Pacific Journal of Mathematics

THE SHORT TIME ASYMPTOTICS OF NASH ENTROPY

GUOYI XU

Volume 266 No. 2 December 2013

PACIFIC JOURNAL OF MATHEMATICS
Vol. 266, No. 2, 2013
dx.doi.org/10.2140/pjm.2013.266.423

THE SHORT TIME ASYMPTOTICS OF NASH ENTROPY

GUOYI XU

Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Rc} \geq -Kg$, $H(x, y, t)$ be the heat kernel on M^n, and $H = (4\pi t)^{-n/2}e^{-f}$. Nash entropy is defined as $N(H, t) = \int_{M^n}(fH)\,d\mu(x) - n/2$. We study the asymptotic behavior of $N(H, t)$ and $\partial N(H, t)/\partial t$ as $t \to 0^+$ and get the asymptotic formulas at $t = 0$. In the appendix, we get a Hamilton-type upper bound for the Laplacian of the positive solution to the heat equation on such manifolds, which is itself interesting.

1. Introduction

On a complete manifold (M^n, g) with $\mathrm{Rc} \geq -Kg$, where $K > 0$ is a constant, for fixed $y \in M^n$, it is well known that the heat kernel $H(x, y, t)$ on (M^n, g) is unique. We assume $H = (4\pi t)^{-n/2}e^{-f}$. As in [Ni 2004b], Nash entropy is defined as follows.

Definition 1.1.
$$N(H, t) = \int_{M^n} (fH)\,d\mu(x) - \frac{n}{2}.$$

Nash entropy is closely related to $\mathcal{W}$-entropy for the linear heat equation, and the large time asymptotics of this entropy reflects the volume growth rate of the manifold; see [Ni 2004a; 2004b; 2010].

In this paper, we study the asymptotic behavior of $N(H, t)$ and $\partial N(H, t)/\partial t$ as $t \to 0^+$, and solve Problem 23.36 of [Chow et al. 2010]. More precisely, we prove:

Theorem 1.2. *Let (M^n, g) be a complete Riemannian manifold with* $\mathrm{Rc} \geq -Kg$, *where $K > 0$ is a constant. Then*

$$N(H, t) = -\tfrac{1}{2}R(y)\cdot t + O(t^{3/2}) \tag{1-1}$$

and

$$\frac{\partial}{\partial t}[N(H, t)] = -\tfrac{1}{2}R(y) + o(1), \tag{1-2}$$

where $\limsup_{t\to 0} O(t^{3/2})t^{-3/2}$ *is bounded*, $\lim_{t\to 0} o(1) = 0$, *and t is small enough.*

MSC2010: 35K15, 53C44.
Keywords: Nash entropy, short time asymptotics.

One motivation to study the short time asymptotics of Nash entropy is Li–Yau–Perelman type estimates for the heat equation on manifolds with Ricci curvature bounded from below. Motivated by Perelman's differential Harnack estimate for Ricci flow on a closed manifold (M^n, g) with $\mathrm{Rc} \geq 0$, Ni [2004a] proved the following Li–Yau–Perelman type estimate for the heat equation when $t > 0$:

$$2\Delta f(x, y, t) - |\nabla f(x, y, t)|^2 + \frac{f(x, y, t) - n}{t} \leq 0, \tag{1-3}$$

where $H(x, y, t) = (4\pi t)^{-n/2} e^{-f}$ is the heat kernel. In fact, (1-3) is also true for the heat kernel on a complete manifold (M^n, g) with $\mathrm{Rc} \geq 0$; see [Chow et al. 2008].

Perelman made the following claim.

Claim 1.3 [Perelman 2002, Remark 9.6]. *If (M^n, g) is a compact Riemannian manifold, $g_{ij}(x, t)$ evolves according to $(g_{ij})_t = A_{ij}(t)$ and $g_{ij}(x, 0) = g_{ij}(x)$, $t \in (-T, 0]$. Define $\Box = \partial/(\partial t) - \Delta$ and its conjugate $\Box^* = -\partial/(\partial t) - \Delta - \frac{1}{2}A$ (where $A = g^{ij} A_{ij}$). Consider the fundamental solution $u = (-4\pi t)^{-n/2} e^{-f}$ for $\Box^*$, starting as a δ-function at some point $(p, 0)$. Then, for general A_{ij}, the function $(\Box \bar{f} + \bar{f}/t)(q, t)$, where $\bar{f} = f - \int_{M^n} f u$, is of order $O(1)$ for (q, t) near $(p, 0)$.*

We focus on the special case where the evolving metrics are the static metric. From Theorem 1.2, it is easy to show that Perelman's claim in the static metric case is equivalent to the following claim on compact manifolds:

$$2\Delta f(x, y, t) - |\nabla f(x, y, t)|^2 + \frac{f(x, y, t) - n}{t} = -R(y) + O(t + d^2(x, y)). \tag{1-4}$$

If (1-4) is true, it is an improvement of (1-3) when $t + d^2(x, y)$ is small enough and $R(y) > 0$. But, using the explicit formula (cf. [Grigor'yan 2009, Section 9.2])

$$H = (4\pi t)^{-3/2} \frac{d}{\sinh d} \exp\Big(-\frac{d^2}{4t} - t\Big)$$

for the heat kernel on a hyperbolic manifold $\mathbb{H}^3$, it is easy to check that (1-4) is not true generally. Hence Claim 1.3 is not generally true for the static metric case on complete manifolds.

As observed in [Ni 2004b], the integrand of $\partial N(H, t)/\partial t$ is simply the expression in Li and Yau's gradient estimate for the heat kernel multiplied with the heat kernel, which is $-(\Delta \ln H + n/(2t))H$. Because so far there is no sharp Li–Yau-type gradient estimate for the heat kernel or solutions to the heat equation on complete manifolds with Ricci curvature bounded from below by a negative constant, we hope that (1-2) will be helpful in better understanding this estimate.

On the other hand, in the case where (M^n, g) is a compact Riemannian manifold, the short time behavior of the logarithm of the heat kernel has been studied by many probabilists. Although the heat kernel $H(x, y, t)$ has an infinite sequence expansion at $t = 0$, generally there is no such expansion of $\ln H$ at $t = 0$, and the singularity of $\ln H$ at $t = 0$ can have many complicated situations. However, Varadhan [1967] proved

$$\lim_{t\to 0} t \ln H(x, y, t) = -\frac{d^2(x, y)}{4}. \tag{1-5}$$

Moreover, using stochastic processes methods, Malliavin and Stroock proved [1996] that the above equation is preserved while taking the first and second spatial derivatives on a domain outside of the cut locus. Using analytic methods, (1-5) was proved for complete Riemannian manifolds by Cheng, Li, and Yau [Cheng et al. 1981]. We hope that Theorem 1.2 will be useful in studying the short time behavior of the logarithm of the heat kernel on complete manifolds by analytic methods.

The strategy to prove (1-1) is to use the infinite sequence expansion $H_N(x, y, t)$ of $H(x, y, t)$ at $t = 0$, although generally $\ln H_N$ does not converge to $\ln H$ near $t = 0$ uniformly. In the integral sense of Definition 1.1, we show there is a uniform convergence in Lemma 3.1 by using an improved estimate of $H - H_N$ obtained in Theorem 2.2. The rest of the calculation of the integral of H_N is standard, but, for completeness, we give the details.

To prove (1-2), because the manifold M^n can be noncompact, we need to be more careful when switching the order of differentiation and integration. A detailed proof of the validity of the switch is given in the beginning of Section 4. We need an upper bound of H_t/H to verify the above switch. This type of bound is known for closed manifolds [Hamilton 1993], and in [Chow et al. 2008] (see also [Ni 2006]) the proof is sketched for complete manifolds with $\mathrm{Rc} \geq 0$ using a strategy similar to that in [Kotschwar 2007]. A detailed proof of this Hamilton-type upper bound for complete manifolds with $\mathrm{Rc} \geq -Kg$ is included in the appendix for completeness.

The paper is organized as follows. In Section 2, we state some preliminary results about the heat kernel and get some improved estimates of $H - H_N$. In Section 3, we prove (1-1). In Section 4, using (1-1) and results in the appendix, we prove (1-2). In the appendix, we prove Hamilton-type upper bound of H_t/H on complete manifolds with Ricci curvature bounded from below.

2. Preliminaries

We first define some notations and functions. In the rest of the paper, we fix $y \in M^n$ and define

$$\Omega_y = \{x \in M^n : d(x, y) < \mathrm{inj}_g(y)\},$$

where $\mathrm{inj}_g(y)$ denotes the injectivity radius of the metric g at y. Define

$$B(\rho) = \{x : d(x, y) \leq \rho\} \quad \text{and} \quad B_z(\rho) = \{x : d(x, z) \leq \rho\}.$$

Hence $B(\rho) = B_y(\rho)$. $V(B_z(\rho))$ is used to denote the volume of $B_z(\rho)$ and $V_{-K}(\rho)$ is the volume of the geodesic ball of radius ρ in the constant $(-K/(n-1))$ sectional curvature space form.

