Pacific Journal of Mathematics

Volume 266 No. 2 December 2013

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 266, No. 2, 2013
dx.doi.org/10.2140/pjm.2013.266.257

RATE OF ATTRACTION FOR A SEMILINEAR WAVE EQUATION WITH VARIABLE COEFFICIENTS AND CRITICAL NONLINEARITIES

FÁGNER DIAS ARARUNA AND FLANK DAVID MORAIS BEZERRA

We study the rate of convergence of global attractors and eigenvalues of the family of dissipative semilinear wave equations with variable coefficients $u_{tt} + \Lambda_\epsilon u + \Lambda_\epsilon^\delta u_t = f(u)$, where Λ_ϵ is the elliptic operator $-\mathrm{div}(a_\epsilon(x)\nabla)$ with $\epsilon \in [0, 1]$ and sufficiently smooth coefficients a_ϵ, and where $\delta \in \left(\frac{1}{2}, 1\right)$ and the nonlinearity f is a continuously differentiable function satisfying suitable growth conditions. We show that the rate of convergence, as $\epsilon \to 0^+$, of the global attractors of these problems, as well as of their eigenvalues, is proportional to the distance of the coefficients $\|a_\epsilon - a_0\|_{L^\infty(\Omega)}$.

1. Introduction and main result

In many theoretical and applied problems, it is important to understand what happens when the solutions varies parameters in the model, and wave equations with variable coefficients arise naturally in mathematical modeling of inhomogeneous media (for example, functionally graded materials or materials with damage induced inhomogeneity) in solid mechanics, electromagnetism, fluid flows through porous media (for example, modeling traveling waves in a inhomogeneous gas; see [Egorov and Shubin 1988; Suggs 2009]), and other areas of physics and engineering.

Nonlinear wave equations arise in quantum mechanics, whereas variants of the form

$$u_{tt} - \mathrm{div}(a\nabla u) + g(u, u_t) = 0$$

appear in the study of vibrating systems with or without damping, and with or without forcing terms.

The first author was partially supported by INCTMat, CAPES and CNPq (Brasil), grants 307893/2011-1, 552758/2011-6 and 477124/2012-7. The second author was partially supported by FAPESP grant 11/04166-5, CAPES, and CNPq (Brasil) grant 552758/2011-6.

MSC2010: primary 35J05; secondary 34D45, 41A25.

Keywords: wave equations, variable coefficients, global attractors, rates of convergence.

In this work, $\epsilon \in [0, 1]$, and we consider the following problem associated with a semilinear dissipative wave equation with variable coefficients:

$$(1\text{-}1)\qquad \begin{cases} u_{tt} + \Lambda_\epsilon u + \Lambda_\epsilon^\delta u_t = f(u), & t > 0,\ x \in \Omega, \\ u(0, x) = u_0(x),\ u_t(0, x) = v_0(x), & x \in \Omega, \\ u(t, x) = 0, & t \geqslant 0,\ x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, is a bounded domain with boundary $\partial\Omega$ sufficiently regular, $\Lambda_\epsilon = -\operatorname{div}(a_\epsilon(x)\nabla)$, and a_ϵ is a real function defined in Ω satisfying

$$(1\text{-}2)\qquad 0 < m_0 \leqslant a_\epsilon(x) \leqslant M_0 \quad \text{for all } x \in \Omega.$$

Moreover, the functions $a_\epsilon \in L^\infty(\Omega)$ converge uniformly to $a_0 \in L^\infty(\Omega)$, as $\epsilon \to 0^+$. Also, we will assume that a_ϵ is smooth for all $\epsilon \in [0, 1]$. For the system (1-1), let us consider $\delta \in \left(\frac{1}{2}, 1\right)$.

The operators $\Lambda_\epsilon^\delta := (\Lambda_\epsilon^{-\delta})^{-1}$ denote the fractional power operators associated with Λ_ϵ. Provided that Λ_ϵ with domain $D(\Lambda_\epsilon) = H^2(\Omega) \cap H_0^1(\Omega)$ is a sectorial operator with $\operatorname{Re}\sigma(\Lambda_\epsilon) > 0$, for any $\alpha \in (0, 1)$, it follows by Theorem 1.4.2 in [Henry 1981] that

$$(1\text{-}3)\qquad \Lambda_\epsilon^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{-\alpha}(\lambda I + \Lambda_\epsilon)^{-1}\, d\lambda.$$

On the nonlinearity $f : \mathbb{R} \to \mathbb{R}$, which is continuously differentiable and bounded, we will give conditions under which the problem (1-7) is globally well posed in $H_0^1(\Omega) \times L^2(\Omega)$ and it has global attractors, in the terminology of [Hale 1988] (following closely Theorem 1.1 and Theorem 1.2 in [Carvalho and Cholewa 2002a]; see also [Carvalho and Cholewa 2002b]): if $\rho \leqslant (N+2)/(N-2)$, there exists a constant $C > 0$, independent of ϵ, such that

$$(1\text{-}4)\qquad |f(s_1) - f(s_2)| \leqslant C|s_1 - s_2|(|s_1|^{\rho-1} + |s_2|^{\rho-1} + 1)$$

and

$$(1\text{-}5)\qquad \limsup_{|s|\to+\infty} \frac{f(s)}{s} \leqslant \mu_{0,1},$$

with $\mu_{0,1}$ being the first eigenvalue of the Λ_0 in Ω.

In the rest of this paper, we will use C to denote a generic positive constant which may change from line to line (unless otherwise stated).

Since the wave equation does not have dissipative character, we have added a "damping" characterized by the term $\Lambda_\epsilon^\delta u_t^\epsilon$ with $\delta \in \left(\frac{1}{2}, 1\right)$. This additional term turns problem (1-1) into a sectorial structure (see [Chen and Triggiani 1989]), however, this gives us an extra difficulty, because it is necessary to perform an analysis of the rate of convergence of fractional derivatives. Although the sectorial structure for (1-1) is preserved when the dissipative term presents the optimal power $\delta = \frac{1}{2}$ (see [Chen and Triggiani 1989]), the convergence (with rate) of attractors is an open problem for this case.

Related to this issue, in [Arrieta et al. 2013] the authors proved that the difference $\|a_\epsilon - a_0\|_{L^\infty(\Omega)}$ can be used to show the rate of convergence of attractors in the context of the heat equation. Nonlinear absorption problems with variable coefficients have been considered by many authors; see [Wu and Li 2011; Suggs 2009] and the references therein. For damped wave equations, several authors have studied existence of global attractors; see [Babin and Vishik 1989; Bruschi et al. 2006; Carvalho and Cholewa 2002a; 2002b; Cholewa and Dlotko 2006; Hale 1988; Webb 1980] and the references therein. We can still cite [Bruschi et al. 2006], where the convergence of attractors was shown, but without explicit rate.

In this work, we will investigate the relationship between the convergence of functions $a_\epsilon \in L^\infty(\Omega)$, which converge uniformly to $a_0 \in L^\infty(\Omega)$, as $\epsilon \to 0^+$, and the proximity between the perturbed and limit attractors, as well as the convergence of the eigenvalues of the operators associated with the problems in (1-1). The difference $\|a_\epsilon - a_0\|_{L^\infty(\Omega)}$ will be our measure.

To better explain the results in the paper, we introduce some terminology. Let us consider the Hilbert spaces $Y = Y^0 := L^2(\Omega)$, $Y^{1/2} := H_0^1(\Omega)$, $Y^1 := D(\Lambda_\epsilon) := \{u \in H_0^1(\Omega) : \Lambda_\epsilon u \in L^2(\Omega)\}$ and the Hilbert energy space $X = X^0 = Y^{1/2} \times Y$ equipped with the inner product

$$\left\langle \begin{bmatrix} \phi \\ \varphi \end{bmatrix}, \begin{bmatrix} \bar{\phi} \\ \bar{\varphi} \end{bmatrix} \right\rangle_X := \int_\Omega a_\epsilon(x) \nabla\phi \nabla\bar{\phi}\, dx + \int_\Omega \varphi\bar{\varphi}\, dx.$$

We define the operator $A_\epsilon : D(A_\epsilon) \subset X \to X$ by

$$A_\epsilon \begin{bmatrix} \phi \\ \varphi \end{bmatrix} = \begin{bmatrix} 0 & -I \\ \Lambda_\epsilon & \Lambda_\epsilon^\delta \end{bmatrix} \begin{bmatrix} \phi \\ \varphi \end{bmatrix} := \begin{bmatrix} -\varphi \\ \Lambda_\epsilon^\delta(\Lambda_\epsilon^{1-\delta}\phi + \varphi) \end{bmatrix}$$

and

$$D(A_\epsilon) = \left\{ \begin{bmatrix} \phi \\ \varphi \end{bmatrix} \in Y^{(3/2)-\delta} \times Y^{1/2};\ \Lambda_\epsilon^{1-\delta}\phi + \varphi \in Y^\delta \right\} =: X^1,$$

with Y^δ denoting the domain of the fractional power operators associated with Λ_ϵ, that is, $Y^\delta := D(\Lambda_\epsilon^\delta)$. Let us consider Y^δ endowed with the graph norm $\|x\|_{Y^\delta} = \|\Lambda_\epsilon^\delta x\|_Y$. Notice that

$$A_\epsilon \begin{bmatrix} \phi \\ \varphi \end{bmatrix} = \begin{bmatrix} -\varphi \\ \Lambda_\epsilon\phi + \Lambda_\epsilon^\delta\varphi \end{bmatrix}, \quad \begin{bmatrix} \phi \\ \varphi \end{bmatrix} \in Y^1 \times Y^\delta, \tag{1-6}$$

where $Y^1 \times Y^\delta$ is a dense subset of $D(A_\epsilon)$.

Notice that the operator A_ϵ with domain $Y^1 \times Y^\delta$ is not a closed operator, unless $\delta = \frac{1}{2}$; see [Chen and Triggiani 1989].

Problem (1-1) can be written as

$$\begin{cases} w_t + A_\epsilon w = F(w), & t > 0, \\ w(0) = w_0 \in X \end{cases} \tag{1-7}$$

with $w = [u \ \ u_t]^T$ and the nonlinear map $F : X \to Y \times Y$ defined by

$$F\begin{bmatrix} \phi \\ \varphi \end{bmatrix} := \begin{bmatrix} 0 \\ f^e(\phi) \end{bmatrix},$$

where $f^e : H_0^1(\Omega) \to H^{-1/2}(\Omega)$ is the Nemytskiĭ operator associated with f.

We will show that these equations define on the space X a nonlinear semigroup $\{T_\epsilon(t) : t \geqslant 0\}$ having global attractors $\mathscr{A}_\epsilon$, $\epsilon \in [0, 1]$, and that the rate of convergence of the attractors in the sense of the symmetric Hausdorff distance is given by the order of $\|a_\epsilon - a_0\|^\theta_{L^\infty(\Omega)}$ with $\theta \in \left(0, \frac{1}{2}\right)$.

It is worth noting that the dependence of regular attractors on parameters is a very well-studied and well-understood topic nowadays, especially for the case when the perturbation is also regular (like in our case). Basically, all the necessary technique to handle such perturbations can be found already in the monograph of Babin and Vishik [1989]. However, the problem considered has some interesting peculiarities in a sense unusual for the attractor theory, namely, the presence of the fractional powers of the elliptic operator Λ_ϵ as well as the necessity to control the dependence of these powers on the parameter ϵ.

The main purpose of this paper is to give a proof of the following result.

Theorem 1.1. *Let $\{T_\epsilon(t) : t \geqslant 0\}$ be the gradient nonlinear semigroup associated with (1-7) and let $\mathscr{A}_\epsilon$ in X be its global attractor*, $\epsilon \in [0, 1]$. *Then there are constants $C > 0$ and $\wp \in \left(0, \frac{1}{2}\right)$, independent of ϵ, such that*

$$\operatorname{dist}(\mathscr{A}_\epsilon, \mathscr{A}_0) + \operatorname{dist}(\mathscr{A}_0, \mathscr{A}_\epsilon) \leqslant C\|a_\epsilon - a_0\|^{\wp}_{L^\infty(\Omega)},$$

where

$$\operatorname{dist}(A, B) := \sup_{x \in A} \inf_{y \in B} \|x - y\|_X, \quad A, B \subset X$$

is the Hausdorff semidistance between A and B in X.

We observe that the Hausdorff semidistance between A and B, $\operatorname{dist}(A, B)$, examines how the set A is contained in the set B. For example, if $\operatorname{dist}(A, B) = 0$ then A is contained in the closure of the set B.

The rest of this paper is organized as follows. In Section 2 we show that the linear semigroups of contractions associated to the problems (1-7) are analytic and compact, and that their nonlinear semigroups have global attractors $\mathscr{A}_\epsilon$ in X. In Section 3 we see that the distance between the semigroups are proportional to a power of the distance between coefficients $a_\epsilon \in L^\infty(\Omega)$. We study the convergence of the operators A_ϵ^{-1} to A_0^{-1}. We also make a spectral analysis and we prove that the convergence of the eigenvalues of the operators associated to (1-7) is proportional to measure $\|a_\epsilon - a_0\|_{L^\infty(\Omega)}$. In Section 4 we analyze the convergence of equilibria. In Section 5 we study some important properties of the Nemytskiĭ operators F. We also study the convergence of the operators $A_\epsilon - F'(w_\epsilon)$, as w_ϵ converges to w_0 in X. In Section 6 we analyze the rate of convergence of equilibria. In Section 7 we study the rate of convergence and attraction of local unstable manifolds of an equilibrium. Finally, in Section 8 we prove the main result of this paper.

2. Functional setting and background results

Our main goal in this section is to prove the well-posedness of problem (1-7) in X and to ensure that the nonlinear semigroup generated by (1-7) has global attractor with uniform bounds in X. Our approach is inspired by a similar idea from [Carvalho and Cholewa 2002a].

Under the assumption above, it is well known that the operator Λ_ϵ is a positive, self-adjoint operator with domain $D(\Lambda_\epsilon) = Y^1$. Let us denote by $\{e^{-\Lambda_\epsilon t} : t \geqslant 0\}$ the analytic linear semigroup generated by $-\Lambda_\epsilon$ on Y, for all $\epsilon \in [0, 1]$.

According to [Henry 1981], we still have

$$\|(\lambda I + \Lambda_\epsilon)^{-1}\|_{\mathscr{L}(Y)} \leqslant C \max\{1, |\lambda|^{-1}\} \tag{2-1}$$

for some $C > 0$ independent of ϵ.

Since Λ_ϵ is a sectorial operator with $\|e^{-\Lambda_\epsilon t}\|_{\mathscr{L}(Y)} \leqslant C$, C independent of ϵ, as a consequence of the moment inequality (see Theorem 1.4.4 in [Henry 1981]), there exists a constant $C > 0$ such that

$$\|\Lambda_\epsilon^\alpha x\|_Y \leqslant C \|\Lambda_\epsilon x\|_Y^\alpha \|x\|_Y^{1-\alpha}, \quad x \in Y^1, \tag{2-2}$$

with $0 \leqslant \alpha \leqslant 1$. The constant C can be chosen uniform with respect to ϵ and α.

In this way, since all operators are selfadjoint, we have that $\sigma(\Lambda_\epsilon) \subset (-\infty, \alpha]$ for some $\alpha < 0$ and, in particular, the set $\Sigma_\phi = \{\lambda \in \mathbb{C} : |\arg\lambda| \leqslant \phi\}$, $\phi \in (\pi/2, \pi)$, is contained in the resolvent sets of Λ_ϵ, for all $\epsilon \in [0, 1]$. Consequently,

$$\|\Lambda_\epsilon(\lambda I + \Lambda_\epsilon)^{-1}\|_{\mathscr{L}(Y^{1/2})} \leqslant C, \quad \lambda \in \Sigma_\phi, \tag{2-3}$$

for some $C \geqslant 1$ independent of ϵ.

We will show that (1-7) is defined on the phase space X, an analytic semigroup.

Proposition 2.1. *Let $\epsilon \in [0, 1]$. The following conditions hold*:

(i) *The operator A_ϵ is closed.*

(ii) *A_ϵ is a maximal accretive operator*, *or equivalently*, *$-A_\epsilon$ is maximal dissipative.*

(iii) *$0 \in \rho(A_\epsilon)$ and A_ϵ has compact resolvent for each $\epsilon \in [0, 1]$.*

(iv) *The semigroup linearly generated by* $-A_\epsilon$ *on* X, $\{e^{-A_\epsilon t} : t \geqslant 0\}$, *is a* C^0 *semigroup of contractions on* X.

(v) A_ϵ *is a sectorial operator in* X *with* $\operatorname{Re}\sigma(A_\epsilon) > 0$. *The semigroup of contractions* $\{e^{-A_\epsilon t} : t \geqslant 0\}$ *is analytic and compact.*

Proof. Note that (i) is immediate from the closedness of Λ_ϵ and Λ_ϵ^δ. For (ii) notice that, given $[\phi \;\; \varphi]^T \in X^1$, we have

$$\begin{aligned}\left\langle A_\epsilon \begin{bmatrix} \phi \\ \varphi \end{bmatrix}, \begin{bmatrix} \phi \\ \varphi \end{bmatrix} \right\rangle_X \\ &= -\langle \varphi, \phi \rangle_{Y^{1/2}} + \langle \Lambda_\epsilon \phi + \Lambda_\epsilon^\delta \varphi, \varphi \rangle_Y \\ &= -\langle \Lambda_\epsilon^{1/2} \varphi, \Lambda_\epsilon^{1/2} \phi \rangle_Y + \langle \Lambda_\epsilon^\delta (\Lambda_\epsilon^{1-\delta} \phi + \varphi), \varphi \rangle_Y \\ &= -\langle \Lambda_\epsilon^{1/2} \varphi, \Lambda_\epsilon^{1/2} \phi \rangle_Y + \overline{\langle \Lambda_\epsilon^{1/2} \varphi, \Lambda_\epsilon^{1/2} \phi \rangle_Y} + \langle \Lambda_\epsilon^{\delta/2} \varphi, \Lambda_\epsilon^{\delta/2} \varphi \rangle_Y,\end{aligned} \tag{2-4}$$

and hence

$$\operatorname{Re}\left\langle A_\epsilon \begin{bmatrix} \phi \\ \varphi \end{bmatrix}, \begin{bmatrix} \phi \\ \varphi \end{bmatrix} \right\rangle_X = \langle \Lambda_\epsilon^{\delta/2} \varphi, \Lambda_\epsilon^{\delta/2} \varphi \rangle_Y \geqslant 0, \quad \begin{bmatrix} \phi \\ \varphi \end{bmatrix} \in X^1,$$

which proves accretivity of A_ϵ.

Furthermore, for each $[\bar{\phi} \;\; \bar{\varphi}]^T \in X$, the linear equation

$$(I + A_\epsilon) \begin{bmatrix} \phi \\ \varphi \end{bmatrix} = \begin{bmatrix} \bar{\phi} \\ \bar{\varphi} \end{bmatrix} \tag{2-5}$$

is equivalent to the system

$$\begin{cases} \phi - \varphi = \bar{\phi}, \\ \Lambda_\epsilon \phi + \varphi + \Lambda_\epsilon^\delta \varphi = \bar{\varphi}, \end{cases}$$

or to the equation

$$\Lambda_\epsilon \phi + \Lambda_\epsilon^\delta \phi + \phi = \bar{\varphi} + \bar{\phi} + \Lambda_\epsilon^\delta \bar{\phi}. \tag{2-6}$$

By elliptic theory, it follows that there exists a unique function $\phi \in Y^{1/2}$ with $\Lambda_\epsilon \phi \in Y$ satisfying (2-6) and, therefore, for each $\epsilon \in [0, 1]$, there exists a unique $[\phi \;\; \varphi]^T \in X^1$ solving (2-5).

Concerning $0 \in \rho(A_\epsilon)$, we recall that there exists a bounded inverse operator $A_\epsilon^{-1} : X \to X$ given by

$$A_\epsilon^{-1} = \begin{bmatrix} \Lambda_\epsilon^{-(1-\delta)} & \Lambda_\epsilon^{-1} \\ -I & 0 \end{bmatrix}, \quad \epsilon \in [0, 1],$$

where $\Lambda_\epsilon^{-\alpha}$ are bounded inverse operators of Λ_ϵ^α. Thus, the resolvent operator A_ϵ^{-1} is compact, because it takes bounded subsets of X into bounded subsets of X^1, which is compactly embedded in X. This shows (iii).

The property (iv) that $-A_\epsilon$, $\epsilon \in [0,1]$, generates a C^0 semigroup of contractions on X follows from the Lummer–Phillips theorem (see [Pazy 1983]) and the observations concerning powers of maximal accretive operators (see [Kato 1976]).

Part (v) follows as a consequence of Theorem 1.1 in [Chen and Triggiani 1989]. Finally, compactness of $\{e^{-A_\epsilon t} : t \geqslant 0\}$, $\epsilon \in [0,1]$, is then a consequence of compactness of the resolvent operators of A_ϵ, and the proof is complete. □

Let us denote by Γ the boundary of Σ_ϕ. The following statements are valid:

$$e^{-A_\epsilon t} = \frac{1}{2\pi \mathrm{i}} \int_\Gamma e^{\lambda t} (\lambda I + A_\epsilon)^{-1} d\lambda, \tag{2-7}$$

$$\|e^{-A_\epsilon t}\|_{\mathscr{L}(X^1,X)} \leqslant C t^{-1/2} e^{-\omega t}, \quad t > 0, \tag{2-8}$$

for some $C > 0$ independent of ϵ.

