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We establish family rigidity and vanishing theorems on the equivariant K-theory level for the Witten type operators on string^c manifolds introduced by Chen, Han, and Zhang.

1. Introduction

Witten [1988] derived a series of elliptic operators on the free loop space $\mathscr{L}M$ of a spin manifold M. In particular, the index of the formal signature operator on the loop space turns out to be exactly the elliptic genus constructed by Landweber and Stong [1988] and Ochanine [1987] in a topological way. Motivated by physics, Witten proposed that these elliptic operators should be rigid with respect to the circle action.

This claim of Witten was first proved by Taubes and Bott [Taubes 1989; Bott and Taubes 1989]. See also [Hirzebruch 1988; Krichever 1990] for other interesting cases. Using the modular invariance property, Kefeng Liu [1995; 1996] presented a simple and unified proof of the above result as well as various further generalizations. In particular, Liu established several new vanishing theorems.

Chen, Han, and Zhang [Chen et al. 2011] introduced a topological condition which they called the string^c condition for even-dimensional spin^c manifolds. Under this string^c condition, they constructed a Witten type genus which is the index of a Witten type operator, a linear combination of twisted spin^c Dirac operators. Furthermore, by applying Liu's method [1995; 1996], Chen, Han, and Zhang established the rigidity and vanishing theorems for this Witten type operator under the relevant anomaly cancellation condition; see [Chen et al. 2011, Theorem 3.2].

In many situations in geometry, it is rather natural and necessary to generalize the rigidity and vanishing theorems to the family case. On the equivariant Chern character level, Liu and Ma [2000; 2002] established several family rigidity and vanishing theorems. In [Liu et al. 2000; Liu et al. 2003], inspired by [Taubes 1989], Liu, Ma, and Zhang established the corresponding family rigidity and vanishing theorems on the equivariant K-theory level. As explained in [Liu et al. 2000; Liu

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et al. 2003], taking the Chern character might kill some torsion elements involved in the index bundle. Therefore, the rigidity and vanishing properties on the K-theory level are more subtle than those on the Chern character level.

The purpose of this paper is to establish the family rigidity and vanishing theorems on the equivariant K-theory level for the Witten type operators introduced in [Chen et al. 2011]. In fact, our main results in Theorem 2.2 may be regarded as an analogue of [Liu et al. 2000, Theorem 2.1; Liu et al. 2003, Theorems 2.1 and 2.2]. In particular, if the base manifold is a point, from our family rigidity theorem, one deduces [Chen et al. 2011, Theorem 3.2(i)]. Both the statement and the proof of Theorem 2.2 are inspired by those of [Liu et al. 2000, Theorem 2.1; Liu et al. 2003, Theorems 2.1 and 2.2], which essentially depend on the techniques developed by Taubes [1989] and Bismut and Lebeau [1991].

This paper is organized as follows. In Section 2, we state (in Theorem 2.2) and prove our main results, providing rigidity and vanishing for the family Witten type operators introduced in [Chen et al. 2011]. Section 3 is devoted to the proofs of two intermediate results, Theorems 2.8 and 2.9, which are used in the proof of Theorem 2.2.

2. Rigidity and vanishing theorems in K-theory

In this section, we establish the main results of this paper, the rigidity and vanishing theorems on the equivariant K-theory level for a family of spin^c manifolds. Such theorems hold under some anomaly cancellation assumption which is inspired by the string^c condition from [Chen et al. 2011]. For the particular case when the base manifold is a point, our results imply Theorem 3.2(i) of that reference.

This section is organized as follows. In Section 2A, we reformulate a K-theory version of the equivariant family index theorem which is proved in [Liu et al. 2003, Theorem 1.2; Liu et al. 2000, Theorem 1.1]. In Section 2B, we state our main results, the rigidity and vanishing theorems on the equivariant K-theory level for a family of spin^c manifolds. In Section 2C, we state two intermediate results on the relations between the family indices on the fixed point set, which are used to prove our main results stated in Section 2A. In Section 2D, we prove the family rigidity and vanishing theorems.

2A. ***A K-theory version of the equivariant family index theorem.*** Let M, B be two compact manifolds, and $\pi : M \to B$ a smooth fibration with compact fiber X such that $\dim X = 2l$. Let TX denote the relative tangent bundle carrying a Riemannian metric g^{TX}. We assume that TX is oriented. Let (W, h^W) be a complex Hermitian vector bundle over M.

Let (V, g^V) and $(V', g^{V'})$ be oriented real Euclidean vector bundles over M, of respective dimensions $2p$ and $2p'$. Let (L, h^L) be a complex Hermitian line bundle over M with the property that the vector bundle $U = TX \oplus V \oplus V'$ satisfies $\omega_2(U) = c_1(L) \mod 2$, where ω_2 denotes the second Stiefel–Whitney class, and c_1 denotes the first Chern class. Then the vector bundle U has a spin^c-structure. Let $S(U, L)$ be the fundamental complex spinor bundle for (U, L); see [Lawson and Michelsohn 1989, Appendix D].

Assume that there is a fiberwise S^1-action on M which lifts to V, V', L, and W, and assume the metrics g^{TX}, g^V, $g^{V'}$, h^L, and h^W are S^1-invariant. Also assume that the S^1-actions on TX, V, V', L lift to $S(U, L)$.

Let ∇^{TX} be the Levi–Civita connection on (TX, g^{TX}) along the fiber X. Let ∇^V and $\nabla^{V'}$ be S^1-invariant Euclidean connections on (V, g^V) and $(V', g^{V'})$, respectively. Let ∇^L and ∇^W be S^1-invariant Hermitian connections on (L, h^L) and (W, h^W), respectively.

The Clifford algebra bundle $C(TX)$ is the bundle of Clifford algebras over X whose fiber at $x \in X$ is the Clifford algebra $C(T_xX)$; see [Lawson and Michelsohn 1989]. Let $C(V)$ and $C(V')$ be the Clifford algebra bundles of (V, g^V) and $(V', g^{V'})$.

Let $\{e_i\}_{i=1}^{2l}$ and $\{f_j\}_{j=1}^{2p}$ be oriented orthonormal bases for (TX, g^{TX}) and (V, g^V), respectively. We denote by $c(\cdot)$ the Clifford action of $C(TX)$, $C(V)$, and $C(V')$ on $S(U, L)$. Let τ be the involution of $S(U, L)$ given by

$$\tau = (\sqrt{-1})^{l+p} c(e_1) \cdots c(e_{2l}) c(f_1) \cdots c(f_{2p}). \tag{2-1}$$

In the rest of the paper, we say that τ is the involution determined by $TX \oplus V$. We decompose $S(U, L) = S_+(U, L) \oplus S_-(U, L)$ corresponding to τ such that $\tau|_{S_\pm(U,L)} = \pm 1$. Let $\nabla^{S(U,L)}$ be the Hermitian connection on $S(U, L)$ induced by ∇^{TX}, ∇^V, $\nabla^{V'}$, and ∇^L; see [Lawson and Michelsohn 1989, Appendix D]. Then $\nabla^{S(U,L)}$ preserves the $\mathbb{Z}_2$-grading of $S(U, L)$ induced by (2-1). Let $\nabla^{S(U,L)\otimes W}$ be the Hermitian connection on $S(U, L) \otimes W$ obtained from the tensor product of $\nabla^{S(U,L)}$ and ∇^W. Let $D^X \otimes W$ be the family twisted spin^c-Dirac operator on the fiber X defined by

$$D^X \otimes W = \sum_{i=1}^{2l} c(e_i) \nabla_{e_i}^{S(U,L)\otimes W}. \tag{2-2}$$

By [Liu and Ma 2000, Proposition 1.1], the index bundle $\mathrm{Ind}_\tau(D^X \otimes W)$ over B is well-defined in the equivariant K-group $K_{S^1}(B)$. Using the same notations as in [Liu et al. 2003, (1.4)–(1.7)], we write, as an identification of virtual S^1-bundles,

$$\mathrm{Ind}_\tau(D^X \otimes W) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Ind}_\tau(D^X \otimes W, n) \otimes [n], \tag{2-3}$$

where by $[n]$ ($n \in \mathbb{Z}$) we mean the one-dimensional complex vector space on which S^1 acts as multiplication by g^n for a generator $g \in S^1$.

Let $F=\{F_\alpha\}$ be the fixed point set of the circle action on M. Then $\pi: F_\alpha\to B$ (respectively $\pi: F\to B$) is a smooth fibration with fiber Y_α (respectively Y). Let $\widetilde{\pi}: N\to F$ denote the normal bundle to F in M. Then $N=TX/TY$. We identify N as the orthogonal complement of TY in $TX|_F$. Then $TX|_F$ admits the following S^1-equivariant decomposition (see [Liu et al. 2003, (1.8)]):

$$TX|_F=\bigoplus_{v\neq 0} N_v\oplus TY, \tag{2-4}$$

where N_v is a complex vector bundle such that $g\in S^1$ acts on it by g^v with $v\in\mathbb{Z}\backslash\{0\}$. Clearly, $N=\bigoplus_{v\neq 0} N_v$. We regard N as a complex vector bundle and write $N_{\mathbb{R}}$ for the underlying real vector bundle of N. For $v\neq 0$, let $N_{v,\mathbb{R}}$ denote the underlying real vector bundle of N_v.

Similarly, let (see [Liu et al. 2003, (1.9) and (1.46)])

$$V|_F=\bigoplus_{v\neq 0} V_v\oplus V_0^{\mathbb{R}},\quad V'|_F=\bigoplus_{v\neq 0} V'_v\oplus V_0'^{\mathbb{R}},\quad W|_F=\bigoplus_{v} W_v, \tag{2-5}$$

be the S^1-equivariant decompositions of the restrictions of V, V', and W over F, respectively, where V_v, V'_v, and W_v $(v\in\mathbb{Z})$ are complex vector bundles over F on which $g\in S^1$ acts by g^v, and $V_0^{\mathbb{R}}$ and $V_0'^{\mathbb{R}}$ are the real subbundles of V and V', respectively, such that S^1 acts as identity. For $v\neq 0$, let $V_{v,\mathbb{R}}$ and $V'_{v,\mathbb{R}}$ denote the underlying real vector bundles of V_v and V'_v. Write $2p_0=\dim V_0^{\mathbb{R}}$ and $2l_0=\dim Y$.

Let us write (compare with [Liu et al. 2003, (1.47)])

$$L_F=L\otimes\Big(\bigotimes_{v\neq 0}\det N_v\otimes\bigotimes_{v\neq 0}\det V_v\otimes\bigotimes_{v\neq 0}\det V'_v\Big)^{-1}. \tag{2-6}$$

Then $TY\oplus V_0^{\mathbb{R}}\oplus V_0'^{\mathbb{R}}$ has a spin^c-structure. Let $S(TY\oplus V_0^{\mathbb{R}}\oplus V_0'^{\mathbb{R}}, L_F)$ be the fundamental spinor bundle for $(TY\oplus V_0^{\mathbb{R}}\oplus V_0'^{\mathbb{R}}, L_F)$. Let R be a Hermitian complex vector bundle equipped with a Hermitian connection over F. We denote by $D^Y\otimes R$ the family (twisted) spin^c Dirac operator on $S(TY\oplus V_0^{\mathbb{R}}\oplus V_0'^{\mathbb{R}}, L_F)\otimes R$ defined as in (2-2) and by $D^{Y_\alpha}\otimes R$ its restriction to Y_α.

Recall that $N_{v,\mathbb{R}}$ and $V_{v,\mathbb{R}}$ are canonically oriented by their complex structures. The decompositions (2-4), (2-5) induce the orientations of TY and $V_0^{\mathbb{R}}$ respectively. Let $\{e_i\}_{i=1}^{2l_0}$, $\{f_j\}_{j=1}^{2p_0}$ be the corresponding oriented orthonormal basis of (TY, g^{TY}) and $(V_0^{\mathbb{R}}, g^{V_0^{\mathbb{R}}})$. The involution of $S(TY\oplus V_0^{\mathbb{R}}\oplus V_0'^{\mathbb{R}}, L_F)$ is canonically associated to that of $S(U,L)$, which we still denote by τ, which is given by

$$\tau=(\sqrt{-1})^{l_0+p_0}c(e_1)\cdots c(e_{2l_0})c(f_1)\cdots c(f_{2p_0}). \tag{2-7}$$

Let $S(TY\oplus V_0^{\mathbb{R}}\oplus V_0'^{\mathbb{R}}, L_F)=S_+(TY\oplus V_0^{\mathbb{R}}\oplus V_0'^{\mathbb{R}}, L_F)\oplus S_-(TY\oplus V_0^{\mathbb{R}}\oplus V_0'^{\mathbb{R}}, L_F)$ be the $\mathbb{Z}_2$-grading of $S(TY\oplus V_0^{\mathbb{R}}\oplus V_0'^{\mathbb{R}}, L_F)$ induced by τ.

Let $C(N_{\mathbb{R}})$ and $C(V_{v,\mathbb{R}})$ be the Clifford algebra bundle of

$$(N_{\mathbb{R}}, g^{TX}|_{N_{\mathbb{R}}}) \quad \text{and} \quad V_{v,\mathbb{R}}, g^{V}|_{V_{v,\mathbb{R}}}),$$

respectively. By [Liu et al. 2003, (1.10)], $\Lambda(\bar{N}^*)$ is a $C(N_{\mathbb{R}})$-Clifford module with the involution $\tau^N|_{\Lambda^{\text{even/odd}}(\bar{N}^*)} = \pm 1$. Similarly to [Liu et al. 2003, (1.10)], we can define the Clifford action of $C(V_{v,\mathbb{R}})$ on $\Lambda(\overline{V_v}^*)$. Then $\Lambda(\overline{V_v}^*)$ is a $C(V_{v,\mathbb{R}})$-Clifford module with the involution $\tau_v^V|_{\Lambda^{\text{even/odd}}(\overline{V_v}^*)} = \pm 1$.