Fix $r \in (0, \frac{1}{4}\,\mathrm{inj}_g(y))$ and let $N_0 = n/2 + 3$. Define

$$E = (4\pi t)^{-n/2} \exp\left(-\frac{d^2(x, y)}{(4t)}\right) \quad \text{and} \quad \widetilde{E} = (4\pi t)^{-n/2} \exp\left(-\frac{d^2(x, y)}{(5t)}\right).$$

When the meaning is clear from context, we sometimes simplify notation by denoting $B(r/2)$ by B and $d(x, y)$ by d.

Assume $\eta : [0, \infty) \to [0, 1]$ is a C^∞ cut-off function with

$$\eta(s) = \begin{cases} 1 & \text{if } s \le r, \\ 0 & \text{if } s \ge 2r. \end{cases} \tag{2-1}$$

The following theorem collects several known results about the heat kernel on complete manifolds; see, for example, [Chow et al. 2010; Garofalo and Lanconelli 1989; Li 2012].

Theorem 2.1. *(M^n, g) is a complete Riemannian manifold with* $\mathrm{Rc} \ge -Kg$, *where $K > 0$ is a constant. Then there exists a unique positive fundamental solution $H(x, y, t)$ to the heat equation, which is called the heat kernel. Moreover,*

$$H(x, y, t) \in C^\infty(M^n \times M^n \times (0, \infty))$$

is symmetric in x and y, and

(i)

$$\int_{M^n} H(x, y, t)\, d\mu(x) \equiv 1; \tag{2-2}$$

(ii)

$$H(x, y, t) = P_{N_0}(x, y, t) + F_{N_0}(x, y, t) \tag{2-3}$$

$$P_{N_0}(x, y, t) = \eta(d(x, y)) H_{N_0}(x, y, t), \tag{2-4}$$

$$H_{N_0}(x, y, t) = (4\pi t)^{-n/2} \exp\left(-\frac{d^2(x, y)}{4t}\right) \cdot \sum_{k=0}^{N_0} \varphi_k(x, y) t^k, \tag{2-5}$$

where $\varphi_k(x, y) \in C^\infty(\Omega_y)$ and $k = 0, 1, \ldots, N_0$. Also, H_{N_0} satisfies

$$\left(\Delta - \frac{\partial}{\partial t}\right) H_{N_0}(x, y, t) = E \Delta \varphi_{N_0} t^{N_0}; \tag{2-6}$$

(iii) *let $\{x^k\}_{k=1}^n$ be exponential normal coordinates centered at $y \in M^n$. Then φ_0 and φ_1 have the asymptotic expansion*

$$\varphi_0(x, y) = 1 + \tfrac{1}{12} R_{pq}(y) x^p x^q + O(d^3(x, y)), \tag{2-7}$$

$$\varphi_1(x, y) = \frac{R(y)}{6} + O(d(x, y)). \tag{2-8}$$

We prove an estimate for F_{N_0}. This estimate is an improvement of the usual estimate of F_{N_0}, which only gives $t^{N_0+1-n/2}$ bound. The improved estimate (2-9) is the key to the proof of Lemma 3.1.

Theorem 2.2. *For $F_{N_0}(x, y, t)$ in Theorem 2.1, we have the following estimates*:

$$|F_{N_0}(x, y, t)| \le C t^4 \exp\Big(-\frac{d^2(x, y)}{5t}\Big), \tag{2-9}$$

$$\Big|\frac{\partial}{\partial t} F_{N_0}(x, y, t)\Big| \le C t^2 \exp\Big(-\frac{d^2(x, y)}{5t}\Big), \tag{2-10}$$

where t is small enough and C is a positive constant independent of x, t.

Remark 2.3. (2-9) was proved in [Garofalo and Lanconelli 1989] for uniformly parabolic operators. Our proof of (2-9) and (2-10) is motivated by an argument in [Li 2012] and is different from the proof in [Garofalo and Lanconelli 1989].

Proof. (Δ). We first prove (2-9). From the definition of $P_{N_0}(x, y, t)$, it is easy to see that $\lim_{t\to 0} P_{N_0}(x, y, t) = \delta_y(x)$. In particular,

$$\begin{aligned} F_{N_0}(x, y, t) &= H(x, y, t) - P_{N_0}(x, y, t) \\ &= -\int_0^t \frac{\partial}{\partial s} \int_{M^n} H(x, z, t-s) P_{N_0}(z, y, s)\, d\mu(z)\, ds \\ &= -\int_0^t \int_{M^n} \Big(\frac{\partial}{\partial s} - \Delta_z\Big) P_{N_0}(z, y, s) \cdot H(x, z, t-s)\, d\mu(z)\, ds, \end{aligned} \tag{2-11}$$

where Δ_z is the Laplacian with respect to the variable z.

From (2-6) and the definition of η, when $z \in B(r)$,

$$\Big|\Big(\frac{\partial}{\partial s} - \Delta_z\Big) P_{N_0}(z, y, s)\Big| \le C_1 s^3 \exp\Big(-\frac{d^2(z, y)}{4s}\Big), \tag{2-12}$$

and when $z \in B(2r)\backslash B(r)$,

$$\Big|\Big(\frac{\partial}{\partial s} - \Delta_z\Big) P_{N_0}(z, y, s)\Big| \le C_2 s^{-n/2-1} \exp\Big(-\frac{d^2(z, y)}{4s}\Big). \tag{2-13}$$

Hence

$$|F_{N_0}(x,y,t)| \le C_1\int_0^t s^3\int_{B(r)} H(x,z,t-s)\exp\Big(-\frac{d^2(z,y)}{4s}\Big)\,d\mu(z)\,ds + C_2\int_0^t s^{-n/2-1}\int_{B(2r)\backslash B(r)} H(x,z,t-s)\exp\Big(-\frac{d^2(z,y)}{4s}\Big)\,d\mu(z)\,ds \le (a)+(b) \tag{2-14}$$

We can find $0<t_1\le 1$ and $k_0>0$ such that if $s\in(0,t_1)$,

$$V(B_p(\sqrt{s}))\ge k_0 s^{n/2}\quad \text{for any } p\in B_y(3r).$$

In the rest of the proof, assume $t\in(0,t_1]$. We have two cases.

Case I. If $x\in B_y(3r)$ and $z\in B_y(2r)$, then, from [Li and Yau 1986] and the above volume lower bound,

$$H(x,z,t-s) \le CV^{-1/2}(B_x(\sqrt{t-s}))V^{-1/2}(B_z(\sqrt{t-s}))\cdot\exp\Big[CK(t-s)-\frac{6d^2(z,x)}{25(t-s)}\Big] \le C(K,k_0,n)(t-s)^{-n/2}\exp\Big(-\frac{6d^2(z,x)}{25(t-s)}\Big) \tag{2-15}$$

Case II. If $x\notin B_y(3r)$ and $z\in B_y(2r)$, using (2-15), $d(x,z)\ge r$, and the volume comparison theorem,

$$H(x,z,t-s)\le CV^{-1}(B_z(\sqrt{t-s}))\cdot\Big[\frac{V_{-K}(\sqrt{t-s}+d(x,z))}{V_{-K}(\sqrt{t-s})}\Big]^{1/2}\cdot\exp\Big[CK(t-s)-\frac{6d^2(z,x)}{25(t-s)}\Big] \le C(K,k_0,n,r)\exp\Big(-\frac{23d^2(z,x)}{100(t-s)}\Big) \tag{2-16}$$

Note that, in Case I, $\mathrm{inj}_g(x)$ has a uniform lower bound. Hence it is easy to get

$$\int_{B_y(r)} s^{-n/2}\exp\Big(-\frac{d^2(z,x)}{100s}\Big)\,d\mu(z)\le C \tag{2-17}$$

for any $s\in(0,t_1]$.

Now, using (2-15), (2-16), (2-17) and the classical inequality

$$\frac{d^2(x,z)}{t-s}+\frac{d^2(y,z)}{s}\ge\frac{d^2(x,y)}{t},$$

we can get

$$\int_{B_y(r)} H(x, z, t-s) \exp\left(-\frac{d^2(z, y)}{4s}\right) d\mu(z) \le C \exp\left(-\frac{23d^2(x, y)}{100t}\right).$$

Hence

$$(a) \le Ct^4 \exp\left(-\frac{23d^2(x, y)}{100t}\right). \tag{2-18}$$

Similarly,

$$\begin{aligned}\int_{B_y(2r)\setminus B_y(r)} H(x, z, t-s) \exp\left(-\frac{d^2(z, y)}{4s}\right) d\mu(z)&\\ \le C \exp\left(-\frac{3r^2}{100s}\right) \exp\left(-\frac{d^2(x, y)}{5t}\right).&\end{aligned}$$

Hence

$$\begin{aligned}(b) &\le C_2\left[\int_0^t s^{-\frac{n}{2}-1} \exp\left(-\frac{3r^2}{100s}\right) ds\right] \exp\left(-\frac{d^2(x, y)}{5t}\right)\\ &\le Ct^4 \exp\left(-\frac{d^2(x, y)}{5t}\right).\end{aligned} \tag{2-19}$$

By (2-18) and (2-19), (2-9) is proved.