Also, we have

$$\|(\lambda I + A_\epsilon)^{-1}\|_{\mathscr{L}(X)} \leqslant \frac{C}{1+|\lambda|}, \quad \lambda \in \Sigma_\phi, \tag{2-9}$$

$$\|(\lambda I + A_\epsilon)^{-1}\|_{\mathscr{L}(X^1)} \leqslant \frac{C}{1+|\lambda|}, \quad \lambda \in \Sigma_\phi, \tag{2-10}$$

where $C = C(\phi) > 0$ independent of ϵ.

Under the assumptions (1-4), problem (1-7) is locally well posed in X; see Theorem 1 in [Carvalho and Cholewa 2002b]. Moreover, under standard dissipative conditions like (1-5), we have the following result.

Theorem 2.2. *Assume (1-4) and (1-5) hold. The nonlinear semigroup $\{T_\epsilon(t) : t \geqslant 0\}$ associated with (1-7) is well defined in X and has a global attractor $\mathscr{A}_\epsilon$ in X. Furthermore*,

$$\sup_{\epsilon \in [0,1]} \sup_{w \in \mathscr{A}_\epsilon} \|w\|_X < \infty.$$

Proof. Problem (1-7) is globally well posed in X due to Theorem 1.1 in [Carvalho and Cholewa 2002a], namely, for any $w_0^\epsilon \in X$, there exists a unique

$$w^\epsilon(\cdot, w_0^\epsilon) \in C([0,\infty), X) \cap C^1((0,\infty), X)$$

with $w^\epsilon(t, w_0^\epsilon) \in D(A_\epsilon)$, for all $t > 0$, which satisfies (1-7) and

$$w(t, w_0^\epsilon) = e^{-A_\epsilon t} w_0^\epsilon + \int_0^t e^{-A_\epsilon (t-s)} f(w(s, w_0^\epsilon))\, ds, \quad t \geqslant 0.$$

Thus $T_\epsilon(t) w_0^\epsilon = u^\epsilon(t, w_0^\epsilon)$, $t \geqslant 0$. To simplify the notation we will denote the solution $w^0(t, w_0^0)$ by $w(t, w_0)$.

The existence of global attractors $\mathscr{A}_\epsilon$ in X for semigroups $\{T_\epsilon(t) : t \geqslant 0\}$ and uniform bounds are also established in Theorem 1.2 in [Carvalho and Cholewa 2002a]. □

3. Resolvent convergence

In this section we will show the convergence of the resolvent operators A_ϵ^{-1} to A_0^{-1}, as $\epsilon \to 0^+$, and we will establish that the rate of this convergence is $\|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}$.

We recall the convergence of the resolvent operators Λ_ϵ^{-1} to Λ_0^{-1}, as $\epsilon \to 0^+$, in terms of the difference $\|a_\epsilon - a_0\|_{L^\infty(\Omega)}$. This was proved in [Arrieta et al. 2013], however, for the sake of completeness, we will sketch a proof.

Lemma 3.1. *For $h \in Y$ and $\epsilon \in [0, 1]$, let us consider $u^\epsilon \in Y^1$ a solution of the problem*

$$\begin{cases} -\operatorname{div}(a_\epsilon(x)\nabla u) = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3-1}$$

Then there is a constant $C > 0$, independent of ϵ, such that

$$\|u^\epsilon\|_{Y^{1/2}} \leqslant C\|h\|_Y \tag{3-2}$$

and

$$\|u^\epsilon - u\|_{Y^{1/2}} \leqslant C\|h\|_Y\|a_\epsilon - a_0\|_\infty. \tag{3-3}$$

Proof. The estimate (3-2) follows from uniform boundedness of a_ϵ and Poincaré's inequality.

The solution of problem (3-1) can be obtained by a minimization procedure. That is, if we define

$$\lambda_\epsilon := \min_{u \in Y^{1/2}} \left\{ \frac{1}{2}\int_\Omega a_\epsilon |\nabla u|^2\, dx - \int_\Omega hu\, dx \right\},$$

then λ_ϵ is attained at u^ϵ. Therefore,

$$\begin{aligned} \lambda_\epsilon &= \frac{1}{2}\int_\Omega a_\epsilon |\nabla u^\epsilon|^2\, dx - \int_\Omega hu^\epsilon\, dx \\ &= \frac{1}{2}\int_\Omega a_\epsilon |\nabla u^\epsilon - \nabla u^0 + \nabla u^0|^2\, dx - \int_\Omega h(u^\epsilon - u^0 + u^0)\, dx, \end{aligned} \tag{3-4}$$

and, evaluating this expression, using that u^ϵ solves Lemma 3.1, we easily obtain

$$\lambda_\epsilon = \lambda_0 - \frac{1}{2}\int_\Omega a_\epsilon(x)|\nabla u^\epsilon - \nabla u^0|^2\, dx + \frac{1}{2}\int_\Omega (a_\epsilon(x) - a_0(x))|\nabla u^0|^2\, dx, \tag{3-5}$$

which implies

$$\lambda_\epsilon - \lambda_0 \leqslant \frac{1}{2}\int_\Omega (a_\epsilon(x) - a_0(x))|\nabla u^0|^2\, dx. \tag{3-6}$$

On the other hand

$$\begin{aligned}\lambda_0 &:= \min_{u\in Y^{1/2}}\left\{\frac{1}{2}\int_\Omega a_0(x)|\nabla u|^2\,dx - \int_\Omega hu\,dx\right\}\\ &\leqslant \frac{1}{2}\int_\Omega a_0(x)|\nabla u^\epsilon|^2\,dx - \int_\Omega hu^\epsilon\,dx\\ &= \lambda_\epsilon + \frac{1}{2}\int_\Omega (a_0(x)-a_\epsilon(x))|\nabla u^\epsilon|^2\,dx.\end{aligned}$$

With this, we obtain

$$\lambda_\epsilon - \lambda_0 \geqslant -\frac{1}{2}\int_\Omega (a_\epsilon(x)-a_0(x))|\nabla u^\epsilon|^2\,dx,$$

which combined with (3-2) and (3-6) gives us

$$|\lambda_\epsilon - \lambda_0| \leqslant \|a_\epsilon - a_0\|_{L^\infty(\Omega)} \sup_{\epsilon\in[0,1]} \|u^\epsilon\|^2_{Y^{1/2}} \leqslant C\|h\|_Y^2\|a_\epsilon - a_0\|_{L^\infty(\Omega)}. \tag{3-7}$$

Finally, the estimate (3-3) is obtained by combining (3-5) and (3-7). □

Corollary 3.2. *The operators $\Lambda_\epsilon^{-1}: Y \to Y^{1/2}$ are uniformly bounded and converge uniformly to $\Lambda_0^{-1}: Y \to Y^{1/2}$, as $\epsilon \to 0^+$. Furthermore, there exists a positive constant $C > 0$, independent of ϵ, such that*

$$\|\Lambda_\epsilon^{-1}\|_{\mathscr{L}(Y,Y^{1/2})} \leqslant C \tag{3-8}$$

and

$$\|\Lambda_\epsilon^{-1} - \Lambda_0^{-1}\|_{\mathscr{L}(Y,Y^{1/2})} \leqslant C\|a_\epsilon - a_0\|_{L^\infty(\Omega)}. \tag{3-9}$$

The uniform convergence of the operators Λ_ϵ^{-1} (see Corollary 3.2 in [Arrieta et al. 2013]) implies the convergence of their spectrum. As a matter of fact, the following result holds.

Proposition 3.3 [Carvalho and Piskarev 2006; Kato 1976]. *The following statements hold*:

(i) *If $\mu_0 \in \sigma(-\Lambda_0)$, there exists a sequence $\epsilon_n \to 0^+$ and $\{\mu_n\}$, with $\mu_n \in \sigma(-\Lambda_{\epsilon_n})$, $n \in \mathbb{N}$, such that $\mu_n \to \mu_0$, as $n \to \infty$;*

(ii) *If for some sequences $\epsilon_n \to 0^+$ and $\mu_n \to \mu_0$, as $n \to \infty$, with $\mu_n \in \sigma(-\Lambda_{\epsilon_n})$, $n \in \mathbb{N}$, then $\mu_0 \in \sigma(-\Lambda_0)$.*

Moreover, from Lemma 3.4 in [Arrieta et al. 2013], there exists $C > 0$, independent of ϵ, such that

$$\|(\lambda I + \Lambda_\epsilon)^{-1} - (\lambda I + \Lambda_0)^{-1}\|_{\mathscr{L}(Y,Y^{1/2})} \leqslant C\|a_\epsilon - a_0\|_{L^\infty(\Omega)}, \tag{3-10}$$

for each $\lambda \in \Sigma_\phi$.

Proposition 3.4. *The operators $\Lambda_\epsilon^{-(1-\delta)} : Y^{1/2} \to Y^{1/2}$ are uniformly bounded and converge uniformly to $\Lambda_0^{-(1-\delta)} : Y^{1/2} \to Y^{1/2}$, as $\epsilon \to 0$. Furthermore, there exists a positive constant $C > 0$, independent of ϵ, such that*

$$\|\Lambda_\epsilon^{-(1-\delta)}\|_{\mathscr{L}(Y^{1/2})} \leqslant C \tag{3-11}$$

and

$$\|\Lambda_\epsilon^{-(1-\delta)} - \Lambda_0^{-(1-\delta)}\|_{\mathscr{L}(Y^{1/2})} \leqslant C\|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}. \tag{3-12}$$

Proof. Notice that, using (2-2), we get

$$\|\Lambda_\epsilon^{-(1-\delta)} u\|_{Y^{1/2}} = \|\Lambda_\epsilon^{\delta-1/2} u\|_Y \leqslant C\|\Lambda_\epsilon^{-1} u\|_{Y^{1/2}}^{1-(\delta-1/2)} \|u\|_{Y^{1/2}}^{\delta-1/2},$$

where $C > 0$ is uniform with respect to ϵ and δ. Thus (3-11) follows by (3-8).

Before we prove (3-12), let us observe that (2-2) and (3-11) imply

$$\begin{aligned} \|(\Lambda_\epsilon^{-(1-\delta)} - \Lambda_0^{-(1-\delta)})h\|_{Y^{1/2}} &= \|\Lambda_\epsilon^{1/2}(\Lambda_\epsilon^{-(1-\delta)} - \Lambda_0^{-(1-\delta)})h\|_Y \\ &\leqslant C\|\Lambda_\epsilon(\Lambda_\epsilon^{-(1-\delta)} - \Lambda_0^{-(1-\delta)})h\|_Y^{1/2} \|(\Lambda_\epsilon^{-(1-\delta)} - \Lambda_0^{-(1-\delta)})h\|_Y^{1/2} \\ &\leqslant C\|\Lambda_\epsilon(\Lambda_\epsilon^{-(1-\delta)} - \Lambda_0^{-(1-\delta)})h\|_Y^{1/2} (\|\Lambda_\epsilon^{-(1-\delta)} h\|_Y^{1/2} + \|\Lambda_0^{-(1-\delta)} h\|_Y^{1/2}) \\ &\leqslant C\|\Lambda_\epsilon(\Lambda_\epsilon^{-(1-\delta)} - \Lambda_0^{-(1-\delta)})h\|_Y^{1/2}, \end{aligned} \tag{3-13}$$

for some $C > 0$ independent of ϵ, and for any $h \in Y$.

To prove (3-12), it follows by (3-13) that it is sufficient to obtain an estimate for the norm $\|\Lambda_\epsilon(\Lambda_\epsilon^{-(1-\delta)} - \Lambda_0^{-(1-\delta)})\|_{\mathscr{L}(Y)}$. In fact, it follows by (1-3) that

$$\Lambda_\epsilon^{-\alpha} - \Lambda_0^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{-\alpha}[(\lambda I + \Lambda_\epsilon)^{-1} - (\lambda I + \Lambda_0)^{-1}]\, d\lambda. \tag{3-14}$$

Using (3-14) (with $\alpha = 1 - \delta$), we can deduce

$$\begin{aligned} &\|\Lambda_\epsilon(\Lambda_\epsilon^{-(1-\delta)} - \Lambda_0^{-(1-\delta)})\|_{\mathscr{L}(Y)} \\ &\qquad \leqslant \int_0^\infty \big\|\lambda^{-(1-\delta)}\Lambda_\epsilon[(\lambda I + \Lambda_\epsilon)^{-1} - (\lambda I + \Lambda_0)^{-1}]\big\|_{\mathscr{L}(Y)} d|\lambda|. \end{aligned} \tag{3-15}$$

Notice that the resolvent identity

$$\begin{aligned} (\lambda I + \Lambda_\epsilon)^{-1} - (\lambda I + \Lambda_0)^{-1} &= (\lambda I + \Lambda_\epsilon)^{-1}[I - (\lambda I + \Lambda_\epsilon)(\lambda I + \Lambda_0)^{-1}] \\ &= (\lambda I + \Lambda_\epsilon)^{-1}[(\lambda I + \Lambda_0) - (\lambda I + \Lambda_\epsilon)](\lambda I + \Lambda_0)^{-1} \\ &= (\lambda I + \Lambda_\epsilon)^{-1}[\Lambda_0 - \Lambda_\epsilon](\lambda I + \Lambda_0)^{-1} \\ &= \Lambda_\epsilon(\lambda I + \Lambda_\epsilon)^{-1}[\Lambda_\epsilon^{-1} - \Lambda_0^{-1}]\Lambda_0(\lambda I + \Lambda_0)^{-1} \end{aligned} \tag{3-16}$$

holds, and, by sectoriality of Λ_ϵ, we have

$$\|\Lambda_\epsilon(\lambda I+\Lambda_\epsilon)^{-1}\|_{\mathscr{L}(Y)}\leqslant C,\quad \text{for all } \epsilon\in[0,1], \tag{3-17}$$

where $C>0$ is independent of ϵ.

Substituting (3-16) into (3-15), we get

$$\begin{aligned}\|\Lambda_\epsilon(\Lambda_\epsilon^{-(1-\delta)}-\Lambda_0^{-(1-\delta)})\|_{\mathscr{L}(Y)}&\\ \leqslant\int_0^\infty \|\lambda^{-(1-\delta)}\Lambda_\epsilon\Lambda_\epsilon(\lambda I+\Lambda_\epsilon)^{-1}[\Lambda_\epsilon^{-1}-\Lambda_0^{-1}]\Lambda_0(\lambda I+\Lambda_0)^{-1}\|_{\mathscr{L}(Y)}d|\lambda|&\\ \leqslant\int_0^\infty |\lambda|^{-(1-\delta)}\|\Lambda_\epsilon^{\gamma}(\lambda I+\Lambda_\epsilon)^{-1}\Lambda_\epsilon^{1+(1-\gamma)}[\Lambda_\epsilon^{-1}-\Lambda_0^{-1}]\Lambda_0(\lambda I+\Lambda_0)^{-1}\|_{\mathscr{L}(Y)}d|\lambda|,&\end{aligned} \tag{3-18}$$

where $\gamma\in(1,2)$ is a constant to be chosen.

Since $Y^{1/2}$ is continuously embedded in $Y^{1+(1-\gamma)}$, by estimates (3-9) and (3-17), we can deduce from (3-18) that

$$\begin{aligned}\|\Lambda_\epsilon(\Lambda_\epsilon^{-(1-\delta)}-\Lambda_0^{-(1-\delta)})\|_{\mathscr{L}(Y)}&\\ \leqslant C\|a_\epsilon-a_0\|_{L^\infty(\Omega)}\int_0^\infty |\lambda|^{-(1-\delta)}\|\Lambda_\epsilon^{\gamma-r}\Lambda_\epsilon^{r}(\lambda I+\Lambda_\epsilon)^{-1}\|_{\mathscr{L}(Y)}d|\lambda|,&\end{aligned} \tag{3-19}$$

where $r\in(\gamma-1,1)$ is a constant to be chosen.

From (2-2) and the fact that Y^1 is continuously embedded in Y^r, it follows by (3-19) that

$$\begin{aligned}\|\Lambda_\epsilon(\Lambda_\epsilon^{-(1-\delta)}-\Lambda_0^{-(1-\delta)})\|_{\mathscr{L}(Y)}&\\ \leqslant C\|a_\epsilon-a_0\|_{L^\infty(\Omega)}\int_0^\infty |\lambda|^{-(1-\delta)}\|\Lambda_\epsilon\Lambda_\epsilon^{r}(\lambda I+\Lambda_\epsilon)^{-1}\|_{\mathscr{L}(Y)}^{\gamma-r}&\\ \times\|\Lambda_\epsilon^{r}(\lambda I+\Lambda_\epsilon)^{-1}\|_{\mathscr{L}(Y)}^{1-(\gamma-r)}d|\lambda|&\\ \leqslant C\|a_\epsilon-a_0\|_{L^\infty(\Omega)}\int_0^\infty |\lambda|^{-(1-\delta)}\|\Lambda_\epsilon(\lambda I+\Lambda_\epsilon)^{-1}\|_{\mathscr{L}(Y)}^{\gamma-r}\|(\lambda I+\Lambda_\epsilon)^{-1}\|_{\mathscr{L}(Y)}^{1-(\gamma-r)}d|\lambda|.&\end{aligned} \tag{3-20}$$

Using (2-1) and (3-17), we get by (3-20) that

$$\begin{aligned}\|\Lambda_\epsilon(\Lambda_\epsilon^{-(1-\delta)}-\Lambda_0^{-(1-\delta)})\|_{\mathscr{L}(Y)}&\\ \leqslant C\|a_\epsilon-a_0\|_{L^\infty(\Omega)}\int_0^\infty |\lambda|^{-(1-\delta)}\|(\lambda I+\Lambda_\epsilon)^{-1}\|_{\mathscr{L}(Y)}^{1-(\gamma-r)}d|\lambda|&\\ \leqslant C\|a_\epsilon-a_0\|_{L^\infty(\Omega)}\int_0^\infty |\lambda|^{-(1-\delta)}(\max\{1,|\lambda|^{-1}\})^{1-(\gamma-r)}d|\lambda|&\\ \leqslant C\|a_\epsilon-a_0\|_{L^\infty(\Omega)}\Big(\int_0^1 |\lambda|^{\delta-1}d|\lambda|+\int_1^\infty |\lambda|^{-2+\delta+\gamma-r}d|\lambda|\Big).&\end{aligned} \tag{3-21}$$

Taking γ and r sufficiently close to 1 such that $\delta+\gamma-r<1$, we can conclude by (3-21) the existence of a positive constant C, independent of ϵ, such that

$$\|\Lambda_\epsilon(\Lambda_\epsilon^{-(1-\delta)}-\Lambda_0^{-(1-\delta)})\|_{\mathscr{L}(Y)}\leqslant C\|a_\epsilon-a_0\|_{L^\infty(\Omega)}. \tag{3-22}$$

Finally, combining (3-13) and (3-22) we obtain the desired estimate (3-12). □

Before we prove analogous result of the last proposition with A_ϵ instead of Λ_ϵ, we present the following general version of the moment inequality (see [Sobolevskii 1961]):

$$\|A_\epsilon w\|_X\leqslant C\|w\|_X^{1-\beta/\alpha}\|A_\epsilon^\alpha w\|_X^{\beta/\alpha},\quad w\in D(A_\epsilon^\alpha), \tag{3-23}$$

where the constant $C>0$ is independent of ϵ.

Proposition 3.5. *The operators $A_\epsilon^{-1}:X\to X^1$ are uniformly bounded and converge in the uniform topology to $A_0^{-1}:X\to X^1$, as $\epsilon\to 0^+$. Furthermore, there exists a positive constant $C>0$, independent of ϵ, such that*

$$\|A_\epsilon^{-1}\|_{\mathscr{L}(X,X^1)}\leqslant C \tag{3-24}$$

and

$$\|A_\epsilon^{-1}-A_0^{-1}\|_{\mathscr{L}(X,X^1)}\leqslant C\|a_\epsilon-a_0\|_{L^\infty(\Omega)}^{1/2}. \tag{3-25}$$

Proof. For $g,h\in L^2(\Omega)$ and $\epsilon\in[0,1]$, let $[\phi_\epsilon\ \ \varphi_\epsilon]^T$ be the solution of the problem

$$A_\epsilon\begin{bmatrix}\phi\\ \varphi\end{bmatrix}=\begin{bmatrix}g\\ h\end{bmatrix},\quad t>0.$$

By (1-6), there exists $C>0$, independent ϵ, such that

$$\left\|\begin{bmatrix}\phi_\epsilon\\ \varphi_\epsilon\end{bmatrix}-\begin{bmatrix}\phi_0\\ \varphi_0\end{bmatrix}\right\|_{X^1}\leqslant C\left\|\begin{bmatrix}\phi_\epsilon\\ \varphi_\epsilon\end{bmatrix}-\begin{bmatrix}\phi_0\\ \varphi_0\end{bmatrix}\right\|_{Y^1\times Y^\delta},$$

and by (3-23), we get

$$\begin{aligned}\left\|\begin{bmatrix}\phi_\epsilon\\ \varphi_\epsilon\end{bmatrix}-\begin{bmatrix}\phi_0\\ \varphi_0\end{bmatrix}\right\|_{Y^1\times Y^\delta}&\\ &=\|\Lambda_\epsilon^{-(1-\delta)}g-\Lambda_0^{-(1-\delta)}g\|_{Y^1}+\|\Lambda_\epsilon^{-1}h-\Lambda_0^{-1}h\|_{Y^\delta}\\ &\leqslant C\|\Lambda_\epsilon^{1/2}(\Lambda_\epsilon^{-(1-\delta)}g-\Lambda_0^{-(1-\delta)}g)\|_Y+\|\Lambda_\epsilon^{-(1-\delta)}h-\Lambda_0^{-(1-\delta)}h\|_Y.\end{aligned}$$

Thus, by Corollary 3.2, we conclude that

$$\begin{aligned}\left\|\begin{bmatrix}\phi_\epsilon\\ \varphi_\epsilon\end{bmatrix}-\begin{bmatrix}\phi_0\\ \varphi_0\end{bmatrix}\right\|_{X^1}&\leqslant C(\|g\|_{Y^{1/2}}+\|h\|_Y)\|a_\epsilon-a_0\|_{L^\infty(\Omega)}^{1/2}\\ &=C\left\|\begin{bmatrix}g\\ h\end{bmatrix}\right\|_X\|a_\epsilon-a_0\|_{L^\infty(\Omega)}^{1/2},\end{aligned}$$

where $C > 0$ is independent of ϵ. □

The next result ensures convergence of the spectrum of operators $-A_\epsilon$.