By restricting to F, one has the isomorphism of $\mathbb{Z}_2$-graded $C(TX)$-Clifford modules over F as follows (compare with [Liu et al. 2003, (1.49)]):

$$\text{(2-8)} \quad (S(U, L), \tau)\big|_F \simeq \big(S(TY \oplus V_0^{\mathbb{R}} \oplus V_0'^{\mathbb{R}}, L_F), \tau\big) \mathbin{\widehat{\otimes}} (\Lambda \bar{N}^*, \tau^N) \mathbin{\widehat{\otimes}} \widehat{\bigotimes_{v\neq 0}} (\Lambda \overline{V_v}^*, \tau_v^V) \mathbin{\widehat{\otimes}} \widehat{\bigotimes_{v\neq 0}} (\Lambda \overline{V_v'}^*, \mathrm{id}),$$

where id denotes the trivial involution and $\widehat{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product (see [Lawson and Michelsohn 1989, p. 11]). Furthermore, the isomorphism (2-8) gives the identifications of the canonical connections on the bundles (compare with [Liu et al. 2003, (1.13)]).

Let S^1 act on $L|_F$ by sending $g \in S^1$ to g^{l_c} ($l_c \in \mathbb{Z}$) on F. Then l_c is locally constant on F. Following [Liu et al. 2003, (1.50)], we define the following elements in $K(F)[\![q^{1/2}]\!]$:

$$\text{(2-9)} \quad R(q) = q^{\frac{1}{2}(\sum_v |v| \dim N_v - \sum_v v \dim V_v - \sum_v v \dim V_v' + l_c)} \bigotimes_{v>0} (\mathrm{Sym}_{q^v}(N_v) \otimes \det N_v) \otimes \bigotimes_{v<0} \mathrm{Sym}_{q^{-v}}(\bar{N}_v) \otimes \bigotimes_{v\neq 0} \Lambda_{-q^v}(V_v) \otimes \bigotimes_{v\neq 0} \Lambda_{q^v}(V_v') \otimes \Big(\sum_v q^v W_v\Big) = \sum_n R_n q^n$$

and

$$\text{(2-10)} \quad R'(q) = q^{1/2(-\sum_v |v| \dim N_v - \sum_v v \dim V_v - \sum_v v \dim V_v' + l_c)} \bigotimes_{v>0} \mathrm{Sym}_{q^{-v}}(\bar{N}_v) \otimes \bigotimes_{v<0} (\mathrm{Sym}_{q^v}(N_v) \otimes \det N_v) \otimes \bigotimes_{v\neq 0} \Lambda_{-q^v}(V_v) \otimes \bigotimes_{v\neq 0} \Lambda_{q^v}(V_v') \otimes \Big(\sum_v q^v W_v\Big) = \sum_n R_n' q^n.$$

As explained in [Liu et al. 2003, p. 139], since $TX \oplus V \oplus V' \oplus L$ is spin, one gets

$$\text{(2-11)} \qquad \sum_v v \dim N_v + \sum_v v \dim V_v + \sum_v v \dim V_v' + l_c \equiv 0 \mod 2.$$

Therefore, $R(q), R'(q) \in K(F)[\![q]\!]$.

The following theorem was essentially proved in [Liu et al. 2003, Theorem 1.2].

Theorem 2.1. *For $n \in \mathbb{Z}$, the following identity holds in $K(B)$:*

$$\operatorname{Ind}_\tau(D^X \otimes W, n) = \sum_\alpha (-1)^{\sum_{0<v} \dim N_v} \operatorname{Ind}_\tau(D^{Y_\alpha} \otimes R_n) = \sum_\alpha (-1)^{\sum_{v<0} \dim N_v} \operatorname{Ind}_\tau(D^{Y_\alpha} \otimes R'_n). \tag{2-12}$$

2B. ***Family rigidity and vanishing theorems.*** Let $\pi : M \to B$ be a fibration of compact manifolds with compact fiber X and $\dim X = 2l$. We assume that S^1 acts fiberwise on M and TX has an S^1-invariant spin^c structure. Let K_X be the S^1-equivariant complex line bundle over M which is induced by the S^1-invariant spin^c structure of TX. Let $S(TX, K_X)$ be the complex spinor bundle of (TX, K_X); see [Lawson and Michelsohn 1989, Appendix D].

Let V be an even-dimensional real vector bundle over M. We assume that V has an S^1-invariant spin structure. Let $S(V) = S^+(V) \oplus S^-(V)$ be the spinor bundle of V. Let W be an S^1-equivariant complex vector bundle over M. Let $K_W = \det(W)$ be the determinant line bundle of W.

We define the following elements in $K(M)[[q^{1/2}]]$:

$$R_1(V) = \big(S^+(V) + S^-(V)\big) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(V),$$

$$R_2(V) = \big(S^+(V) - S^-(V)\big) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^n}(V), \tag{2-13}$$

$$R_3(V) = \bigotimes_{n=1}^{\infty} \Lambda_{-q^{n-1/2}}(V), \qquad R_4(V) = \bigotimes_{n=1}^{\infty} \Lambda_{q^{n-1/2}}(V),$$

$$Q_1(W) = \bigotimes_{n=0}^{\infty} \Lambda_{q^n}(\overline{W}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(W) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^{n-1/2}}(\overline{W}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^{n-1/2}}(W) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^{n-1/2}}(\overline{W}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^{n-1/2}}(W).$$

For $N \in \mathbb{Z}$, $N \geq 1$, let $y = e^{2\pi i/N} \in \mathbb{C}$. Let G_y be the multiplicative group generated by y. Following [Witten 1988], as in [Liu et al. 2000, Section 2.1], we consider the fiberwise action G_y on W and $\overline{W}$ by sending $y \in G_y$ to y on W and y^{-1} on $\overline{W}$. Then G_y acts naturally on $Q_1(W)$.

Let $H^*_{S^1}(M, \mathbb{Z}) = H^*(M \times_{S^1} ES^1, \mathbb{Z})$ denote the S^1-equivariant cohomology group of M, where ES^1 is the universal S^1-principal bundle over the classifying space BS^1 of S^1. So $H^*_{S^1}(M, \mathbb{Z})$ is a module over $H^*(BS^1, \mathbb{Z})$ induced by the projection $\bar{\pi} : M \times_{S^1} ES^1 \to BS^1$. Let $p_1(\cdot)_{S^1}$ denote the first S^1-equivariant

Pontryagin class and $\omega_2(\cdot)_{S^1}$ the second S^1-equivariant Stiefel–Whitney class. As $V \times_{S^1} ES^1$ is spin over $M \times_{S^1} ES^1$, one knows that $\frac{1}{2}p_1(V)_{S^1}$ is well-defined in $H^*_{S^1}(M, \mathbb{Z})$; see [Taubes 1989, pp. 456–457]. Recall that

$$H^*(BS^1, \mathbb{Z}) = \mathbb{Z}[\![u]\!] \tag{2-14}$$

with u a generator of degree 2.

In the following, we denote by $D^X \otimes R$ the family twisted spin^c Dirac operator acting fiberwise on $S(TX, K_X) \otimes R$. Recall that if $\mathrm{Ind}(D^X \otimes R, n)$ vanishes for $n \neq 0$, we say that $D^X \otimes R$ is rigid on the equivariant K-theory level for the S^1-action.

Now we can state the main results of this paper, which can be thought of as analogous to [Liu et al. 2000, Theorem 2.1].

Theorem 2.2. *Assume $w_2(W)_{S^1} = w_2(TX)_{S^1}$, $\frac{1}{2}p_1(V + 3W - TX)_{S^1} = e \cdot \bar{\pi}^* u^2$ ($e \in \mathbb{Z}$) in $H^*_{S^1}(M, \mathbb{Z})$, and $c_1(W) = 0 \mod N$. For $i = 1, 2, 3, 4$, consider the family of $G_y \times S^1$-equivariant twisted spin^c Dirac operators*

$$D^X \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \bigotimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX) \otimes R_i(V) \otimes Q_1(W). \tag{2-15}$$

(i) *If $e = 0$, these operators are rigid on the equivariant K-theory level for the S^1-action.*

(ii) *If $e < 0$, the index bundles of these operators are zero in $K_{G_y \times S^1}(B)$. In particular, these index bundles are zero in $K_{G_y}(B)$.*

Remark 2.3. As explained in [Liu et al. 2000, Remark 2.1], $w_2(W)_{S^1} = w_2(TX)_{S^1}$ means that $\frac{1}{2}p_1(3W - TX)_{S^1}$ is well defined and that $c_1(K_W \otimes K_X^{-1})_{S^1} = 0 \mod 2$. By [Hattori and Yoshida 1976, Corollary 1.2], the S^1-action on M can be lifted to $(K_W \otimes K_X^{-1})^{1/2}$ and is compatible with the S^1-action on $K_W \otimes K_X^{-1}$.

Take $N = 1$, that is, we forget the G_y-action on W and remove the corresponding assumption $c_1(W) = 0 \mod N$. Furthermore, take $W = K_X$ and $V = 0$. Then an interesting consequence of Theorem 2.2 is the following family rigidity and vanishing property, which may be thought of as an extension of [Liu et al. 2003, Theorem 2.3] to the spin^c case. When the base manifold is a point, it turns out to be exactly [Chen et al. 2011, Theorem 3.2(i)].

Corollary 2.4. *Assume $\frac{1}{2}p_1(3K_X - TX)_{S^1} = e \cdot \bar{\pi}^* u^2$ ($e \in \mathbb{Z}$) in $H^*_{S^1}(M, \mathbb{Z})$. Consider the family of S^1-equivariant twisted spin^c Dirac operators*

$$D^X \otimes \bigotimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX) \otimes Q_1(K_X). \tag{2-16}$$

(i) *If* $e = 0$, *these operators are rigid on the equivariant* K*-theory level for the* S^1*-action.*

(ii) *If* $e < 0$, *the index bundles of these operators are zero in* $K_{S^1}(B)$. *In particular, these index bundles are zero in* $K(B)$.

Remark 2.5. The operators in (2-16) are the Witten type operators introduced in [Chen et al. 2011]. By taking $N = 1$, $W = K_X$, $V = 0$, and letting the base manifold B be a point in [Liu et al. 2000, Theorem 2.1], we get [Chen et al. 2011, Theorem 3.2(ii)]. It is rather natural to establish an analogue of [Liu et al. 2000, Theorem 2.1], which corresponds to [Chen et al. 2011, Theorem 3.2(i)]. That is one of the motivations of Theorem 2.2.

Actually, as in [Liu et al. 2000; Liu et al. 2003], our proof of Theorem 2.2 works under the following slightly weaker hypothesis. Let us first explain some notations.

For each $n > 1$, consider $\mathbb{Z}_n \subset S^1$, the cyclic subgroup of order n. We have the $\mathbb{Z}_n$-equivariant cohomology of M defined by

$$H^*_{\mathbb{Z}_n}(M, \mathbb{Z}) = H^*(M \times_{\mathbb{Z}_n} ES^1, \mathbb{Z}),$$

and there is a natural "forgetful" map

$$\alpha(S^1, \mathbb{Z}_n) : M \times_{\mathbb{Z}_n} ES^1 \to M \times_{S^1} ES^1$$

which induces a pullback

$$\alpha(S^1, \mathbb{Z}_n)^* : H^*_{S^1}(M, \mathbb{Z}) \to H^*_{\mathbb{Z}_n}(M, \mathbb{Z}).$$

We denote by $\alpha(S^1, 1)$ the arrow which forgets the S^1-action. Thus

$$\alpha(S^1, 1)^* : H^*_{S^1}(M, \mathbb{Z}) \to H^*(M, \mathbb{Z})$$

is induced by the inclusion of M into $M \times_{S^1} ES^1$ as a fiber over BS^1.

Finally, note that if $\mathbb{Z}_n$ acts trivially on a space Y, then there is a new arrow $t^* : H^*(Y, \mathbb{Z}) \to H^*_{\mathbb{Z}_n}(Y, \mathbb{Z})$ induced by the projection $t : Y \times_{\mathbb{Z}_n} ES^1 = Y \times B\mathbb{Z}_n \to Y$.

Let $\mathbb{Z}_\infty = S^1$. For each $1 < n \le +\infty$, let $i : M(n) \to M$ be the inclusion of the fixed point set of $\mathbb{Z}_n \subset S^1$ in M, and so i induces $i_{S^1} : M(n) \times_{S^1} ES^1 \to M \times_{S^1} ES^1$.

In the rest of this paper, we suppose that there exists some integer $e \in \mathbb{Z}$ such that, for $1 < n \le +\infty$,

$$\tag{2-17} \begin{aligned} \alpha(S^1, \mathbb{Z}_n)^* \circ i^*_{S^1}\left(\tfrac{1}{2} p_1(V + 3W - TX)_{S^1} - e \cdot \bar{\pi}^* u^2\right) \\ = t^* \circ \alpha(S^1, 1)^* \circ i^*_{S^1}\left(\tfrac{1}{2} p_1(V + 3W - TX)_{S^1}\right). \end{aligned}$$

As indicated in [Liu et al. 2000, Remark 2.4], the relation (2-17) clearly follows from the hypothesis of Theorem 2.2 by pulling back and forgetting. Thus it is a weaker hypothesis.

We can now state a slightly more general version of Theorem 2.2.

Theorem 2.6. *Let the hypothesis be as in (2-17).*

(i) *If $e = 0$, the index bundles of the twisted spin^c Dirac operators in Theorem 2.2 are rigid on the equivariant K-theory level for the S^1-action.*

(ii) *If $e < 0$, the index bundles of the twisted spin^c Dirac operators in Theorem 2.2 are zero as elements in $K_{G_y \times S^1}(B)$, and, in particular, these index bundles are zero in $K_{G_y}(B)$.*

The rest of this section is devoted to a proof of Theorem 2.6.

2C. ***Two recursive formulas.*** Let $F = \{F_\alpha\}$ be the fixed point set of the circle action. Then $\pi : F \to B$ is a fibration with compact fiber denoted by $Y = \{Y_\alpha\}$.

As in [Liu et al. 2000, (2.5)], we may and we will assume that

$$\begin{aligned} TX|_F &= TY \oplus \bigoplus_{v>0} N_v, \\ TX|_F \otimes_{\mathbb{R}} \mathbb{C} &= TY \otimes_{\mathbb{R}} \mathbb{C} \oplus \bigoplus_{v>0} (N_v \oplus \bar{N}_v), \end{aligned} \tag{2-18}$$

where N_v are complex vector bundles on which S^1 acts by sending $g \in S^1$ to g^v. We also assume that (see [Liu et al. 2000, (2.6)])

$$V|_F = V_0^{\mathbb{R}} \oplus \bigoplus_{v>0} V_v, \qquad W|_F = \bigoplus_v W_v, \tag{2-19}$$

where V_v, W_v are complex vector bundles on which S^1 acts by sending g to g^v, and $V_0^{\mathbb{R}}$ is a real vector bundle on which S^1 acts as identity.

