac{n}{3} + \frac{1}{2}\right)R(y) + o(1).$$

Proof. We use a strategy similar to that used in the proof of (1-1).

$$\begin{aligned} &\frac{1}{4t^2} \int_B H(-f) \cdot d^2 d\mu(x) \\ &= \frac{1}{4t^2} \int_B \left[\ln H_{N_0} + \frac{n}{2} \ln(4\pi t) \right] H d^2 d\mu(x) + \frac{1}{4t^2} \int_B \left[\ln \frac{H}{H_{N_0}} \right] H d^2 d\mu(x). \end{aligned}$$

From (3-6),

$$\left[\ln \frac{H}{H_{N_0}} \right] H = O(t^4).$$

Hence,

$$\begin{aligned} & \frac{1}{4t^2} \int_B H(-f) \cdot d^2 d\mu(x) \\ &= \frac{1}{4t^2} \int_B \left[-\frac{d^2}{4t} + \ln \varphi_0 + \frac{\varphi_1}{\varphi_0} t + O(t^2) \right] H d^2 \cdot \alpha dx + o(1) \\ &= \frac{1}{4t^2} \int_B \left(-\frac{d^2}{4t} + \frac{1}{12} R_{pq}(y) x^p x^q + \frac{1}{6} R(y) t - \frac{R(y)}{24} d^2 + \frac{1}{48t} R_{pq}(y) x^p x^q \cdot d^2 \right) \\ & \qquad \qquad \qquad \cdot E d^2 dx + o(1) \\ &= -\frac{n(n+2)}{4t} + \frac{-n^2 + 2n + 4}{24} R(y) + \frac{1}{192t^3} \int_{\mathbb{R}^n} E R_{pq}(y) x^p x^q \cdot d^4 dx + o(1). \end{aligned}$$

Define

$$Q_n = \int_{\mathbb{R}^n} E R_{pq}(y) x^p x^q \cdot d^4 dx = \int_{\mathbb{R}^n} E \cdot \left(\sum_{i=1}^n \lambda_i x_i^2 \right) \cdot \left(\sum_{j=1}^n x_j^2 \right) dx,$$

where we diagonalize $R_{pq}(y)$ and let $\lambda_i = R_{ii}(y)$. We can get $Q_1 = 120\lambda_1 t^3$ and the induction formula

$$Q_n = Q_{n-1} + 8(2n+5) \left(\sum_{i=1}^n \lambda_i \right) t^3 + 8(n^2 + 4n + 3) \lambda_n \cdot t^3.$$

Then it is easy to get $Q_n = 8(n^2 + 6n + 8)R(y) \cdot t^3$. Hence

$$\frac{1}{4t^2} \int_B H(-f) \cdot d^2 d\mu(x) = -\frac{n(n+2)}{4t} + \left(\frac{n}{3} + \frac{1}{2} \right) R(y) + o(1). \quad \square$$

Appendix

Richard Hamilton [1993] established an upper bound of the Laplacian of the positive solution to the heat equation on closed manifolds. We generalize his theorem to complete manifolds with Ricci curvature bounded below. Our proof follows a strategy similar to that in [Kotschwar 2007]. We firstly establish a preliminary estimate on $t|\Delta u|$ so that the maximum principle of Ni and Tam [2004] may be applied to the quantity of interest in Hamilton's second derivative estimate.

We introduce a cut-off function ϕ defined on \mathbb{R} , which is a smooth nonnegative nonincreasing function which is 1 on $(-\infty, 1)$ and 0 on $[2, +\infty)$. We can further assume

$$(A-1) \qquad |\phi'| \leq 2, \quad |\phi''| + \frac{(\phi')^2}{\phi} \leq 16.$$

To prove the following Bernstein-type local estimate, we employ a technique of W.-X. Shi [1989] from the estimation of derivatives of curvature under the Ricci

flow (see also [Chow et al. 2008]). Define $F = (C + t|\nabla u|^2)t^2|\Delta u|^2$ and consider the evolution of F .

Lemma A.6. *Suppose (M^n, g) is a complete Riemannian manifold. If $|u(x, t)| \leq \mathcal{M}$ is a solution to the heat equation on $B_p(4\rho) \times [0, T]$ for some $p \in M^n$, constants $\mathcal{M}, \rho, T, K > 0$, and $\text{Rc} \geq -Kg$ on $B_p(4\rho)$,*

$$(A-2) \quad t|\Delta u| \leq C(n, K, \mathcal{M}) \left[1 + T \left(1 + \frac{1}{\rho^2} \right) \right] \cdot \left(\frac{1}{\rho} + 1 \right) \cdot \left[T + \coth \left(\sqrt{\frac{K}{n-1}} \rho \right) \right]$$

holds on $B_p(\rho) \times [0, T]$.