Proposition 3.6 [Carvalho and Piskarev 2006; Kato 1976]. *The following statements hold*:

(i) *If* $\mu_0 \in \sigma(-A_0)$, *there exists a sequence* $\epsilon_n \to 0^+$ *and* $\{\mu_n\}$, *with* $\mu_n \in \sigma(-A_{\epsilon_n})$, $n \in \mathbb{N}$ *such that* $\mu_n \to \mu_0$ *as* $n \to \infty$.

(ii) *If for some sequences* $\epsilon_n \to 0^+$ *and* $\mu_n \to \mu_0$ *as* $n \to \infty$, *with* $\mu_n \in \sigma(-A_{\epsilon_n})$, $n \in \mathbb{N}$, *then* $\mu_0 \in \sigma(-A_0)$.

Now let us establish the result which treats convergence for resolvent operators.

Proposition 3.7. *For each* $\phi \in (\pi/2, \pi)$, *there exists a constant* $C = C(\phi) > 0$ *such that*

$$\sup_{\lambda \in \Sigma_\phi} \|(\lambda I + A_\epsilon)^{-1} - (\lambda I + A_0)^{-1}\|_{\mathscr{L}(X,X^1)} \leqslant C\|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}.$$

Proof. We can see that

$$(\lambda I + A_\epsilon)^{-1} - (\lambda I - A_0)^{-1} = (\lambda I + A_\epsilon)^{-1} A_\epsilon [A_0^{-1} - A_\epsilon^{-1}] A_0 (\lambda I + A_0)^{-1}$$
$$= A_\epsilon (\lambda I + A_\epsilon)^{-1} [A_0^{-1} - A_\epsilon^{-1}] A_0 (\lambda I + A_0)^{-1}. \tag{3-26}$$

Notice that, for $\lambda \in \Sigma_\phi \subset \rho(-A_\epsilon)$, we have

$$A_\epsilon(\lambda I + A_\epsilon)^{-1} = [(\lambda I + A_\epsilon)A_\epsilon^{-1}]^{-1} = [\lambda A_\epsilon^{-1} - I]^{-1},$$

and, therefore,

$$\|A_\epsilon(\lambda I + A_\epsilon)^{-1}\|_{\mathscr{L}(X^1)} \leqslant C, \ \lambda \in \Sigma_\phi, \tag{3-27}$$

for some $C > 0$ independent of ϵ.

By (3-25)–(3-27), we have the existence of a constant $C > 0$ (independent of ϵ and of $\lambda \in \Sigma_\phi$) such that

$$\|(\lambda I + A_\epsilon)^{-1} - (\lambda I + A_0)^{-1}\|_{\mathscr{L}(X,X^1)} \leqslant C\|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}. \quad \square$$

To finish this section, we will make a spectral analysis, where we will give a characterization, as well as a rate of convergence, as $\epsilon \to 0$, for the eigenvalues associated with the operators A_ϵ.

Let γ be a closed, rectifiable, simple and oriented counterclockwise curve in $\rho(-A_0)$ around $\mu_0 \in \sigma(A_0)$ which has index 1 relative to γ. From part (ii) of Proposition 3.6, it is easy to see that there is an $\epsilon_\gamma > 0$ such that the trace of γ is in $\rho(A_\epsilon)$, for $\epsilon \in [0, \epsilon_\gamma]$. We define the spectral projection in X

$$Q_\epsilon(\mu_0) = \frac{1}{2\pi i}\int_\gamma (\lambda I + A_\epsilon)^{-1}\, d\lambda,$$

and, for $\mu \in \mathbb{C}$ such that $(1/(2\pi i)) \int_\gamma (\lambda - \mu)^{-1}\, d\lambda = 1$, we define the generalized eigenspace associated with μ, $W(\mu, -A_\epsilon) = Q_\epsilon(\mu_0)(X)$, $\epsilon \in [0, \epsilon_\gamma]$. Furthermore, $Q_\epsilon(\mu_0)$ is compact and $\dim W(\mu, -A_\epsilon) = \operatorname{rank}(Q_\epsilon(\mu_0)) < \infty$.

Related to the rate of convergence, the following result holds.

Proposition 3.8. *The family of operators $Q_\epsilon(\mu_0) : X \to X$ converges uniformly to $Q_0(\mu_0) : X \to X$, as $\epsilon \to 0^+$. Moreover,*

$$\|Q_\epsilon(\mu_0) - Q_0(\mu_0)\|_{\mathscr{L}(X)} \leqslant C \|a_\epsilon - a_0\|^{1/2}_{L^\infty(\Omega)}, \tag{3-28}$$

and

$$\|A_\epsilon Q_\epsilon(\mu_0) - A_0 Q_0(\mu_0)\|_{\mathscr{L}(X)} \leqslant C \|a_\epsilon - a_0\|^{1/2}_{L^\infty(\Omega)}, \tag{3-29}$$

where $C > 0$ independent of ϵ.

Proof. Since

$$Q_\epsilon(\mu_0) - Q_0(\mu_0) = \frac{1}{2\pi i} \int_\gamma [(\lambda I + A_\epsilon)^{-1} - (\lambda I + A_0)^{-1}]\, d\lambda,$$

we can use Proposition 3.7 to guarantee the estimate (3-28).

To prove (3-29), it is sufficient to use (3-26) and (3-28). □

Remark 3.9. If μ_0 is an isolated eigenvalue for A_0, we may define $Q_\epsilon(\mu_0)$ as above and it follows from Proposition 3.6 that there exists μ_ϵ, which is an eigenvalue of A_ϵ such that $\mu_\epsilon \to \mu_0$, as $\epsilon \to 0^+$. Hence $Q_\epsilon(\mu_0) = Q_\epsilon(\mu_\epsilon)$. We still have from Proposition 3.8 that

$$\|Q_\epsilon(\mu_\epsilon) Q_0(\mu_0) - Q_0(\mu_0)\|_{\mathscr{L}(X)} \leqslant C \|a_\epsilon - a_0\|^{1/2}_{L^\infty(\Omega)}$$

and that $Q_\epsilon(\mu_\epsilon) Q_0(\mu_0)$ is an isomorphism between $R(Q_0(\mu_0))$ and $R(Q_\epsilon(\mu_\epsilon))$.

The next result deals with the characterization and rate of convergence of the eigenvalues associated to operators A_ϵ.

Theorem 3.10. *For each $\epsilon \in [0, 1]$, the eigenvalues of the operator A_ϵ are given by*

$$\lambda^\pm_{\epsilon,n} = \frac{\mu^\delta_{\epsilon,n}}{2} \pm i \frac{\sqrt{4\mu_{\epsilon,n} - \mu^{2\delta}_{\epsilon,n}}}{2}, \quad n \in \mathbb{N}, \tag{3-30}$$

where $\mu_{\epsilon,n}$, $n \in \mathbb{N}$, denotes the eigenvalues of the operator Λ_ϵ. Furthermore, if $\operatorname{Ker}(\lambda^\pm_{0,n} I - A_0) = R(Q_0(\lambda^\pm_{0,n}))$, then

$$|\lambda^\pm_{\epsilon,n} - \lambda^\pm_{0,n}| \leqslant C_n \|a_\epsilon - a_0\|^{1/2}_{L^\infty(\Omega)}, \quad n \in \mathbb{N},$$

for some constant $C_n > 0$, independent of ϵ.

Proof. To study the spectral problem for the operator A_ϵ, we consider the equation

$$A_\epsilon \begin{bmatrix} \phi \\ \varphi \end{bmatrix} = \lambda \begin{bmatrix} \phi \\ \varphi \end{bmatrix}, \tag{3-31}$$

that is,

$$\lambda^2 \phi - \lambda \Lambda_\epsilon^\delta \phi + \Lambda_\epsilon \phi = 0, \tag{3-32}$$

whose solutions are the eigenvectors $\{\phi_{\epsilon,n}\}$ of Λ_ϵ:

$$\lambda^2 \phi_{\epsilon,n} - \mu_{\epsilon,n}^\delta \lambda \phi_{\epsilon,n} + \mu_{\epsilon,n} \phi_{\epsilon,n} = 0. \tag{3-33}$$

In this way, the corresponding eigenvalues $\{\lambda_{\epsilon,n}^\pm\}$ of A_ϵ are the solutions of the equation

$$\lambda^2 - \mu_{\epsilon,n}^\delta \lambda + \mu_{\epsilon,n} = 0$$

and they are given by (3-30).

Moreover, by the above remark and Proposition 3.8, we have that, for each $\epsilon > 0$, there exists $[\phi \ \ \varphi]^T \in R(Q_0)$, $\|[\phi \ \ \varphi]^T\|_{X^1} = 1$, such that $Q_\epsilon [\phi \ \ \varphi]^T$ is an eigenvector of A_ϵ associated to λ_ϵ and

$$|\lambda_{\epsilon,n}^\pm - \lambda_{0,n}^\pm|$$

$$\leqslant \left\| \lambda_{\epsilon,n}^\pm Q_0 \begin{bmatrix} \phi \\ \varphi \end{bmatrix} - \lambda_{\epsilon,n}^\pm Q_\epsilon \begin{bmatrix} \phi \\ \varphi \end{bmatrix} \right\|_{X^1} + \left\| \lambda_{\epsilon,n}^\pm Q_\epsilon \begin{bmatrix} \phi \\ \varphi \end{bmatrix} - \lambda_{0,n}^\pm Q_0 \begin{bmatrix} \phi \\ \varphi \end{bmatrix} \right\|_{X^1} \tag{3-34}$$

$$\leqslant C \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}, \tag{3-35}$$

and the proof is completed. □

4. Rate of convergence of resolvents of linearized operators

In this section we will study the rate of convergence of the resolvents of operators which corresponds to linearizations of (1-7) around equilibria.

It is known that the Nemytskiĭ map $f^e(u) := f(u)$, $u \in Y^{1/2}$, is Fréchet continuously differentiable. Moreover, if $\{u_\epsilon\}$ converges to u_0 in $Y^{1/2}$ and $0 \notin \sigma(\Lambda_0 - (f^e)'(u_0))$, then $((f^e)'(u_\epsilon))\Lambda_\epsilon^{-1}$ converges to $((f^e)'(u_0))\Lambda_0^{-1}$ in the uniform operator topology of $\mathscr{L}(Y)$; see, for instance, [Arrieta et al. 2013]. Hence the Nemytskiĭ map F is Fréchet continuously differentiable. Moreover, if $u_\epsilon \to u_0$ in X and $0 \notin \sigma(A_0 - F'(u_0))$, then

$$(F'(u_\epsilon))A_\epsilon^{-1} \to (F'(u_0))A_0^{-1} \quad \text{in } \mathscr{L}(X). \tag{4-1}$$

Lemma 4.1. *We assume $u_\epsilon \to u_0$ in X and $0 \notin \sigma(A_0 - F'(u_0))$. Then there exists $\epsilon_0 > 0$ such that the net of operators*

$$\{A_\epsilon^1/2(A_\epsilon - F'(u_\epsilon))^{-1} : \epsilon \in [0, 1]\}$$

is uniformly bounded in $\mathscr{L}(X)$ and

$$\|A_\epsilon^1/2(A_\epsilon - F'(u_\epsilon))^{-1} - A_0^1/2(A_0 - F'(u_0))^{-1}\|_{\mathscr{L}(X)} \leqslant C\|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2},$$

where $C > 0$ is independent of ϵ.

Proof. The proof follows from the identity

$$A_\epsilon^{1/2}(A_\epsilon - F'(u_\epsilon))^{-1} = A_\epsilon^{-1/2}(I - F'(u_\epsilon)A_\epsilon^{-1})^{-1}$$

and by (4-1). □

5. Rate of convergence of the linear and nonlinear semigroups

Since the operators A_ϵ, $\epsilon \in [0, 1]$, are self-adjoint and A_ϵ^{-1} converges uniformly to A_0^{-1} as $\epsilon \to 0^+$, for each $\alpha < \lambda_1^0$ (λ_1^0 the first eigenvalue of A_0), there exists $C > 0$, independent of $\epsilon \in [0, 1]$, such that

$$\|e^{-A_\epsilon t}\|_{\mathscr{L}(X)} \leqslant Ce^{-\alpha t}t^{-1/2}, \quad t > 0,\ \epsilon \in [0, 1]. \tag{5-1}$$

Theorem 5.1. *If $\theta \in \left(0, \frac{1}{2}\right]$ and $\alpha < \lambda_1^0$, there exists $C > 0$, independent of ϵ, such that*

$$\|e^{-A_\epsilon t} - e^{-A_0 t}\|_{\mathscr{L}(X)} \leqslant Ce^{-\alpha t}\|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{2\theta}t^{-(1/2+\theta)} \tag{5-2}$$

for all $t > 0$ and $\epsilon \in [0, 1]$.

Proof. Considering the linear semigroup

$$e^{-A_\epsilon t} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda I + A_\epsilon)^{-1}\,d\lambda, \quad \epsilon \in [0, 1],$$

where Γ is the boundary of sector $\Sigma_{-\omega,\phi} = \{\lambda \in \mathbb{C} : |\arg(\lambda + \omega)| \leqslant \phi\}$ with $\pi/2 < \phi < \pi$, oriented in such a way that the imaginary part of λ increases as λ runs in Γ.

The estimate

$$\|e^{-A_\epsilon t} - e^{-A_0 t}\|_{\mathscr{L}(X)} \leqslant \|e^{-A_\epsilon t}\|_{\mathscr{L}(X)} + \|e^{-A_0 t}\|_{\mathscr{L}(X)} \leqslant Ce^{-\alpha t}t^{-1/2} \tag{5-3}$$

follows by (5-1).

On the other hand, using Proposition 3.7, we have

$$\|e^{-A_\epsilon t} - e^{-A_0 t}\|_{\mathscr{L}(X)} \leqslant Ce^{-\alpha t}\|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}t^{-1}. \tag{5-4}$$

Therefore, for $\theta \in \left(0, \frac{1}{2}\right]$, we obtain

$$\begin{aligned}\|e^{-A_\epsilon t} - e^{-A_0 t}\|_{\mathscr{L}(X)} &\leqslant Ce^{-\alpha(1-2\theta)t}t^{-1/2(1-2\theta)}e^{-\alpha(2\theta)t}\|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{2\theta}t^{-2\theta}\\ &\leqslant Ce^{-\alpha t}\|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{2\theta}t^{-(1/2+\theta)}.\end{aligned}$$

where $C > 0$ is independent of ϵ. □

Theorem 5.2. *Let $[u_\epsilon \ \ v_\epsilon]^T, [u \ \ v]^T \in X$, and $\theta \in \left(0, \frac{1}{2}\right)$. Then there are positive constants C and L_f such that*

$$\left\| T_\epsilon(t)\begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - T_0(t)\begin{bmatrix} u \\ v \end{bmatrix} \right\|_X \leqslant C e^{Lt} t^{-(1/2+\theta)} \left(\left\| \begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - \begin{bmatrix} u \\ v \end{bmatrix} \right\|_X + \|a_\epsilon - a_0\|^\theta_{L^\infty(\Omega)} \right)$$

for all $t \geqslant 0$.

Proof. For $t \geqslant 0$ and $[u_\epsilon \ \ v_\epsilon]^T \in X$ we have

$$T_\epsilon(t)\begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} = e^{-A_\epsilon t}\begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} + \int_0^t e^{-A_\epsilon(t-s)} f\left(T_\epsilon(s)\begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} \right) ds, \quad \epsilon \in [0, 1],$$

and therefore

$$\begin{aligned} \text{(5-5)} \quad & \left\| T_\epsilon(t)\begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - T_0(t)\begin{bmatrix} u \\ v \end{bmatrix} \right\|_X \\ & \leqslant \left\| e^{-A_\epsilon t}\begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - e^{-A_0 t}\begin{bmatrix} u \\ v \end{bmatrix} \right\|_X + \int_0^t \left\| e^{-A_\epsilon(t-s)} f\left(T_\epsilon(s)\begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} \right) \right. \\ & \qquad \left. - e^{-A_0(t-s)} f\left(T_0(s)\begin{bmatrix} u \\ v \end{bmatrix} \right) \right\|_X ds. \end{aligned}$$

From (5-1) and (5-2) we get

$$\begin{aligned} \text{(5-6)} \quad & \left\| e^{-A_\epsilon t}\begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - e^{-A_0 t}\begin{bmatrix} u \\ v \end{bmatrix} \right\|_X \\ & \leqslant C t^{-(1/2+\theta)} \left\| \begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - \begin{bmatrix} u \\ v \end{bmatrix} \right\|_X + C \|a_\epsilon - a_0\|^{2\theta}_{L^\infty(\Omega)} t^{-(1/2+\theta)}. \end{aligned}$$

We still have

$$\begin{aligned} \text{(5-7)} \quad & \int_0^t \left\| e^{-A_\epsilon(t-s)} f\left(T_\epsilon(s)\begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} \right) - e^{-A_0(t-s)} f\left(T_0(s)\begin{bmatrix} u \\ v \end{bmatrix} \right) \right\|_X ds \\ & \leqslant C L_f \int_0^t (t-s)^{-1/2} e^{-\alpha(t-s)} \left\| T_\epsilon(s)\begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - T_0(s)\begin{bmatrix} u \\ v \end{bmatrix} \right\|_X ds \\ & \qquad + C \|a_\epsilon - a_0\|^{2\theta}_{L^\infty(\Omega)} \int_0^t (t-s)^{-(1/2+\theta)} e^{-\alpha(t-s)} ds. \end{aligned}$$

Substituting (5-6) and (5-7) in (5-5), it follows that

$$\begin{aligned}\Big\| T_\epsilon(t)\begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - T_0(t)\begin{bmatrix} u \\ v \end{bmatrix}\Big\|_X \\ \leqslant C\Big(\Big\|\begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - \begin{bmatrix} u \\ v \end{bmatrix}\Big\|_X + \|a_\epsilon - a_0\|^{2\theta}_{L^\infty(\Omega)}\Big) t^{-(1/2+\theta)} e^{-\alpha t} \\ + CL_f \int_0^t (t-s)^{-1/2} e^{-\alpha(t-s)} \Big\| T_\epsilon(s)\begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - T_0(s)\begin{bmatrix} u \\ v \end{bmatrix}\Big\|_X ds.\end{aligned}$$

Thus the singular Gronwall inequality (see Lemma 7.1.1 in [Henry 1981]) guarantees the existence of a constant $L > 0$ such that

$$\begin{aligned}\Big\| T_\epsilon(t)\begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - T_0(t)\begin{bmatrix} u \\ v \end{bmatrix}\Big\|_X \\ \leqslant Ce^{L_f t} t^{-(1/2+\theta)}\Big(\Big\|\begin{bmatrix} u_\epsilon \\ v_\epsilon \end{bmatrix} - \begin{bmatrix} u \\ v \end{bmatrix}\Big\|_X + \|a_\epsilon - a_0\|^{\theta}_{L^\infty(\Omega)}\Big).\end{aligned}$$

□

6. Rate of convergence of the equilibria and of the linearizations

Now we will work to control the behavior of equilibria in terms of $\|a_\epsilon - a_0\|^{1/2}_{L^\infty(\Omega)}$. First, we will give the definition of equilibrium of problem (1-7).

Definition 6.1. The equilibrium solutions of (1-7) are the functions that solve the stationary

$$A_\epsilon w^\epsilon = F(w^\epsilon), \quad \epsilon \in [0, 1]. \tag{6-1}$$

For each $\epsilon \in [0, 1]$, we denote by $\mathscr{E}_\epsilon$ the set of the equilibrium solutions of (1-7). We say that an equilibrium w^ϵ_* of (1-7) is hyperbolic if the spectrum $\sigma(A_\epsilon - F'(w^\epsilon_*))$ of $A_\epsilon - F'(w^\epsilon_*)$ is disjoint from the imaginary axis.

We start by proving the upper semicontinuity of the family of equilibria.