By (2-18), as in [Liu et al. 2000, (2.7)] , there is a natural isomorphism between the $\mathbb{Z}_2$-graded $C(TX)$-Clifford modules over F,

$$S(TX, K_X)|_F \simeq S\Big(TY, K_X \otimes \bigotimes_{v>0} (\det N_v)^{-1}\Big) \widehat{\otimes} \widehat{\bigotimes_{v>0}} \Lambda N_v. \tag{2-20}$$

For a complex vector bundle R over F, let $D^Y \otimes R$ and $D^{Y_\alpha} \otimes R$ be the twisted spin^c Dirac operators on $S\big(TY, K_X \bigotimes_{v>0} (\det N_v)^{-1}\big) \otimes R$ over F and F_α, respectively.

We introduce the following locally constant functions on F (see [Liu et al. 2000, (2.8)]):

$$\begin{aligned} e(N) &= \sum_{v>0} v^2 \dim N_v, & d'(N) &= \sum_{v>0} v \dim N_v, \\ e(V) &= \sum_{v>0} v^2 \dim V_v, & d'(V) &= \sum_{v>0} v \dim V_v, \\ e(W) &= \sum_{v} v^2 \dim W_v, & d'(W) &= \sum_{v} v \dim W_v. \end{aligned} \tag{2-21}$$

As in [Liu et al. 2000, (2.9)], we write

$$
\begin{aligned}
&L(N)=\bigotimes_{v>0}(\det N_v)^v, \quad L(V)=\bigotimes_{v>0}(\det V_v)^v,\\
&L(W)=\bigotimes_{v\neq 0}(\det W_v)^v, \quad L=L(N)^{-1}\otimes L(V)\otimes L(W)^3.
\end{aligned}
\tag{2-22}
$$

By using (2-17) and computing as in [Liu et al. 2000, (2.10)–(2.11)], we know that

$$
c_1(L)=0, \quad e(V)+3\cdot e(W)-e(N)=2e,
\tag{2-23}
$$

which means L is a trivial complex line bundle over each component F_α of F, and S^1 acts on L by sending g to g^{2e}, and G_y acts on L by sending y to $y^{3d'(W)}$. From [Liu et al. 2000, Lemma 2.1], we know that $d'(W) \bmod N$ is constant on each connected component of M. Thus we can extend L to a trivial complex line bundle over M, and we extend the S^1-action on it by sending $g\in S^1$ on the canonical section 1 of L to $g^{2e}\cdot 1$, and G_y acts on L by sending y to $y^{3d'(W)}$.

In what follows, if $R(q)=\sum_{m\in\frac{1}{2}\mathbb{Z}} q^m R_m \in K_{S^1}(M)[\![q^{1/2}]\!]$, we also denote $\operatorname{Ind}(D^X\otimes R_m, h)$ by $\operatorname{Ind}(D^X\otimes R(q), m, h)$. For $i=1,2,3,4$, set

$$
R_{i1}=(K_W\otimes K_X^{-1})^{1/2}\otimes R_i(V)\otimes Q_1(W).
\tag{2-24}
$$

As in [Liu et al. 2000, Proposition 2.1], by using Theorem 2.1, we first express the global equivariant family index via the family indices on the fixed point set.

Proposition 2.7. *For $m\in\frac{1}{2}\mathbb{Z}$, $h\in\mathbb{Z}$, $1\le i\le 4$, we have the following identity in $K_{G_y}(B)$:*

$$
\begin{aligned}
&\operatorname{Ind}\Big(D^X\otimes\bigotimes_{n=1}^{\infty}\operatorname{Sym}_{q^n}(TX)\otimes R_{i1}, m, h\Big)\\
&\quad=\sum_\alpha(-1)^{\sum_{v>0}\dim N_v}\operatorname{Ind}\Big(D^{Y_\alpha}\otimes\bigotimes_{n=1}^{\infty}\operatorname{Sym}_{q^n}(TX|_F)\otimes R_{i1}\\
&\qquad\qquad\otimes\operatorname{Sym}\Big(\bigoplus_{v>0}N_v\Big)\bigotimes_{v>0}\det N_v, m, h\Big).
\end{aligned}
\tag{2-25}
$$

To simplify the notation, we use the same convention as in [Liu et al. 2000, p. 945]. For $n_0\in\mathbb{N}^*$, we define a number operator P on $K_{S^1}(M)[\![q^{1/n_0}]\!]$ in the following way: if $R(q)=\bigoplus_{n\in(1/n_0)\mathbb{Z}} R_n q^n\in K_{S^1}(M)[\![q^{1/n_0}]\!]$, P acts on $R(q)$ by multiplication by n on R_n. From now on, we simply denote $\operatorname{Sym}_{q^n}(TX)$, $\Lambda_{q^n}(V)$, and $\Lambda_{q^n}(W)$ by $\operatorname{Sym}(TX_n)$, $\Lambda(V_n)$, and $\Lambda(W_n)$. In this way, P acts on TX_n, V_n, and W_n by multiplication by n, and the action of P on $\operatorname{Sym}(TX_n)$, $\Lambda(V_n)$, and $\Lambda(W_n)$ is naturally induced by the corresponding action of P on TX_n, V_n, and W_n. So the eigenspace of $P=n$ is just given by the coefficient of q^n of the corresponding element $R(q)$. For $R(q)=\bigoplus_{n\in(1/n_0)\mathbb{Z}} R_n q^n\in K_{S^1}(M)[\![q^{1/n_0}]\!]$, we also denote $\operatorname{Ind}(D^X\otimes R_m, h)$ by $\operatorname{Ind}(D^X\otimes R(q), m, h)$.

For $p \in \mathbb{N}$, we introduce the following elements in $K_{S^1}(F)[\![q]\!]$ (see [Liu et al. 2000, (3.6)]):

$$
\begin{aligned}
\mathscr{F}_p(X) &= \bigotimes_{n=1}^{\infty} \operatorname{Sym}(TY_n) \otimes \bigotimes_{v>0}\Big(\bigotimes_{n=1}^{\infty} \operatorname{Sym}(N_{v,n}) \bigotimes_{n>pv} \operatorname{Sym}(\bar{N}_{v,n})\Big),\\
\mathscr{F}'_p(X) &= \bigotimes_{v>0} \bigotimes_{0\le n\le pv} (\operatorname{Sym}(N_{v,-n}) \otimes \det N_v),\\
\mathscr{F}^{-p}(X) &= \mathscr{F}_p(X) \otimes \mathscr{F}'_p(X).
\end{aligned}
\tag{2-26}
$$

Then, from (2-18), over F, we have

$$
\mathscr{F}^0(X) = \bigotimes_{n=1}^{\infty} \operatorname{Sym}_{q^n}(TX|_F) \otimes \operatorname{Sym}\Big(\bigoplus_{v>0} N_v\Big) \otimes \bigotimes_{v>0} \det N_v. \tag{2-27}
$$

We now state two intermediate results on the relations between the family indices on the fixed point set. These two recursive formulas are used in the next subsection to prove Theorem 2.6.

Theorem 2.8 (compare with [Liu et al. 2000, Theorem 2.3]). *For $1 \le i \le 4$, h, $p \in \mathbb{Z}$, $p > 0$, $m \in \frac{1}{2}\mathbb{Z}$, the following identity holds in $K_{G_y}(B)$:*

$$
\begin{aligned}
\sum_\alpha (-1)^{\sum_{v>0} \dim N_v} \operatorname{Ind}\big(D^{Y_\alpha} \otimes \mathscr{F}^0(X) \otimes R_{i1}, m, h\big)
&= \sum_\alpha (-1)^{pd'(N)+\sum_{v>0}\dim N_v}\\
&\qquad \times \operatorname{Ind}\big(D^{Y_\alpha} \otimes \mathscr{F}^{-p}(X) \otimes R_{i1}, m + \tfrac{1}{2}p^2 e(N) + \tfrac{1}{2}pd'(N), h\big).
\end{aligned}
\tag{2-28}
$$

The proof of Theorem 2.8 will be given in Sections 3B–3D.

Theorem 2.9 (compare with [Liu et al. 2000, Theorem 2.4]). *For each α, $1 \le i \le 4$, h, $p \in \mathbb{Z}$, $p > 0$, $m \in \frac{1}{2}\mathbb{Z}$, the following identity holds in $K_{G_y}(B)$:*

$$
\begin{aligned}
\operatorname{Ind}\big(D^{Y_\alpha} \otimes \mathscr{F}^{-p}(X) \otimes R_{i1}, m + \tfrac{1}{2}p^2 e(N) + \tfrac{1}{2}pd'(N), h\big)&\\
= (-1)^{pd'(W)} \operatorname{Ind}(D^{Y_\alpha} \otimes \mathscr{F}^0(X) \otimes R_{i1} \otimes L^{-p}, m + ph + p^2 e, h)&.
\end{aligned}
\tag{2-29}
$$

The proof of Theorem 2.9 will be given in Section 3A.

2D. *A proof of Theorem 2.6.*

Proof. As $\frac{1}{2}p_1(3W - TX)_{S^1} \in H^*_{S^1}(X, \mathbb{Z})$ is well defined, one has the same identity as in [Liu et al. 2000, (2.27)]:

$$
d'(N) + d'(W) = 0 \mod 2. \tag{2-30}
$$

From Proposition 2.7, Theorems 2.8 and 2.9, and (2-30), for $1 \le i \le 4$, h, $p \in \mathbb{Z}$, $p > 0$, $m \in \frac{1}{2}\mathbb{Z}$, we get the following identity (compare with [Liu et al. 2000, (2.28)]):

$$\operatorname{Ind}\Big(D^X \otimes \bigotimes_{n=1}^{\infty} \operatorname{Sym}_{q^n}(TX) \otimes R_{i1}, m, h\Big) = \operatorname{Ind}\Big(D^X \otimes \bigotimes_{n=1}^{\infty} \operatorname{Sym}_{q^n}(TX) \otimes R_{i1} \otimes L^{-p}, m', h\Big), \tag{2-31}$$

with

$$m' = m + ph + p^2 e. \tag{2-32}$$

By (2-13) and (2-24), if $m < 0$ or $m' < 0$, either side of (2-31) is identically zero, which completes the proof of Theorem 2.6. In fact:

(i) Assume that $e = 0$. Let $h \in \mathbb{Z}$, $m_0 \in \frac{1}{2}\mathbb{Z}$, $h \neq 0$ be fixed. If $h > 0$, we take $m' = m_0$. Then, for p large enough, we get $m < 0$ in (2-31). If $h < 0$, we take $m = m_0$. Then, for p large enough, we get $m' < 0$ in (2-31).

(ii) Assume that $e < 0$. For $h \in \mathbb{Z}$, $m_0 \in \frac{1}{2}\mathbb{Z}$, we take $m = m_0$. Then, for p large enough, we get $m' < 0$ in (2-31). □

Remark 2.10. We point out here that there is a $\mathbb{Z}/k$ version of Theorem 2.6, which is an analogue of [Liu and Yu ≥ 2013, Theorem 4.4]. In fact, by using the mod k localization formula for $\mathbb{Z}/k$ circle actions on $\mathbb{Z}/k$ spin^c manifolds established in [Liu and Yu ≥ 2013, Theorem 2.7] (see also [Zhang 2003, Theorem 2.1] for the spin case), our proof of Theorem 2.6 can be applied to the case of $\mathbb{Z}/k$ manifolds with little modification.

Remark 2.11 (compare with [Liu et al. 2000, Remark 2.5]). If M is connected, by (2-31), for $1 \leq i \leq 4$, in $K_{G_y}(B)$, we get

$$\operatorname{Ind}\Big(D^X \otimes \bigotimes_{n=1}^{\infty} \operatorname{Sym}_{q^n}(TX) \otimes R_{i1}\Big) = \operatorname{Ind}\Big(D^X \otimes \bigotimes_{n=1}^{\infty} \operatorname{Sym}_{q^n}(TX) \otimes R_{i1}\Big) \otimes [3d'(W)], \tag{2-33}$$

where by $[3d'(W)]$ we mean the one-dimensional complex vector space on which $y \in G_y$ acts by multiplication by $y^{3d'(W)}$. In particular, if B is a point and $3d'(W) \neq 0$ mod N, we get the vanishing theorem for string^c manifolds analogue to the result of [Hirzebruch 1988, Section 10].

3. Proofs of Theorems 2.8 and 2.9

In this section, we prove the two intermediate results stated in Section 2C and used in Section 2D to prove our main results.

In Section 3A, following [Liu et al. 2000, Section 3.2], we prove Theorem 2.9. In Section 3B, we introduce the same refined shift operators as in [Liu et al. 2000, Section 4.2]. In Section 3C, we construct the twisted spin^c Dirac operator on

$M(n_j)$, the fixed point set of the naturally induced $\mathbb{Z}_{n_j}$-action on M. In Section 3D, by applying the S^1-equivariant index theorem in Section 2A, we finally prove Theorem 2.8.

3A. ***A proof of Theorem 2.9.*** We start with some notation and conventions.

Let H be the canonical basis of $\mathrm{Lie}(S^1) = \mathbb{R}$, that is,

$$\exp(tH) = \exp(2\sqrt{-1}\pi t),$$

for $t \in \mathbb{R}$. On the fixed point F, let $\boldsymbol{J}_H$ denote the operator which computes the weight of the S^1-action on $\Gamma(F, E|_F)$ for any S^1-equivariant vector bundle E over M. Then $\boldsymbol{J}_H$ can be explicitly given by (see [Liu et al. 2003, (3.2)])

$$\boldsymbol{J}_H = \frac{1}{2\pi\sqrt{-1}} \mathscr{L}_H|_{\Gamma(F, E|_F)}, \tag{3-1}$$

where $\mathscr{L}_H$ denotes the infinitesimal action of H on $\Gamma(M, E)$.