Proof. From [Kotschwar 2007], we get that

$$(A-3) \quad t|\nabla u|^2 \leq C_1 \left[1 + T \left(1 + \frac{1}{\rho^2} \right) \right] =: C_2$$

holds on $B_p(2\rho) \times [0, T]$, where $C_1 = C_1(K, \mathcal{M})$. Define $C_3 = 4C_2$, and

$$F(x, t) = (C_3 + t|\nabla u(x, t)|^2)t^2|\Delta u(x, t)|^2.$$

A long but straightforward computation gives

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta \right) F \\ &= -2(C_3 + t|\nabla u|^2)|\nabla \Delta u|^2 - 8t^3 \sum_{i,j} \nabla_i \nabla_j u \nabla_i \Delta u \nabla_j u \Delta u - 2t^3 |\nabla^2 u|^2 \cdot |\Delta u|^2 \\ & \quad + 2t(C_3 + t|\nabla u|^2)|\Delta u|^2 + [|\nabla u|^2 - 2t \text{Rc}(\nabla u, \nabla u)]t^2|\Delta u|^2. \end{aligned}$$

When $x \in B_p(4\rho)$, using $t|\nabla u|^2 \leq C_2 = \frac{1}{4}C_3$ and $\text{Rc} \geq -Kg$,

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta \right) F &\leq -10t^3|\nabla u|^2 \cdot |\nabla \Delta u|^2 + 8t^3|\nabla u| \cdot |\nabla \Delta u| \cdot |\nabla^2 u| \cdot |\Delta u| \\ & \quad - 2t^3|\nabla^2 u|^2 \cdot |\Delta u|^2 + C_4t|\Delta u|^2 \\ &\leq -\frac{2}{5}t^3|\nabla^2 u|^2 \cdot |\Delta u|^2 + C_4t|\Delta u|^2, \end{aligned}$$

where $C_4 = (2KT + 11)C_2$. The term with the coefficient $-\frac{2}{5}$ arose from the inequality $-10x^2 + 8xy - 2y^2 \leq -\frac{2}{5}y^2$. On the other hand, we know that $|\nabla^2 u|^2 \geq (1/n)|\Delta u|^2$. Hence

$$\left(\frac{\partial}{\partial t} - \Delta \right) F \leq -\frac{2}{5n}t^3|\Delta u|^4 + C_4t|\Delta u|^2 \leq -\frac{1}{5nt}[t^2|\Delta u|^2]^2 + \frac{5n}{4t}C_4 \leq -\frac{C_6}{t}F^2 + \frac{C_5}{t}.$$

In the last equality we used $F \leq (C_3 + C_2)t^2|\Delta u|^2 = 5C_2t^2|\Delta u|^2$, and

$$(A-4) \quad C_5 = C(n, K, \mathcal{M})(1 + T) \left[1 + T \left(1 + \frac{1}{\rho^2} \right) \right],$$

$$(A-5) \quad C_6 = C(n, K, \mathcal{M}) \left[1 + T \left(1 + \frac{1}{\rho^2} \right) \right]^{-2}.$$

Define $\gamma(x) = \phi(d(x, p)/\rho)$. Then $\gamma(x)F(x, t)$ attains its maximum at a point $(x_0, t_0) \in B_p(2\rho) \times [0, T]$. The rest of the computation is at (x_0, t_0) ;

$$0 \leq \left(\frac{\partial}{\partial t} - \Delta\right)(\gamma F) \leq \gamma \left(-\frac{C_6}{t}F^2 + \frac{C_5}{t}\right) - \Delta\gamma \cdot F - 2\nabla\gamma \nabla F.$$