(Θ). The strategy to prove (2-10) is similar.

$$\begin{aligned}\frac{\partial}{\partial t} F_{N_0}(x, y, t) &= \frac{\partial}{\partial t}\left[-\int_0^t \int_{M^n} \left(\frac{\partial}{\partial s} - \Delta_z\right) P_{N_0}(z, y, s) \cdot H(x, z, t-s)\, d\mu(z)\, ds\right]\\ &= -\int_0^t \int_{M^n} \left(\frac{\partial}{\partial s} - \Delta_z\right) P_{N_0}(z, y, s) \cdot \left(\frac{\partial}{\partial t} H(x, z, t-s)\right) d\mu(z)\, ds\\ &\qquad + \left(\Delta_x - \frac{\partial}{\partial t}\right) P_{N_0}(x, y, t)\\ &= (\gamma) + (\tau)\end{aligned}$$

From (2-12), (2-13), and $P_{N_0}(x, y, t) = 0$, when $x \notin B(2r)$,

$$(\tau) \le Ct^4 \exp\left(-\frac{d^2(x, y)}{5t}\right). \tag{2-20}$$

Now we estimate (γ).

$$\begin{aligned}(\gamma) &= -\int_0^t \int_{M^n} \left(\frac{\partial}{\partial s} - \Delta_z\right) P_{N_0}(z, y, s) \cdot (\Delta_z H(x, z, t-s))\, d\mu(z)\, ds\\ &= -\int_0^t \int_{M^n} \left[\Delta_z \left(\frac{\partial}{\partial s} - \Delta_z\right) P_{N_0}(z, y, s)\right] \cdot H(x, z, t-s)\, d\mu(z)\, ds.\end{aligned}$$

Similarly as with (2-12) and (2-13), from (2-6), when $z \in B(r)$,

$$\left|\Delta_z\Big(\frac{\partial}{\partial s} - \Delta_z\Big) P_{N_0}(z, y, s)\right| \le C_3 s \exp\Big(-\frac{d^2(z, y)}{4s}\Big), \tag{2-21}$$

and when $z \in B(2r)\setminus B(r)$,

$$\left|\Delta_z\Big(\frac{\partial}{\partial s} - \Delta_z\Big) P_{N_0}(z, y, s)\right| \le C_4 s^{-n/2-3} \exp\Big(-\frac{d^2(z, y)}{4s}\Big). \tag{2-22}$$

Following a similar argument as in the proof of (2-9), using (2-21) and (2-22) instead of (2-12) and (2-13),

$$(\gamma) \le Ct^2 \exp\Big(-\frac{d^2(x, y)}{5t}\Big). \tag{2-23}$$

From (2-20) and (2-23),

$$\left|\frac{\partial}{\partial t} F_{N_0}(x, y, t)\right| \le (\gamma) + (\tau) \le Ct^2 \exp\Big(-\frac{d^2(x, y)}{5t}\Big). \qquad \square$$

3. The short time asymptotics of $N(H, t)$

From (2-5) and (2-7) in Theorem 2.1, there exists $0 < t_0 \le 1$ such that

$$\tfrac{1}{2} \le (4\pi t)^{n/2} \exp\Big(\frac{d^2(x, y)}{4t}\Big) H_{N_0}(x, y, t) \le 2 \tag{3-1}$$

holds when $x \in B(r/2)$ and $0 < t \le t_0$. In Sections 3 and 4, we assume that $t \in (0, t_0]$ and (M^n, g) and H are from Theorem 2.1.

Lemma 3.1.

$$\int_{B(r/2)} \Big[\ln \frac{H(x, y, t)}{H_{N_0}(x, y, t)}\Big] \cdot H(x, y, t)\, d\mu(x) = O(t^2). \tag{3-2}$$

Proof. Assume $x \in B(r/2)$, $t \le t_0$. Then $P_{N_0}(x, y, t) = H_{N_0}(x, y, t)$. Hence

$$F_{N_0}(x, y, t) = H(x, y, t) - H_{N_0}(x, y, t).$$

From (2-9),

$$|F_{N_0}(x, y, t)| \le Ct^{N_0+1-n/2} \exp\Big(-\frac{d^2(x, y)}{5t}\Big). \tag{3-3}$$

If $F_{N_0}(x, y, t) > 0$,

$$\begin{aligned}\left|\ln \frac{H}{H_{N_0}} \cdot H\right|(x, y, t) &= \ln\Big(1 + \frac{F_{N_0}}{H_{N_0}}\Big) \cdot H \le \frac{F_{N_0}}{H_{N_0}} \cdot H \\ &\le Ct^{N_0+1} \exp\Big(\frac{d^2(x, y)}{20t}\Big) \cdot H(x, y, t).\end{aligned}$$

If $F_{N_0}(x,y,t)\le 0$, $H(x,y,t)\le H_{N_0}(x,y,t)$ and

$$\left|\ln\frac{H}{H_{N_0}}\cdot H\right|(x,y,t)=|\ln H(x,y,t)-\ln H_{N_0}(x,y,t)|\cdot H(x,y,t)$$
$$=\left|\frac{1}{\xi}[H(x,y,t)-H_{N_0}(x,y,t)]\right|\cdot H(x,y,t),$$

where $H(x,y,t)\le\xi\le H_{N_0}(x,y,t)$. Hence

$$\left|\ln\frac{H}{H_{N_0}}\cdot H\right|(x,y,t)\le\left|\frac{F_{N_0}}{H}\right|\cdot H=F_{N_0}\le Ct^{N_0+1-n/2}\exp\left(-\frac{d^2(x,y)}{5t}\right).$$

By the above,

$$\left|\ln\frac{H}{H_{N_0}}\cdot H\right|(x,y,t)\le Ct^4\left[t^{n/2}\exp\left(\frac{d^2(x,y)}{20t}\right)\cdot H+\exp\left(-\frac{d^2(x,y)}{5t}\right)\right]. \tag{3-4}$$

From (3-1) and (3-3),

$$\begin{aligned}H(x,y,t)&\le|H_{N_0}|+|F_{N_0}|\\&\le 2(4\pi t)^{-n/2}\exp\left(-\frac{d^2(x,y)}{4t}\right)+Ct^4\cdot\exp\left(-\frac{d^2(x,y)}{5t}\right).\end{aligned} \tag{3-5}$$

By (3-4) and (3-5),

$$\left|\ln\frac{H}{H_{N_0}}\cdot H\right|(x,y,t)\le Ct^4. \tag{3-6}$$

Hence

$$\int_{B(r/2)}\left[\ln\frac{H(x,y,t)}{H_{N_0}(x,y,t)}\right]\cdot H(x,y,t)\,d\mu(x)=O(t^2).$$ □

Proof of (1-1).

$$-\int_{M^n}fH\,d\mu=\int_{M^n\setminus B(r/2)}(-fH)\,d\mu+\int_{B(r/2)}(-fH)\,d\mu=(\mathrm{I})+(\mathrm{II}).$$

First we estimate (I). From [Li and Yau 1986], we have

$$H(x,y,t)\le CV^{-1/2}(B_x(\sqrt{t}))V^{-1/2}(B_y(\sqrt{t}))\cdot\exp\left[CKt-\frac{d^2(x,y)}{5t}\right].$$

If $x\in M^n\setminus B(r/2)$ and t is small enough, using the volume comparison theorem,

$$\begin{aligned}H(x,y,t)&\le CV^{-1}(B_y(\sqrt{t}))\cdot\frac{V_{-K}(\sqrt{t}+d)}{V_{-K}(\sqrt{t})}\cdot\exp\left[CKt-\frac{d}{5t}\right]\\&\le Ct^2\exp\left(-\frac{d^2}{6t}\right),\end{aligned} \tag{3-7}$$

where C depends on n, K, r, and the metric g near y. Choose t small enough such that $H \le Ct^2 \le e^{-1}$. Then, by the monotonicity of $h(x) = \ln x \cdot x$ on $(0, e^{-1}]$,

$$|\ln H(x, y, t) \cdot H(x, y, t)| \le \left|\ln\Big[Ct^2 \exp\Big(-\frac{d^2}{6t}\Big)\Big] \cdot \Big[Ct^2 \exp\Big(-\frac{d^2}{6t}\Big)\Big]\right|.$$

Hence

$$\begin{aligned}
|(\mathrm{I})| &= \left|\int_{M^n\setminus B(r/2)} \Big[\ln H + \frac{n}{2}\ln(4\pi t)\Big] \cdot H\, d\mu(x)\right| \\
&\le \int_{M^n\setminus B(r/2)} \left|\ln\Big[Ct^2 \exp\Big(-\frac{d^2}{6t}\Big)\Big] \cdot \Big[Ct^2 \exp\Big(-\frac{d^2}{6t}\Big)\Big]\right| d\mu(x) \\
&\qquad\qquad + \frac{n}{2}\int_{M^n\setminus B(r/2)} \left|\ln(4\pi t) \cdot \Big[Ct^2 \exp\Big(-\frac{d^2}{6t}\Big)\Big]\right| d\mu(x) \\
&\le O(t^{3/2}).
\end{aligned} \tag{3-8}$$

In the last inequality, we used $\mathrm{Rc} \ge -Kg$ and the volume comparison theorem.