Proposition 6.2. *The family $\{\mathscr{E}_\epsilon : \epsilon \in [0, 1]\}$ is upper semicontinuous at $\epsilon = 0$.*

Proof. Since $\mathscr{E}_\epsilon$ is contained in $\mathscr{A}_\epsilon$, $\sup\{\|w^\epsilon\|_X : w^\epsilon \in \mathscr{E}_\epsilon,\ \epsilon \in [0, 1]\} < \infty$. Using the fact that $F : X \to Y \times Y$ is bounded, for each $w^\epsilon \in \mathscr{E}_\epsilon$, we have that $w^\epsilon = A_\epsilon^{-1} F(w^\epsilon)$, and the result follows from the uniform convergence of A_ϵ^{-1} to A_0^{-1}. □

The proof of lower semicontinuity requires additional assumptions. We need to assume that the equilibrium points of (1-7) are stable under perturbation. This stability under perturbation will be given by the hyperbolicity.

Proposition 6.3. *Any hyperbolic point of $\mathscr{E}_0$ is isolated.*

Proof. We note that $w_* \in \mathscr{E}_0$ is a solution of (6-1) if and only if w_* is a fixed point of

$$\Psi(w) := (A_0 - F'(w_*))^{-1}(F(w) - F'(w_*)w).$$

If we show that, for some $r > 0$, $\Psi : \bar{B}_r(w_*) \to \bar{B}_r(w_*)$ is a contraction, where $\bar{B}_r(w_*) := \{w \in X : \|w - w_*\|_X \leqslant r\}$, then w_* is a unique element in $\bar{B}_r(w_*) \cap \mathscr{E}_0$ and, consequently, is isolated. In fact, letting $r > 0$ and $u, v \in \bar{B}_r(u_*)$, we observe by (4-1) that

$$\begin{aligned}\|\Psi(u) - \Psi(v)\|_X &\leqslant \|(A_0 - F'(w_*))^{-1}\|_{\mathscr{L}(X)} \|(F(u) - F(v) - F'(w_*)(u - v))\|_X \\ &\leqslant Cr\|u - v\|_X.\end{aligned}$$

Thus, choosing r such that $Cr < 1$, we have Ψ is a contraction. We can see that, if $v \in \bar{B}_r(w_*)$, then $\|\Psi(v) - w_*\|_X = \|\Psi(v) - \Psi(w_*)\|_X \leqslant C\|v - w_*\|_X < r$, for some constant $C \in [0, 1)$. Then $\Psi(\bar{B}_r(w_*)) \subset \bar{B}_r(w_*)$. This implies that Ψ has a unique fixed point in $\bar{B}_r(w_*)$ and the proof is complete. □

Corollary 6.4. *The set $\mathscr{E}_0$ has at most a finite number of hyperbolic points.*

Proof. It follows directly of the compactness of $\mathscr{E}_\epsilon$. □

Now we are going to study the convergence properties of resolvent operators of the form $(A_\epsilon + V_\epsilon)^{-1}$, with $V_\epsilon \in \mathscr{L}(X^1, X)$. This is because we are interested in comparing the resolvent operators of the linearization around equilibrium.

The convergence of resolvents of $A_\epsilon + V_0$ follows from the convergence of resolvents of A_ϵ (see Proposition 3.5) and the lemma below, whose proof is immediate.

Lemma 6.5. *The operator $A_\epsilon + V_0$, $\epsilon \in [0, 1]$, satisfies the identity*

$$\begin{aligned}(A_\epsilon + V_0)^{-1} &- (A_0 + V_0)^{-1} \\ &= [I - (A_\epsilon + V_0)^{-1} V_0](A_\epsilon^{-1} - A_0^{-1})[I - V_0(A_0 + V_0)^{-1}].\end{aligned} \tag{6-2}$$

Theorem 6.6. *Let us consider w_* a hyperbolic of $\mathscr{E}_0$ with $0 \notin \sigma(A_0 - f'(w_*))$. Then there exist $\epsilon_1 > 0$ and $r > 0$ such that problem (1-7) has exactly one equilibrium solution w_*^ϵ in $\bar{B}_r(w_*) := \{w \in X : \|w - w_*\|_X \leqslant r\}$ for $\epsilon \in [0, \epsilon_1]$. Furthermore, $\|w_*^\epsilon - w_*\|_X \leqslant C\|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}$ for some $C > 0$ independent of ϵ.*

Proof. The hyperbolicity of w_* means that $\sigma(A_\epsilon - f'(w_*))$ is disjoint from the imaginary axis. Thus, by Lemma 4.1, we can guarantee the existence of a constant $C > 0$ such that

$$\|(A_\epsilon - F'(w_*))^{-1}\|_{\mathscr{L}(X)} \leqslant C, \quad \epsilon \in [0, 1].$$

We have that w^ϵ is a solution of (6-1) if and only if it is a fixed point of the map

$$\Psi_\epsilon(\omega) := (A_\epsilon - F'(w_*))^{-1}(F(w) - F'(w_*)w).$$

From Lemma 4.1, we get that $A_\epsilon^{1/2}(A_\epsilon - F'(w_*))^{-1}$ converges uniformly to $A_0^{1/2}(A_0 - F'(w_*))^{-1}$, which implies

$$\Psi_\epsilon(w_*) \to \Psi_0(w_*) \quad \text{in } X.$$

Now we will prove the existence of $r > 0$ and $\epsilon_1 \in [0, 1]$ such that Ψ_ϵ is a contraction of $\bar{B}_r(w_*) = \{w \in X : \|w - w_*\|_X < r\}$ into itself, uniformly in $[0, \epsilon_1]$. In fact, first we will see that Ψ_ϵ is a contraction map. For this, we take u^ϵ and v^ϵ in $\bar{B}_r(w_*)$. In this way,

$$\begin{aligned}
&\|\Psi_\epsilon(u^\epsilon) - \Psi_\epsilon(v^\epsilon)\|_X \\
&\quad = \|(A_\epsilon - F'(w_*))^{-1}[F(u^\epsilon) - F(v^\epsilon) - F'(w_*)(u^\epsilon - v^\epsilon)]\|_X \\
&\quad \leqslant \|(A_\epsilon - F'(w_*))^{-1}\|_{\mathscr{L}(X,X^1)}\|F(u^\epsilon) - F(v^\epsilon) - F'(u_*)(u^\epsilon - v^\epsilon)\|_X \\
&\quad = \|A_\epsilon^{-1}(I - F'(w_*)^{-1}A_\epsilon^{-1})\|_{\mathscr{L}(X,X^1)}\|F(u^\epsilon) - F(v^\epsilon) - F'(u_*)(u^\epsilon - v^\epsilon)\|_X,
\end{aligned}$$

and, according to Proposition 3.5 and (4-1), there exist $C > 0$ and $\epsilon_1 > 0$ such that

$$\|\Psi_\epsilon(u^\epsilon) - \Psi_\epsilon(v^\epsilon)\|_{X^1} \leqslant C\delta\|u^\epsilon - v^\epsilon\|_X, \quad \text{for all } \delta > 0 \text{ and all } \epsilon \in [0, \epsilon_1]. \tag{6-3}$$

Therefore, choosing δ such that $C\delta \leqslant a < 1$, it follows that Ψ_ϵ is a contraction as claimed.

Let us show now that $\Psi_\epsilon(\bar{B}_r(w_*)) \subset \bar{B}_r(w_*)$. Taking $u^\epsilon \in \bar{B}_r(w_*)$, we obtain by (6-3) that

$$\begin{aligned}
\|\Psi_\epsilon(u^\epsilon) - w_*\|_X &\leqslant \|\Psi_\epsilon(u^\epsilon) - \Psi_\epsilon(w_*)\|_X + \|\Psi_\epsilon(w_*) - w_*\|_X \\
&\leqslant a\|u^\epsilon - w_*\|_X + \|\Psi_\epsilon(w_*) - w_*\|_X \\
&\leqslant ar + \|\Psi_\epsilon(w_*) - w_*\|_X, \quad \text{for all } \epsilon \in (0, \bar{\epsilon}].
\end{aligned} \tag{6-4}$$

It follows from Lemma 4.1 that there exists $\epsilon_1 > 0$ such that

$$\|\Psi_\epsilon(w_*) - w_*\|_X \leqslant r/2, \quad \text{for all } \epsilon \in [0, \epsilon_1]. \tag{6-5}$$

Combining (6-4) and (6-5), and considering $a \leqslant 1/2$, we deduce that

$$\|\Psi_\epsilon(u^\epsilon) - w_*\|_X \leqslant r, \quad \text{for all } \epsilon \in [0, \epsilon_1],$$

and, therefore, $\Psi_\epsilon : \bar{B}_r(w_*) \to \bar{B}_r(w_*)$ is a contraction, for all $\epsilon \in [0, \epsilon_1]$. Hence, there exists a fixed point of Ψ_ϵ in $\bar{B}_r(w_*)$, which we will call w_*^ϵ.

Finally, we will find an estimate of the difference $w_*^\epsilon - w_*$ in terms of $\|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}$.

Observe that $w_*^\epsilon = \Psi_\epsilon(w_*^\epsilon)$ and $w_* = \Psi_0(w_*)$. If we denote $F'(w_*) = V_0$, we have

$$\begin{aligned}
\|w_*^\epsilon - w_*\|_X \leqslant \|((A_\epsilon + V_0)^{-1} - (A_0 + V_0)^{-1})[F(w_*^\epsilon) + V_0 w_*^\epsilon] \\
+ (A_0 + V_0)^{-1}[F(w_*^\epsilon) - F(w_*) + V_0(w_*^\epsilon - w_*)]\|_X.
\end{aligned} \tag{6-6}$$

Identity (6-2) and Proposition 3.5 give us

$$\|(A_\epsilon+V_0)^{-1}-(A_0+V_0)^{-1}\|_{\mathscr{L}(X)} \leqslant C\|A_\epsilon^{-1}-A_0^{-1}\|_{\mathscr{L}(X)} \leqslant C\|a_\epsilon-a_0\|_{L^\infty(\Omega)}^{1/2}, \tag{6-7}$$

where the constant $C > 0$ is independent of ϵ.

On the other hand, denoting $z_*^\epsilon = F(w_*^\epsilon) - F(w_*) + V_0(w_*^\epsilon - w_*)$ and using the differentiability of the map $F : X \to Y \times Y$ (see (4-1)), we get that, for every $r > 0$, $\|z_\epsilon\|_X \leqslant r\|w_*^\epsilon - w_*\|_X$. Hence,

$$\|(A_0 + V_0)^{-1} z_*^\epsilon\|_X \leqslant r\|(A_0 + V_0)^{-1}\|_{\mathscr{L}(X)}\|w_*^\epsilon - w_*\|_X. \tag{6-8}$$

Substituting (6-7) and (6-8) in (6-5) and choosing $r > 0$ such that $r\|(A_0 + V_0)^{-1}\|_{\mathscr{L}(X)} \leqslant 1/2$, we obtain

$$\|w_*^\epsilon - w_*\|_X \leqslant C\|F(w_*^\epsilon) + V_0 w_*^\epsilon\|_X \|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2} + \tfrac{1}{2}\|w_*^\epsilon - w_*\|_X,$$

which, combined with the fact that f and its derivative are limited, allows us to conclude

$$\|w_*^\epsilon - w_*\|_X \leqslant C\|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}. \qquad \square$$

Remark 6.7. Notice that, by assuming that elements of $\mathscr{E}_0 = \{w_*^{1,0}, \dots, w_*^{\mathfrak{n},0}\}$ are hyperbolic, we have that the points of $\mathscr{E}_\epsilon = \{w_*^{1,\epsilon}, \dots, w_*^{\mathfrak{n},\epsilon}\}$, with $\epsilon \in (0, 1]$, satisfy the estimate $\|w_*^{i,\epsilon} - w_*^{i,0}\|_X \leqslant C\|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}$. We still have by (4-1) that, writing $V_\epsilon = F'(w_*^\epsilon)$ with $w_*^\epsilon \in \mathscr{E}_\epsilon$, V_ϵ converges to V_0 in the uniform topology.

Lemma 6.8. *There exists a constant $C > 0$, independent of ϵ, such that*

$$\|V_\epsilon A_\epsilon^{-1} - V_0 A_0^{-1}\|_{\mathscr{L}(X)} \leqslant C\|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}.$$

Proof. The estimate follows by the decomposition

$$V_\epsilon A_\epsilon^{-1} - V_0 A_0^{-1} = V_\epsilon(A_\epsilon^{-1} - A_0^{-1}) + (V_\epsilon - V_0)A_0^{-1},$$

(3-25), and Theorem 6.6. $\square$

The next result shows an analogous property found in Proposition 3.5 with $A_\epsilon + V_\epsilon$ instead of A_ϵ. This will be important in the analysis in the next section.

Proposition 6.9. *Let us consider $\bar{A}_\epsilon = A_\epsilon + V_\epsilon$ for all $\epsilon \in [0, 1]$. If $0 \in \rho(\bar{A}_0)$, then $0 \in \rho(\bar{A}_\epsilon)$, for all $\epsilon \in (0, 1]$, and the following identity holds*:

$$\begin{aligned} &\bar{A}_\epsilon^{-1} - \bar{A}_0^{-1} \\ &= (A_\epsilon^{-1} - A_0^{-1})(I + V_0 A_0^{-1})^{-1} - A_\epsilon^{-1}(I + V_0 A_0^{-1})^{-1}(V_\epsilon A_\epsilon^{-1} - V_0 A_0^{-1})(I + V_\epsilon A_\epsilon^{-1})^{-1}. \end{aligned} \tag{6-9}$$

Furthermore

$$\|\bar{A}_\epsilon^{-1} - \bar{A}_0^{-1}\|_{\mathscr{L}(X)} \leqslant C\|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}, \tag{6-10}$$

for some $C > 0$ independent of ϵ.

Proof. The first part follows from Proposition 2.1. The identity (6-9) is immediate, and (6-10) follows using (6-9), Proposition 3.5, and Lemma 6.8. ☐

The last proposition enables us to prove similar results as Proposition 3.6, Proposition 3.7, Theorem 5.1 and Theorem 5.2 for $A_\epsilon + V_\epsilon$ instead of A_ϵ.

7. Rate of convergence and attraction of local unstable manifolds

The main aim of this section is the proof of the existence unstable local manifolds as a graph of a Lipschitz function, its convergence, and exponential attraction.

For each $\epsilon \in [0, \epsilon_1]$, let us consider w_*^ϵ to be an equilibrium solution for (1-7). We assume the existence of a constant $C > 0$ such that $\|w_*^\epsilon - w_*\|_X \leqslant C\|a_\epsilon - a_0\|_{L^\infty(\Omega)}^{1/2}$, for all $\epsilon \in [0, \epsilon_1]$, and that $w_* := w_*^0$ is hyperbolic. To deal with a neighborhood of the equilibrium point w_*^ϵ, we rewrite the problems (1-7) as

$$z_t^\epsilon + \bar{A}_\epsilon z^\epsilon = F(u^\epsilon + w_*^\epsilon) - F(w_*^\epsilon) - F'(w_*^\epsilon)u^\epsilon, \tag{7-1}$$

where $z^\epsilon = u^\epsilon - w_*^\epsilon$ and $\bar{A}_\epsilon = A_\epsilon - F'(w_*^\epsilon)$. With this, one can look for Proposition 3.7 with $\bar{A}_\epsilon$ instead of A_ϵ.

Let γ be a smooth, closed, simple, rectifiable curve in $\{z \in \mathbb{C} : \mathrm{Re}z > 0\}$, oriented counterclockwise and such that the bounded connected component of $\mathbb{C}\backslash\{\gamma\}$ (here $\{\gamma\}$ denotes the trace of γ) contains $\{z \in \sigma(-\bar{A}_0) : \mathrm{Re}z > 0\}$. From part (ii) of Proposition 3.6, there exists $\epsilon_1 > 0$ such that $\{\gamma\} \subset \rho(-\bar{A}_\epsilon)$ for all $\epsilon \in [0, \epsilon_1]$. We define $\bar{Q}_\epsilon$ by

$$\bar{Q}_\epsilon = \frac{1}{2\pi i}\int_\gamma (\lambda I - \bar{A}_\epsilon)^{-1} d\lambda, \quad \text{for all } \epsilon \in [0, \epsilon_1].$$

The operator $\bar{A}_\epsilon$ is self-adjoint and there exist $\beta > 0$ and $C \geqslant 1$ such that

$$\|e^{-\bar{A}_\epsilon t}\bar{Q}_\epsilon\|_{\mathscr{L}(X)} \leqslant Ce^{-\beta t}, \quad \text{for all } t \geqslant 0 \text{ and all } \epsilon \in [0, \epsilon_1]$$

and

$$\|e^{-\bar{A}_\epsilon t}(I - \bar{Q}_\epsilon)\|_{\mathscr{L}(X)} \leqslant Ct^{-1/2}e^{-\beta t}, \quad t > 0.$$

Using the decomposition $X = \bar{Q}_\epsilon X \oplus (I - \bar{Q}_\epsilon)X$ (the solution z^ϵ of (7-1) can be decomposed as $z^\epsilon = \bar{Q}_\epsilon z^\epsilon + (I - \bar{Q}_\epsilon)z^\epsilon$), we rewrite (7-1) as

$$\begin{cases} \dfrac{d}{dt}(\bar{Q}_\epsilon z^\epsilon) + \bar{A}_\epsilon\bar{Q}_\epsilon z^\epsilon = H_\epsilon(\bar{Q}_\epsilon z^\epsilon, (I - \bar{Q}_\epsilon)z^\epsilon), \\ \dfrac{d}{dt}[(I - \bar{Q}_\epsilon)z^\epsilon] + \bar{A}_\epsilon(I - \bar{Q}_\epsilon)z^\epsilon = G_\epsilon(\bar{Q}_\epsilon z^\epsilon, (I - \bar{Q}_\epsilon)z^\epsilon), \end{cases} \tag{7-2}$$

where

$$\begin{aligned} &H_\epsilon(\bar{Q}_\epsilon z^\epsilon, (I - \bar{Q}_\epsilon)z^\epsilon) \\ &\quad := \bar{Q}_\epsilon[F(\bar{Q}_\epsilon z^\epsilon + (I - \bar{Q}_\epsilon)z^\epsilon + w_*^\epsilon) - F(w_*^\epsilon) - F'(w_*^\epsilon)(\bar{Q}_\epsilon z^\epsilon + (I - \bar{Q}_\epsilon)z^\epsilon)], \end{aligned} \tag{7-3}$$

and

$$(7\text{-}4)\quad G_\epsilon(\bar{Q}_\epsilon z^\epsilon, (I-\bar{Q}_\epsilon)z^\epsilon)$$
$$:= (I-\bar{Q}_\epsilon)[F(\bar{Q}_\epsilon z^\epsilon + (I-\bar{Q}_\epsilon)z^\epsilon + w_*^\epsilon) - F(w_*^\epsilon) - F'(w_*^\epsilon)(\bar{Q}_\epsilon z^\epsilon + (I-\bar{Q}_\epsilon)z^\epsilon)].$$

The functions H_ϵ and G_ϵ are continuously differentiable with

$$H_\epsilon(0,0) = G_\epsilon(0,0) = 0$$

and $H'_\epsilon(0,0) = 0 = G'_\epsilon(0,0) \in \mathscr{L}(X)$. For simplicity of notation, we write $\omega^\epsilon = \bar{Q}_\epsilon z^\epsilon$ and $\vartheta^\epsilon = (I-\bar{Q}_\epsilon)z^\epsilon$. Hence, given $\rho > 0$, there exist $\epsilon_1 > 0$ and $r > 0$ such that if $\|\omega^\epsilon\|_{\bar{Q}_\epsilon X} + \|\vartheta^\epsilon\|_{(I-\bar{Q}_\epsilon)X} < r$ and $\epsilon \in [0,\epsilon_1]$, then

$$(7\text{-}5)\qquad \|H_\epsilon(\omega^\epsilon,\vartheta^\epsilon)\|_{\bar{Q}_\epsilon X} \leqslant \rho \quad\text{and}\quad \|G_\epsilon(\omega^\epsilon,\vartheta^\epsilon)\|_{(I-\bar{Q}_\epsilon)X} \leqslant \rho,$$

$$(7\text{-}6)\quad \|H_\epsilon(\omega^\epsilon,\vartheta^\epsilon) - H_\epsilon(\bar{\omega}^\epsilon,\bar{\vartheta}^\epsilon)\|_{\bar{Q}_\epsilon X} \leqslant \rho(\|\omega^\epsilon - \bar{\omega}^\epsilon\|_{\bar{Q}_\epsilon X} + \|\vartheta^\epsilon - \bar{\vartheta}^\epsilon\|_{(I-\bar{Q}_\epsilon)X})$$

and

$$(7\text{-}7)\quad \|G_\epsilon(\omega^\epsilon,\vartheta^\epsilon) - G_\epsilon(\bar{\omega}^\epsilon,\bar{\vartheta}^\epsilon)\|_{(I-\bar{Q}_\epsilon)X} \leqslant \rho(\|\omega^\epsilon - \bar{\omega}^\epsilon\|_{\bar{Q}_\epsilon X} + \|\vartheta^\epsilon - \bar{\vartheta}^\epsilon\|_{(I-\bar{Q}_\epsilon)X}).$$

Considering the coupled system (7-2), we can show an unstable manifold theorem using similar arguments to those in the results of Chapter 6 in [Henry 1981].