Note that at (x_0, t_0) , $\nabla(\gamma F) = 0$. Letting $G = (\gamma F)(x_0, t_0)$, we get

$$(A-6) \quad 0 \leq -\frac{C_6}{t}G^2 + \left(2\frac{|\nabla\gamma|^2}{\gamma} - \Delta\gamma\right)G + \frac{C_5}{t}$$

and

$$(A-7) \quad \begin{aligned} \left(2\frac{|\nabla\gamma|^2}{\gamma} - \Delta\gamma\right) &= \frac{2}{\rho^2} \cdot \frac{|\phi'|^2}{\phi} - \frac{\phi''}{\rho^2} - \frac{\phi'}{\rho} \Delta d(x, p) \\ &\leq \frac{32}{\rho^2} + \frac{2}{\rho} \cdot \coth\left(\sqrt{\frac{K}{n-1}}\rho\right). \end{aligned}$$

In the last inequality we used (A-1), $\text{Rc} \geq -Kg$, and the Laplacian comparison theorem. From (A-4)–(A-7),

$$\begin{aligned} 0 \leq -G^2 + C(n, K, \mathcal{M}) \left[1 + T\left(1 + \frac{1}{\rho^2}\right)\right]^2 T \cdot \left[\frac{1}{\rho^2} + \frac{1}{\rho} \coth\left(\sqrt{\frac{K}{n-1}}\rho\right)\right] G \\ + C(n, K, \mathcal{M}) \left[1 + T\left(1 + \frac{1}{\rho^2}\right)\right]^3 (1 + T). \end{aligned}$$

Then it is easy to get

$$G \leq C(n, K, \mathcal{M}) \cdot \left[1 + T\left(1 + \frac{1}{\rho^2}\right)\right]^2 (1 + T) \cdot \left[\left(\frac{1}{\rho^2} + \frac{1}{\rho}\right) \coth\left(\sqrt{\frac{K}{n-1}}\rho\right) + 1 + T\left(1 + \frac{1}{\rho^2}\right)\right].$$

Hence, on $B_p(\rho)$,

$$\begin{aligned} t^2 |\Delta u|^2 &\leq C_3^{-1} F \leq C_3^{-1} G \\ &\leq C(n, K, \mathcal{M}) \cdot \left[1 + T\left(1 + \frac{1}{\rho^2}\right)\right]^2 \cdot \left[\left(\frac{1}{\rho^2} + 1\right) \cdot \left(T + \coth\left(\sqrt{\frac{K}{n-1}}\rho\right)\right) + 1\right] \end{aligned}$$

Taking the square root in the above inequality, we get our conclusion. □

Letting $\rho \rightarrow \infty$, we get the following global estimate.

Corollary A.7. *Suppose (M^n, g) is a complete Riemannian manifold with $\text{Rc} \geq -Kg$, and $|u(x, t)| \leq \mathcal{M}$ is a solution to the heat equation on $M^n \times [0, T]$, where K, \mathcal{M}, T are positive constants. Then*

$$(A-8) \quad t|\Delta u| \leq C(n, K, \mathcal{M})(1 + T)^2$$

holds on $M^n \times [0, T]$.

We also need a maximum principle, due originally to Karp and Li [1982], which was stated more generally by Ni and Tam.

Theorem A.8 [Ni and Tam 2004, Theorem 1.2]. *Suppose (M^n, g) is a complete Riemannian manifold and $h(x, t)$ is a smooth function on $M^n \times [0, T]$ such that*

$$\left(\frac{\partial}{\partial t} - \Delta\right)f(x, t) \leq 0$$

whenever $f(x, t) \geq 0$. Assume that

$$\int_0^T \int_{M^n} e^{-a \cdot d^2(x,p)} f_+^2(x, s) d\mu(x) ds < \infty$$

for some $a > 0$, where p is a fixed point on M^n and $f_+(x, t) := \max\{f(x, t), 0\}$. If $f(x, 0) \leq 0$ for all $x \in M^n$, $f(x, t) \leq 0$ for all $(x, t) \in M^n \times [0, T]$.

Now we are ready to prove Hamilton’s theorem in the complete case.