$$\begin{aligned}
|(\mathrm{II})| &= \int_{B(r/2)} \Big[\ln H + \frac{n}{2}\ln(4\pi t)\Big] \cdot H\, d\mu \\
&= \int_{B(r/2)} \ln\frac{H}{H_{N_0}} \cdot H\, d\mu(x) + \int_{B(r/2)} \Big[\ln H_{N_0} + \frac{n}{2}\ln(4\pi t)\Big] \cdot H\, d\mu(x) \\
&= (\mathrm{III}) + (\mathrm{IV}).
\end{aligned}$$

By Lemma 3.1, $(\mathrm{III}) = O(t^2)$. From Lemma 3.2, which follows,

$$(\mathrm{IV}) = -\frac{n}{2} + \frac{1}{2}R(y) \cdot t + O(t^{3/2}).$$ □

Lemma 3.2.

$$\int_{B(r/2)} \Big[\ln H_{N_0} + \frac{n}{2}\ln(4\pi t)\Big] \cdot H d\mu(x) = -\frac{n}{2} + \tfrac{1}{2}R(y) \cdot t + O(t^{3/2}). \tag{3-9}$$

Proof. Set $(\mathrm{I}) := \int_{B(r/2)} [\ln H_{N_0} + (n/2)\ln(4\pi t)] \cdot H\, d\mu(x)$. From Theorem 2.1,

$$\ln H_{N_0} = -\frac{n}{2}\ln(4\pi t) - \frac{d^2(x, y)}{4t} + \ln\Big(\sum_{k=0}^{N_0} \varphi_k t^k\Big)$$

and

$$\ln\Big(\sum_{k=0}^{N_0} \varphi_k t^k\Big) = \ln\varphi_0 + \frac{\varphi_1}{\varphi_0} \cdot t + O(t^2).$$

Hence

$$(\mathrm{I}) = \int_{B(r/2)} \Big[-\frac{d^2(x, y)}{4t} + \ln\varphi_0 + \frac{\varphi_1}{\varphi_0} \cdot t + O(t^2)\Big] \cdot H\, d\mu(x).$$

Now using Theorem 2.1(iii),

$$\begin{aligned}\text{(3-10)}\qquad \text{(I)} &= \int_{B(r/2)}\Big[-\frac{d^2}{4t}+\tfrac{1}{12}R_{pq}(y)x^px^q+O(d^3)+\Big(\frac{R(y)}{6}+O(d)\Big)t\Big] \\ &\qquad\qquad \cdot H\,d\mu(x)+O(t^2) \\ &= \text{(II)}+\text{(III)}+\text{(IV)}+\text{(V)}+\text{(VI)}+O(t^2),\end{aligned}$$

where

$$\begin{aligned}\text{(II)} &= \int_{B(r/2)}\Big(-\frac{d^2(x,y)}{4t}\Big)\cdot H\,d\mu(x),\\ \text{(III)} &= \frac{1}{12}\int_{B(r/2)}\big(R_{pq}(y)x^px^q\big)\cdot H\,d\mu(x),\\ \text{(IV)} &= C\int_{B(r/2)} d^3(x,y)\cdot H(x,y,t)\,d\mu(x),\\ \text{(V)} &= \frac{R(y)}{6}t\cdot\int_{B(r/2)} H(x,y,t)\,d\mu(x),\\ \text{(VI)} &= Ct\cdot\int_{B(r/2)} d(x,y)\cdot H(x,y,t)\,d\mu(x).\end{aligned}$$

From (3-7),

$$\int_{B(r/2)} H=\int_{M^n} H-\int_{M^n\setminus B(r/2)} H=1+O(t^2).$$

Hence

$$\text{(V)}=\tfrac{1}{6}R(y)\cdot t+O(t^2).$$

Using (3-5) and the fact that

$$\int_{R^n} O(|x|^k)(4\pi t)^{-n/2}\exp\Big(-\frac{|x|^2}{4t}\Big)\,dx=O(t^{k/2}),$$

where k is any nonnegative integer, we can get (IV) $=O(t^{3/2})$ and (VI) $=O(t^{3/2})$. Similarly,

$$\text{(III)}=\tfrac{1}{6}R(y)\cdot t+O(t^2)$$

Finally, from Lemma 3.3, which follows,

$$\text{(II)}=-\frac{n}{2}+\tfrac{1}{6}R(y)\cdot t+O(t^{3/2}). \qquad \square$$

Lemma 3.3.

$$\text{(3-11)}\qquad -\frac{1}{4t}\int_{B(r/2)} d^2(x,y)\cdot H\,d\mu(x)=-\frac{n}{2}+\tfrac{1}{6}R(y)\cdot t+O(t^{3/2}).$$

Proof. (II) $:= -(1/(4t))\int_{B(r/2)} d^2(x,y)\cdot H\,d\mu(x)$. Then

$$\text{(II)} = -\frac{1}{4t}\int_{B(r/2)} d^2(x,y)\cdot(H_{N_0}+F_{N_0})\cdot\alpha\,dx, \tag{3-12}$$

where dx in the integral of (3-12) is the volume element of Euclidean space $\mathbb{R}^n$, and

$$\alpha = \sqrt{\det(g)} = 1-\tfrac{1}{6}R_{pq}(y)x^p x^q + O(d^3(x,y)).$$

Then

$$\begin{aligned}
\text{(II)} &= -\frac{1}{4t}\int_{B(r/2)} d^2(x,y)\cdot(4\pi t)^{-n/2}\exp\Big(-\frac{d^2(x,y)}{4t}\Big)(\varphi_0+\varphi_1 t)\cdot\alpha\,dx + O(t^2)\\
&= \Big[-\frac{1}{4t}-\frac{1}{24}R(y)\Big]\cdot\int_{B(r/2)} d^2\cdot(4\pi t)^{-n/2}\exp\Big(-\frac{d^2}{4t}\Big)\,dx\\
&\qquad + \frac{1}{48t}\int_{B(r/2)}(4\pi t)^{-n/2}(R_{pq}(y)x^p x^q)d^2\cdot\exp\Big(-\frac{d^2}{4t}\Big)\,dx + O(t^{3/2})\\
&= \Big[-1/4t - 1/24R(y)\Big]\cdot 2nt + \frac{1}{48t}I_n + O(t^{3/2}),
\end{aligned}$$

where

$$I_n = \int_{\mathbb{R}^n}(4\pi t)^{-n/2}\Big(\sum_{k=1}^n \lambda_k x_k^2\Big)\cdot\Big(\sum_{i=1}^n x_i^2\Big)\exp\Big(-\frac{1}{4t}\cdot\sum_{j=1}^n x_j^2\Big)\,dx.$$

In the above we diagonalize $R_{pq}(y)$ and let $\lambda_k = R_{kk}(y)$.

We can get $I_1 = 12\lambda_1 t^2$ and the induction formula

$$I_n = I_{n-1} + 4\Big(\sum_{i=1}^n \lambda_i\Big)t^2 + 4(n+1)\lambda_n t^2.$$

Then it is easy to get

$$I_n = 4(n+2)\Big(\sum_{i=1}^n \lambda_i\Big)t^2 = 4(n+2)R(y)t^2. \tag{3-13}$$

By all the above (II) $= -n/2 + (R(y)/6)t + O(t^{3/2})$. □

4. The short time asymptotics of $\partial N(H,t)/\partial t$

To study $\partial N(H,t)/\partial t$, we must first switch the order of differentiation with integration. Because the manifold M^n can be noncompact, we need to be more careful when doing this. The following lemma justifies this switch in our case.

Lemma 4.1.

$$\frac{\partial}{\partial t}\left[\int_{M^n} H(-f)\,d\mu(x)\right]=\int_{M^n}[H(-f)]_t\,d\mu(x). \tag{4-1}$$

Proof. Define $\varphi_\rho(x)=\phi(d(x,y)/\rho)$, where ϕ is defined in the appendix, $\rho>1$ is a constant. Fix $t>0$ and define $G(x,t)=[H(-f)](x,y,t)$. For any $\epsilon>0$, assume $1>l>0$ (if $l<0$, a similar argument works). Then

$$\begin{aligned}
&\left|\int_{M^n}\frac{G(x,t+l)-G(x,t)}{l}\,d\mu(x)-\int_{M^n}G_t\varphi_\rho\,d\mu(x)\right|\\
&\quad\le\int_{B(\rho)}|G_t(x,t+\xi_x l)-G_t(x,t)|\,d\mu(x)+2\int_{M^n\setminus B(\rho)}\sup_{s\in[t,t+l]}|G_t(x,s)|\,d\mu(x)\\
&\quad\le\int_{B(\rho)}\left|\frac{\partial^2}{\partial^2 t}G(x,t+\zeta_x l)\right|d\mu(x)\cdot l+2\int_{M^n\setminus B(\rho)}\Big(\sup_{s\in[t,t+l]}|G_t(x,s)|\Big)\,d\mu(x)\\
&\quad\le(\mathrm{I})+(\mathrm{II}).
\end{aligned}$$