Theorem 7.1. *There exists a map $s_*^\epsilon : \bar{Q}_\epsilon X \to (I-\bar{Q}_\epsilon)X$ such that the unstable manifold of w_*^ϵ is given by*

$$W^u(w_*^\epsilon) = \{(\omega,\vartheta) \in X : \vartheta = s_*^\epsilon(\omega),\ \omega \in \bar{Q}_\epsilon X\}.$$

The map s_*^ϵ satisfies

$$|||s_*^\epsilon||| := \sup_{\omega\in\bar{Q}_\epsilon X} \|s_*^\epsilon(\omega)\|_X \leqslant C, \qquad \|s_*^\epsilon(\omega) - s_*^\epsilon(\tilde{\omega})\|_X \leqslant \bar{C}\|\omega - \tilde{\omega}\|_{\bar{Q}_\epsilon X},$$

where $C > 0$ is a constant independent of ϵ, and for $\theta \in \left(0,\frac{1}{2}\right)$ there exists a $C > 0$, independent of ϵ, such that

$$(7\text{-}8)\qquad |||s_*^\epsilon - s_*^0||| \leqslant C\|a_\epsilon - a_0\|^{2\theta}_{L^\infty(\Omega)}.$$

Furthermore, there exists $\rho_1 > 0$, $C > 0$ (independent of ϵ), and $t_0 > 0$ such that, for any solution $(\omega^\epsilon(t),\vartheta^\epsilon(t)) \in X$, $t \in [t_0,\infty)$, of (7-2), we have

$$(7\text{-}9)\quad \|\vartheta^\epsilon(t) - s_*^\epsilon(\omega^\epsilon(t))\|_X \leqslant Ce^{-\rho_1(t-t_0)}\|\vartheta^\epsilon(t_0) - s_*^\epsilon(\omega^\epsilon(t_0))\|_X, \quad \textit{for all } t \geqslant t_0.$$

Proof. We consider the set

$$\Sigma_\epsilon = \{s : \bar{Q}_\epsilon X^1 \to (I-\bar{Q}_\epsilon)X : |||s||| \leqslant C,\ \|s(\omega) - s(\tilde{\omega})\|_X \leqslant C\|\omega - \tilde{\omega}\|_{\bar{Q}_\epsilon X}\}.$$

It is not difficult to see that $(\Sigma_\epsilon, |||\cdot|||)$ is a complete metric space.

Given $s \in \Sigma_\epsilon$ and $\eta \in \bar{Q}_\epsilon X$, we denote by $\omega^\epsilon(t) = \psi(t, \tau, \eta, s)$ the solution of

$$\begin{cases} \omega_t^\epsilon(t) + B_\epsilon \omega^\epsilon(t) = H_\epsilon(\omega^\epsilon(t), s(\omega^\epsilon(t))), & t < \tau \\ \omega^\epsilon(\tau) = \eta. \end{cases}$$

We define $\Psi_\epsilon : \Sigma_\epsilon \to \Sigma_\epsilon$ by

$$\Psi_\epsilon(s)\eta = \int_{-\infty}^{\tau} e^{-\widetilde{A}_\epsilon(\tau-\xi)} G_\epsilon(\omega^\epsilon(\xi), s(\omega^\epsilon(\xi)))\, d\xi.$$

According to Theorem 7.1 in [Arrieta et al. 2013], we can deduce that Ψ_ϵ is a contraction. Therefore, there is a fixed point $s_*^\epsilon = \Psi(s_*^\epsilon)$ in Σ_ϵ.

Now we shall prove that the graph of s_*^ϵ $\{(\omega^\epsilon, s_*^\epsilon(\omega^\epsilon)) : \omega^\epsilon \in \bar{Q}_\epsilon X\}$ is invariant for (7-2), in the sense that initial data for (7-2) in $\{(\omega^\epsilon, s_*^\epsilon(\omega^\epsilon)) : \omega^\epsilon \in \bar{Q}_\epsilon X\}$ lead to solutions in this space. In fact, we take $(\omega_0^\epsilon, \vartheta_0^\epsilon) \in W^u(w_*^\epsilon)$ $(\vartheta_0^\epsilon = s_*^\epsilon(\omega_0^\epsilon))$. We denote by $\omega_*^\epsilon(t)$ the solution of the initial value problems

$$\begin{cases} \dfrac{d}{dt}(\bar{Q}_\epsilon z^\epsilon) + \bar{A}_\epsilon \bar{Q}_\epsilon z^\epsilon = H_\epsilon(\omega^\epsilon, s_*^\epsilon(\omega^\epsilon)), \\ \omega^\epsilon(0) = \omega_0^\epsilon, \end{cases}$$

where $z^\epsilon = \omega^\epsilon + \vartheta^\epsilon \in \bar{Q}_\epsilon X \oplus (I - \bar{Q}_\epsilon)X$. This defines a curve $(\omega_*^\epsilon(t), s_*^\epsilon(\omega_*^\epsilon(t))) \in W^u(w_*^\epsilon)$, $t \in \mathbb{R}$. Also, the unique solution of

$$\frac{d}{dt}[(I - \bar{Q}_\epsilon)z^\epsilon] + \bar{A}_\epsilon(I - \bar{Q}_\epsilon)z^\epsilon = G_\epsilon(\omega^\epsilon, s_*^\epsilon(\omega^\epsilon)),$$

which remains bounded as $t \to -\infty$, is

$$\vartheta_*^\epsilon(t) = (I - \bar{Q}_\epsilon)z_*^\epsilon(t) = \int_{-\infty}^{t} e^{\bar{A}_\epsilon(I-\bar{Q}_\epsilon)(t-\xi)} G_\epsilon(\omega_*^\epsilon(\xi), s_*^\epsilon(\omega_*^\epsilon(\xi)))\, d\xi = s_*^\epsilon(\omega_*^\epsilon(t)).$$

Therefore $(\omega_*^\epsilon(t), s_*^\epsilon(\omega_*^\epsilon(t)))$ is a solution of the system (7-2) through the point $(\omega_0^\epsilon, \vartheta_0^\epsilon)$, proving the invariance of the graph of s_*^ϵ.

To show (7-8), we can proceed as in the proof of Proposition 6.1 in [Arrieta et al. 2009].

Finally, the proof that the graph of s_*^ϵ is the unstable manifold that attracts exponentially, uniformly in ϵ, that is, the inequality (7-9) holds, follows by similar arguments to those in the proof of (A.8) in [Bruschi et al. 2006]. □

Now we are able to prove our main result.

8. Proof of Theorem 1.1

The purpose of this section is to emphasize the proof of our main result. For this, we return to Theorem 1.1 to establish its proof.

Proof of Theorem 1.1. This proof follows by Theorem 5.2, Theorem 6.6, and Theorem 7.1 jointly with Theorems 2.1 and 2.2, and Corollary 2.1 in Chapter 8 of [Babin and Vishik 1989]. □

Acknowledgments

The authors thank A.N. Carvalho for constructive conversations about this problem.

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Received October 3, 2012. Revised March 4, 2013.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 266, No. 2, 2013
dx.doi.org/10.2140/pjm.2013.266.283

THE BRIN–THOMPSON GROUPS sV ARE OF TYPE F_∞

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We prove that the Brin–Thompson groups sV, also called higher-dimensional Thompson's groups, are of type F_∞ for all $s \in \mathbb{N}$. This result was previously shown for $s \leq 3$, by considering the action of sV on a naturally associated space. Our key step is to replace this space by a subspace sX that is easier to analyze.

Recall that a group is of *type* F_∞ if it admits a classifying space with finitely many cells in each dimension. Well-known examples of groups of type F_∞ include Thompson's groups F, T, and V. Some generalizations of V were introduced by Brin [2004; 2005] and shown to be simple. We denote these groups sV, for $s \in \mathbb{N}$, with $1V = V$. These groups are usually termed higher-dimensional Thompson's groups or Brin–Thompson groups. All of the groups sV are known to be finitely presented [Hennig and Matucci 2012], and Kochloukova, Martínez-Pérez, and Nucinkis [Kochloukova et al. 2013] showed that $2V$ and $3V$ are of type F_∞. We prove that this result extends to all dimensions.

Main Theorem. *The Brin–Thompson group sV is of type F_∞ for all s.*

Fix some s. There is a natural poset $\mathcal{P}_1$ associated to sV. The realization $|\mathcal{P}_1|$ of this poset is contractible and the action of sV is proper but not cocompact. To prove the Main Theorem it suffices to produce a cocompact filtration of $|\mathcal{P}_1|$ whose connectivity tends to infinity. The tool to study relative connectivity is discrete Morse theory. This was carried out for $s = 2, 3$ in [Kochloukova et al. 2013]. However, for larger s this space quickly becomes cumbersome.

We therefore consider a subspace sX of $|\mathcal{P}_1|$, which we call the *Stein space* for sV. As before, the Stein space is contractible and the action is not cocompact. The advantage of the Stein space is that the Morse theory becomes easier to handle.

In Section 1 we recall the definition of sV. The Stein space sX is defined in Section 2 and some basic properties are verified. In Section 3 we analyze the connectivity of the subspaces in the filtration and deduce the Main Theorem.

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MSC2010: primary 20F65; secondary 57Q12.

Keywords: Thompson's groups, finiteness properties.

1. The Brin–Thompson groups

The elements of the Brin–Thompson group sV can be described as dyadic self-maps of s-dimensional cubes. We will first give a brief intuition for these maps, and then delve into some formalism.

To get an intuition for the elements of sV for arbitrary s, recall first that elements of Thompson's group $V = 1V$ can be thought of as left-continuous, piecewise linear maps from the unit interval [0, 1] to itself, where the slope of any linear piece is a positive dyadic rational. An equivalent description of such an element is obtained as follows: First divide the unit interval representing the domain into two halves and iterate this procedure by further subdividing some of the resulting pieces. Then similarly cut up the unit interval representing the codomain into the same number of pieces as the domain, and finally identify the pieces of the domain and codomain via a permutation. Note that the intervals identified in the last step will usually have different lengths. For more details see [Cannon et al. 1996].

To describe elements of sV, we no longer think of the unit interval but the unit s-cube $[0, 1]^s$. The unit s-cube can be halved by dyadic hyperplanes in s different directions, as can any iterated piece obtained this way. As with V, an element of sV can be described as a sequence of halvings of the domain and codomain and an identification of the resulting pieces by a permutation. Again the identification will affinely deform the individual pieces. Alternatively we can describe an element by a dyadic map from the s-cube to itself. A sequence of halvings of the s-cube will be modeled by "dyadic coverings". To get an intuition, the reader might want to look at Figure 1 (the map f_1 represents an element of $2V$). It may also be helpful to read Section 1 of [Burillo and Cleary 2010], which additionally details the *paired trees* model for elements of sV.

1A. ***Dyadic maps and the group sV.*** We now describe more formally the notions needed to define the group sV, and also a certain poset $\mathcal{P}_1$, which will then be used to define the space sX for our main argument.

A real number is called *dyadic* if it is of the form $k/2^l$ for some $k \in \mathbb{Z}$ and $l \in \mathbb{N}_0$. We denote by I the subspace of [0, 1] of nondyadic numbers. By a *dyadic interval* we mean a set of the form $[k/2^l, (k+1)/2^l] \cap I$ with $k, l \in \mathbb{N}_0$, and the *length* of the dyadic interval is defined to be $1/2^l$. A bijection $A \to B$ between dyadic intervals is called a *simple dyadic map* if it is affine of positive slope. Note that this slope will necessarily be a power of two.

In general we consider the unit s-cube I^s (or rather, the set of nondyadic points in the unit s-cube), which is the s-fold product of I. A *brick* is a subset C of I^s that is a product of s dyadic intervals, called the *edges* of C, and the *volume* of C is the product of the lengths of its edges. Note that the volume of a brick is always a power of two. A *dyadic covering* is a finite set of bricks that disjointly cover I^s.

Note that by our definition the set I does not contain any dyadic numbers.

For a natural number m, denote by $I^s(m)$ the disjoint union

$$I^s(m) := B_1 \sqcup \cdots \sqcup B_m,$$

where each B_i is a copy of I^s. Note that I^s is the same as $I^s(1)$. We call B_i the *i-th block* of $I^s(m)$. A covering $\mathcal{U}$ of $I^s(m)$ is called *dyadic* if it is a disjoint union $\mathcal{U} = \mathcal{U}_1 \sqcup \cdots \sqcup \mathcal{U}_m$, where $\mathcal{U}_i$ is a dyadic covering of the block B_i. We denote by $\mathcal{T}_m := \{B_1, \dots, B_m\}$ the *trivial* dyadic covering of $I^s(m)$, in which the bricks are just the blocks themselves.

Observation 1.1. The set of dyadic coverings of $I^s(m)$ is a lattice with respect to the refinement relation.

Proof. Existence of joins (that is, coarsest common refinements) as well as existence of a unique minimum (namely, $\mathcal{T}_m$) are clear. The statement now follows from standard order theory. □

Let $\mathcal{U}$ and $\mathcal{V}$ be dyadic coverings of $I^s(m)$ and $I^s(n)$, respectively, and let $f : I^s(m) \to I^s(n)$ be a map. We say that the pair of dyadic coverings $(\mathcal{U}, \mathcal{V})$ is *compatible* with f if for every $C \in \mathcal{U}$, $f|_C$ is a product of simple dyadic maps and $f(C) \in \mathcal{V}$. Less formally, this means that every brick in the domain maps in an affine way to a brick in the codomain. If such a pair of dyadic coverings exists, then we say that f is a *dyadic map*. It is easy to see that composition of two dyadic maps is again a dyadic map, that every dyadic map is invertible, and that the inverse of a dyadic map is dyadic.

There is a combinatorial description of dyadic maps. If $f : I^s(m) \to I^s(n)$ is a dyadic map and $(\mathcal{U}_1, \mathcal{U}_2)$ is a compatible covering, then f induces a bijection of dyadic coverings $\mathcal{U}_1 \to \mathcal{U}_2$. Conversely, every bijection of dyadic coverings $\mathcal{U}_1 \to \mathcal{U}_2$ induces a dyadic map $I^s(m) \to I^s(n)$.

Note that two bijections $\mathcal{U}_1 \to \mathcal{V}_1$ and $\mathcal{U}_2 \to \mathcal{V}_2$ induce the same map $I^s(m) \to I^s(n)$ if and only if there are common refinements $\mathcal{U}$ and $\mathcal{V}$ such that the induced bijections $\mathcal{U} \to \mathcal{V}$ coincide.

Definition 1.2. The Brin–Thompson group sV is the group of all dyadic self maps of I^s with the multiplication given by composition, $gh := g \circ h$.

1B. *The poset* $\mathcal{P}_1$. In order to define the poset $\mathcal{P}_1$ on which sV acts we need some more notation.

Denote by $\widetilde{\mathcal{P}}_{m,n}$ the set of all dyadic maps $f : I^s(m) \to I^s(n)$, so for example $\widetilde{\mathcal{P}}_{1,1} = sV$. Let $\widetilde{\mathcal{P}}$ be the union of the $\widetilde{\mathcal{P}}_{m,n}$, where m and n range over all positive integers. Also denote by $\widetilde{\mathcal{P}}_m$ the subset of $\widetilde{\mathcal{P}}$ where the domain of the maps consists of m blocks.

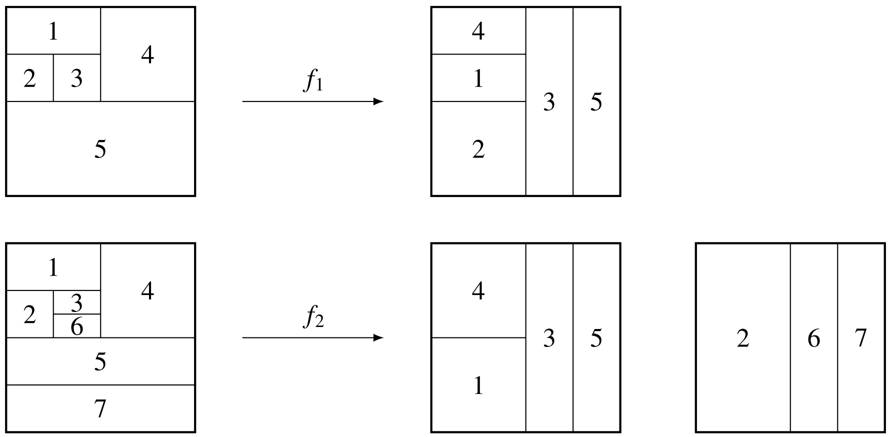

Figure 1. An example of a dyadic map $f_1 \colon I^2(1) \to I^2(1)$ and a dyadic map $f_2 \colon I^2(1) \to I^2(2)$, obtained from f_1 by splitting along a horizontal line. The map f_2 is equivalent in $\mathcal{P}_1$ to the one where the blocks on the right are interchanged.

There is a natural action of sV on $\widetilde{\mathcal{P}}_1$ given by precomposition: $f^g := f \circ g$ for $g \in sV$ and $f \in \widetilde{\mathcal{P}}_1$. For each positive n there is also an action of the symmetric group S_n on $\widetilde{\mathcal{P}}_{m,n}$ by permuting the blocks of the codomain. We denote the quotient $\widetilde{\mathcal{P}}_{m,n}/S_n$ by $\mathcal{P}_{m,n}$. In other words, an element of $\mathcal{P}_{m,n}$ is obtained from $\widetilde{\mathcal{P}}_{m,n}$ by forgetting the order of the blocks in the codomain. We set

$$\mathcal{P} := \bigcup_{n,m \geq 1} \mathcal{P}_{m,n} \quad \text{and} \quad \mathcal{P}_1 := \bigcup_{n \geq 1} \mathcal{P}_{1,n}.$$

Note that $\widetilde{\mathcal{P}}_{1,n}$ is an sV-invariant subset of $\widetilde{\mathcal{P}}_1$, and the action of sV on $\widetilde{\mathcal{P}}_{1,n}$ commutes with the action of the symmetric group S_n, so we get an action of sV on $\mathcal{P}_{1,n}$ for every n. In particular the sV-action on $\widetilde{\mathcal{P}}_1$ induces an action of sV on $\mathcal{P}_1$.

Definition 1.3. The function $t \colon \mathcal{P} \to \mathbb{N}$ assigns to each $x \in \mathcal{P}$ the number of blocks in the codomain of x, that is, if $x \in \mathcal{P}_{m,n}$ for some m, then $t(x) = n$.

Next we define a poset structure on $\mathcal{P}$ using the notion of "splitting". A dyadic map $z \colon I^s(m) \to I^s(n)$ is called a *splitting* (*along* $\mathcal{U}$) if z is compatible with a pair of dyadic coverings of the form $(\mathcal{U}, \mathcal{T}_n)$. The splitting z is called *nontrivial* if $n > m$. Colloquially then, as the name suggests, a nontrivial splitting is given by splitting up some cubes (and then not sticking any resulting cubes together). The inverse of a splitting (along $\mathcal{U}$) is called a *merging* (along $\mathcal{U}$).

We define an order $\leq$ on $\widetilde{\mathcal{P}}$ by saying that $x < y$ if there exists a nontrivial splitting z such that $y = z \circ x$, that is, if y is obtained from x by nontrivial splitting.

We also denote the induced order on $\mathcal{P}$ by $\leq$. In particular, $\mathcal{P}_1$ is ordered by $\leq$. See Figure 1 for an example of dyadic maps and splitting.

The poset $\mathcal{P}_1$ is filtered by the t-sublevel sets

$$\mathcal{P}_1^{\leq n} = \bigcup_{1\leq k\leq n} \mathcal{P}_{1,k}.$$

We make the following easy observations:

Observation 1.4. The poset $\widetilde{\mathcal{P}}_1$ is directed (that is, any two elements have a common upper bound). Therefore, $|\widetilde{\mathcal{P}}_1|$ and $|\mathcal{P}_1|$ are contractible.

Observation 1.5. The action of sV on $\widetilde{\mathcal{P}}_1$ is free. Thus, for any vertex x in $|\mathcal{P}_1|$, the stabilizer $\mathrm{Stab}_{sV}(x)$ is finite. Hence all cell stabilizers are finite and of type F_∞.

Observation 1.6. The action of sV on $\mathcal{P}_1^{\leq 1}$ is transitive, and for each $n \geq 1$ the sublevel set $|\mathcal{P}_1^{\leq n}|$ is locally finite. Hence $|\mathcal{P}_1^{\leq n}|$ is finite modulo sV.

These observations suggest that the filtration $(|\mathcal{P}_1^{\leq n}|)_n$ of $|\mathcal{P}_1|$ could be used to show that sV is of type F_∞, using Brown's criterion.

Brown's criterion [Brown 1987, Corollary 3.3]. *Let G be a group and X a contractible G-CW-complex such that the stabilizer of every cell is of type* F_∞. *Let $\{X_j\}_{j\geq 1}$ be a filtration of X such that each X_j is finite* mod G. *Suppose that the connectivity of the pair (X_{j+1}, X_j) tends to ∞ as j tends to ∞. Then G is of type* F_∞.

It would suffice now to show that the connectivity of the pair $(|\mathcal{P}_1^{\leq n+1}|, |\mathcal{P}_1^{\leq n}|)$ tends to ∞ as n tends to ∞. This was proved for the cases $s = 2, 3$ [Kochloukova et al. 2013]. However, it becomes increasingly difficult to verify for higher s. The main difference of our approach here is that we consider a certain subcomplex sX of $|\mathcal{P}_1|$. Analyzing the relative connectivity in sX turns out to be substantially easier than in $|\mathcal{P}_1|$.