Theorem A.9. *Suppose (M^n, g) is a complete Riemannian manifold with $Rc \geq -Kg$, and $0 < u(x, t) \leq \mathcal{M}$ is a solution to the heat equation on $M^n \times [0, T]$, where K, \mathcal{M}, T are positive constants. Then*

$$(A-9) \quad t \left(\frac{\Delta u}{u} + \frac{|\nabla u|^2}{u^2} \right) \leq n + (4 + 2Kt) \ln \frac{\mathcal{M}}{u}.$$

Proof. Defining $u_\epsilon = u + \epsilon$ for $\epsilon > 0$, we obtain a solution satisfying $\epsilon < u_\epsilon \leq \mathcal{M} + \epsilon =: \mathcal{M}_\epsilon$. Once the estimate has been proved for u_ϵ , the theorem follows by letting $\epsilon \rightarrow 0$. Consider the function

$$F(x, t) = t \left(\Delta u_\epsilon + \frac{|\nabla u_\epsilon|^2}{u_\epsilon} \right) - u_\epsilon \left[n + (4 + 2Kt) \ln \frac{\mathcal{M}_\epsilon}{u_\epsilon} \right].$$

A long but straightforward computation gives

$$(A-10) \quad \begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right)F(x, t) &= u_\epsilon \left[-2t|\nabla^2 \ln u_\epsilon|^2 + \Delta \ln u_\epsilon - (2 + 2Kt)|\nabla \ln u_\epsilon|^2 \right. \\ &\quad \left. - 2t Rc(\nabla \ln u_\epsilon, \nabla \ln u_\epsilon) - 2K \ln \frac{\mathcal{M}_\epsilon}{u_\epsilon} \right] \\ &\leq u_\epsilon \left[-\frac{2t}{n} |\Delta \ln u_\epsilon|^2 + \Delta \ln u_\epsilon - 2|\nabla \ln u_\epsilon|^2 \right]. \end{aligned}$$

If $F(x, t) \geq 0$ at (x, t) ,

$$(A-11) \quad -2|\nabla \ln u_\epsilon|^2 \leq \Delta \ln u_\epsilon - \frac{n}{t}.$$

From (A-10) and (A-11),

$$(A-12) \quad \left(\frac{\partial}{\partial t} - \Delta\right)F(x, t) \leq u_\epsilon \left[-\frac{2t}{n} |\Delta \ln u_\epsilon|^2 + 2\Delta \ln u_\epsilon - \frac{n}{t} \right] \leq -\frac{n}{2t} < 0.$$

In (A-3) let $\rho \rightarrow \infty$. Then

$$(A-13) \quad t|\nabla u|^2 \leq C(K, \mathcal{M}_\epsilon, T).$$

From (A-13) and (A-8),

$$(A-14) \quad F_+^2(x, t) \leq \left[t \left(\Delta u_\epsilon + \frac{|\nabla u_\epsilon|^2}{u_\epsilon} \right) \right]^2 \leq C(\epsilon, n, K, \mathcal{M}_\epsilon, T).$$

Using (A-14), for any $p \in M^n$ and $\rho > 0$,

$$(A-15) \quad \int_0^T \int_{B_p(\rho)} \exp(-d^2(x, p)) F_+^2(x, t) d\mu(x) dt \leq C(\epsilon, n, K, \mathcal{M}_\epsilon, T) \int_{M^n} \exp(-d^2(x, p)) d\mu(x) \leq C.$$

In the last equality we used the volume comparison theorem and $Rc \geq -Kg$. Letting $\rho \rightarrow \infty$,

$$(A-16) \quad \int_0^T \int_{M^n} \exp(-d^2(x, p)) F_+^2(x, t) d\mu(x) dt \leq C < \infty.$$

From (A-12) and (A-16), using Theorem A.8, we get $F(x, t) \leq 0$ for all $0 \leq t \leq T$, completing the proof. \square

We now give an the upper bound for the Laplacian of the heat kernel.

Corollary A.10. *Suppose (M^n, g) is a complete Riemannian manifold such that $Rc \geq -Kg$, $H(x, y, t)$ is the heat kernel on M^n , and $0 < t \leq T$, where K, T are positive constants. Then*

$$\left(\Delta H + \frac{|\nabla H|^2}{H} \right)(x, y, t) \leq \frac{2H(x, y, t)}{t} \left\{ n + (4 + Kt) \ln \frac{C(K, T)}{H(x, y, t) V^{1/2}(B_x(\sqrt{t/2})) V^{1/2}(B_y(\sqrt{t/2}))} \right\}.$$

Proof. Note that if $s \in [t/2, t]$, from [Li and Yau 1986],

$$H(x, y, t) \leq C(K, T) \cdot V^{1/2}(B_x(\sqrt{t/2})) V^{1/2}(B_y(\sqrt{t/2})).$$

Then apply Theorem A.9 on $u(x, s) = H(x, y, s + t/2)$ and $M^n \times [0, t/2]$. The conclusion follows from (A-9). \square

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
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