Recall that the $\mathbb{Z}_2$-grading on

$$S(TX, K_X) \otimes \bigotimes_{n=1}^{\infty} \mathrm{Sym}(TX_n)$$

is induced by the $\mathbb{Z}_2$-grading on $S(TX, K_X)$, and the $\mathbb{Z}_2$-grading on

$$S\Big(TY, K_X \otimes \bigotimes_{v>0} (\det N_v)^{-1}\Big) \otimes \mathscr{F}^{-p}(X)$$

is induced by the one on $S\big(TY, K_X \otimes \bigotimes_{v>0} (\det N_v)^{-1}\big)$. Write

$$\begin{aligned} Q_W^1 &= \bigotimes_{n=0}^{\infty} \Lambda(\overline{W}_n) \otimes \bigotimes_{n=1}^{\infty} \Lambda(W_n), & Q_W^2 &= \bigotimes_{n\in\mathbb{N}+\frac{1}{2}} \Lambda(\overline{W}_n) \otimes \bigotimes_{n\in\mathbb{N}+\frac{1}{2}} \Lambda(W_n), \\ F_V^1 &= S(V) \otimes \bigotimes_{n=1}^{\infty} \Lambda(V_n), & F_V^2 &= \bigotimes_{n\in\mathbb{N}+\frac{1}{2}} \Lambda(V_n). \end{aligned} \tag{3-2}$$

There are two natural $\mathbb{Z}_2$-gradings on F_V^1, F_V^2 (respectively Q_W^1, Q_W^2). The first grading is induced by the $\mathbb{Z}_2$-grading of $S(V)$ and the forms of homogeneous degrees in $\bigotimes_{n=1}^{\infty} \Lambda(V_n)$, $\bigotimes_{n\in\mathbb{N}+\frac{1}{2}} \Lambda(V_n)$ (respectively Q_W^2). We define $\tau_e|_{F_V^{i\pm}} = \pm 1$ $(i = 1, 2)$ (respectively $\tau_e|_{Q_W^{2\pm}} = \pm 1$) to be the involution defined by this $\mathbb{Z}_2$-grading. The second grading is the one for which F_V^i and Q_W^i $(i = 1, 2)$ are purely even, that is, $F_V^{i+} = F_V^i$, $Q_W^{i+} = Q_W^i$. We denote by $\tau_s = \mathrm{id}$ the involution defined by this $\mathbb{Z}_2$-grading. Set $Q(W) = Q_W^1 \otimes Q_W^2 \otimes Q_W^2$. We denote by τ_1 the $\mathbb{Z}_2$-grading on $Q(W)$ defined by

$$(Q(W), \tau_1) = (Q_W^1, \tau_s) \,\widehat{\otimes}\, (Q_W^2, \tau_e) \,\widehat{\otimes}\, (Q_W^2, \tau_s). \tag{3-3}$$

Then the coefficients of q^n ($n \in \frac{1}{2}\mathbb{Z}$) in (2-13) of $R_1(V)$, $R_2(V)$, $R_3(V)$, $R_4(V)$, $Q_1(W)$ are exactly the $\mathbb{Z}_2$-graded vector subbundles of (F_V^1, τ_s), (F_V^1, τ_e), (F_V^2, τ_e), (F_V^2, τ_s), $(Q(W), \tau_1)$, respectively, on which P acts by multiplication by n.

Furthermore, we denote by τ_e (respectively τ_s) the $\mathbb{Z}_2$-grading on

$$S(TX, K_X) \otimes \bigotimes_{n=1}^{\infty} \mathrm{Sym}(TX_n) \otimes F_V^i$$

($i = 1, 2$) induced by the above $\mathbb{Z}_2$-gradings. We denote by τ_{e1} (respectively τ_{s1}) the $\mathbb{Z}_2$-grading on $S(TX, K_X) \otimes \bigotimes_{n=1}^{\infty} \mathrm{Sym}(TX_n) \otimes F_V^i \otimes Q(W)$ ($i = 1, 2$) defined by

$$\tau_{e1} = \tau_e \mathbin{\widehat{\otimes}} \tau_1, \qquad \tau_{s1} = \tau_s \mathbin{\widehat{\otimes}} \tau_1. \tag{3-4}$$

We still denote by τ_{e1} (respectively τ_{s1}) the $\mathbb{Z}_2$-grading on

$$S\Big(TY, K_X \bigotimes_{v>0} (\det N_v)^{-1}\Big) \otimes \mathscr{F}^{-p}(X) \otimes F_V^i \otimes Q(W)$$

($i = 1, 2$) which is induced as in (3-4).

By (2-19), as in (2-20), there is a natural isomorphism between the $\mathbb{Z}_2$-graded $C(V)$-Clifford modules over F,

$$S(V)|_F \simeq S\Big(V_0^{\mathbb{R}}, \bigotimes_{v>0} (\det V_v)^{-1}\Big) \otimes \widehat{\bigotimes_{v>0}} \Lambda V_v. \tag{3-5}$$

Let $V_0 = V_0^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Using (2-19) and (3-5), we rewrite (3-2) on the fixed point set F as follows:

$$\begin{aligned}
Q_W^1 &= \bigotimes_{n=0}^{\infty} \Lambda\Big(\bigoplus_v \overline{W}_{v,n}\Big) \otimes \bigotimes_{n=1}^{\infty} \Lambda\Big(\bigoplus_v W_{v,n}\Big),\\
Q_W^2 &= \bigotimes_{n\in\mathbb{N}+\frac{1}{2}} \Lambda\Big(\bigoplus_v \overline{W}_{v,n}\Big) \otimes \bigotimes_{n\in\mathbb{N}+\frac{1}{2}} \Lambda\Big(\bigoplus_v W_{v,n}\Big),\\
F_V^1 &= \bigotimes_{n=1}^{\infty} \Lambda\Big(V_{0,n} \oplus \bigoplus_{v>0} (V_{v,n} \oplus \overline{V}_{v,n})\Big) \otimes S\Big(V_0^{\mathbb{R}}, \bigotimes_{v>0} (\det V_v)^{-1}\Big) \otimes \bigotimes_{v>0} \Lambda V_{v,0},\\
F_V^2 &= \bigotimes_{n\in\mathbb{N}+\frac{1}{2}} \Lambda\Big(V_{0,n} \oplus \bigoplus_{v>0} (V_{v,n} \oplus \overline{V}_{v,n})\Big).
\end{aligned} \tag{3-6}$$

We can reformulate Theorem 2.9 as follows.

Theorem 3.1. *For each $\alpha, h, p \in \mathbb{Z}$, $p > 0$, $m \in \frac{1}{2}\mathbb{Z}$, for $i = 1, 2$, $\tau = \tau_{e1}$ or τ_{s1}, the following identity holds in $K_{G_y}(B)$:*

$$\begin{aligned}
&\mathrm{Ind}_\tau\big(D^{Y_\alpha} \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \mathscr{F}^{-p}(X) \otimes F_V^i \otimes Q(W),\\
&\qquad\qquad m + \tfrac{1}{2}p^2 e(N) + \tfrac{1}{2}p d'(N), h\big)\\
&= (-1)^{pd'(W)} \mathrm{Ind}_\tau\big(D^{Y_\alpha} \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \mathscr{F}^0(X) \otimes F_V^i \otimes Q(W) \otimes L^{-p},\\
&\qquad\qquad m + ph + p^2 e, h\big).
\end{aligned} \tag{3-7}$$

Following [Taubes 1989] in spirit, we introduce the same shift operators as in [Liu et al. 2000, (3.9)]. For $p \in \mathbb{N}$, we set

$$
\begin{aligned}
&r_* : N_{v,n} \to N_{v,n+pv}, \quad && r_* : \overline{N}_{v,n} \to \overline{N}_{v,n-pv},\\
&r_* : V_{v,n} \to V_{v,n+pv}, \quad && r_* : \overline{V}_{v,n} \to \overline{V}_{v,n-pv},\\
&r_* : W_{v,n} \to W_{v,n+pv}, \quad && r_* : \overline{W}_{v,n} \to \overline{W}_{v,n-pv}.
\end{aligned}
\tag{3-8}
$$

Proposition 3.2. *For $p \in \mathbb{Z}$, $p > 0$, $i = 1, 2$, there are natural isomorphisms of vector bundles over F*:

$$
r_*(\mathscr{F}^{-p}(X)) \simeq \mathscr{F}^0(X) \otimes L(N)^p, \qquad r_*(F_V^i) \simeq F_V^i \otimes L(V)^{-p}. \tag{3-9}
$$

For any $p \in \mathbb{Z}$, $p > 0$, $i = 1, 2$, there are natural $G_y \times S^1$-equivariant isomorphisms of vector bundles over F,

$$
r_*(Q_W^i) \simeq Q_W^i \otimes L(W)^{-p}. \tag{3-10}
$$

In particular, one gets the $G_y \times S^1$-equivariant bundle isomorphism

$$
r_*(Q(W)) \simeq Q(W) \otimes L(W)^{-3p}. \tag{3-11}
$$

Proof. By Proposition 3.1 of [Liu et al. 2000], only the $i = 2$ case in (3-10) needs to be proved.

Using Equations (3.14)–(3.16) of the same reference, we have a natural $G_y \times S^1$-equivariant isomorphisms of vector bundles over F:

$$
\begin{aligned}
\bigotimes_{\substack{n\in\mathbb{N}+\frac12, v>0\\ 0<n<pv}} \Lambda^{i_n}(\overline{W}_{v,n-pv}) &\simeq \bigotimes_{\substack{n\in\mathbb{N}+\frac12, v>0\\ 0<n<pv}} \Lambda^{\dim W_v - i_n}(W_{v,-n+pv}) \otimes \bigotimes_{v>0} (\det \overline{W}_v)^{pv},\\
\bigotimes_{\substack{n\in\mathbb{N}+\frac12, v<0\\ 0<n<-pv}} \Lambda^{i'_n}(W_{v,n+pv}) &\simeq \bigotimes_{\substack{n\in\mathbb{N}+\frac12, v<0\\ 0<n<-pv}} \Lambda^{\dim W_v - i'_n}(\overline{W}_{v,-n-pv}) \otimes \bigotimes_{v>0} (\det W_v)^{-pv}.
\end{aligned}
\tag{3-12}
$$

From (2-22) and (3-12), we get (3-10) for the case $i = 2$. □

The following proposition, which is an analogue of [Liu et al. 2000, Proposition 3.2], is deduced from Proposition 3.2.

Proposition 3.3. *For $p \in \mathbb{Z}$, $p > 0$, $i = 1, 2$, the G_y-equivariant isomorphism of vector bundles over F induced by (3-9), (3-11), denoted by*

$$
\begin{aligned}
r_* : S\Big(TY, K_X \otimes \bigotimes_{v>0} (\det N_v)^{-1}\Big) \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \mathscr{F}^{-p}(X) \otimes F_V^i \otimes Q(W)\\
\longrightarrow S\Big(TY, K_X \otimes \bigotimes_{v>0} (\det N_v)^{-1}\Big) \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \mathscr{F}^0(X) \otimes F_V^i \otimes Q(W) \otimes L^{-p},
\end{aligned}
\tag{3-13}
$$

satisfies the identities

$$
\begin{aligned}
&r_*^{-1}\cdot \boldsymbol{J}_H \cdot r_* = \boldsymbol{J}_H,\\
&r_*^{-1}\cdot P\cdot r_* = P + p\boldsymbol{J}_H + p^2 e - \tfrac{1}{2}p^2 e(N) - \tfrac{1}{2}pd'(N).
\end{aligned}
\tag{3-14}
$$

For the $\mathbb{Z}_2$-gradings, we have

$$
r_*^{-1}\tau_e r_* = \tau_e,\quad r_*^{-1}\tau_s r_* = \tau_s,\quad r_*^{-1}\tau_1 r_* = (-1)^{pd'(W)}\tau_1. \tag{3-15}
$$

Proof. By the proof of [Liu et al. 2000, Proposition 3.2], we need to compute the action of $r_*^{-1}\cdot P\cdot r_*$ on

$$
\bigotimes_{\substack{n\in\mathbb{N}+\frac{1}{2},v>0\\ 0<n<pv}} \Lambda^{i_n}(\overline{W}_{v,n}) \otimes \bigotimes_{\substack{n\in\mathbb{N}+\frac{1}{2},v<0\\ 0<n<-pv}} \Lambda^{i'_n}(W_{v,n}).
$$

In fact, by (3-12),

$$
\begin{aligned}
r_*^{-1}\cdot P\cdot r_*
&= \sum_{\substack{n\in\mathbb{N}+\frac{1}{2},\, v>0\\ 0<n<pv}} (\dim W_v - i_n)(-n+pv) + \sum_{\substack{n\in\mathbb{N}+\frac{1}{2},\, v<0\\ 0<n<-pv}} (\dim W_v - i'_n)(-n-pv)\\
&= P + p\boldsymbol{J}_H + \tfrac{1}{2}p^2 e(W).
\end{aligned}
\tag{3-16}
$$

By [Liu et al. 2000, (3.21)–(3.23)], (2-21)–(2-23), and (3-16), we deduce the second line of (3-14). The first line of (3-14) is obvious.

Consider the $\mathbb{Z}_2$-gradings. The first two identities of (3-15) were proved in [Liu et al. 2003, (3.18)]. τ_1 changes only on

$$
\bigotimes_{\substack{n\in\mathbb{N}+\frac{1}{2},v>0\\ 0<n<pv}} \Lambda^{i_n}(\overline{W}_{v,n}) \otimes \bigotimes_{\substack{n\in\mathbb{N}+\frac{1}{2},v<0\\ 0<n<-pv}} \Lambda^{i'_n}(W_{v,n}).
$$

From (2-21) and (3-12), we get the third identity of (3-15). This completes the proof of Proposition 3.3. □

Theorem 3.1 is a direct consequence of Proposition 3.3. This also completes the proof of Theorem 2.9. □

The rest of this section is devoted to a proof of Theorem 2.8.