2. The Stein space for sV

The idea of passing to what we are calling a "Stein space" was first introduced by Stein [1992], and in particular was used to obtain a new proof that F is of type F_∞. This construction generalizes nicely to deal with some more complicated versions of Thompson's groups. For example Stein spaces were used in [Bux et al. 2012] to prove that braided Thompson's groups are of type F_∞. The key idea is that the splitting establishing a relation $x \leq y$ can be obtained from "elementary splittings" that give rise to elementary relations $x \preceq x_1 \preceq \cdots \preceq x_r \preceq y$, and these small steps are much easier to understand locally. Heuristically, an elementary splitting amounts to halving an s-cube at most once in any given direction. We now describe more rigorously the construction of the Stein space.

Definition 2.1. Call a brick C *elementary* if every edge of C has length at least $\frac{1}{2}$. Call an elementary brick *very elementary* if it has volume at least $\frac{1}{2}$. A dyadic covering $\mathcal{U}$ is called (*very*) *elementary* if every brick in $\mathcal{U}$ has this property. Likewise, a splitting or merging along $\mathcal{U}$ is (*very*) *elementary* if $\mathcal{U}$ is.

For $x, y \in \mathcal{P}$, if y can be obtained from x by an elementary splitting, write $x \preceq y$; if moreover $x \neq y$ then we write $x \prec y$. If y is obtained from x by a very elementary splitting, write $x \sqsubseteq y$; if moreover $x \neq y$, then we write $x \sqsubset y$. Note that the relations $\preceq$ and $\sqsubseteq$ are not transitive. In particular, the length of a chain of very elementary splittings is bounded by the number of blocks. However, if $x_1 \leq x_2 \leq x_3$ and $x_1 \preceq x_3$ then $x_1 \preceq x_2$ and $x_2 \preceq x_3$, and analogously for $\sqsubseteq$. It is clear that the action of sV respects the relations $\leq$, $\preceq$ and $\sqsubseteq$.

Clearly $I^s(m)$ has a unique maximal elementary covering $\mathcal{E}$ by $m \cdot 2^s$ bricks all of which have volume 2^{-s}. A covering is elementary if and only if $\mathcal{E}$ is a refinement of it.

The closed interval $[x, y]$ in $\mathcal{P}_1$ is defined to be $[x, y] := \{w \in \mathcal{P}_1 \mid x \leq w \leq y\}$; the open and half-open intervals are defined analogously. Call an interval $[x, y]$ in $|\mathcal{P}_1|$ *elementary* if $x \preceq y$, and *very elementary* if $x \sqsubseteq y$. A simplex of $|\mathcal{P}_1|$ is (*very*) *elementary* if there is a (very) elementary interval that contains all of its vertices.

Definition 2.2. The *Stein space for sV*, denoted sX, is the subcomplex of $|\mathcal{P}_1|$ consisting of elementary simplices.

The following statement is the key to showing the contractibility of the Stein space:

Lemma 2.3. *Let $x, y \in \mathcal{P}_1$ with $x \leq y$. There exists a unique $y_0 \in [x, y]$ such that $x \preceq y_0$ and for any $x \preceq w \leq y$, we have $w \leq y_0$. If $x < y$, then $x < y_0$.*

Proof. Set $m := t(x)$ and $n := t(y)$. Let $\tilde{x}$ be a representative in $\widetilde{\mathcal{P}}_1$ for x. Let $\mathcal{U}$ be the dyadic covering of $I^s(m)$ such that y is obtained from $\tilde{x}$ by splitting along $\mathcal{U}$. Let $\mathcal{E}$ be the maximal elementary covering of $I^s(m)$. The element y_0 is obtained from $\tilde{x}$ by splitting along the finest common coarsening $\mathcal{E} \wedge \mathcal{U}$. The desired properties follow from Observation 1.1. □

For $x \leq y$, call the y_0 from the lemma the *elementary core of y with respect to x*, and denote it $\mathrm{core}_x(y) := y_0$. When x is understood we omit the subscript. Observe that if $y_1 \leq y_2$ then $\mathrm{core}(y_1) \leq \mathrm{core}(y_2)$, that is, taking elementary cores respects the poset relation. Figure 2 gives an example of an elementary core.

Lemma 2.4. *For $x < y$ with $x \not\preceq y$, $|(x, y)|$ is contractible.*

The proof is essentially the same as the proof of the lemma in Section 4 of [Brown 1992].

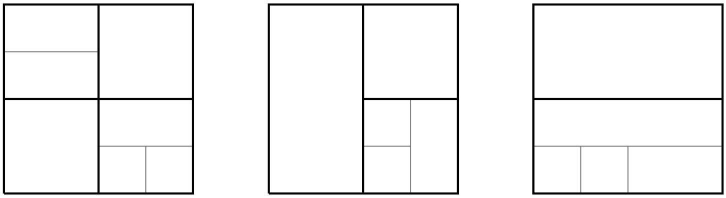

Figure 2. A nonelementary dyadic covering, for $s = 2$. The thick lines indicate the elementary core.

Proof. If $w \in (x, y]$, then $\mathrm{core}(w) \in [x, y)$ because $x \not< y$, and $\mathrm{core}(w) \in (x, y]$ because $x < w$. So in fact $\mathrm{core}(w) \in (x, y)$. Also, $\mathrm{core}(w) \le \mathrm{core}(y)$ by the previous discussion. The inequalities $w \ge \mathrm{core}(w) \le \mathrm{core}(y)$ provide a contraction of $|(x, y)|$, by Section 1.5 of [Quillen 1978]. □

As was done in [Brown 1992] for the Stein space of V, we can build up from sX to $|\mathcal{P}_1|$ to show that sX is contractible.

Corollary 2.5. *The Stein space sX is contractible for all s.*

Proof. By Observation 1.4, $|\mathcal{P}_1|$ is contractible. We build up from sX to $|\mathcal{P}_1|$ without changing the homotopy type.

Given a closed interval $[x, y]$, define $r([x, y]) := t(y) - t(x)$. We attach the contractible subcomplexes $|[x, y]|$ for $x \not< y$ to sX in increasing order of r-value. In particular, when we attach $|[x, y]|$, we attach it along $|[x, y)| \cup |(x, y]|$. But this is the suspension of $|(x, y)|$, and so is contractible by the previous lemma. We conclude that attaching $|[x, y]|$ does not change the homotopy type, and since $|\mathcal{P}_1|$ is contractible, so is sX. □

For each $n \ge 1$ let $sX^{\le n}$ be the full subcomplex of sX spanned by vertices x with $t(x) \le n$. Similarly define $sX^{<n}$, and let $sX^{=n}$ be the set of vertices x with $t(x) = n$. Note that all of these sets are invariant under the action of sV. We will show that the filtration $(sX^{\le n})_n$ of sX satisfies the assumptions of Brown's criterion.

Thanks to Observations 1.5 and 1.6 and to Corollary 2.5, the only remaining feature of the filtration $(sX^{\le n})_n$ of sX that we need to verify is that the connectivity of the pair $(sX^{\le n+1}, sX^{\le n})$ tends to ∞ as n tends to ∞. This is exactly the condition that proved difficult to verify for the filtration of $|\mathcal{P}_1|$ in [Kochloukova et al. 2013].

We will verify the relative connectivity in the next section using discrete Morse theory. The idea is to treat t as a height function on sX and inspect descending links.

3. Connectivity of the descending links and proof of the Main Theorem

We will use the following Morse-theoretic tools: Fix a vertex x in sX, say with $t(x) = n$, and call n the *height* of x. The *descending link* $\mathrm{lk}{\downarrow}(x)$ of x is defined to be the intersection of $\mathrm{lk}(x)$ with $sX^{<n}$. The fact that vertices with equal heights

cannot share an edge means that we can obtain $sX^{\leq n}$ from $sX^{<n}$ by "gluing in" each vertex at height n along its descending link. This is made rigorous by the Morse lemma (compare Corollary 2.6 of [Bestvina and Brady 1997]):

Lemma 3.1. *Let X be a simplicial complex and let $f\colon X^{(0)} \to \mathbb{Z}$ be such that $f(x) \neq f(y)$ for adjacent vertices x and y of X. If $\operatorname{lk}{\downarrow}(x)$ is $(k-1)$-connected for every vertex $x \in X^{=n}$, then the pair $(X^{\leq n}, X^{<n})$ is k-connected, that is, the inclusion $X^{<n} \hookrightarrow X^{\leq n}$ induces an isomorphism in π_j, $j < k$ and an epimorphism in π_k.*

Fix a vertex x in sX and consider $L(x) := \operatorname{lk}{\downarrow}(x)$. As a subcomplex of $|\mathcal{P}_1|$, $L(x)$ is the collection of simplices given by chains $y_k < \cdots < y_0 < x$ with $y_k \prec x$. We first consider the subcomplex $L_0(x)$ of $L(x)$ consisting of such chains with $y_k \sqsubset x$.

The complex $L_0(x)$ naturally projects onto a matching complex.

Definition 3.2. Let Γ be a graph. The *matching complex* $\mathcal{M}(\Gamma)$ of Γ is the simplicial complex with a k-simplex for every collection $\{e_0, \dots, e_k\}$ of $k+1$ pairwise disjoint edges, with the face relation given by inclusion. If we regard every edge as consisting of two oriented edges (effectively doubling each edge), we get the *oriented matching complex* $\mathcal{M}^o(\Gamma)$.

The specific graphs that we will need are generalizations of complete graphs. For $s \in \mathbb{N}$, let sK_n be the graph with n nodes and s edges between any two distinct nodes. In particular $1K_n$ is just K_n, the complete graph on n nodes. Color the edges from 1 to s so that any two distinct nodes have precisely one edge of each color between them. For a fixed labeling 1 through n of the nodes of each sK_n, we have a projection $s\pi : sK_n \to K_n$ for each s, given by sending an edge with endpoints i and j to the unique edge of K_n with endpoints i and j. Since disjoint edges map to disjoint edges, this induces a map $\mathcal{M}(s\pi)\colon \mathcal{M}(sK_n) \to \mathcal{M}(K_n)$.

For any $l \in \mathbb{Z}$, define $\nu(l) := \lfloor (l-2)/3 \rfloor$.

Lemma 3.3. *$\mathcal{M}(sK_n)$ is $(\nu(n)-1)$-connected, as is $\mathcal{M}^o(sK_n)$.*

Proof. It is well known that $\mathcal{M}(K_n)$ is $(\nu(n)-1)$-connected; see for example [Athanasiadis 2004; Bux et al. 2012; Björner et al. 1994]. For any k-simplex σ in $\mathcal{M}(K_n)$, the fiber $\mathcal{M}(s\pi)^{-1}(\sigma)$ is the join of the fibers of the vertices of σ, so it is homotopy equivalent to a wedge of spheres of dimension k. It is clear also that links in $\mathcal{M}(K_n)$ are themselves matching complexes of complete graphs. Therefore the hypotheses of Theorem 9.1 in [Quillen 1978] are satisfied, and we conclude that $\mathcal{M}(sK_n)$ is $(\nu(n)-1)$-connected. We also have an obvious map $\mathcal{M}^o(sK_n) \twoheadrightarrow \mathcal{M}(sK_n)$ obtained by forgetting the orientation on the edges. The fibers of this map are similarly spherical of the right dimension, so again using Theorem 9.1 of [Quillen 1978] we conclude that $\mathcal{M}^o(sK_n)$ is $(\nu(n)-1)$-connected. $\square$

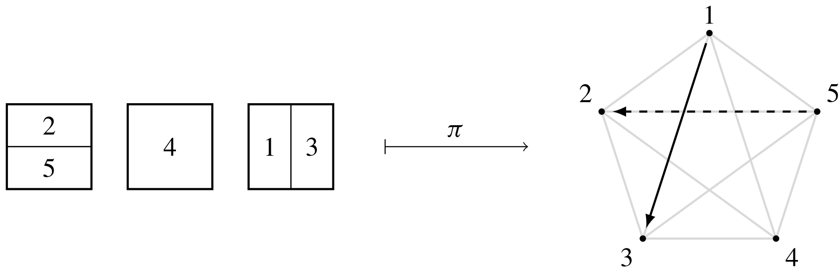

Figure 3. An example of $\pi : VE_n \to \mathcal{M}^o(sK_n)$ in the case $n = 5$ and $s = 2$. The solid arrow corresponds to a merge along a vertical face, and the dashed arrow corresponds to a merge along a horizontal face.

Every vertex $y \in L_0(x)$, say with $t(y) = m$, is obtained from x by applying a nontrivial very elementary merging. The merging is given by a very elementary covering $\mathcal{U}$ of m blocks whose n bricks are indexed by the blocks of x. Two such mergings define the same element y if and only if they differ by a permutation of the blocks. Consequently, denoting by VE_n the set of very elementary coverings by n labeled bricks up to permutation of the blocks, we get a one-to-one correspondence between $L_0(x)$ and VE_n. We obtain a partial order VE_n from the partial order on $\mathcal{P}_1$ via this identification.

Corollary 3.4. *VE_n, and therefore $L_0(x)$, is isomorphic to $\mathcal{M}^o(sK_n)$. Hence, both are $(\nu(n)-1)$-connected.*

Proof. Consider a nontrivial very elementary dyadic covering $\mathcal{U}$ of $I^s(m)$ with n bricks labeled 1 to n. Since $\mathcal{U}$ is very elementary, each block consists of at most two bricks. If it does consist of two bricks, then it defines an oriented edge in the graph sK_n as follows. The two bricks are

$$I^{k-1} \times \left(I \cap \left[0, \tfrac{1}{2}\right]\right) \times I^{s-k} \quad \text{and} \quad I^{k-1} \times \left(I \cap \left[\tfrac{1}{2}, 1\right]\right) \times I^{s-k}$$

for some $1 \le k \le s$. Say the first brick is labeled i and the second brick is labeled j. Then the edge in sK_n defined by this block points from i to j and has color k. See Figure 3 for an example.

This procedure defines an isomorphism of ordered sets $VE_n \to \mathcal{M}^o(sK_n)$. The connectivity statement now follows from Lemma 3.3. □

The next step is to show that $L(x)$ is highly connected by building up from $L_0(x)$ to $L(x)$ along highly connected links. If $s = 1$, then $L_0(x) = L(x)$ so we may assume $s > 1$ in what follows.

We start by giving a combinatorial description of $L(x)$ similar to the one given for $L_0(x)$ before. Every vertex in $L(x)$ is obtained from x via a nontrivial elementary merging. We can therefore replace "very elementary" by "elementary" in the discussion of VE_n above. We get that the poset E_n of elementary mergings of n labeled bricks up to permutation of blocks is isomorphic to $L(x)$.

We now describe the Morse function that determines in which order we build up from $L_0(x)$ to $L(x)$. For any $\mathcal{U} \in E_n$, the volume of any brick in $\mathcal{U}$ is at least $1/2^s$. For each $0 \le i \le s$ define c_i to be the number of bricks in $\mathcal{U}$ with volume $1/2^i$. Then define c to be the lexicographically ordered function $c = (c_s, c_{s-1}, \ldots, c_3, c_2)$. Note that we *do not* include c_1 or c_0 in this tuple; this will be crucial to our arguments. Denote by b the number of blocks of $\mathcal{U}$. The *height* h of $\mathcal{U}$ is now defined to be $h = (c, b)$, ordered lexicographically.

Observation 3.5. Let $\mathcal{X}$ and $\mathcal{Y}$ be vertices in E_n with $\mathcal{X} < \mathcal{Y}$. Then $c(\mathcal{X}) \ge c(\mathcal{Y})$ and $b(\mathcal{X}) < b(\mathcal{Y})$, so in particular $h(\mathcal{X}) < h(\mathcal{Y})$ if and only if $c(\mathcal{X}) = c(\mathcal{Y})$, and $h(\mathcal{X}) > h(\mathcal{Y})$ if and only if $c(\mathcal{X}) > c(\mathcal{Y})$.

Fix a vertex $\mathcal{U}$ in $E_n \setminus VE_n$. The descending link of $\mathcal{U}$ with respect to h will be denoted $\mathrm{lk}{\downarrow_h}(\mathcal{U})$. There are two types of vertices $\mathcal{V}$ in $\mathrm{lk}{\downarrow_h}(\mathcal{U})$. First, we could have $\mathcal{U} > \mathcal{V}$ and $h(\mathcal{U}) > h(\mathcal{V})$, which by the above observation implies that $c(\mathcal{U}) = c(\mathcal{V})$. The full subcomplex of $\mathrm{lk}{\downarrow_h}(\mathcal{U})$ spanned by such vertices will be called the (*descending*) *down-link*. Second, we could have $\mathcal{U} < \mathcal{V}$ and $h(\mathcal{U}) > h(\mathcal{V})$, which implies that $c(\mathcal{U}) > c(\mathcal{V})$. The full subcomplex of $\mathrm{lk}{\downarrow_h}(\mathcal{U})$ spanned by these vertices will be called the (*descending*) *up-link*.

Observation 3.6. Vertices $\mathcal{V}$ in the down-link and $\mathcal{W}$ in the up-link automatically satisfy $\mathcal{V} < \mathcal{W}$. Therefore $\mathrm{lk}{\downarrow_h}(\mathcal{U})$ is a join of the down-link and the up-link.

This allows us to study the up-link and the down-link separately.

Lemma 3.7. *If $\mathcal{U}$ has a block with precisely two bricks, then the up-link of $\mathcal{U}$ is contractible, and hence so is* $\mathrm{lk}{\downarrow_h}(\mathcal{U})$.

Proof. Let B be a block in $\mathcal{U}$ with two bricks. Note that splitting only B does not yield a vertex with lower height. For a vertex $\mathcal{V}$ of the up-link we define a vertex $\mathcal{V}_0$ as follows (see Figure 4): Since $\mathcal{V}$ is in the up-link, it is obtained from $\mathcal{U}$ by splitting. Let $\mathcal{V}_0$ be the covering obtained from $\mathcal{U}$ by doing all the same splittings as for $\mathcal{V}$, except that B is not split (whether or not it was split for $\mathcal{V}$). Then $\mathcal{V}_0 > \mathcal{U}$, since $\mathcal{V}$ was obtained by splitting more than just B, as observed above. It is also clear that $c(\mathcal{V}_0) < c(\mathcal{U})$, and so $\mathcal{V}_0$ is again in the up-link of $\mathcal{U}$. Now let $\mathcal{Z}_B$ be the maximal elementary splitting of $\mathcal{U}$ that does not split B. Then for all $\mathcal{V}$ in the up-link, we have $\mathcal{V}_0 \le \mathcal{Z}_B$. Hence we have the inequalities $\mathcal{V} \ge \mathcal{V}_0 \le \mathcal{Z}_B$, which provide a contraction of the up-link of $\mathcal{U}$, by Section 1.5 of [Quillen 1978]. $\square$

For $l \in \mathbb{Z}$, define $\eta(l) := \lfloor (l-2)/2^s \rfloor$. Note that, for a fixed s, as $n \to \infty$, $\eta(n)$ increases monotonically to ∞.

Lemma 3.8. *If $\mathcal{U}$ has no block with precisely two bricks, then* $\mathrm{lk}{\downarrow_h}(\mathcal{U})$ *is at least* $(\eta(n)-2)$*-connected.*

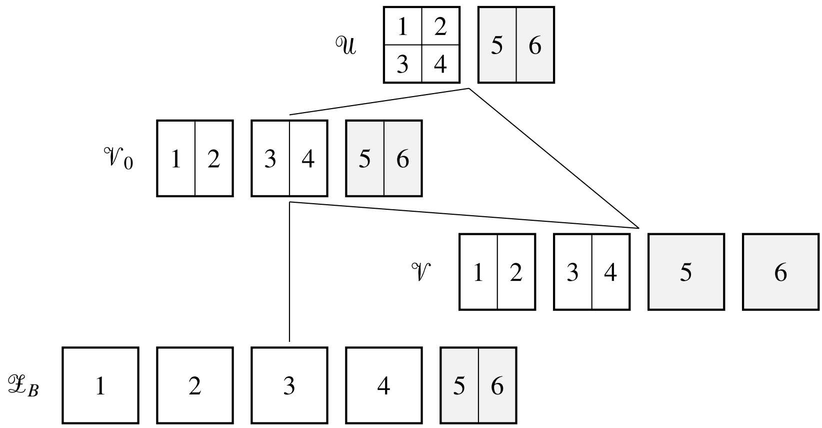

Figure 4. A step in building up from VE_6 to E_6 as described in the proof of Lemma 3.7. The block B of the covering $\mathcal{U}$ and its images under the various splittings are highlighted.

Proof. Call a block in $\mathcal{U}$ with more than two bricks *big*, and a block with only one brick *small*. Let k_b be the number of big blocks and k_s the number of small blocks. By assumption $k_b + k_s$ is the number m of blocks in $\mathcal{U}$.

The up-link of $\mathcal{U}$ is clearly at least (k_b-2)-connected, since splitting a big block in any way produces a vertex with lower height, and so each big block contributes a nonempty join factor to the up-link. The down-link of $\mathcal{U}$ is isomorphic to VE_{k_s}, and therefore is $(\nu(k_s)-1)$-connected by Corollary 3.4. This implies that $\mathrm{lk}{\downarrow}_h(\mathcal{U})$ is $(k_b+\nu(k_s)-1)$-connected. Also, n is the number of bricks in $\mathcal{U}$, so $n \le 2^s k_b + k_s$.