We first estimate (II). From [Li and Yau 1986], for $s\in[t,t+l]$,

$$\frac{H_t}{H}(x,y,s)\ge\tfrac12\left[\frac{|\nabla H|^2}{H^2}-\frac{2n}{s}-CK\right]\ge-\frac{C}{s}, \tag{4-2}$$

where $C=C(K,n)$. From Corollary A.10,

$$\begin{aligned}
\frac{H_t}{H}&(x,s)\\
&\le\frac{2}{s}\left\{n+(4+Ks)\ln\frac{C(K,t+1)}{H(x,y,s)V^{1/2}(B_x(\sqrt{s/2}))V^{1/2}(B_y(\sqrt{s/2}))}\right\}\\
&\le\frac{C}{s}\left(1+|\ln H|+\left|\ln\left[V\left(B_x\left(\sqrt{\frac{s}{2}}\right)\right)\cdot V\left(B_y\left(\sqrt{\frac{s}{2}}\right)\right)\right]\right|\right).
\end{aligned} \tag{4-3}$$

When $x\in M^n\setminus B(\rho)$, using the volume comparison theorem,

$$\begin{aligned}
&\left|\ln\left[V\left(B_x\left(\sqrt{\frac{s}{2}}\right)\right)\cdot V\left(B_y\left(\sqrt{\frac{s}{2}}\right)\right)\right]\right|\\
&\qquad\le2\left|\ln V\left(B_y\left(\sqrt{\frac{s}{2}}\right)\right)\right|+\left|\ln V_{-K}\left(\sqrt{\frac{s}{2}}\right)\right|+|\ln V_{-K}(s+d(x,y))|\\
&\qquad\le C(|\ln s|+s+d),
\end{aligned} \tag{4-4}$$

where C is independent of ρ. From (4-2), (4-3), and (4-4),

$$\left|\frac{H_t}{H}\right|(x,s)\le\frac{C}{s}(|\ln H|+|\ln s|+s+d) \tag{4-5}$$

when $x\in M^n\setminus B(\rho)$.

From (4-5), on $M^n \backslash B(\rho)$,

$$
\begin{aligned}
(4\text{-}6)\quad |(-f)H_t|(x,s) &\le \Big[|\ln H| + \frac{n}{2}|\ln(4\pi s)|\Big] \cdot |H_t|(x,s) \\
&\le \Big[|\ln H| + \frac{n}{2}|\ln(4\pi s)|\Big] \cdot C|H| \cdot s^{-1}(|\ln H| + |\ln s| + s + d) \\
&\le \frac{C}{s} \cdot H[|\ln H|^2 + |\ln s|^2 + s^2 + d^2].
\end{aligned}
$$

From (4-5) and (4-6), if $s \in [t, t+l]$ and $x \in M^n \backslash B(\rho)$,

$$
\begin{aligned}
(4\text{-}7)\quad & |G_t(x,s)| \\
&\le \Big[|H_t| + \frac{n}{2s}|H| + |(-f)H_t|\Big](x,s) \\
&\le \frac{C}{s} H \cdot (|\ln H| + |\ln s| + s + d) + \frac{n}{2s}|H| + \frac{C}{s} H \cdot (|\ln H|^2 + |\ln s|^2 + s^2 + d^2) \\
&\le \frac{C}{s} H \cdot (|\ln H|^2 + |\ln s|^2 + s^2 + d^2),
\end{aligned}
$$

where C is independent of ρ. We can choose l smooth enough such that $(t+l) \le 2t$. Then, using (3-7) and (4-7), on $x \in M^n \backslash B(\rho)$,

$$
\begin{aligned}
(4\text{-}8)\quad |G_t(x,s)| &\le Cs \exp\Big(-\frac{d^2}{6s}\Big) \cdot \Big[\Big|C + 2\ln s - \frac{d^2}{6s}\Big|^2 + |\ln s|^2 + s^2 + d^2\Big] \\
&\le C(t+l) \exp\Big(-\frac{d^2}{6(t+l)}\Big) \cdot \Big[t^2 + d^2 + |\ln t|^2 + \Big(\frac{d^2}{t}\Big)^2\Big] \\
&\le Ct \exp\Big(-\frac{d^2}{12t}\Big) \cdot \Big[t^2 + |\ln t|^2 + \Big(\frac{d^2}{t}\Big)^2\Big].
\end{aligned}
$$

Hence, for any $\epsilon > 0$, we can find $\rho_0 > 1$ such that if $\rho \ge \rho_0$,

$$
(4\text{-}9)\qquad \int_{M^n \backslash B(\rho)} \big(\sup_{s \in [t,t+l]} |G_t(x,s)|\big)\, d\mu(x) < \frac{\epsilon}{4}
$$

On the other hand, because $0 < l < 1$,

$$
(4\text{-}10)\quad \int_{B(\rho)} |G_{tt}(x, t + \zeta_x l)|\, d\mu(x) \le \int_{B(\rho)} \sup_{s \in [t,t+1]} |G_{tt}(x,s)|\, d\mu(x) \le C(\rho).
$$

Choose $l \le \epsilon/(4C(\rho))$. From (4-9) and (4-10), if $\rho > \rho_0$,

$$
(4\text{-}11)\qquad \Big| \int_{M^n} \frac{G(x,t+l) - G(x,t)}{l}\, d\mu(x) - \int_{M^n} G_t \varphi_\rho\, d\mu(x) \Big| < \epsilon.
$$

It is easy to see from Lemma 4.3 and its proof that

$$
\lim_{\rho \to \infty} \int_{M^n} G_t \phi_\rho
$$

exists and

$$\lim_{\rho\to\infty}\int_{M^n} G_t\phi_\rho = \int_{M^n} G_t. \tag{4-12}$$

From (4-11) and (4-12), we get our conclusion. □

From results in [Cheng et al. 1981], $\lim_{t\to 0} t\ln H = -d^2/4$ and the limit is uniform for any x in $B(r)$. Hence we can assume

$$t\ln H(x, y, t) = -\frac{d^2(x, y)}{4} + \epsilon(t, x, y).$$

We sometimes simplify notation by denoting $\epsilon(t, x, y)$ by ϵ. Then

$$t(-f) = \frac{n}{2}t\ln(4\pi t) - \frac{d^2}{4} + \epsilon, \tag{4-13}$$

where $\lim_{t\to 0}\epsilon(t, x, y) = 0$, and the limit is uniform for any x in $B(r)$. Without loss of generality, we can assume that $\varphi_0(x, y) \geq 1/2$ when $x \in B(r/2)$.

Lemma 4.2.

$$\int_{B(r/2)} E(-f)\,d\mu(x) = -\frac{n}{2} + \tfrac{1}{3}R(y)\cdot t + o(t), \tag{4-14}$$

$$\int_{B(r/2)} E(-f)O(d(x, y))\,d\mu(x) = o(1), \tag{4-15}$$

where $\lim_{t\to 0} o(t)/t = 0$.

Proof.

$$\begin{aligned}\int_B E(-f)\,d\mu(x) &= \int_B \frac{H_{N_0}}{\sum_{k=0}^{N_0}\varphi_k t^k}\cdot(-f)\,d\mu(x)\\ &= \int_B\left(\frac{1}{\varphi_0} - \frac{\varphi_1}{\varphi_0^2}t\right)H(-f)\,d\mu(x) + o(t)\\ &= \int_B\left(1 + \tfrac{1}{12}R_{pq}(y)x^p x^q - \frac{R(y)}{6}t\right)H(-f) + o(t)\\ &= -\frac{n}{2} + \left(\frac{1}{2} + \frac{n}{12}\right)R(y)t + \frac{1}{12}\int_B R_{pq}(y)x^p x^q\cdot H(-f)\,d\mu(x) + o(t).\end{aligned} \tag{4-16}$$

In the last equation, we used (1-1).

We estimate the third term on the right side of (4-16).

$$
\begin{aligned}
\text{(I)} &:= \frac{1}{12}\int_B R_{pq}(y)x^p x^q \cdot H(-f)\,d\mu(x) \\
&= \frac{1}{12}\int_B R_{pq}(y)x^p x^q \cdot H\Big[\ln H_{N_0} + \frac{n}{2}\ln(4\pi t)\Big]d\mu(x) \\
&= \frac{1}{12}\int_B R_{pq}(y)x^p x^q \cdot H_{N_0}\Big[-\frac{d^2}{4t} + \ln\varphi_0\Big]\cdot \alpha\,dx + o(t) \\
&= -\frac{1}{48t}\int_B E\cdot d^2\cdot R_{pq}(y)x^p x^q\,dx + o(t) \\
&= -\frac{n+2}{12}R(y)t + o(t).
\end{aligned} \tag{4-17}
$$

In the last equation above, we used (3-13). From (4-16) and (4-17), we get (4-14). To prove (4-15), we follow a similar strategy.

$$
\begin{aligned}
\int_B E(-f)O(d)\,d\mu(x) &= \int_B \Big(\frac{1}{\varphi_0} - \frac{\varphi_1}{\varphi_0^2}t\Big)H(-f)O(d)\,d\mu(x) + o(1) \\
&= \int_B H_{N_0}\Big[\ln H_{N_0} + \frac{n}{2}\ln(4\pi t)\Big]O(d)\,d\mu(x) + o(1) \\
&= \int_B E\Big(-\frac{d^2}{4t} + \ln\varphi_0\Big)O(d)\,d\mu(x) + o(1) = o(1). \qquad \square
\end{aligned}
$$

Lemma 4.3.

$$
\int_{M^n\setminus B} |(-f)H_t|\,d\mu(x) = O(t^{1/2}),
$$

where $t \ll 1$ is small enough.