3B. ***The refined shift operators.*** We first introduce a partition of [0, 1] as in [Liu et al. 2000, pp. 942–943]. Set

$$
J = \{v\in\mathbb{N} \mid \text{there exists } \alpha \text{ such that } N_v \neq 0 \text{ on } F_\alpha\}
$$

and

$$
\Phi = \big\{\beta\in(0,1] \mid \text{there exists } v\in J \text{ such that } \beta v\in\mathbb{Z}\big\}. \tag{3-17}
$$

We order the elements in Φ so that

$$\Phi = \{\beta_i \mid 1 \le i \le J_0,\ J_0 \in \mathbb{N} \text{ and } \beta_i < \beta_{i+1}\}.$$

Then, for any integer $1 \le i \le J_0$, there exist $p_i, n_i \in \mathbb{N}$, $0 < p_i \le n_i$, with $(p_i, n_i) = 1$ such that

$$\beta_i = p_i/n_i. \tag{3-18}$$

Clearly, $\beta_{J_0} = 1$. We also set $p_0 = 0$ and $\beta_0 = 0$.

For $0 \le j \le J_0$, $p \in \mathbb{N}^*$, we write

$$\begin{aligned} I_j^p &= \Big\{(v,n) \in \mathbb{N}\times\mathbb{N} \,\Big|\, v \in J,\ (p-1)v < n \le pv,\ \frac{n}{v} = p-1+\frac{p_j}{n_j}\Big\},\\ \bar{I}_j^p &= \Big\{(v,n) \in \mathbb{N}\times\mathbb{N} \,\Big|\, v \in J,\ (p-1)v < n \le pv,\ \frac{n}{v} > p-1+\frac{p_j}{n_j}\Big\}. \end{aligned} \tag{3-19}$$

Clearly, I_0^p is the empty set. We define $\mathcal{F}_{p,j}(X)$ as in [Liu et al. 2000, (2.21)], analogously to (2-26). More specifically, we set

$$\begin{aligned} \mathcal{F}_{p,j}(X) &\\ &= \bigotimes_{n=1}^{\infty} \mathrm{Sym}(TY_n) \otimes \bigotimes_{v>0}\Big(\bigotimes_{n=1}^{\infty} \mathrm{Sym}(N_{v,n}) \otimes \bigotimes_{n>(p-1)v+\frac{p_j}{n_j}v} \mathrm{Sym}(\bar{N}_{v,n})\Big)\\ &\qquad\qquad \otimes \bigotimes_{\substack{v>0\\ 0\le n\le (p-1)v+\lfloor \frac{p_j}{n_j}v\rfloor}} (\mathrm{Sym}(N_{v,-n}) \otimes \det N_v)\\ &= \mathcal{F}_p(X) \otimes \mathcal{F}'_{p-1}(X) \otimes \bigotimes_{(v,n)\in \bar{I}_j^p} \mathrm{Sym}(\bar{N}_{v,n}) \otimes \bigotimes_{(v,n)\in \bigcup_{i=0}^{j} I_i^p} (\mathrm{Sym}(N_{v,-n}) \otimes \det N_v), \end{aligned} \tag{3-20}$$

where, for $s \in \mathbb{R}$, the notation $\lfloor s \rfloor$ denotes the greatest integer not exceeding s. Then

$$\mathcal{F}_{p,0}(X) = \mathcal{F}^{-p+1}(X), \quad \mathcal{F}_{p,J_0}(X) = \mathcal{F}^{-p}(X). \tag{3-21}$$

From the construction of β_i, we know that, for $v \in J$, there is no integer in $((p_{j-1}/n_{j-1})v, (p_j/n_j)v)$. Furthermore (see [Liu et al. 2000, (4.24)]),

$$\Big\lfloor \frac{p_{j-1}}{n_{j-1}} v \Big\rfloor = \begin{cases} \lfloor (p_j/n_j)v \rfloor - 1 & \text{if } v \equiv 0 \mod (n_j),\\ \lfloor (p_j/n_j)v \rfloor & \text{if } v \not\equiv 0 \mod (n_j). \end{cases} \tag{3-22}$$

We use the same shift operators r_{j*}, $1 \le j \le J_0$ as in [Liu et al. 2000, (4.21)], which refine the shift operator r_* defined in (3-8). For $p \in \mathbb{N}\backslash\{0\}$, set

$$\begin{aligned} r_{j*} &: N_{v,n} \to N_{v,n+(p-1)v+p_j v/n_j}, & r_{j*} &: \bar{N}_{v,n} \to \bar{N}_{v,n-(p-1)v-p_j v/n_j},\\ r_{j*} &: V_{v,n} \to V_{v,n+(p-1)v+p_j v/n_j}, & r_{j*} &: \bar{V}_{v,n} \to \bar{V}_{v,n-(p-1)v-p_j v/n_j},\\ r_{j*} &: W_{v,n} \to W_{v,n+(p-1)v+p_j v/n_j}, & r_{j*} &: \bar{W}_{v,n} \to \bar{W}_{v,n-(p-1)v-p_j v/n_j}. \end{aligned} \tag{3-23}$$

For $1 \le j \le J_0$, we define $\mathscr{F}(\beta_j)$, $F_V^1(\beta_j)$, $F_V^2(\beta_j)$, $Q_W^1(\beta_j)$, and $Q_W^2(\beta_j)$ over F as follows (compare with [Liu et al. 2000, (4.13)]):

(3-24)

$$\mathscr{F}(\beta_j) = \bigotimes_{0<n\in\mathbb{Z}} \operatorname{Sym}(TY_n) \otimes \bigotimes_{\substack{v>0 \\ v\equiv 0, n_j/2 \bmod n_j}} \bigotimes_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v} \operatorname{Sym}(N_{v,n} \oplus \overline{N}_{v,n})$$
$$\otimes \bigotimes_{0<v'<n_j/2} \operatorname{Sym}\Big(\bigoplus_{v\equiv v', -v' \bmod n_j} \Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v} N_{v,n} \oplus \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v} \overline{N}_{v,n} \Big)\Big),$$

$$F_V^1(\beta_j) = \Lambda\Big(\bigoplus_{0<n\in\mathbb{Z}} V_{0,n} \bigoplus_{\substack{v>0 \\ v\equiv 0, n_j/2 \bmod n_j}} \Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v} V_{v,n} \oplus \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v} \overline{V}_{v,n} \Big)$$
$$\bigoplus_{0<v'<n_j/2} \Big(\bigoplus_{v\equiv v', -v' \bmod n_j} \Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v} V_{v,n} \oplus \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v} \overline{V}_{v,n} \Big)\Big)\Big),$$

$$F_V^2(\beta_j) = \Lambda\Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{1}{2}} V_{0,n} \bigoplus_{\substack{v>0 \\ v\equiv 0, n_j/2 \bmod n_j}} \Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v+\frac{1}{2}} V_{v,n} \oplus \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v+\frac{1}{2}} \overline{V}_{v,n} \Big)$$
$$\bigoplus_{0<v'<n_j/2} \Big(\bigoplus_{v\equiv v', -v' \bmod n_j} \Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v+\frac{1}{2}} V_{v,n} \oplus \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v+\frac{1}{2}} \overline{V}_{v,n} \Big)\Big)\Big),$$

$$Q_W^1(\beta_j) = \Lambda\Big(\bigoplus_{v} \Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v} W_{v,n} \oplus \bigoplus_{0\le n\in\mathbb{Z}-\frac{p_j}{n_j}v} \overline{W}_{v,n} \Big)\Big),$$
$$Q_W^2(\beta_j) = \Lambda\Big(\bigoplus_{v} \Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v+\frac{1}{2}} W_{v,n} \oplus \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v+\frac{1}{2}} \overline{W}_{v,n} \Big)\Big).$$

Using (3-22), Equations (3-24), and computing directly, we get an analogue of [Liu et al. 2000, Proposition 4.1] which refines Proposition 3.2:

Proposition 3.4. *For $p \in \mathbb{Z}$, $p > 0$, $1 \le j \le J_0$, there are natural isomorphisms of vector bundles over F:*

$$r_{j*}(\mathscr{F}_{p,j-1}(X)) \simeq$$
$$\mathscr{F}(\beta_j) \otimes \bigotimes_{\substack{v>0 \\ v\equiv 0 \bmod n_j}} \operatorname{Sym}(\overline{N}_{v,0}) \otimes \bigotimes_{v>0} (\det N_v)^{\lfloor \frac{p_j}{n_j} v \rfloor + (p-1)v+1} \otimes \bigotimes_{\substack{v>0 \\ v\equiv 0 \bmod n_j}} (\det N_v)^{-1},$$

$$r_{j*}(\mathscr{F}_{p,j}(X)) \simeq \mathscr{F}(\beta_j) \otimes \bigotimes_{\substack{v>0 \\ v\equiv 0 \bmod n_j}} \operatorname{Sym}(N_{v,0}) \otimes \bigotimes_{v>0} (\det N_v)^{\lfloor \frac{p_j}{n_j} v \rfloor + (p-1)v+1},$$

$$r_{j*}(F_V^1) \simeq S\Big(V_0^{\mathbb{R}}, \bigotimes_{v>0} (\det V_v)^{-1} \Big) \otimes F_V^1(\beta_j)$$
$$\otimes \bigotimes_{\substack{v>0 \\ v\equiv 0 \bmod n_j}} \Lambda(V_{v,0}) \otimes \bigotimes_{v>0} (\det \overline{V}_v)^{\lfloor \frac{p_j}{n_j} v \rfloor + (p-1)v},$$

$$r_{j*}(F_V^2) \simeq F_V^2(\beta_j) \otimes \bigotimes_{\substack{v>0 \\ v\equiv n_j/2 \bmod n_j}} \Lambda(V_{v,0}) \otimes \bigotimes_{v>0} (\det \overline{V}_v)^{\lfloor \frac{p_j}{n_j} v + \frac{1}{2} \rfloor + (p-1)v}.$$

For $p \in \mathbb{Z}$, $p > 0$, $1 \le j \le J_0$, there are natural $G_y \times S^1$-equivariant isomorphisms of vector bundles over F,

$$\text{(3-25)}\quad r_{j*}(Q_W^1) \simeq Q_W^1(\beta_j) \otimes \bigotimes_{\substack{v>0 \\ v\equiv 0 \bmod n_j}} \det W_v \otimes \bigotimes_{v>0} (\det \overline{W}_v)^{\lfloor \frac{p_j}{n_j} v \rfloor + (p-1)v + 1} \otimes \bigotimes_{v<0} (\det W_v)^{\lfloor -\frac{p_j}{n_j} v \rfloor - (p-1)v},$$

$$r_{j*}(Q_W^2) \simeq Q_W^2(\beta_j) \otimes \bigotimes_{\substack{v>0 \\ v\equiv n_j/2 \bmod n_j}} \Lambda(W_{v,0}) \otimes \bigotimes_{\substack{v<0 \\ v\equiv n_j/2 \bmod n_j}} \Lambda(\overline{W}_{v,0}) \otimes \bigotimes_{v>0} (\det \overline{W}_v)^{\lfloor \frac{p_j}{n_j} v + \frac{1}{2} \rfloor + (p-1)v} \otimes \bigotimes_{v<0} (\det W_v)^{\lfloor -\frac{p_j}{n_j} v + \frac{1}{2} \rfloor - (p-1)v}.$$

Proof. By [Liu et al. 2000, Proposition 4.1], we need only prove the second isomorphism in (3-25). In fact, using [Liu et al. 2000, (3.14)], we have the natural $G_y \times S^1$-equivariant isomorphisms of vector bundles over F:

$$\text{(3-26)}\quad \bigotimes_{\substack{0<n\in\mathbb{Z}+\frac{1}{2},\, v>0 \\ n-(p-1)v-(p_j/n_j)v\le 0}} \Lambda^{i_n}(\overline{W}_{v,n-(p-1)v-(p_j/n_j)v}) \simeq \bigotimes_{v>0} (\det \overline{W}_v)^{\lfloor \frac{p_j}{n_j} v + \frac{1}{2} \rfloor + (p-1)v} \otimes \bigotimes_{\substack{0<n\in\mathbb{Z}+\frac{1}{2},\, v>0 \\ n-(p-1)v-(p_j/n_j)v\le 0}} \Lambda^{\dim W_v - i_n}(W_{v,-n+(p-1)v+(p_j/n_j)v}),$$

$$\text{(3-27)}\quad \bigotimes_{\substack{0<n\in\mathbb{Z}+\frac{1}{2},\, v<0 \\ n+(p-1)v+(p_j/n_j)v\le 0}} \Lambda^{i'_n}(W_{v,n+(p-1)v+(p_j/n_j)v}) \simeq \bigotimes_{v<0} (\det W_v)^{\lfloor -\frac{p_j}{n_j} v + \frac{1}{2} \rfloor - (p-1)v} \otimes \bigotimes_{\substack{0<n\in\mathbb{Z}+\frac{1}{2},\, v<0 \\ n+(p-1)v+(p_j/n_j)v\le 0}} \Lambda^{\dim W_v - i'_n}(\overline{W}_{v,-n-(p-1)v-(p_j/n_j)v}).$$

From the last equation in (3-24), together with (3-26) and (3-27), we get the second isomorphism in (3-25). The proof of Proposition 3.4 is complete. □

3C. ***The spin^c Dirac operators on $M(n_j)$.*** Recall that there is a nontrivial circle action on M which can be lifted to the circle actions on V and W.

For $n \in \mathbb{N}\backslash\{0\}$, let $\mathbb{Z}_n \subset S^1$ denote the cyclic subgroup of order n. Let $M(n_j)$ be the fixed point set of the induced $\mathbb{Z}_{n_j}$ action on M. Then

$$\pi : M(n_j) \to B$$

is a fibration with compact fiber $X(n_j)$. Let $N(n_j) \to M(n_j)$ be the normal bundle to $M(n_j)$ in M. As in [Bott and Taubes 1989, p. 151] (see also [Liu et al. 2000, Section 4.1; Liu et al. 2003, Section 4.1; Taubes 1989]), we see that $N(n_j)$ and V can be decomposed, as real vector bundles over $M(n_j)$, into

$$
\begin{aligned}
N(n_j) &= \bigoplus_{0<v<n_j/2} N(n_j)_v \oplus N(n_j)^{\mathbb{R}}_{n_j/2},\\
V|_{M(n_j)} &= V(n_j)^{\mathbb{R}}_0 \oplus \bigoplus_{0<v<n_j/2} V(n_j)_v \oplus V(n_j)^{\mathbb{R}}_{n_j/2},
\end{aligned}
\tag{3-28}
$$

where $V(n_j)^{\mathbb{R}}_0$ is the real vector bundle on which $\mathbb{Z}_{n_j}$ acts by identity, and $N(n_j)^{\mathbb{R}}_{n_j/2}$ and $V(n_j)^{\mathbb{R}}_{n_j/2}$ are defined to be zero if n_j is odd. Moreover, for $0 < v < n_j/2$, $N(n_j)_v$ and $V(n_j)_v$ each admit a unique complex structure making them into complex vector bundles on which $g \in \mathbb{Z}_{n_j}$ acts by g^v. We also denote by $V(n_j)_0$, $V(n_j)_{n_j/2}$, and $N(n_j)_{n_j/2}$ the corresponding complexification of $V(n_j)^{\mathbb{R}}_0$, $V(n_j)^{\mathbb{R}}_{n_j/2}$, and $N(n_j)^{\mathbb{R}}_{n_j/2}$.