Since $s > 1$, $2^s > 3$, so we have

$$\begin{aligned} k_b + \nu(k_s) - 1 &\ge k_b + \left\lfloor \frac{k_s-2}{2^s} \right\rfloor - 1 \ge k_b + \frac{k_s-2}{2^s} - 2 \\ &= \frac{2^s k_b + k_s - 2}{2^s} - 2 \ge \frac{n-2}{2^s} - 2 \ge \eta(n) - 2. \end{aligned}$$

We conclude that $\mathrm{lk}{\downarrow}_h(\mathcal{U})$ is at least $(\eta(n)-2)$-connected. $\square$

Corollary 3.9. *If $s = 1$ then E_n, and hence $L(x)$ is $(\nu(n)-1)$-connected. If $s > 1$, then E_n, and hence $L(x)$ is $(\eta(n)-1)$-connected.*

Proof. The $s = 1$ case is already done, since then $E_n = VE_n$. Now suppose $s > 1$. Then $\eta \le \nu$, so VE_n is at least $(\eta(n)-1)$-connected. Also, for $\mathcal{U} \in E_n \setminus VE_n$, $\mathrm{lk}{\downarrow}_h(\mathcal{U})$ is $(\eta(n)-2)$-connected by Lemmas 3.7 and 3.8. It follows from Lemma 3.1 that E_n is at least $(\eta(n)-1)$-connected. $\square$

Proposition 3.10. *For each $n \ge 1$, the pair $(sX^{\le n}, sX^{<n})$ is $\eta(n)$-connected for $s > 1$, and the pair $(1X^{\le n}, 1X^{<n})$ is $\nu(n)$-connected.*

Proof. Let x be a vertex in $sX^{=n}$. By Corollary 3.9, the descending link $\mathrm{lk}{\downarrow}(x)$ of x in sX is at least $(\eta(n)-1)$-connected for $s > 1$, or $(\nu(n)-1)$-connected for $s = 1$. The result now follows from Lemma 3.1. □

We are now in a position to apply Brown's criterion.

Proof of Main Theorem. Consider the action of sV on sX. By Corollary 2.5, sX is contractible, by Observation 1.5, the stabilizer of every cell is finite, and by Observation 1.6, each $sX^{\leq n}$ is finite modulo sV. By Proposition 3.10, the connectivity of the pairs $(sX^{\leq n}, sX^{<n})$ tends to ∞ as n tends to ∞. Hence, sV is of type F_∞ by Brown's criterion. □

Acknowledgments

We would like to thank Kai-Uwe Bux for suggesting this project and for many helpful discussions. We also gratefully acknowledge support through the SFB 701 in Bielefeld (first, second, and fourth author) and the SFB 878 in Münster (third author). The research for this article was carried out during the 2012 PCMI Summer Session in Geometric Group Theory and we thank the organizers and the PCMI for this opportunity. Finally, we thank Matt Brin for his helpful suggestions and remarks.

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Received September 12, 2012. Revised November 20, 2012.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 266, No. 2, 2013
dx.doi.org/10.2140/pjm.2013.266.297

IDEAL DECOMPOSITIONS OF A TERNARY RING OF OPERATORS WITH PREDUAL

MASAYOSHI KANEDA

We show that any TRO (ternary ring of operators) with predual can be decomposed into the direct sum of a two-sided ideal, a left ideal, and a right ideal in some von Neumann algebra using an extreme point of the unit ball of the TRO.

Recall that an operator space X is called a *triple system* or a *ternary ring of operators* (*TRO* for short) if there exists a complete isometry ι from X into a C^*-algebra such that $\iota(x)\iota(y)^*\iota(z) \in \iota(X)$ for all $x, y, z \in X$. Our main result is that any TRO with predual can be decomposed into the direct sum of a two-sided ideal, a left ideal, and a right ideal in some von Neumann algebra:

Theorem. *Let X be a TRO which is also a dual Banach space. Then X can be decomposed into the direct sum of TROs X_T, X_L, and X_R,*

$$X = X_T \overset{\infty}{\oplus} X_L \overset{\infty}{\oplus} X_R,$$

so that there is a complete isometry ι from X into a von Neumann algebra in which $\iota(X_T)$, $\iota(X_L)$, and $\iota(X_R)$ are a weak*-closed two-sided, left, and right ideal, respectively, and

$$\iota(X) = \iota(X_T) \overset{\infty}{\oplus} \iota(X_L) \overset{\infty}{\oplus} \iota(X_R).$$

In the special case that the TRO is finite-dimensional, the decomposition is into a direct sum of rectangular matrices, as first proved essentially by R. R. Smith [2000]. In the Appendix we give a short proof of that result. The following lemma is a version of Kadison's theorem [1951, Theorem 1] as found in [Pedersen 1979, Proposition 1.4.8] or [Sakai 1971, Proposition 1.6.5]. Together with the idea of embedding an off-diagonal corner into a diagonal corner developed in [Blecher and Kaneda 2004, Section 2] (see also [Kaneda 2003, Section 2.2]), it plays a key role in the proof of our theorem.

The author was supported by a research fund from Department of Mathematics, University of California, Irvine.

MSC2010: primary 46L45, 47L20; secondary 46L07, 47L25, 47L50, 47L07.

Keywords: extreme points, dual operator spaces, von Neumann algebras, ideals, ternary rings of operators.

Lemma (Kadison's theorem). *Let $\mathcal{A}$ be a C^*-algebra, and let p, q be orthogonal projections in $\mathcal{A}$. Then an element $x \in p\mathcal{A}q$ is an extreme point of* $\mathrm{Ball}(p\mathcal{A}q)$ *if and only if $(p - xx^*)\mathcal{A}(q - x^*x) = \{0\}$. In this case, x is a partial isometry.*

Proof of the Theorem. By [Effros et al. 2001, Theorem 2.6], we may regard X as a weak*-closed subspace of $\mathbb{B}(\mathcal{K}, \mathcal{H})$ for some Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ such that $XX^*X \subset X$. We may assume that $[X\mathcal{K}] = \mathcal{H}$ and $[X^*\mathcal{H}] = \mathcal{K}$. We also identify $\mathbb{B}(\mathcal{K}, \mathcal{H})$ with the (1, 2)-corner of $\mathbb{B}(\mathcal{H} \oplus \mathcal{K})$, and let $1_{\mathcal{H}} \in \mathbb{B}(\mathcal{H} \oplus \mathcal{K})$ and $1_{\mathcal{K}} \in \mathbb{B}(\mathcal{H} \oplus \mathcal{K})$ denote the orthogonal projections on $\mathcal{H}$ and $\mathcal{K}$. Then

$$\mathcal{L}(X) := \begin{bmatrix} \overline{XX^*}^{\mathrm{w}^*} & X \\ X^* & \overline{X^*X}^{\mathrm{w}^*} \end{bmatrix}$$

is the linking von Neumann algebra, $1_{\mathcal{H}}, 1_{\mathcal{K}} \in \mathcal{L}(X)$, and $X = 1_{\mathcal{H}}\mathcal{L}(X)1_{\mathcal{K}}$. Since $\mathrm{Ball}(X)$ is weak*-closed in $\mathbb{B}(\mathcal{K}, \mathcal{H})$, there is an extreme point $e \in \mathrm{Ball}(X)$. By Kadison's theorem above,

$$(1_{\mathcal{H}} - ee^*)X(1_{\mathcal{K}} - e^*e) = \{0\}, \tag{1}$$

and e is a partial isometry. Let $p \in \overline{X(1_{\mathcal{K}} - e^*e)X^*}^{\mathrm{w}^*}$ and $q \in \overline{X^*(1_{\mathcal{H}} - ee^*)X}^{\mathrm{w}^*}$ be the identities of these two von Neumann algebras. Then by the adjoint of (1), it follows that

$$pXq = \{0\}, \tag{2}$$

$$p = pee^* = ee^*p = pee^*p \quad \text{and} \quad q = e^*eq = qe^*e = qe^*eq. \tag{3}$$

Noting that $pxy^* \in \overline{X(1_{\mathcal{K}} - e^*e)X^*}^{\mathrm{w}^*}$ and $qx^*y \in \overline{X^*(1_{\mathcal{H}} - ee^*)X}^{\mathrm{w}^*}$, we also get

$$pxy^* = pxy^*p = xy^*p \quad \text{and} \quad qx^*y = qx^*yq = x^*yq \quad \text{for all } x, y \in X. \tag{4}$$

Put

$$q_1 := e^*(1_{\mathcal{H}} - p)e(1_{\mathcal{K}} - q) \quad \text{and} \quad q_2 := 1_{\mathcal{K}} - q - q_1.$$

We claim that q_1 and q_2 are orthogonal projections. Indeed, (4) and the fact that $pe \in X$ yield

$$\begin{aligned} q_1^* &= (1_{\mathcal{K}} - q)e^*(1_{\mathcal{H}} - p)e = e^*e - e^*pe - qe^*e + qe^*pe \\ &= e^*e - e^*pe - e^*eq + e^*peq = q_1 \end{aligned}$$

and

$$\begin{aligned} q_1^2 &= e^*(1_{\mathcal{H}} - p)e(1_{\mathcal{K}} - q)e^*(1_{\mathcal{H}} - p)e(1_{\mathcal{K}} - q) = e^*(1_{\mathcal{H}} - p)eq_1^*(1_{\mathcal{K}} - q) \\ &= e^*(1_{\mathcal{H}} - p)eq_1(1_{\mathcal{K}} - q) = e^*(1_{\mathcal{H}} - p)ee^*(1_{\mathcal{H}} - p)e(1_{\mathcal{K}} - q)(1_{\mathcal{K}} - q) \\ &= e^*ee^*(1_{\mathcal{H}} - p)(1_{\mathcal{H}} - p)e(1_{\mathcal{K}} - q)(1_{\mathcal{K}} - q) = e^*(1_{\mathcal{H}} - p)e(1_{\mathcal{K}} - q) \\ &= q_1. \end{aligned}$$

Noting that $q_1 q = 0$, we have $q_2^2 = q_2 = q_2^*$.

To see that

$$(1_{\mathcal{H}} - p)X(1_{\mathcal{K}} - e^*e) = \{0\}, \tag{5}$$

let $\{u_\alpha\}$ be an approximate identity of the C^*-algebra X^*X. Then for each $x \in X$, $px(1_{\mathcal{K}} - e^*e)u_\alpha = x(1_{\mathcal{K}} - e^*e)u_\alpha$. Taking the limit $\alpha \to \infty$ yields that

$$px(1_{\mathcal{K}} - e^*e) = x(1_{\mathcal{K}} - e^*e)$$

for $x \in X$, and hence (5) holds. Similarly,

$$(1_{\mathcal{H}} - ee^*)X(1_{\mathcal{K}} - q) = \{0\} \tag{6}$$

also holds.

Let $x, y \in X$. Then

$$\begin{aligned} q_1 x^* y &= e^*(1_{\mathcal{H}} - p)e(1_{\mathcal{K}} - q)x^*y \\ &= e^*(1_{\mathcal{H}} - p)ex^*y(1_{\mathcal{K}} - q) && \text{by (4)} \\ &= e^*ex^*(1_{\mathcal{H}} - p)y(1_{\mathcal{K}} - q) && \text{by (4)} \\ &= x^*(1_{\mathcal{H}} - p)y(1_{\mathcal{K}} - q) && \text{by the adjoint of (5)} \\ &= x^*(1_{\mathcal{H}} - p)ye^*e(1_{\mathcal{K}} - q) && \text{by (5)} \\ &= x^*ye^*(1_{\mathcal{H}} - p)e(1_{\mathcal{K}} - q) && \text{by (4)} \\ &= x^*yq_1, \end{aligned}$$

and so we have

$$q_1 x^* y = x^* y q_1 = q_1 x^* y q_1 \quad \text{for all } x, y \in X. \tag{7}$$

Put $X_T := Xq_1$, $X_L := Xq$, and $X_R := Xq_2$. Then these are weak*-closed TROs, and $X = X_T \oplus X_L \oplus X_R$. Using (4) and (7) and noting that q_1, q, and q_2 are mutually disjoint, we have

$$X_T^* X_L = X_T^* X_R = X_L^* X_T = X_L^* X_R = X_R^* X_T = X_R^* X_L = \{0\}$$

and

$$X^*X = X_T^* X_T \overset{\infty}{\oplus} X_L^* X_L \overset{\infty}{\oplus} X_R^* X_R.$$

This proves that $X = X_T \overset{\infty}{\oplus} X_L \overset{\infty}{\oplus} X_R$.

Define

$$\iota : X \to \overline{XX^*}^{\mathrm{w}^*} \overset{\infty}{\oplus} \overline{X^*X}^{\mathrm{w}^*}$$

by

$$\iota(x) := (x_T + x_L)e^* \oplus e^* x_R,$$

where $x = x_T + x_L + x_R$ is the unique decomposition of $x \in X$ such that $x_T \in X_T$, $x_L \in X_L$, and $x_R \in X_R$. First note that $\iota(X_T) \cap \iota(X_L) = \{0\}$. Indeed, assume that $\iota(x_T) + \iota(x_L) = 0$, that is, $xq_1e^* + xqe^* = 0$. Then by multiplying both sides by e on the right and using (3) and (7), we obtain that $xe^*eq_1 + xq = 0$. Multiplying both sides by q on the right noting that $q_1q = 0$ yields that $xq = 0$, and hence $xq_1e^* = xqe^* = 0$, that is, $\iota(x_T) = \iota(x_L) = 0$. Since $\iota(X_T)^*\iota(X_L) = eX_T^*X_Le^* = \{0\}$ and $\iota(X_L)^*\iota(X_T) = eX_L^*X_Te^* = \{0\}$, we obtain

$$(\iota(X_T) \oplus \iota(X_L))^*(\iota(X_T) \oplus \iota(X_L)) = \iota(X_T)^*\iota(X_T) \overset{\infty}{\oplus} \iota(X_L)^*\iota(X_L)$$

noting that $\iota(X_T)^*\iota(X_T) = q_1X_T^*X_Tq_1$ and $\iota(X_L)^*\iota(X_L) = qX_L^*X_Lq$. Thus $\iota(X) = \iota(X_T) \overset{\infty}{\oplus} \iota(X_L) \overset{\infty}{\oplus} \iota(X_R)$. To show that ι is a complete isometry, it suffices to show that each of $\iota_{|X_T}$, $\iota_{|X_L}$, and $\iota_{|X_R}$ is a complete isometry. Since $e^*eq_1 = q_1$,

$$\|\iota(x_T)\|^2 = \|\iota(x_T)\iota(x_T)^*\| = \|xq_1e^*eq_1x^*\| = \|xq_1x^*\| = \|xq_1\|^2 = \|x_T\|^2.$$

A similar calculation works at the matrix level, which concludes that $\iota_{|X_T}$ is a complete isometry. Similarly, (3) yields that $\iota_{|X_L}$ is a complete isometry.

$$\begin{aligned}\|\iota(x_R)\|^2 &= \|\iota(x_R)^*\iota(x_R)\| = \|q_2x^*ee^*xq_2\| = \|q_2x^*ee^*x(1_{\mathcal{K}} - q - q_1)\| \\ &= \|q_2x^*x(1_{\mathcal{K}} - q)\| = \|q_2x^*x(1_{\mathcal{K}} - q - q_1)\| = \|q_2x^*xq_2\| = \|x_R\|^2,\end{aligned}$$

where we used (6) and (7) as well as the fact that $q_2q_1 = 0$ in the fourth equality, and (7) together with the fact that $q_2q_1 = 0$ in the fifth equality. A similar calculation works at the matrix level, which concludes that $\iota_{|X_R}$ is a complete isometry.

By [Blecher 2001, Lemma 1.5(3)] or [Blecher and Le Merdy 2004, Theorem A.2.5(3)] for example, $\iota(X_T)$, $\iota(X_L)$, and $\iota(X_R)$ are weak*-closed. Clearly, $\iota(X_T)$ and $\iota(X_L)$ are left ideals and $\iota(X_R)$ is a right ideal in the von Neumann algebra $\overline{XX^*}^{\mathrm{w}^*} \overset{\infty}{\oplus} \overline{X^*X}^{\mathrm{w}^*}$. To see that $\iota(X_T)$ is a right ideal as well, it suffices to show that $\iota(X_T)^* \subset \iota(X_T)$, in which case necessarily $\iota(X_T)^* = \iota(X_T)$. To show this, first note that it follows from the adjoint of (6) that

$$q_1x^* = e^*(1_{\mathcal{H}} - p)e(1_{\mathcal{K}} - q)x^* = e^*(1_{\mathcal{H}} - p)e(1_{\mathcal{K}} - q)x^*ee^* = q_1x^*ee^* \quad \text{for all } x \in X.$$

Therefore, together with (7), we obtain

$$\iota(x_T)^* = eq_1x^* = eq_1x^*ee^* = ex^*eq_1e^* \in Xq_1e^* = \iota(X_T) \quad \text{for all } x \in X. \qquad \square$$

Definition. We call the decomposition $X = X_T \overset{\infty}{\oplus} X_L \overset{\infty}{\oplus} X_R$ obtained in the proof of Theorem the *ideal decomposition* of the TRO X with predual with respect to an extreme point e of $\mathrm{Ball}(X)$.

Remarks. (A) The reader should distinguish ideal decompositions from Peirce decompositions in the literature of Jordan triples. In fact, a TRO can be regarded as a Jordan triple with the canonical symmetrization of the triple product. However, an ideal decomposition and a Peirce decomposition give totally different decompositions.

(B) It is also possible to define $\iota : X \to \overline{XX^*}^{\mathrm{w}^*} \overset{\infty}{\oplus} \overline{X^*X}^{\mathrm{w}^*}$ by

$$\iota(x) := x_L e^* \oplus e^*(x_R + x_T) \quad \text{for } x \in X.$$

(C) Simpler expressions for X_T and X_R are $X_T = \{x - px - xq \mid x \in X\}$ and $X_R = pX$, which would be more helpful in understanding what is going on in the decomposition. To see the equivalences of expressions, let $x \in X$. Then, using (4), (5), and (2), we have

$$\begin{aligned} x_T : &= xq_1 = xe^*(1_{\mathcal{H}} - p)e(1_{\mathcal{K}} - q) = (1_{\mathcal{H}} - p)xe^*e(1_{\mathcal{K}} - q) \\ &= (1_{\mathcal{H}} - p)x(1_{\mathcal{K}} - q) = x - px - xq. \end{aligned}$$

Accordingly, it follows that

$$x_R := xq_2 = x(1_{\mathcal{K}} - q - q_1) = x(1_{\mathcal{K}} - q) - xq_1 = x(1_{\mathcal{K}} - q) - (x - px - xq) = px.$$

(D) The ideal decomposition highly depends on the extreme point chosen. Indeed, let X be a von Neumann algebra, $u \in X$ be a unitary element, and $w \in X$ be an isometry which is not unitary. Then the ideal decomposition with respect to u is just $X = X_T$, while the one with respect to w is $X = X_T \overset{\infty}{\oplus} X_L$.

Appendix: A short proof of Smith's result

The following theorem was proved in [Smith 2000] (also see [Effros and Ruan 2000, Lemma 6.1.7 and Corollary 6.1.8]). We observed it independently in 2000, together with Corollary A.2. Since these results are a special case of this paper's Theorem, and our proof is short enough to understand the essence of the results transparently, it seems worthwhile to present them here. The key to the shortness of the proof is the obvious fact that if a TRO X is finite-dimensional, then so are the C^*-algebras XX^* and X^*X.

Theorem A.1 [Smith 2000]. *If X is a finite-dimensional TRO, then there exist a finite-dimensional C^*-algebra $\mathcal{A}$ and an orthogonal projection $p \in \mathcal{A}$ such that $X \cong p\mathcal{A}p^{\perp}$ completely isometrically.*

Proof. Let $X \subset \mathbb{B}(\mathcal{K}, \mathcal{H})$ be a finite-dimensional TRO and $\{x_1, \dots, x_n\} \subset X$ be its base. We may assume that $[X\mathcal{K}] = \mathcal{H}$ and $[X^*\mathcal{H}] = \mathcal{K}$. Then the C^*-algebra $XX^* := \operatorname{span}\{xy^* \mid x, y \in X\}$ is equal to the set $\operatorname{span}\{x_i x_j^* \mid 1 \le i, j \le n\}$, and the latter is obviously a finite-dimensional vector space. Similarly, $X^*X := \operatorname{span}\{x^*y \mid$

$x, y \in X\}$ is a finite-dimensional C^*-algebra. Let $\mathcal{L}(X)$ be the linking C^*-algebra for X, that is,

$$\mathcal{L}(X) := \begin{bmatrix} XX^* & X \\ X^* & X^*X \end{bmatrix} (\subset B(\mathcal{H} \oplus \mathcal{K}))\,.$$

Let e, f be the identities of the C^*-algebras XX^* and X^*X, respectively, and let

$$p := \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{L}(X).$$

Then

$$p^\perp = \begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix}$$

and $X \cong p\mathcal{L}(X)p^\perp$ completely isometrically. □

Corollary A.2. *A finite-dimensional TRO is completely isometric to the direct sum of rectangular matrices*: $\mathbb{M}_{l_1,k_1}(\mathbb{C}) \overset{\infty}{\oplus} \cdots \overset{\infty}{\oplus} \mathbb{M}_{l_m,k_m}(\mathbb{C})$.