Proof. Similarly as with (4-6), on $M^n\setminus B$,

$$
|(-f)H_t| \le \frac{C}{t}\cdot H[|\ln H|^2 + |\ln t|^2 + t^2 + d^2]. \tag{4-18}
$$

Hence

$$
\begin{aligned}
\int_{M^n\setminus B} |(-f)H_t| &\le \frac{C}{t}\int_{M^n\setminus B} H\cdot|\ln H|^2 + \frac{C}{t}\int_{M^n\setminus B} H(|\ln t|^2 + t^2 + d^2) \\
&= \text{(I)} + \text{(II)}.
\end{aligned}
$$

Similarly as in the proof of (3-8), using (3-7), the volume comparison theorem, and the monotonicity of $h(x) = x(\ln x)^2$, when $x \in (0, e^{-2}]$,

$$
\text{(I)} \le O(t^{1/2}).
$$

Using (2-9), when $x \in M^n \backslash B$,

$$H \le |\eta H_{N_0}| + |F_{N_0}| \le C\Big[t^{-n/2}\exp\Big(-\frac{d^2}{4t}\Big) + t^4 \cdot \exp\Big(-\frac{d^2}{5t}\Big)\Big] = O(t^2)\widetilde{E}. \tag{4-19}$$

From (4-19), it is easy to get

$$(\mathrm{II}) \le O(t). \qquad \square$$

Proof of (1-2).

$$\begin{aligned}\frac{\partial}{\partial t}\Big[\int_{M^n} H(-f)\,d\mu(x)\Big] &= \int_{M^n}\Big[H_t + \frac{n}{2t}H + (-f)H_t\Big]\,d\mu(x)\\ &= \frac{n}{2t} + \int_{M^n\backslash B(r/2)} (-f)H_t\,d\mu(x) + \int_{B(\frac{r}{2})}(-f)H_t\,d\mu(x)\\ &= \frac{n}{2t} + (\mathrm{I}) + (\mathrm{II}).\end{aligned}$$

From Lemma 4.3, we have

$$(\mathrm{I}) = O(t^{1/2}).$$

From Lemma 4.4, which follows, we get

$$(\mathrm{II}) = -\frac{n}{2t} + \frac{1}{2}R(y) + o(1). \qquad \square$$

Lemma 4.4.

$$\int_B (-f)H_t\,d\mu(x) = -\frac{n}{2t} + \frac{1}{2}R(y) + o(1).$$

Proof. From (2-10) and (4-13),

$$\int_B (-f)H_t d\mu(x) = \int_B (-f)\cdot (H_{N_0})_t + O(t)$$

and

$$\begin{aligned}&\int_B (-f)(H_{N_0})_t\,d\mu(x)\\ &= \int_B \Big(\frac{d^2}{4t^2} - \frac{n}{2t}\Big)H_{N_0}\cdot(-f)\,d\mu(x) + \int_B E\varphi_1(-f)\,d\mu(x) + o(1)\\ &= \frac{1}{4t^2}\int_B H_{N_0}(-f)d^2\,d\mu(x) - \frac{n}{2t}\int_B H_{N_0}(-f)\,d\mu(x) + \int_B E\varphi_1(-f)\,d\mu(x) + o(1)\\ &= (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + o(1).\end{aligned}$$

Using Lemma 4.2,

$$(\mathrm{III}) = \int_B E\varphi_1(-f)\,d\mu(x) = \tfrac{1}{6}R(y)\int_B E(-f)\,d\mu(x) + \int_B E(-f)\cdot O(d)$$
$$= -\frac{n}{12}R(y) + o(1).$$

From (2-9) and (1-1),

$$\begin{aligned}(\mathrm{II}) &= -\frac{n}{2t}\int_B H_{N_0}(-f)\,d\mu(x)\\ &= -\frac{n}{2t}\int_B H(-f)\,d\mu(x) - \frac{n}{2t}\int_B O(t^{N_0+1})\widetilde{E}(-f)\,d\mu(x)\\ &= \frac{n^2}{4t} - \frac{n}{4}R(y) + o(1).\end{aligned}$$

Similarly, by Lemma 4.5, which follows,

$$(\mathrm{I}) = \frac{1}{4t^2}\int_B (H + O(t^{N_0+1})\widetilde{E})(-f)\cdot d^2\,d\mu(x) = \frac{1}{4t^2}\int_B H(-f)\cdot d^2\,d\mu(x) + o(1)$$
$$= -\frac{n(n+2)}{4t} + \Big(\frac{n}{3}+\frac{1}{2}\Big)R(y) + o(1).$$

From all the above,

$$\int_B (-f)H_t d\mu(x) = -\frac{n}{2t} + \frac{1}{2}R(y) + o(1) \qquad \square$$

Lemma 4.5.

$$\frac{1}{4t^2}\int_B H(-f)\cdot d^2 d\mu(x) = -\frac{n(n+2)}{4t} + \Big(\frac{n}{3}+\frac{1}{2}\Big)R(y) + o(1).$$

Proof. We use a strategy similar to that used in the proof of (1-1).

$$\frac{1}{4t^2}\int_B H(-f)\cdot d^2 d\mu(x)$$
$$= \frac{1}{4t^2}\int_B \Big[\ln H_{N_0} + \frac{n}{2}\ln(4\pi t)\Big] H d^2 d\mu(x) + \frac{1}{4t^2}\int_B \Big[\ln\frac{H}{H_{N_0}}\Big] H d^2 d\mu(x).$$

From (3-6),

$$\Big[\ln\frac{H}{H_{N_0}}\Big]H = O(t^4).$$

Hence,

$$
\begin{aligned}
&\frac{1}{4t^2}\int_B H(-f)\cdot d^2\,d\mu(x)\\
&=\frac{1}{4t^2}\int_B\Big[-\frac{d^2}{4t}+\ln\varphi_0+\frac{\varphi_1}{\varphi_0}t+O(t^2)\Big]Hd^2\cdot\alpha\,dx+o(1)\\
&=\frac{1}{4t^2}\int_B\Big(-\frac{d^2}{4t}+\frac{1}{12}R_{pq}(y)x^px^q+\frac{1}{6}R(y)t-\frac{R(y)}{24}d^2+\frac{1}{48t}R_{pq}(y)x^px^q\cdot d^2\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\cdot Ed^2\,dx+o(1)\\
&=-\frac{n(n+2)}{4t}+\frac{-n^2+2n+4}{24}R(y)+\frac{1}{192t^3}\int_{\mathbb{R}^n}ER_{pq}(y)x^px^q\cdot d^4\,dx+o(1).
\end{aligned}
$$

Define

$$Q_n=\int_{\mathbb{R}^n}ER_{pq}(y)x^px^q\cdot d^4\,dx=\int_{\mathbb{R}^n}E\cdot\Big(\sum_{i=1}^n\lambda_ix_i^2\Big)\cdot\Big(\sum_{j=1}^n x_j^2\Big)\,dx,$$

where we diagonalize $R_{pq}(y)$ and let $\lambda_i=R_{ii}(y)$. We can get $Q_1=120\lambda_1t^3$ and the induction formula

$$Q_n=Q_{n-1}+8(2n+5)\Big(\sum_{i=1}^n\lambda_i\Big)t^3+8(n^2+4n+3)\lambda_n\cdot t^3.$$

Then it is easy to get $Q_n=8(n^2+6n+8)R(y)\cdot t^3$. Hence

$$\frac{1}{4t^2}\int_B H(-f)\cdot d^2\,d\mu(x)=-\frac{n(n+2)}{4t}+\Big(\frac{n}{3}+\frac{1}{2}\Big)R(y)+o(1).\qquad\square$$

Appendix

Richard Hamilton [1993] established an upper bound of the Laplacian of the positive solution to the heat equation on closed manifolds. We generalize his theorem to complete manifolds with Ricci curvature bounded below. Our proof follows a strategy similar to that in [Kotschwar 2007]. We firstly establish a preliminary estimate on $t|\Delta u|$ so that the maximum principle of Ni and Tam [2004] may be applied to the quantity of interest in Hamilton's second derivative estimate.

We introduce a cut-off function ϕ defined on $\mathbb{R}$, which is a smooth nonnegative nonincreasing function which is 1 on $(-\infty, 1)$ and 0 on $[2, +\infty)$. We can further assume

$$|\phi'|\le 2,\quad |\phi''|+\frac{(\phi')^2}{\phi}\le 16. \tag{A-1}$$

To prove the following Bernstein-type local estimate, we employ a technique of W.-X. Shi [1989] from the estimation of derivatives of curvature under the Ricci flow (see also [Chow et al. 2008]). Define $F=(C+t|\nabla u|^2)t^2|\Delta u|^2$ and consider the evolution of F.

Lemma A.6. *Suppose (M^n, g) is a complete Riemannian manifold. If $|u(x,t)|\le \mathcal{M}$ is a solution to the heat equation on $B_p(4\rho)\times[0,T]$ for some $p\in M^n$, constants $\mathcal{M}, \rho, T, K>0$, and $\mathrm{Rc}\ge -Kg$ on $B_p(4\rho)$,*

$$t|\Delta u|\le C(n,K,\mathcal{M})\Big[1+T\Big(1+\frac{1}{\rho^2}\Big)\Big]\cdot\Big(\frac{1}{\rho}+1\Big)\cdot\Big[T+\coth\Big(\sqrt{\frac{K}{n-1}}\rho\Big)\Big] \tag{A-2}$$

holds on $B_p(\rho)\times[0,T]$.