Similarly, we also have the following $\mathbb{Z}_{n_j}$-equivariant decomposition of W over $M(n_j)$ into complex vector bundles:

$$
W|_{M(n_j)} = \bigoplus_{0\le v<n_j} W(n_j)_v,
\tag{3-29}
$$

where for $0 \le v < n_j$, $g \in \mathbb{Z}_{n_j}$ acts on $W(n_j)_v$ by sending g to g^v.

By [Liu et al. 2000, Lemma 4.1] (which generalizes [Bott and Taubes 1989, Lemmas 9.4 and 10.1] and [Taubes 1989, Lemma 5.1]), we know that the vector bundles $TX(n_j)$ and $V(n_j)^{\mathbb{R}}_0$ are orientable and even-dimensional. Thus $N(n_j)$ is orientable over $M(n_j)$. By (3-28), $V(n_j)^{\mathbb{R}}_{n_j/2}$ and $N(n_j)^{\mathbb{R}}_{n_j/2}$ are also orientable and even-dimensional. In what follows, we fix the orientations of $N(n_j)^{\mathbb{R}}_{n_j/2}$ and $V(n_j)^{\mathbb{R}}_{n_j/2}$. Then $TX(n_j)$ and $V(n_j)^{\mathbb{R}}_0$ are naturally oriented by (3-28) and the orientations of TX, V, $N(n_j)^{\mathbb{R}}_{n_j/2}$ and, $V(n_j)^{\mathbb{R}}_{n_j/2}$. Let $W(n_j)^{\mathbb{R}}_{n_j/2}$ be the underlying real vector bundle of $W(n_j)_{n_j/2}$, which are canonically oriented by its complex structure.

By (2-18), (2-19), (3-28), and (3-29), we get identifications of complex vector bundles over F (see [Liu et al. 2000, (4.9) and (4.12)]): for $0 < v \le n_j/2$,

$$
\begin{aligned}
N(n_j)_v|_F &= \bigoplus_{\substack{v'>0\\ v'\equiv v \bmod n_j}} N_{v'} \oplus \bigoplus_{\substack{v'>0\\ v'\equiv -v \bmod n_j}} \overline{N}_{v'},\\
V(n_j)_v|_F &= \bigoplus_{\substack{v'>0\\ v'\equiv v \bmod n_j}} V_{v'} \oplus \bigoplus_{\substack{v'>0\\ v'\equiv -v \bmod n_j}} \overline{V}_{v'},
\end{aligned}
\tag{3-30}
$$

and for $0 \le v < n_j$,

$$
W(n_j)_v|_F = \bigoplus_{v'\equiv v \bmod n_j} W_{v'}.
\tag{3-31}
$$

We also get identifications of real vector bundles over F (see [Liu et al. 2000, (4.11)]):

$$TX(n_j)|_F = TY \oplus \bigoplus_{\substack{v>0\\ v\equiv 0 \bmod n_j}} N_v, \qquad N(n_j)^{\mathbb{R}}_{n_j/2}|_F = \bigoplus_{\substack{v>0\\ v\equiv n_j/2 \bmod n_j}} N_v,$$

$$V(n_j)^{\mathbb{R}}_0|_F = V_0^{\mathbb{R}} \oplus \bigoplus_{\substack{v>0\\ v\equiv 0 \bmod n_j}} V_v, \qquad V(n_j)^{\mathbb{R}}_{n_j/2}|_F = \bigoplus_{\substack{v>0\\ v\equiv n_j/2 \bmod n_j}} V_v. \tag{3-32}$$

Moreover, we have an identifications of complex vector bundles over F:

$$TX(n_j)|_F \otimes_{\mathbb{R}} \mathbb{C} = TY \otimes_{\mathbb{R}} \mathbb{C} \oplus \bigoplus_{\substack{v>0\\ v\equiv 0 \bmod n_j}} (N_v \oplus \overline{N}_v),$$

$$V(n_j)_0|_F = V_0^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \oplus \bigoplus_{\substack{v>0\\ v\equiv 0 \bmod n_j}} (V_v \oplus \overline{V}_v). \tag{3-33}$$

As $(p_j, n_j) = 1$, we know that, for $v \in \mathbb{Z}$, $(p_j/n_j)v \in \mathbb{Z}$ if and only if $(v/n_j) \in \mathbb{Z}$. Also, $(p_j/n_j)v \in \mathbb{Z} + \frac{1}{2}$ if and only if $(v/n_j) \in \mathbb{Z} + \frac{1}{2}$. Also, if $v \equiv -v' \bmod n_j$, then

$$\{n \mid 0 < n \in \mathbb{Z} + (p_j/n_j)v\} = \{n \mid 0 < n \in \mathbb{Z} - (p_j/n_j)v'\}.$$

Using the identifications (3-30), (3-31), and (3-33), we can rewrite $\mathscr{F}(\beta_j)$, $F_V^1(\beta_j)$, $F_V^2(\beta_j)$, $Q_W^1(\beta_j)$, and $Q_W^2(\beta_j)$ over F defined in (3-24) as follows (compare with [Liu et al. 2000, (4.7)]):

$$\mathscr{F}(\beta_j) = \bigotimes_{0<n\in\mathbb{Z}} \mathrm{Sym}(TX(n_j)_n) \otimes \bigotimes_{0<v<n_j/2} \mathrm{Sym}\Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v} N(n_j)_{v,n} \oplus \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v} \overline{N(n_j)}_{v,n} \Big) \otimes \bigoplus_{0<n\in\mathbb{Z}+\frac{1}{2}} \mathrm{Sym}(N(n_j)_{n_j/2,n}), \tag{3-34}$$

$$F_V^1(\beta_j) = \Lambda\Big(\bigoplus_{0<n\in\mathbb{Z}} V(n_j)_{0,n} \oplus \bigoplus_{0<v<n_j/2} \Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v} V(n_j)_{v,n} \oplus \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v} \overline{V(n_j)}_{v,n} \Big) \oplus \bigoplus_{0<n\in\mathbb{Z}+\frac{1}{2}} V(n_j)_{n_j/2,n} \Big), \tag{3-35}$$

$$F_V^2(\beta_j) = \Lambda\Big(\bigoplus_{0<n\in\mathbb{Z}} V(n_j)_{n_j/2,n} \oplus \bigoplus_{0<v<n_j/2} \Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v+\frac{1}{2}} V(n_j)_{v,n} \oplus \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v+\frac{1}{2}} \overline{V(n_j)}_{v,n} \Big) \oplus \bigoplus_{0<n\in\mathbb{Z}+\frac{1}{2}} V(n_j)_{0,n} \Big), \tag{3-36}$$

$$Q_W^1(\beta_j) = \Lambda\Big(\bigoplus_{0\le v<n_j}\Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v} W(n_j)_{v,n} \oplus \bigoplus_{0\le n\in\mathbb{Z}-\frac{p_j}{n_j}v} \overline{W(n_j)}_{v,n}\Big)\Big), \tag{3-37}$$

$$Q_W^2(\beta_j) = \Lambda\Big(\bigoplus_{0\le v<n_j}\Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v+\frac{1}{2}} W(n_j)_{v,n} \oplus \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v+\frac{1}{2}} \overline{W(n_j)}_{v,n}\Big)\Big). \tag{3-38}$$

We indicate here that $\mathscr{F}(\beta_j)$, $F_V^1(\beta_j)$, $F_V^2(\beta_j)$, $Q_W^1(\beta_j)$, and $Q_W^2(\beta_j)$ in (3-24) are the restrictions of the corresponding vector bundles in the right side of (3-34)–(3-38) over $M(n_j)$, which will still be denoted as $\mathscr{F}(\beta_j)$, $F_V^1(\beta_j)$, $F_V^2(\beta_j)$, $Q_W^1(\beta_j)$, and $Q_W^2(\beta_j)$. Write

$$Q_W(\beta_j) = Q_W^1(\beta_j) \otimes Q_W^2(\beta_j) \otimes Q_W^2(\beta_j), \tag{3-39}$$

which we now think of as a vector bundle over $M(n_j)$.

We now define the spin^c Dirac operators on $M(n_j)$. The following lemma follows from the proof of [Bott and Taubes 1989, Lemmas 11.3 and 11.4].

Lemma 3.5 (compare with [Liu et al. 2000, Lemma 4.2]). *Assume that (2-17) holds. Let*

$$L(n_j) = \bigotimes_{0<v<n_j/2}\Big(\det(N(n_j)_v) \otimes \det(\overline{V(n_j)}_v) \otimes \big(\det(\overline{W(n_j)}_v) \otimes \det(W(n_j)_{n_j-v})\big)^3\Big)^{(r(n_j)+1)v} \tag{3-40}$$

be the complex line bundle over $M(n_j)$. Then $L(n_j)$ has an n_j-th root over $M(n_j)$.

Moreover, $U_1 := TX(n_j) \oplus V(n_j)_0^{\mathbb{R}} \oplus W(n_j)_{n_j/2}^{\mathbb{R}} \oplus W(n_j)_{n_j/2}^{\mathbb{R}}$ has a spin^c structure defined by

$$L_1 := K_X \otimes \bigotimes_{0<v<n_j/2}\big(\det(N(n_j)_v) \otimes \det(\overline{V(n_j)}_v)\big) \otimes (\det(W(n_j)_{n_j/2}))^3 \otimes L(n_j)^{r(n_j)/n_j},$$

and $U_2 := TX(n_j) \oplus V(n_j)_{n_j/2}^{\mathbb{R}} \oplus W(n_j)_{n_j/2}^{\mathbb{R}} \oplus W(n_j)_{n_j/2}^{\mathbb{R}}$ has a spin^c structure defined by

$$L_2 := K_X \otimes \bigotimes_{0<v<n_j/2} \det(N(n_j)_v) \otimes (\det(W(n_j)_{n_j/2}))^3 \otimes L(n_j)^{r(n_j)/n_j}.$$

We remark that in order to define an S^1- or G_y- action on $L(n_j)^{r(n_j)/n_j}$, we must replace the S^1- or G_y-action by its n_j-fold action. Here, by abusing notation, we still speak of an S^1- or G_y-action without causing any confusion.

Let $S(U_1, L_1)$ and $S(U_2, L_2)$ be the fundamental complex spinor bundles for (U_1, L_1) and (U_2, L_2); see [Lawson and Michelsohn 1989, Appendix D]. There are two $\mathbb{Z}_2$-gradings on these bundles. The first grading, denoted by τ_s, is induced by the involutions on $S(U_1, L_1)$ and $S(U_2, L_2)$ determined by $TX(n_j) \oplus W(n_j)_{n_j/2}^{\mathbb{R}}$ as in (2-1). The second grading, which we denote by τ_e, is induced by the involution on $S(U_1, L_1)$ determined by $TX(n_j) \oplus V(n_j)_0^{\mathbb{R}} \oplus W(n_j)_{n_j/2}^{\mathbb{R}}$, and by the involution on $S(U_2, L_2)$ determined by $U_2 = TX(n_j) \oplus V(n_j)_{n_j/2}^{\mathbb{R}} \oplus W(n_j)_{n_j/2}^{\mathbb{R}}$, as in (2-1).

In what follows, by $D^{X(n_j)}$ we mean the S^1-equivariant spin^c Dirac operator on $S(U_1, L_1)$ or $S(U_2, L_2)$ over $M(n_j)$.

Corresponding to (2-8), by (3-30) and (3-31), we define $S(U_1, L_1)'$ and $S(U_2, L_2)'$ equipped with involutions τ_s' and τ_e' as follows (compare with [Liu et al. 2000, (4.16)]):

$$(3\text{-}41)\quad (S(U_1, L_1)', \tau_s'/\tau_e') = \Big(S\Big(TY \oplus V_0^{\mathbb{R}}, L_1 \otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} (\det N_v \otimes \det V_v)^{-1} \otimes \bigotimes_{v\equiv n_j/2 \bmod n_j} (\det W_v)^{-2}\Big), \tau_s'/\tau_e'\Big) \otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} \Lambda_{\pm 1}(V_v) \otimes \bigotimes_{v\equiv n_j/2 \bmod n_j} \Lambda_{-1}(W_v) \otimes \bigotimes_{v\equiv n_j/2 \bmod n_j} \Lambda(W_v)$$

and

$$(3\text{-}42)\quad (S(U_2, L_2)', \tau_s'/\tau_e') = S\Big(TY, L_2 \otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} (\det N_v)^{-1} \otimes \bigotimes_{\substack{v>0\\ v\equiv n_j/2 \bmod n_j}} (\det V_v)^{-1} \otimes \bigotimes_{v\equiv n_j/2 \bmod n_j} (\det W_v)^{-2}\Big) \otimes \bigotimes_{\substack{v>0\\ v\equiv n_j/2 \bmod n_j}} \Lambda_{\pm 1}(V_v) \otimes \bigotimes_{v\equiv n_j/2 \bmod n_j} \Lambda_{-1}(W_v) \otimes \bigotimes_{v\equiv n_j/2 \bmod n_j} \Lambda(W_v).$$

Then, by (2-8), for $i = 1, 2$, we have the following isomorphisms of Clifford modules over F preserving the $\mathbb{Z}_2$-gradings (compare with [Liu et al. 2000, (4.17)]):

$$(3\text{-}43)\qquad (S(U_i, L_i), \tau_s/\tau_e)|_F \simeq (S(U_i, L_i)', \tau_s'/\tau_e') \otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} \Lambda_{-1}(N_v).$$

As in [Liu et al. 2000, pp. 952], we introduce formally the following complex line bundles over F:

$$(3\text{-}44)\quad L_1' = \Big(L_1^{-1} \otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} (\det N_v \otimes \det V_v) \otimes \bigotimes_{v\equiv n_j/2 \bmod n_j} (\det W_v)^2 \otimes \bigotimes_{v>0} (\det N_v \otimes \det V_v)^{-1} \otimes K_X\Big)^{1/2}$$

and

$$(3\text{-}45)\quad L_2' = \Big(L_2^{-1} \otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} \det N_v \otimes \bigotimes_{\substack{v>0\\ v\equiv n_j/2 \bmod n_j}} \det V_v \otimes \bigotimes_{v\equiv n_j/2 \bmod n_j} (\det W_v)^2 \otimes \bigotimes_{v>0} (\det N_v)^{-1} \otimes K_X\Big)^{1/2}.$$

In fact, from (2-8), Lemma 3.5, and the assumption that V is spin, one verifies easily that $c_1(L_i'^2)=0 \mod 2$ for $i=1,2$, which implies that L_1' and L_2' are well-defined complex line bundles over F.

Then, by [Liu et al. 2000, (3.14)], (3-41)–(3-45), and the definitions of L_1, L_2, we get the following identifications of Clifford modules over F (compare with [Liu et al. 2000, (4.19)]):

$$\begin{aligned}
(3\text{-}46)\quad & (S(U_1,L_1)'\otimes L_1', (\tau_s'/\tau_e')\otimes \mathrm{id})\\
&= S\Big(TY, K_X\otimes \bigotimes_{v>0}(\det N_v)^{-1}\Big)\otimes (S(V_0^{\mathbb{R}}, \bigotimes_{v>0}(\det V_v)^{-1}), \mathrm{id}/\tau)\\
&\otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} \Lambda_{\pm1}(V_v)\otimes \bigotimes_{\substack{v>0\\ v\equiv n_j/2 \bmod n_j}} \Lambda_{-1}(W_v)\otimes \bigotimes_{\substack{v<0\\ v\equiv n_j/2 \bmod n_j}} \Lambda_{-1}(\overline{W}_v)\\
&\otimes \bigotimes_{\substack{v>0\\ v\equiv n_j/2 \bmod n_j}} \Lambda(W_v)\otimes \bigotimes_{\substack{v<0\\ v\equiv n_j/2 \bmod n_j}} \Lambda(\overline{W}_v)\otimes \bigotimes_{\substack{v<0\\ v\equiv n_j/2 \bmod n_j}} (\det W_v)^2
\end{aligned}$$

and

$$\begin{aligned}
(3\text{-}47)\quad & (S(U_2,L_2)'\otimes L_2', (\tau_s'/\tau_e')\otimes \mathrm{id})\\
&= S\Big(TY, K_X\otimes \bigotimes_{v>0}(\det N_v)^{-1}\Big)\otimes \bigotimes_{\substack{v>0\\ v\equiv n_j/2 \bmod n_j}} \Lambda_{\pm1}(V_v)\otimes \bigotimes_{\substack{v>0\\ v\equiv n_j/2 \bmod n_j}} \Lambda_{-1}(W_v)\\
&\otimes \bigotimes_{\substack{v<0\\ v\equiv n_j/2 \bmod n_j}} \Lambda_{-1}(\overline{W}_v)\otimes \bigotimes_{\substack{v>0\\ v\equiv n_j/2 \bmod n_j}} \Lambda(W_v)\otimes \bigotimes_{\substack{v<0\\ v\equiv n_j/2 \bmod n_j}} \Lambda(\overline{W}_v)\otimes \bigotimes_{\substack{v<0\\ v\equiv n_j/2 \bmod n_j}} (\det W_v)^2.
\end{aligned}$$

Now we compare the $\mathbb{Z}_2$-gradings in (3-46) and (3-47). Set (compare with [Liu et al. 2000, (4.20)])

$$\begin{aligned}
&\Delta(n_j,N) = \sum_{n_j/2<v'<n_j}\ \sum_{0<v,\ v\equiv v' \bmod n_j} \dim N_v + o(N(n_j)^{\mathbb{R}}_{n_j/2}),\\
(3\text{-}48)\qquad &\Delta(n_j,V) = \sum_{n_j/2<v'<n_j}\ \sum_{0<v,\ v\equiv v' \bmod n_j} \dim V_v + o(V(n_j)^{\mathbb{R}}_{n_j/2}),\\
&\Delta(n_j,W) = \sum_{v<0,\ v\equiv n_j/2 \bmod n_j} \dim W_v,
\end{aligned}$$

where $o(N(n_j)^{\mathbb{R}}_{n_j/2})$ and $o(V(n_j)^{\mathbb{R}}_{n_j/2})$ equal 0 or 1 depending on whether the given orientation on $N(n_j)^{\mathbb{R}}_{n_j/2}$ and $V(n_j)^{\mathbb{R}}_{n_j/2}$ agrees or disagrees with the complex orientation of

$$\bigoplus_{\substack{v>0\\ v\equiv n_j/2 \mod n_j}} N_v \quad\text{and}\quad \bigoplus_{\substack{v>0\\ v\equiv n_j/2 \mod n_j}} V_v,$$

respectively.

As explained in [Liu et al. 2003, p. 166], for the $\mathbb{Z}_2$-gradings induced by τ_s, the differences of the $\mathbb{Z}_2$-gradings of (3-46) and (3-47) are both

$$(-1)^{\Delta(n_j,N)+\Delta(n_j,W)};$$

for the $\mathbb{Z}_2$-gradings induced by τ_e, the difference of the $\mathbb{Z}_2$-gradings of (3-46) (respectively (3-47)) is

$$(-1)^{\Delta(n_j,N)+\Delta(n_j,V)+\Delta(n_j,W)}$$

(respectively $(-1)^{\Delta(n_j,N)+o(V(n_j)^{\mathbb{R}}_{n_j/2})+\Delta(n_j,W)}$).

Lemma 3.6 (compare with [Liu et al. 2000, Lemma 4.3]). *Let us write*

$$\begin{aligned}L(\beta_j)_1 = L'_1 \otimes \bigotimes_{v>0} (\det N_v)^{\lfloor \frac{p_j}{n_j} v\rfloor+(p-1)v+1} \otimes \bigotimes_{v>0} (\det \overline{V}_v)^{\lfloor \frac{p_j}{n_j} v\rfloor+(p-1)v} \\ \otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} (\det N_v)^{-1} \otimes \bigotimes_{v<0} (\det W_v)^{\lfloor -\frac{p_j}{n_j} v\rfloor+2\lfloor -\frac{p_j}{n_j} v+\frac{1}{2}\rfloor-3(p-1)v} \\ \otimes \bigotimes_{v>0} (\det \overline{W}_v)^{\lfloor \frac{p_j}{n_j} v\rfloor+2\lfloor \frac{p_j}{n_j} v+\frac{1}{2}\rfloor+3(p-1)v+1} \\ \otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} \det W_v \otimes \bigotimes_{\substack{v<0\\ v\equiv n_j/2 \bmod n_j}} (\det \overline{W}_v)^2\end{aligned}$$

and

$$\begin{aligned}L(\beta_j)_2 = L'_2 \otimes \bigotimes_{v>0} (\det N_v)^{\lfloor \frac{p_j}{n_j} v\rfloor+(p-1)v+1} \otimes \bigotimes_{v>0} (\det \overline{V}_v)^{\lfloor \frac{p_j}{n_j} v+\frac{1}{2}\rfloor+(p-1)v} \\ \otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} (\det N_v)^{-1} \otimes \bigotimes_{v<0} (\det W_v)^{\lfloor -\frac{p_j}{n_j} v\rfloor+2\lfloor -\frac{p_j}{n_j} v+\frac{1}{2}\rfloor-3(p-1)v} \\ \otimes \bigotimes_{v>0} (\det \overline{W}_v)^{\lfloor \frac{p_j}{n_j} v\rfloor+2\lfloor \frac{p_j}{n_j} v+\frac{1}{2}\rfloor+3(p-1)v+1} \\ \otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} \det W_v \otimes \bigotimes_{\substack{v<0\\ v\equiv n_j/2 \bmod n_j}} (\det \overline{W}_v)^2.\end{aligned}$$

Then $L(\beta_j)_1$ and $L(\beta_j)_2$ can be extended naturally to $G_y \times S^1$-equivariant complex line bundles over $M(n_j)$ which we will still denote by $L(\beta_j)_1$ and $L(\beta_j)_2$.

Proof. We introduce the following line bundle over $M(n_j)$:

$$\begin{aligned}L^{\omega}(\beta_j) = \bigotimes_{0<v<n_j/2} \Big(\det(N(n_j)_v) \otimes \det(\overline{V(n_j)}_v) \\ \otimes (\det(\overline{W(n_j)}_v) \otimes \det(W(n_j)_{n_j-v}))^3\Big)^{-\omega(v)-r(n_j)v}.\end{aligned} \tag{3-49}$$

where, as in [Liu et al. 2003, (4.35)], we define ω by

$$\left\lfloor \frac{p_j}{n_j} v \right\rfloor = \frac{p_j}{n_j} v - \frac{\omega(v)}{n_j}.$$

As in [Liu et al. 2003, (4.38); Liu et al. 2000, (4.28)], Lemma 3.5 implies that $L^{\omega}(\beta_j)^{1/n_j}$ is well-defined over $M(n_j)$. Direct calculation shows that

$$L(\beta_j)_1 = L^{-(p-1)-p_j/n_j} \otimes L^{\omega}(\beta_j)^{1/n_j} \otimes \bigotimes_{0<v<n_j/2} \det(\overline{W(n_j)_v}) \otimes (\det(\overline{W(n_j)_{n_j/2}}))^2 \\ \otimes \bigotimes_{1\le m\le p_j/2} \ \bigotimes_{m-\frac{1}{2}<(p_j/n_j)v<m} (\det(\overline{W(n_j)_v}) \otimes \det(W(n_j)_{n_j-v}))^2$$

and

$$L(\beta_j)_2 = L^{-(p-1)-p_j/n_j} \otimes L^{\omega}(\beta_j)^{1/n_j} \otimes \bigotimes_{0<v<n_j/2} \det(\overline{W(n_j)_v}) \otimes (\det(\overline{W(n_j)_{n_j/2}}))^2 \\ \otimes \bigotimes_{1\le m\le p_j/2} \ \bigotimes_{m-\frac{1}{2}<(p_j/n_j)v<m} \big((\det(\overline{W(n_j)_v}) \otimes \det(W(n_j)_{n_j-v}))^2 \otimes \det(\overline{V(n_j)_v})\big).$$

The proof of Lemma 3.6 is complete. □

To simplify the notation, we introduce the following locally constant functions on F (compare with [Liu et al. 2003, (4.45); Liu et al. 2000, (4.30)]):

$$\varepsilon_W^1 = -\frac{1}{2}\sum_{v>0}(\dim W_v)\cdot\bigg(\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v\Big)\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v+1\Big) \\ -\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v\Big)\Big(2\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v\Big)+1\Big)\bigg) \\ -\frac{1}{2}\sum_{v<0}(\dim W_v)\cdot\bigg(\Big(-\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor-(p-1)v\Big)\Big(-\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor-(p-1)v+1\Big) \\ +\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v\Big)\Big(2\Big(-\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor-(p-1)v\Big)+1\Big)\bigg), \tag{3-50}$$

$$\varepsilon_W^2 = -\frac{1}{2}\sum_{v>0}(\dim W_v)\cdot\bigg(\Big(\Big\lfloor\frac{p_j}{n_j}v+\frac{1}{2}\Big\rfloor+(p-1)v\Big)^2 \\ -2\Big(\frac{p_j}{n_j}v+(p-1)v\Big)\Big(\Big\lfloor\frac{p_j}{n_j}v+\frac{1}{2}\Big\rfloor+(p-1)v\Big)\bigg) \\ -\frac{1}{2}\sum_{v<0}(\dim W_v)\cdot\bigg(\Big(\Big\lfloor-\frac{p_j}{n_j}v+\frac{1}{2}\Big\rfloor-(p-1)v\Big)^2 \\ +2\Big(\frac{p_j}{n_j}v+(p-1)v\Big)\Big(\Big\lfloor-\frac{p_j}{n_j}v+\frac{1}{2}\Big\rfloor-(p-1)v\Big)\bigg), \tag{3-51}$$

$$\varepsilon_1 = \frac{1}{2}\sum_{v>0}(\dim N_v-\dim V_v)\bigg(\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v\Big)\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v+1\Big) \\ -\Big(\frac{p_j}{n_j}v+(p-1)v\Big)\Big(2\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v\Big)+1\Big)\bigg), \tag{3-52}$$

$$\begin{aligned}\varepsilon_2 = \frac{1}{2}\sum_{v>0}(\dim N_v)\cdot\Big(&\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v\Big)\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v+1\Big)\\ &-\Big(\frac{p_j}{n_j}v+(p-1)v\Big)\Big(2\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v\Big)+1\Big)\Big)\\ &-\frac{1}{2}\sum_{v>0}(\dim V_v)\cdot\Big(\Big(\Big\lfloor\frac{p_j}{n_j}+\frac{1}{2}\Big\rfloor+(p-1)v\Big)^2\\ &-2\Big(\frac{p_j}{n_j}v+(p-1)v\Big)\Big(\Big\lfloor\frac{p_j}{n_j}+\frac{1}{2}\Big\rfloor+(p-1)v\Big)\Big).\end{aligned}\tag{3-53}$$

As in [Liu et al. 2000, (2.23)], for $0 \le j \le J_0$, we set

$$\begin{aligned} e(p,\beta_j,N) &= \frac{1}{2}\sum_{v>0}(\dim N_v)\cdot\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v\Big)\\ &\qquad\qquad\times\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v+1\Big),\\ d'(p,\beta_j,N) &= \sum_{v>0}(\dim N_v)\cdot\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v\Big).\end{aligned}\tag{3-54}$$