Proof. From [Kotschwar 2007], we get that

$$t|\nabla u|^2\le C_1\Big[1+T\Big(1+\frac{1}{\rho^2}\Big)\Big]=:C_2 \tag{A-3}$$

holds on $B_p(2\rho)\times[0,T]$, where $C_1=C_1(K,\mathcal{M})$. Define $C_3=4C_2$, and

$$F(x,t)=(C_3+t|\nabla u(x,t)|^2)t^2|\Delta u(x,t)|^2.$$

A long but straightforward computation gives

$$\begin{aligned}\Big(\frac{\partial}{\partial t}-\Delta\Big)F&\\ =-2(C_3+t|\nabla u|^2)|\nabla\Delta u|^2-8t^3\sum_{i,j}\nabla_i\nabla_j u\nabla_i\Delta u\nabla_j u\Delta u-2t^3|\nabla^2u|^2\cdot|\Delta u|^2&\\ +2t(C_3+t|\nabla u|^2)|\Delta u|^2+[|\nabla u|^2-2t\,\mathrm{Rc}(\nabla u,\nabla u)]t^2|\Delta u|^2.&\end{aligned}$$

When $x\in B_p(4\rho)$, using $t|\nabla u|^2\le C_2=\frac14 C_3$ and $\mathrm{Rc}\ge -Kg$,

$$\begin{aligned}\Big(\frac{\partial}{\partial t}-\Delta\Big)F&\le -10t^3|\nabla u|^2\cdot|\nabla\Delta u|^2+8t^3|\nabla u|\cdot|\nabla\Delta u|\cdot|\nabla^2u|\cdot|\Delta u|\\ &\qquad -2t^3|\nabla^2u|^2\cdot|\Delta u|^2+C_4t|\Delta u|^2\\ &\le -\tfrac25 t^3|\nabla^2u|^2\cdot|\Delta u|^2+C_4t|\Delta u|^2,\end{aligned}$$

where $C_4=(2KT+11)C_2$. The term with the coefficient $-\frac25$ arose from the inequality $-10x^2+8xy-2y^2\le-\frac25y^2$. On the other hand, we know that $|\nabla^2u|^2\ge(1/n)|\Delta u|^2$. Hence

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)F\le-\frac{2}{5n}t^3|\Delta u|^4+C_4t|\Delta u|^2\le-\frac{1}{5nt}[t^2|\Delta u|^2]^2+\frac{5n}{4t}C_4\le-\frac{C_6}{t}F^2+\frac{C_5}{t}.$$

In the last equality we used $F\le(C_3+C_2)t^2|\Delta u|^2=5C_2t^2|\Delta u|^2$, and

$$C_5=C(n,K,\mathcal{M})(1+T)\Big[1+T\Big(1+\frac{1}{\rho^2}\Big)\Big], \tag{A-4}$$

$$C_6=C(n,K,\mathcal{M})\Big[1+T\Big(1+\frac{1}{\rho^2}\Big)\Big]^{-2}. \tag{A-5}$$

Define $\gamma(x) = \phi(d(x, p)/\rho)$. Then $\gamma(x)F(x,t)$ attains its maximum at a point $(x_0, t_0) \in B_p(2\rho) \times [0, T]$. The rest of the computation is at (x_0, t_0);

$$0 \le \left(\frac{\partial}{\partial t} - \Delta\right)(\gamma F) \le \gamma\left(-\frac{C_6}{t}F^2 + \frac{C_5}{t}\right) - \Delta\gamma \cdot F - 2\nabla\gamma\nabla F.$$

Note that at (x_0, t_0), $\nabla(\gamma F) = 0$. Letting $G = (\gamma F)(x_0, t_0)$, we get

$$0 \le -\frac{C_6}{t}G^2 + \left(2\frac{|\nabla\gamma|^2}{\gamma} - \Delta\gamma\right)G + \frac{C_5}{t} \tag{A-6}$$

and

$$\begin{aligned}\left(2\frac{|\nabla\gamma|^2}{\gamma} - \Delta\gamma\right) &= \frac{2}{\rho^2}\cdot\frac{|\phi'|^2}{\phi} - \frac{\phi''}{\rho^2} - \frac{\phi'}{\rho}\Delta d(x,p)\\ &\le \frac{32}{\rho^2} + \frac{2}{\rho}\cdot\coth\left(\sqrt{\frac{K}{n-1}}\rho\right).\end{aligned} \tag{A-7}$$

In the last inequality we used (A-1), $\mathrm{Rc} \ge -Kg$, and the Laplacian comparison theorem. From (A-4)–(A-7),

$$\begin{aligned}0 \le -G^2 + C(n, K, \mathcal{M})\left[1 + T\left(1+\frac{1}{\rho^2}\right)\right]^2 T\cdot\left[\frac{1}{\rho^2} + \frac{1}{\rho}\coth\left(\sqrt{\frac{K}{n-1}}\rho\right)\right]G\\ + C(n, K, \mathcal{M})\left[1 + T\left(1+\frac{1}{\rho^2}\right)\right]^3(1+T).\end{aligned}$$

Then it is easy to get

$$G \le C(n,K,\mathcal{M})\cdot\left[1+T\left(1+\frac{1}{\rho^2}\right)\right]^2(1+T)\cdot\left[\left(\frac{1}{\rho^2}+\frac{1}{\rho}\right)\coth\left(\sqrt{\frac{K}{n-1}}\rho\right)+1+T\left(1+\frac{1}{\rho^2}\right)\right].$$

Hence, on $B_p(\rho)$,

$$\begin{aligned}t^2|\Delta u|^2 &\le C_3^{-1}F \le C_3^{-1}G\\ &\le C(n,K,\mathcal{M})\cdot\left[1+T\left(1+\frac{1}{\rho^2}\right)\right]^2\cdot\left[\left(\frac{1}{\rho^2}+1\right)\cdot\left(T+\coth\left(\sqrt{\frac{K}{n-1}}\rho\right)\right)+1\right]\end{aligned}$$

Taking the square root in the above inequality, we get our conclusion. □

Letting $\rho \to \infty$, we get the following global estimate.

Corollary A.7. *Suppose (M^n, g) is a complete Riemannian manifold with* $\mathrm{Rc} \ge -Kg$, *and $|u(x,t)| \le \mathcal{M}$ is a solution to the heat equation on $M^n \times [0, T]$, where $K, \mathcal{M}, T$ are positive constants. Then*

$$t|\Delta u| \le C(n, K, \mathcal{M})(1+T)^2 \tag{A-8}$$

holds on $M^n \times [0, T]$.

We also need a maximum principle, due originally to Karp and Li [1982], which was stated more generally by Ni and Tam.

Theorem A.8 [Ni and Tam 2004, Theorem 1.2]. *Suppose (M^n, g) is a complete Riemannian manifold and $h(x,t)$ is a smooth function on $M^n \times [0,T]$ such that*

$$\Big(\frac{\partial}{\partial t} - \Delta\Big) f(x,t) \le 0$$

whenever $f(x,t) \ge 0$. Assume that

$$\int_0^T \int_{M^n} e^{-a\cdot d^2(x,p)} f_+^2(x,s)\, d\mu(x)\, ds < \infty$$

for some $a > 0$, where p is a fixed point on M^n and $f_+(x,t) := \max\{f(x,t), 0\}$. If $f(x,0) \le 0$ for all $x \in M^n$, $f(x,t) \le 0$ for all $(x,t) \in M^n \times [0,T]$.

Now we are ready to prove Hamilton's theorem in the complete case.

Theorem A.9. *Suppose (M^n, g) is a complete Riemannian manifold with $\mathrm{Rc} \ge -Kg$, and $0 < u(x,t) \le \mathcal{M}$ is a solution to the heat equation on $M^n \times [0,T]$, where $K, \mathcal{M}, T$ are positive constants. Then*

$$t\Big(\frac{\Delta u}{u} + \frac{|\nabla u|^2}{u^2}\Big) \le n + (4 + 2Kt) \ln \frac{\mathcal{M}}{u}. \tag{A-9}$$

Proof. Defining $u_\epsilon = u + \epsilon$ for $\epsilon > 0$, we obtain a solution satisfying $\epsilon < u_\epsilon \le \mathcal{M} + \epsilon =: \mathcal{M}_\epsilon$. Once the estimate has been proved for u_ϵ, the theorem follows by letting $\epsilon \to 0$. Consider the function

$$F(x,t) = t\Big(\Delta u_\epsilon + \frac{|\nabla u_\epsilon|^2}{u_\epsilon}\Big) - u_\epsilon\Big[n + (4 + 2Kt) \ln \frac{\mathcal{M}_\epsilon}{u_\epsilon}\Big].$$