Then $e(p,\beta_j,N)$ and $d'(p,\beta_j,N)$ are locally constant functions on F. In particular, we have

$$\begin{aligned} e(p,\beta_0,N) &= \tfrac{1}{2}(p-1)^2 e(N)+\tfrac{1}{2}(p-1)d'(N),\\ e(p,\beta_{J_0},N) &= \tfrac{1}{2}p^2 e(N)+\tfrac{1}{2}pd'(N),\\ d'(p,\beta_{J_0},N) &= d'(p+1,\beta_0,N) = pd'(N).\end{aligned}\tag{3-55}$$

Proposition 3.7 (compare with [Liu et al. 2000, Proposition 4.2]). *For $i=1,2$, the G_y-equivariant isomorphisms of complex vector bundles over F induced by Proposition 3.4 and (3-46)–(3-47),*

$$\begin{aligned} r_{i1} : S\Big(TY, K_X\otimes\bigotimes_{v>0}(\det N_v)^{-1}\Big)&\otimes(K_W\otimes K_X^{-1})^{1/2}\otimes\mathscr{F}_{p,j-1}(X)\otimes F_V^i\otimes Q(W)\\ &\longrightarrow S(U_i,L_i)'\otimes(K_W\otimes K_X^{-1})^{1/2}\otimes\mathscr{F}(\beta_j)\otimes F_V^i(\beta_j)\\ &\qquad\otimes Q_W(\beta_j)\otimes L(\beta_j)_i\otimes\bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}}\mathrm{Sym}(\bar N_{v,0})\end{aligned}$$

and

$$\begin{aligned} r_{i2} : S\Big(TY, K_X\otimes\bigotimes_{v>0}(\det N_v)^{-1}\Big)&\otimes(K_W\otimes K_X^{-1})^{1/2}\otimes\mathscr{F}_{p,j}(X)\otimes F_V^i\otimes Q(W)\\ &\longrightarrow S(U_i,L_i)'\otimes(K_W\otimes K_X^{-1})^{1/2}\otimes\mathscr{F}(\beta_j)\otimes F_V^i(\beta_j)\\ &\qquad\otimes Q_W(\beta_j)\otimes L(\beta_j)_i\otimes\bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}}(\mathrm{Sym}(N_{v,0})\otimes\det N_v)\end{aligned}$$

have the following properties:

(i) *For* $i = 1, 2$ *and* $\gamma = 1, 2$, *we have*

$$r_{i\gamma}^{-1} \cdot \boldsymbol{J}_H \cdot r_{i\gamma} = \boldsymbol{J}_H, \quad r_{i\gamma}^{-1} \cdot P \cdot r_{i\gamma} = P + \left(\frac{p_j}{n_j} + (p-1)\right)\boldsymbol{J}_H + \varepsilon_{i\gamma}, \tag{3-56}$$

where the $\varepsilon_{i\gamma}$ *are given by*

$$\begin{aligned} \varepsilon_{i1} &= \varepsilon_i + \varepsilon_W^1 + 2\varepsilon_W^2 - e(p, \beta_{j-1}, N), \\ \varepsilon_{i2} &= \varepsilon_i + \varepsilon_W^1 + 2\varepsilon_W^2 - e(p, \beta_j, N). \end{aligned} \tag{3-57}$$

(ii) *For* $i = 1, 2$ *and* $\gamma = 1, 2$, *we have*

$$r_{i\gamma}^{-1}\tau_e r_{i\gamma} = (-1)^{\mu_i}\tau_e, \quad r_{i\gamma}^{-1}\tau_s r_{i\gamma} = (-1)^{\mu_3}\tau_s, \quad r_{i\gamma}^{-1}\tau_1 r_{i\gamma} = (-1)^{\mu_4}\tau_1, \tag{3-58}$$

where the μ_i *are given by*

$$\begin{aligned}
\mu_1 &= -\sum_{v>0}(\dim V_v)\Big\lfloor \frac{p_j}{n_j}v \Big\rfloor + \Delta(n_j, N) + \Delta(n_j, V) + \Delta(n_j, W) \bmod 2, \\
\mu_2 &= -\sum_{v>0}(\dim V_v)\cdot\Big\lfloor \frac{p_j}{n_j}v + \frac{1}{2}\Big\rfloor + \Delta(n_j, N) + o(V(n_j)^{\mathbb{R}}_{n_j/2}) + \Delta(n_j, W) \bmod 2, \\
\mu_3 &= \Delta(n_j, N) + \Delta(n_j, W) \bmod 2, \\
\mu_4 &= \sum_{v>0}(\dim W_v)\cdot\Big(\Big\lfloor \frac{p_j}{n_j}v + \frac{1}{2}\Big\rfloor + (p-1)v\Big) \\
&\qquad + \sum_{v<0}(\dim W_v)\cdot\Big(\Big\lfloor -\frac{p_j}{n_j}v + \frac{1}{2}\Big\rfloor - (p-1)v\Big) \bmod 2.
\end{aligned}$$

Proof. By the proof of [Liu et al. 2000, Proposition 4.2], we need to compute the action of $r_*^{-1} \cdot P \cdot r_*$ on

$$\bigotimes_{\substack{0<n\in\mathbb{Z}+\frac{1}{2},\, v>0 \\ n-(p-1)v-(p_j/n_j)v\le 0}} \Lambda^{i_n}(\overline{W}_{v,n}) \otimes \bigotimes_{\substack{0<n\in\mathbb{Z}+\frac{1}{2},\, v<0 \\ n+(p-1)v+(p_j/n_j)v\le 0}} \Lambda^{i'_n}(W_{v,n}).$$

In fact, by (3-26) and (3-27), as in (3-16), we get

$$\begin{aligned}
r_*^{-1} \cdot P \cdot r_* &= \sum_{\substack{0<n\in\mathbb{Z}+\frac{1}{2},\, v>0 \\ n-(p-1)v-(p_j/n_j)v\le 0}} (\dim W_v - i_n)\Big(-n + (p-1)v + \frac{p_j}{n_j}v\Big) \\
&\quad + \sum_{\substack{0<n\in\mathbb{Z}+\frac{1}{2},\, v<0 \\ n+(p-1)v+(p_j/n_j)v\le 0}} (\dim W_v - i'_n)\Big(-n - (p-1)v - \frac{p_j}{n_j}v\Big) \\
&= P + \Big(p - 1 + \frac{p_j}{n_j}\Big)\boldsymbol{J}_H + \varepsilon_W^2.
\end{aligned} \tag{3-59}$$

By [Liu et al. 2000, (4.36)–(4.38)] and (3-59), we deduce the second identity in (3-56). The first identity in (3-56) is obvious.

Consider the $\mathbb{Z}_2$-gradings. By [Liu et al. 2003, (4.49)–(4.50)] and the discussion following (3-48), we get the first two identities in (3-58). Observe that τ_1 changes only on

$$\bigotimes_{\substack{0<n\in\mathbb{Z}+\frac{1}{2},\, v>0\\ n-(p-1)v-(p_j/n_j)v\le 0}} \Lambda^{i_n}(\overline{W}_{v,n}) \otimes \bigotimes_{\substack{0<n\in\mathbb{Z}+\frac{1}{2},\, v<0\\ n+(p-1)v+(p_j/n_j)v\le 0}} \Lambda^{i'_n}(W_{v,n}).$$

From (3-26) and (3-27), we get the third identity in (3-58). $\square$

3D. *A proof of Theorem 2.8.*

Lemma 3.8 (compare with [Liu et al. 2000, Lemmas 4.4 and 4.6]). *For each connected component M' of $M(n_j)$, the following functions are independent on the connected components of F in M'*:

$$\begin{aligned}
&\varepsilon_i+\varepsilon_W^1+2\varepsilon_W^2, && i=1,2,\\
&d'(p,\beta_j,N)+\mu_i+\mu_4 \bmod 2, && i=1,2,3,\\
&d'(p,\beta_{j-1},N)+\sum_{0<v}\dim N_v+\mu_i+\mu_4 \bmod 2, && i=1,2,3.
\end{aligned}\tag{3-60}$$

Proof. Recall that $\lfloor \frac{p_j}{n_j}v\rfloor=\frac{p_j}{n_j}v-\frac{\omega(v)}{n_j}$. By using (3-31), we explicitly express ε_W^1 and ε_W^2 defined in (3-50)–(3-51) as follows:

$$\begin{aligned}
\varepsilon_W^1 = {}&\tfrac{1}{2}(p-1+p_j/n_j)^2e(W)+\tfrac{1}{8}\dim W(n_j)_{n_j/2}\\
&+\tfrac{1}{2}\sum_{0<v<n_j/2}\frac{\omega(v)\omega(-v)}{n_j^2}(\dim W(n_j)_v+\dim W(n_j)_{n_j-v}),
\end{aligned}\tag{3-61}$$

and

$$\begin{aligned}
\varepsilon_W^2 = {}&\tfrac{1}{2}(p-1+p_j/n_j)^2e(W)-\tfrac{1}{8}\dim W(n_j)_{n_j/2}\\
&-\tfrac{1}{2}\sum_{0\le m\le (p_j-1)/2}\ \sum_{m<\frac{p_j}{n_j}v<m+\frac{1}{2}}\Bigl(\frac{\omega(v)}{n_j}\Bigr)^2(\dim W(n_j)_v+\dim W(n_j)_{n_j-v})\\
&-\tfrac{1}{2}\sum_{0\le m\le p_j/2}\ \sum_{m-\frac{1}{2}<\frac{p_j}{n_j}v<m}\Bigl(\frac{\omega(-v)}{n_j}\Bigr)^2(\dim W(n_j)_v+\dim W(n_j)_{n_j-v}).
\end{aligned}\tag{3-62}$$

By using (2-23), (3-61), (3-62), and the explicit expressions of ε_i given in [Liu et al. 2003, (4.56)–(4.57)], we know the functions in the first line of (3-60) are independent on the connected components of F in M'.

Now consider the functions in the rest of the lines of (3-60). By (2-30), (3-30), (3-32), (3-48) and [Liu et al. 2000, Lemma 4.5], we get

$$d'(p,\beta_j,N)+\mu_i+\mu_4 \equiv \sum_{0<m\le p_j/2} \sum_{\substack{0<v<n_j/2 \\ m-\frac12<\frac{p_i}{n_j}v<m}} \dim N(n_j)_v + \tfrac12 \dim_{\mathbb{R}} N(n_j)^{\mathbb{R}}_{n_j/2} + \sum_{v>0}(\dim N_v)\Big\lfloor \frac{p_j}{n_j}v+\frac12\Big\rfloor + \sum_{v>0}(\dim W_v)\Big\lfloor \frac{p_j}{n_j}v+\frac12\Big\rfloor + \sum_{v<0}(\dim W_v)\Big\lfloor -\frac{p_j}{n_j}v+\frac12\Big\rfloor + o\big(N(n_j)^{\mathbb{R}}_{n_j/2}\big)+\Delta(n_j,W) \mod 2. \tag{3-63}$$

But, by [Liu et al. 2000, Lemma 4.5], as $w_2(W\oplus TX)_{S^1}=0$, we know that, modulo 2,

$$\sum_{v>0}(\dim N_v)\Big\lfloor \frac{p_j}{n_j}v+\frac12\Big\rfloor + \sum_{v>0}(\dim W_v)\Big\lfloor \frac{p_j}{n_j}v+\frac12\Big\rfloor + \sum_{v<0}(\dim W_v)\Big\lfloor -\frac{p_j}{n_j}v+\frac12\Big\rfloor + o(N(n_j)^{\mathbb{R}}_{n_j/2})+\Delta(n_j,W) \tag{3-64}$$

is independent on the connected components of F in M'. Thus, the independence on the connected components of F in M' of the functions in the second line of (3-60) is proved, which, combined with [Liu et al. 2000, (4.42)], implies the same independent property of the functions in the third line of (3-60). □

By (3-34)–(3-39) and Lemma 3.6, we know that the Dirac operator

$$D^{X(n_j)}\otimes \mathscr{F}(\beta_j)\otimes F_V^i(\beta_j)\otimes Q_W(\beta_j)\otimes L(\beta_j)_i$$

($i=1,2$) is well-defined on $M(n_j)$. Observe that (2-12) in Theorem 2.1 is compatible with the G_y-action. Thus, by using Proposition 3.7, Lemma 3.8 and applying Theorem 2.1 to each connected component of $M(n_j)$ separately, we deduce that, for $i=1,2$, $1\le j\le J_0$, $m\in(1/2)\mathbb{Z}$, $h\in\mathbb{Z}$, $\tau=\tau_{e1}$ or τ_{s1},

$$\begin{aligned}\sum_\alpha(-1)^{d'(p,\beta_{j-1},N)+\sum_{v>0}\dim N_v}\operatorname{Ind}_\tau\big(D^{Y_\alpha}\otimes(K_W\otimes K_X^{-1})^{1/2} &\otimes \mathscr{F}_{p,j-1}(X)\otimes F_V^i\otimes Q(W), m+e(p,\beta_{j-1},N),h\big)\\
=\sum_\beta(-1)^{d'(p,\beta_{j-1},N)+\sum_{v>0}\dim N_v+\mu}\operatorname{Ind}_\tau\bigg(D^{X(n_j)}&\otimes(K_W\otimes K_X^{-1})^{1/2}\otimes\mathscr{F}(\beta_j)\\
\otimes F_V^i(\beta_j)\otimes Q_W(\beta_j)\otimes L(\beta_j)_i, m+\varepsilon_i+\varepsilon_W^1+2\varepsilon_W^2&+\Big(\frac{p_j}{n_j}+(p-1)\Big)h,h\bigg)\\
=\sum_\alpha(-1)^{d'(p,\beta_j,N)+\sum_{v>0}\dim N_v}\operatorname{Ind}_\tau\big(D^{Y_\alpha}\otimes(K_W\otimes K_X^{-1})^{1/2}&\otimes\mathscr{F}_{p,j}(X)\\
&\otimes F_V^i\otimes Q(W), m+e(p,\beta_j,N),h\big),\end{aligned} \tag{3-65}$$

where $\sum_\beta$ means the sum over all the connected components of $M(n_j)$. In (3-65), if $\tau = \tau_{s1}$, $\mu = \mu_3 + \mu_4$; if $\tau = \tau_{e1}$, $\mu = \mu_i + \mu_4$. Combining (3-55) with (3-65), we get (2-28). The proof of Theorem 2.8 is complete.

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