A long but straightforward computation gives

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta\Big) F(x,t) &= u_\epsilon\Big[-2t|\nabla^2 \ln u_\epsilon|^2 + \Delta \ln u_\epsilon - (2 + 2Kt)|\nabla \ln u_\epsilon|^2 \\ &\qquad - 2t\,\mathrm{Rc}(\nabla \ln u_\epsilon, \nabla \ln u_\epsilon) - 2K \ln \frac{\mathcal{M}_\epsilon}{u_\epsilon}\Big] \\ &\le u_\epsilon\Big[-\frac{2t}{n}|\Delta \ln u_\epsilon|^2 + \Delta \ln u_\epsilon - 2|\nabla \ln u_\epsilon|^2\Big].\end{aligned} \tag{A-10}$$

If $F(x,t) \ge 0$ at (x,t),

$$-2|\nabla \ln u_\epsilon|^2 \le \Delta \ln u_\epsilon - \frac{n}{t}. \tag{A-11}$$

From (A-10) and (A-11),

$$\Big(\frac{\partial}{\partial t} - \Delta\Big) F(x,t) \le u_\epsilon\Big[-\frac{2t}{n}|\Delta \ln u_\epsilon|^2 + 2\Delta \ln u_\epsilon - \frac{n}{t}\Big] \le -\frac{n}{2t} < 0. \tag{A-12}$$

In (A-3) let $\rho \to \infty$. Then

$$t|\nabla u|^2 \le C(K, \mathcal{M}_\epsilon, T). \tag{A-13}$$

From (A-13) and (A-8),

$$F_+^2(x,t) \le \left[t\left(\Delta u_\epsilon + \frac{|\nabla u_\epsilon|^2}{u_\epsilon} \right) \right]^2 \le C(\epsilon, n, K, \mathcal{M}_\epsilon, T). \tag{A-14}$$

Using (A-14), for any $p \in M^n$ and $\rho > 0$,

$$\begin{aligned} &\int_0^T \int_{B_p(\rho)} \exp\big(-d^2(x,p)\big) F_+^2(x,t)\, d\mu(x)\, dt \\ &\qquad \le C(\epsilon, n, K, \mathcal{M}_\epsilon, T) \int_{M^n} \exp\big(-d^2(x,p)\big)\, d\mu(x) \le C. \end{aligned} \tag{A-15}$$

In the last equality we used the volume comparison theorem and $\mathrm{Rc} \ge -Kg$. Letting $\rho \to \infty$,

$$\int_0^T \int_{M^n} \exp -d^2(x,p) F_+^2(x,t)\, d\mu(x)\, dt \le C < \infty. \tag{A-16}$$

From (A-12) and (A-16), using Theorem A.8, we get $F(x,t) \le 0$ for all $0 \le t \le T$, completing the proof. □

We now give an the upper bound for the Laplacian of the heat kernel.

Corollary A.10. *Suppose (M^n, g) is a complete Riemannian manifold such that* $\mathrm{Rc} \ge -Kg$, *$H(x,y,t)$ is the heat kernel on M^n, and* $0 < t \le T$, *where K, T are positive constants. Then*

$$\begin{aligned} &\left(\Delta H + \frac{|\nabla H|^2}{H} \right)(x,y,t) \\ &\qquad \le \frac{2H(x,y,t)}{t} \left\{ n + (4 + Kt) \ln \frac{C(K,T)}{H(x,y,t) V^{1/2}(B_x(\sqrt{t/2})) V^{1/2}(B_y(\sqrt{t/2}))} \right\}. \end{aligned}$$

Proof. Note that if $s \in [t/2, t]$, from [Li and Yau 1986],

$$H(x,y,t) \le C(K,T) \cdot V^{1/2}\big(B_x\big(\sqrt{t/2}\big)\big) V^{1/2}\big(B_y\big(\sqrt{t/2}\big)\big).$$

Then apply Theorem A.9 on $u(x,s) = H(x,y,s+t/2)$ and $M^n \times [0, t/2]$. The conclusion follows from (A-9). □

Acknowledgements

The author thanks Zhiqin Lu, Brett Kotschwar for interest and suggestions and Peter Li and Jiaping Wang for their interest. After this paper was posted on arXiv, Jia-yong Wu kindly informed us that he had independently proved a Hamilton-type upper bound for complete manifolds with $\mathrm{Rc} \ge 0$ in detail; see [Wu 2013].

References

[Cheng et al. 1981] S. Y. Cheng, P. Li, and S. T. Yau, "On the upper estimate of the heat kernel of a complete Riemannian manifold", *Amer. J. Math.* **103**:5 (1981), 1021–1063. MR 83c:58083 Zbl 0484.53035

[Chow et al. 2008] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni, *The Ricci flow: techniques and applications, II: Analytic aspects*, Mathematical Surveys and Monographs **144**, American Mathematical Society, Providence, RI, 2008. MR 2008j:53114 Zbl 1157.53035

[Chow et al. 2010] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni, *The Ricci flow: techniques and applications, III: Geometric-analytic aspects*, Mathematical Surveys and Monographs **163**, American Mathematical Society, Providence, RI, 2010. MR 2011g:53142 Zbl 1216.53057

[Garofalo and Lanconelli 1989] N. Garofalo and E. Lanconelli, "Asymptotic behavior of fundamental solutions and potential theory of parabolic operators with variable coefficients", *Math. Ann.* **283**:2 (1989), 211–239. MR 90i:35040 Zbl 0638.35003

[Grigor'yan 2009] A. Grigor'yan, *Heat kernel and analysis on manifolds*, AMS/IP Studies in Advanced Mathematics **47**, American Mathematical Society, Providence, RI, 2009. MR 2011e:58041 Zbl 1206.58008

[Hamilton 1993] R. S. Hamilton, "A matrix Harnack estimate for the heat equation", *Comm. Anal. Geom.* **1**:1 (1993), 113–126. MR 94g:58215 Zbl 0799.53048

[Karp and Li 1982] L. Karp and P. Li, "The heat equation on complete Riemannian manifolds", unpublished paper, 1982, http://math.uci.edu/∼pli/heat.pdf.

[Kotschwar 2007] B. L. Kotschwar, "Hamilton's gradient estimate for the heat kernel on complete manifolds", *Proc. Amer. Math. Soc.* **135**:9 (2007), 3013–3019. MR 2008e:58034 Zbl 1127.58021

[Li 2012] P. Li, *Geometric analysis*, Cambridge Studies in Advanced Mathematics **134**, Cambridge University Press, 2012. MR 2962229 Zbl 1246.53002

[Li and Yau 1986] P. Li and S.-T. Yau, "On the parabolic kernel of the Schrödinger operator", *Acta Math.* **156**:3-4 (1986), 153–201. MR 87f:58156 Zbl 0611.58045

[Malliavin and Stroock 1996] P. Malliavin and D. W. Stroock, "Short time behavior of the heat kernel and its logarithmic derivatives", *J. Differential Geom.* **44**:3 (1996), 550–570. MR 98c:58164 Zbl 0873.58063

[Ni 2004a] L. Ni, "The entropy formula for linear heat equation", *J. Geom. Anal.* **14**:1 (2004), 87–100. MR 2004m:53118a Zbl 1044.58030

[Ni 2004b] L. Ni, "Addenda to 'The entropy formula for linear heat equation'", *J. Geom. Anal.* **14**:2 (2004), 369–374. MR 2004m:53118b Zbl 1062.58028

[Ni 2006] L. Ni, "A note on Perelman's LYH-type inequality", *Comm. Anal. Geom.* **14**:5 (2006), 883–905. MR 2007h:53103 Zbl 1116.58031

[Ni 2010] L. Ni, "The large time asymptotics of the entropy", pp. 301–306 in *Complex analysis*, edited by P. Ebenfelt et al., Birkhäuser, Basel, 2010. MR 2885123 Zbl 1223.58021

[Ni and Tam 2004] L. Ni and L.-F. Tam, "Kähler–Ricci flow and the Poincaré–Lelong equation", *Comm. Anal. Geom.* **12**:1-2 (2004), 111–141. MR 2005f:53108 Zbl 1067.53054

[Perelman 2002] G. Perelman, "The entropy formula for the Ricci flow and its geometric applications", preprint, 2002. Zbl 1130.53001 arXiv math/0211159

[Shi 1989] W.-X. Shi, "Deforming the metric on complete Riemannian manifolds", *J. Differential Geom.* **30**:1 (1989), 223–301. MR 90i:58202 Zbl 0676.53044

[Varadhan 1967] S. R. S. Varadhan, "On the behavior of the fundamental solution of the heat equation with variable coefficients", *Comm. Pure Appl. Math.* **20** (1967), 431–455. MR 34 #8001 Zbl 0155.16503

[Wu 2013] J.-Y. Wu, "Sharp Hamilton's Laplacian estimate for the heat kernel on complete manifolds", *Proc. Amer. Math. Soc.* **141**:12 (2013), 4401–4409. MR 3105882 Zbl 06218168

Received October 8, 2012. Revised November 16, 2012.

GUOYI XU
MATHEMATICS DEPARTMENT
UNIVERSITY OF CALIFORNIA, IRVINE
340 ROWLAND HALL
IRVINE, CA 92697
UNITED STATES
Current address:
MATHEMATICAL SCIENCES CENTER
TSINGHUA UNIVERSITY
100084 BEIJING
CHINA
gyxu@math.tsinghua.edu.cn

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2013 is US $400/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 266 No. 2 December 2013