Proof. Let X be a finite-dimensional TRO. By Theorem A.1, we may assume that $X = p\left(\bigoplus_{i=1}^m \mathbb{M}_{n_i}(\mathbb{C})\right) p^\perp$, where p is an orthogonal projection in $\bigoplus_{i=1}^m \mathbb{M}_{n_i}(\mathbb{C})$. For each $1 \le i \le m$, let us denote by 1_i the identity of $\mathbb{M}_{n_i}(\mathbb{C})$ which is identified with an element of $\bigoplus_{i=1}^m \mathbb{M}_{n_i}(\mathbb{C})$ in the obvious way, and let $p_i := p1_i$. Then $X = \bigoplus_{i=1}^m p_i \mathbb{M}_{n_i}(\mathbb{C}) p_i^\perp$. By a unitary transform which is a complete isometry, we may assume that

$$p_i = \operatorname{diag}\{\overbrace{1, \dots, 1}^{l_i \text{ times}}, \overbrace{0, \dots, 0}^{(n_i - l_i) \text{ times}}\} \quad \text{and} \quad p_i^\perp = \operatorname{diag}\{\overbrace{0, \dots, 0}^{l_i \text{ times}}, \overbrace{1, \dots, 1}^{(n_i - l_i) \text{ times}}\}$$

for each $1 \le i \le m$. □

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Received September 30, 2012.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 266, No. 2, 2013
dx.doi.org/10.2140/pjm.2013.266.305

A STUDY OF REAL HYPERSURFACES WITH RICCI OPERATORS IN 2-DIMENSIONAL COMPLEX SPACE FORMS

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We prove that a real hypersurface M in complex projective space $P_2(\mathbb{C})$ or complex hyperbolic space $H_2(\mathbb{C})$, whose Ricci operator is η-parallel and commutes with the structure tensor on the holomorphic distribution, is a Hopf hypersurface. We also give a characterization of this hypersurface.

1. Introduction

A complex n-dimensional Kählerian manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. As is well known, a complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n(\mathbb{C})$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n(\mathbb{C})$, according to $c > 0$, $c = 0$ or $c < 0$.

In this paper we consider a real hypersurface M in a complex space form $M_2(c)$, $c \neq 0$. Then M has an almost contact metric structure (ϕ, g, ξ, η) induced from the Kähler metric and complex structure J on $M_n(c)$. The structure vector field ξ is said to be *principal* if $A\xi = \alpha\xi$ is satisfied, where A is the shape operator of M and $\alpha = \eta(A\xi)$. In this case, it is known that α is locally constant [Ki and Suh 1990] and that M is called a *Hopf hypersurface*.

Takagi [1973] classified homogeneous real hypersurfaces in $P_n(\mathbb{C})$ into six model spaces A_1, A_2, B, C, D and E of Hopf hypersurfaces with constant principal curvatures. Berndt [1989] classified all homogeneous Hopf hypersurfaces in $H_n(\mathbb{C})$ as four model spaces, which are said to be A_0, A_1, A_2 and B. A real hypersurface M of type A_1 or A_2 in $P_n(\mathbb{C})$ or type A_0, A_1 or A_2 in $H_n(\mathbb{C})$ is said to be *of type A* for simplicity.

As a typical characterization of real hypersurfaces of type A, the following is due to Okumura [1975] for $c > 0$, and Montiel and Romero [1986] for $c < 0$.

Theorem A [Montiel and Romero 1986; Okumura 1975]. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. It satisfies $A\phi - \phi A = 0$ on M if and only if M is locally congruent to one of the model spaces of type A.*

MSC2010: primary 53C40; secondary 53C15.
Keywords: real hypersurface, η-parallel Ricci operator, Hopf hypersurface.

The Ricci operator of M will be denoted by S, and the shape operator or the second fundamental tensor field of M by A. The holomorphic distribution T_0 of a real hypersurface M in $M_n(c)$ is defined by

$$T_0(p) = \{X \in T_p(M) \mid g(X, \xi)_p = 0\}, \tag{1-1}$$

where $T_p(M)$ is the tangent space of M at $p \in M$. The Ricci operator S is said to be η-parallel if

$$g((\nabla_X S)Y, Z) = 0 \tag{1-2}$$

for any vector fields X, Y and Z in T_0.

Theorem B [Kimura and Maeda 1989; Suh 1990]. *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. Then the Ricci operator of M is η-parallel and the structure vector field ξ is a principal if and only if M is locally congruent to one of the model spaces of type A or type B.*

I.-B. Kim, K. H. Kim and one of the present authors [Kim et al. 2006; 2007] studied real hypersurfaces with certain conditions related to the Ricci operator and the structure tensor field ϕ in $M_n(c)$. As for the Ricci operator and structure tensor field ϕ, one of the present authors proved the following.

Theorem C [Sohn 2007]. *Let M be a real hypersurface with η-parallel Ricci operator in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$. If M satisfies*

$$g((S\phi - \phi S)X, Y) = 0 \tag{1-3}$$

for any X and Y in T_0, then M is locally congruent to one of the model spaces of type A or type B.

The purpose of this paper is to complete the results of [Sohn 2007] and characterize real hypersurfaces with η-parallel Ricci operator such that the Ricci operator and structure tensor field commute in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 2$. Namely, we prove:

Theorem. *A real hypersurface in a complex space form $M_2(c)$, $c \neq 0$ satisfies (1-2) and (1-3) if and only if it is pseudo-Einstein.*

The pseudo-Einstein hypersurfaces are classified by Kim and Ryan [2008] and Ivey and Ryan [2009] and are described in detail in these papers. In view of their results, we can state the following.

Corollary. *Let M be a real hypersurface with an η-parallel Ricci operator in a complex space form $M_2(c)$, $c \neq 0$. If M satisfies (1-3) then M is locally congruent to either a Hopf hypersurface with $A\xi = 0$ or one of the model spaces of type A.*

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M_2(c)$, and N be a unit normal vector field of M. By $\widetilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini–Study metric tensor $\tilde{g}$ of $M_2(c)$. Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N \quad \text{and} \quad \widetilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M, where g denotes the Riemannian metric tensor of M induced from $\tilde{g}$, and A is the shape operator of M in $M_2(c)$.

For any vector field X on M we put

$$JX = \phi X + \eta(X)N, \qquad JN = -\xi,$$

where J is the almost complex structure of $M_2(c)$. Then we see that M induces an almost contact metric structure (ϕ, g, ξ, η), that is,

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi)$$

for any vector fields X and Y on M. Since the almost complex structure J is parallel, we can verify from the Gauss formula that

$$\nabla_X \xi = \phi AX. \tag{2-1}$$

Since the ambient manifold is of constant holomorphic sectional curvature c, we have the Gauss equation

$$\begin{aligned} R(X, Y)Z &= \frac{c}{4}\big(g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\big) \\ &\qquad + g(AY, Z)AX - g(AX, Z)AY \end{aligned} \tag{2-2}$$

for any vector fields X, Y and Z on M, where R denotes the Riemannian curvature tensor of M.

From (1-3) the Ricci operator S of M is expressed by

$$SX = \frac{c}{4}\big((2n+1)X - 3\eta(X)\xi\big) + mAX - A^2 X, \tag{2-3}$$

where $m = \operatorname{trace} A$ is the mean curvature of M, and the covariant derivative of (2-3) is given by

$$\begin{aligned}(\nabla_X S)Y = -\frac{3c}{4}\big(g(\phi AX, Y)\xi + \eta(Y)\phi AX\big) \\ + (Xm)AY + m(\nabla_X A)Y - (\nabla_X A)AY - A(\nabla_X A)Y.\end{aligned}$$

Let U be a unit vector field on M with the same direction of the vector field $-\phi\nabla_\xi\xi$, and let β be the length of the vector field $-\phi\nabla_\xi\xi$ if it does not vanish. It is not possible to define U without specifying that $\beta \neq 0$. Then it is easily seen from (2-1) that

$$A\xi = \alpha\xi + \beta U, \tag{2-4}$$

where $\alpha = \eta(A\xi)$. We notice here that U is orthogonal to ξ.

We put

$$\Omega = \{p \in M \mid \beta(p) \neq 0\}.$$

Then Ω is an open subset of M.

3. η-parallel Ricci operators

In this section, we assume that Ω is not empty. Then there are scalar fields γ, ε and δ and a unit vector field U and ϕU orthogonal to ξ such that

$$AU = \beta\xi + \gamma U + \varepsilon\phi U, \quad A\phi U = \varepsilon U + \delta\phi U \tag{3-1}$$

and

$$m = \operatorname{trace} A = \alpha + \gamma + \delta \tag{3-2}$$

in $M_2(c)$.

We shall prove the following lemmas.

Lemma 3.1. *Let M be a real hypersurface in a complex space form $M_2(c)$, $c \neq 0$. If M satisfies (1-3), then we have $AU = \beta\xi + \gamma U$, $A\phi U = \delta\phi U$ and $\beta^2 = \alpha(\gamma - \delta)$.*

Proof. If we put $X = \xi$ into (2-3), we have

$$S\xi = \Big(\frac{c}{2} + \alpha\gamma + \alpha\delta - \beta^2\Big)\xi + \beta\delta U - \beta\varepsilon\phi U. \tag{3-3}$$

Putting $X = U$ into (2-3) and taking account of (3-1) yields

$$SU = \beta\delta\xi + \Big(\frac{5c}{4} + \alpha\gamma + \gamma\delta - \beta^2 - \varepsilon^2\Big) + \alpha\varepsilon\phi U. \tag{3-4}$$

Putting $X = \phi U$ into (2-3) and using (3-1), we obtain

$$S\phi U = -\beta\varepsilon\xi + \alpha\varepsilon U + \Big(\frac{5c}{4} + \alpha\delta + \gamma\delta - \varepsilon^2\Big)\phi U. \tag{3-5}$$

If we apply ϕ to (3-4), then we have

$$(S\phi - \phi S)U = -\beta\varepsilon\xi + 2\alpha\varepsilon U + (\alpha\delta - \alpha\gamma + \beta^2)\phi U. \tag{3-6}$$

From condition (1-3), we have, for all $X \in T_0$,

$$(S\phi - \phi S)X = -\beta g(\varepsilon U + \delta\phi U, X)\xi \tag{3-7}$$

If we substitute $X = U$ into (3-7), then we obtain

$$(S\phi - \phi S)U = -\beta\varepsilon\xi. \tag{3-8}$$

Comparing (3-6) and (3-8), we get $\varepsilon = 0$ and $\beta^2 = \alpha(\gamma - \delta)$. It follows that AU is expressed in terms of ξ and U only and $A\phi U$ is given by ϕU. □

It follows from (2-3) and (3-1) that

$$S\xi = \Big(\frac{c}{2} + 2\alpha\delta\Big)\xi + \beta\delta U, \tag{3-9}$$

$$SU = \beta\delta\xi + \Big(\frac{5c}{4} + \gamma\delta + \alpha\delta\Big)U, \tag{3-10}$$

$$S\phi U = \Big(\frac{5c}{4} + \gamma\delta + \alpha\delta\Big)\phi U. \tag{3-11}$$

Lemma 3.2. *Under the assumptions of Lemma 3.1, if M has the η-parallel Ricci operator S, then we have $AU = \beta\xi + \gamma U$, $A\phi U = 0$ and $\beta^2 = \alpha\gamma$.*

Proof. Differentiating (3-10) covariantly along vector field X in T_0, we obtain

$$(\nabla_X S)U = \Big(\Big(\frac{5c}{4}+\gamma\delta+\alpha\delta\Big)I - S\Big)\nabla_X U + \beta\delta\phi AX + X(\beta\delta)\xi + X\Big(\frac{5c}{4}+\gamma\delta+\alpha\delta\Big)U.$$

Taking the inner product of this equation with U and ϕU and making use of (3-9)–(3-11) and Lemma 3.1, we obtain

$$(\alpha + \gamma)\nabla\delta + \delta(\nabla\gamma + \nabla\alpha) = 2\beta\delta^2\phi U \tag{3-12}$$

and

$$\delta\gamma = 0.$$

If we differentiate this along the vector field X in T_0, then (3-12) is reduced to

$$\alpha\nabla\delta + \delta\nabla\alpha = 2\beta\delta^2\phi U. \tag{3-13}$$

Differentiating (3-11) covariantly along vector field X in T_0, we obtain

$$(\nabla_X S)\phi U = \Big(\Big(\frac{5c}{4} + \gamma\delta + \alpha\delta\Big)I - S\Big)\nabla_X\phi U + \Big(X\Big(\frac{5c}{4} + \gamma\delta + \alpha\delta\Big)\Big)\phi U. \tag{3-14}$$

If we take the inner product of (3-14) with ϕU and use (3-9)–(3-11), then we have

$$\alpha\nabla\delta + \delta\nabla\alpha = 0. \tag{3-15}$$

Comparing (3-13) and (3-15), we obtain $\delta = 0$ and $\beta^2 = \alpha\gamma$ from Lemma 3.1. From this and Lemma 3.1 we conclude that AU is expressed in terms of ξ and U only and $A\phi U = 0$. □

4. Proof of the main theorem

Assume that M satisfies (1-2) and (1-3). We first show that M is Hopf. If the open set Ω is not empty, then Lemma 3.2 yields $\delta = 0$. Thus the Ricci operator, as expressed in (3-9)–(3-11), has the property that ξ, U and ϕU are eigenvectors and that U and ϕU have the same eigenvalue. That is, M is pseudo-Einstein with

$$SX = \frac{5c}{4}X - \frac{3c}{4}g(X,\xi)\xi.$$

This contradicts a result from [Kim and Ryan 2008]. Thus we conclude that any hypersurface satisfying (1-2) and (1-3) must be Hopf.

Since M is Hopf, condition (1-3) yields $\alpha(\gamma - \delta) = 0$ and that the criteria for Proposition 2.21 in [Kim and Ryan 2008] are satisfied. Thus M is pseudo-Einstein.

Conversely, if M is pseudo-Einstein, observe that (1-2) and (1-3) must be satisfied. □

Acknowledgments

The authors would like to express their sincere gratitude to the referee who gave them valuable suggestions and comments.

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Received September 20, 2012. Revised May 6, 2013.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 266, No. 2, 2013
dx.doi.org/10.2140/pjm.2013.266.313

ON COMMENSURABILITY OF FIBRATIONS ON A HYPERBOLIC 3-MANIFOLD

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We discuss fibered commensurability of fibrations on hyperbolic 3-manifolds, a notion introduced by Calegari, Sun, and Wang (*Pacific J. Math.* 250:2 (2011), 287–317). We construct manifolds with nonsymmetric but commensurable fibrations on the same fibered face, and prove that if a given manifold M does not have hidden symmetries, then M does not admit nonsymmetric but commensurable fibrations.

It was also proved by Calegari et al that every hyperbolic fibered commensurability class contains a unique minimal element. Here we provide a detailed discussion on the proof of the theorem in the cusped case.

1. Introduction

In this paper, we are mainly interested in fibered hyperbolic 3-manifolds with the first Betti number greater than or equal to 2. Thurston [1986] showed that such a manifold admits infinitely many distinct fibrations (see also Section 4). It is an interesting question to investigate the relationship between such fibrations.

Calegari, Sun, and Wang defined the notion of fibered commensurability, which gives rise to an equivalence relation on fibrations. An *automorphism* on a surface is an isotopy class of self-homeomorphisms of the surface. For any fibration on a 3-manifold, we have the pair (F, ϕ) of the fiber surface F, and the monodromy automorphism ϕ. Since the monodromy is determined up to conjugacy in the mapping class group of F, we use the notation (F, ϕ) to denote the conjugacy class. Then commensurability of fibrations is defined as follows.

Definition 1.1 [Calegari et al. 2011]. A pair $(\widetilde{F}, \widetilde{\phi})$ *covers* (F, ϕ) if there is a finite cover $\pi : \widetilde{F} \to F$ and representative homeomorphisms $\widetilde{f}$ of $\widetilde{\phi}$ and f of ϕ so that $\pi \widetilde{f} = f\pi$ as maps $\widetilde{F} \to F$.

Definition 1.2 [Calegari et al. 2011]. Two pairs (F_1, ϕ_1) and (F_2, ϕ_2) are *commensurable* if there is a surface $\widetilde{F}$, automorphisms $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$, and nonzero integers k_1

This work was partially supported by JSPS Research Fellowship for Young Scientists.
MSC2010: primary 57M50; secondary 37B40.
Keywords: hyperbolic manifold, fibration, commensurability.

and k_2, so that $(\widetilde{F}, \widetilde{\phi}_i)$ covers (F_i, ϕ_i) for $i = 1, 2$ and if $\widetilde{\phi}_1^{k_1} = \widetilde{\phi}_2^{k_2}$ as automorphisms of $\widetilde{F}$.

For the remainder of the paper, we consider fibrations on hyperbolic 3-manifolds. In this case, the monodromy of each fibration is always *pseudo-Anosov* (see Definition 2.5 for the definition). The *normalized entropy* of a conjugacy class (F, ϕ) is defined as $\chi(F) \log \lambda(\phi)$, where $\chi(F)$ is the Euler characteristic of F and $\lambda(\phi)$ is the dilatation of ϕ. In Section 2, we observe that the normalized entropies of commensurable fibrations on the same hyperbolic 3-manifold agree. Then we offer an example of a manifold such that two of its fibrations are commensurable if and only if they share the same normalized entropy. We also give an example of a manifold with two noncommensurable fibrations of the same normalized entropy.

In this paper, we study commensurable fibrations on a hyperbolic 3-manifold in the context of a fibered face. A *fibered face* is a face of the Thurston norm ball whose rational points correspond to fibrations of the 3-manifold and a *fibered cone* is a cone over a fibered face (see Section 3 for details). Two fibrations on M are said to be *symmetric* if there exists a self-homeomorphism $\varphi : M \to M$ that maps one to the other. In [Calegari et al. 2011, Remark 3.9], Calegari, Sun, and Wang asked if there is an example of two fibrations on the same closed manifold, which are commensurable but have fibers distinguished by their genera. The following theorem provides such a construction in the cusped case. In this theorem fibers are distinguished by their Euler characteristics (see Section 4 for a proof).

Theorem 1.3. *There are hyperbolic* 3*-manifolds with nonsymmetric but commensurable fibrations whose corresponding elements in $H^1(M; \mathbb{Z})$ are in the same fibered cone.*

On the other hand, if M has no hidden symmetries, then such fibrations do not exist. Here, a (finite-volume) hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ is said to have *hidden symmetries* if $[C^+(\Gamma) : N^+(\Gamma)] > 1$, where $C^+(\Gamma)$ and $N^+(\Gamma)$ are the commensurator and normalizer of Γ; see Section 4 for details.

Theorem 1.4. *Suppose that M is a hyperbolic* 3*-manifold that does not have hidden symmetries. Then*, *any pair of fibrations of M is either symmetric or noncommensurable*, *but not both.*

Theorems 1.3 and 1.4 are motivated by the fact that up to isotopy, there are only finitely many commensurable fibrations on a hyperbolic 3-manifold. This fact is a corollary of the following:

Theorem 1.5 (see also Theorem 3.1 of [Calegari et al. 2011]). *Every commensurability class of hyperbolic fibered pairs contains a unique* (*orbifold*) *minimal element.*

Here the notion of a fibered pair is a generalization of the notion of a pair (F, ϕ), see Section 2 for details. The proof in [Calegari et al. 2011] works for the closed case. In Section 2 we extend it to the case where the manifolds have boundary (Theorem 2.6). Further, as a corollary of this extension, we show examples of manifolds such that every fibration is the minimal element in its commensurability class (Corollary 2.8).

Commensurability classes are defined using the transitive hull of the relation in Definition 1.2. In Section 2 we also discuss the transitivity of commensurability. We show that if the automorphisms are pseudo-Anosov (that is to say, in the hyperbolic case), then commensurability is transitive.

2. Preliminaries

In this section, we recall the definitions and basic facts about commensurability of fibrations. Most of the contents in this section are discussed in [Calegari et al. 2011]. In this paper, unless otherwise stated, by a *surface* and a *hyperbolic 3-manifold*, we mean a compact connected orientable 2-manifold possibly with boundary and of negative Euler characteristic, and a connected, orientable, complete hyperbolic 3-manifold of finite volume respectively.

Fibered pairs. Given a homeomorphism $f : F \to F$, the *mapping torus* of f is the 3-manifold

$$M = F \times [0, 1]/\big((f(x), 0) \sim (x, 1)\big).$$

Mapping tori of conjugate automorphisms are homeomorphic, so if ϕ is a conjugacy class of homeomorphisms we obtain a homeomorphism class of mapping tori, which we denote by $[F, \phi]$. We call F the *fiber* and ϕ the *monodromy* of $[F, \phi]$.

We will focus on fibrations of a fixed hyperbolic 3-manifold M. Each fibration on M over the circle determines an element of $H^1(M; \mathbb{Z})$, and if $\omega \in H^1(M; \mathbb{Z})$ corresponds to a fibration, then there is an associated pair (F, ϕ), in the sense that $[F, \phi]$ is homeomorphic to M. This correspondence of ω and (F, ϕ) is well defined up to the conjugation of (F, ϕ).

Later in this section (page 317), we discuss Theorem 3.1 of [Calegari et al. 2011] for the case of fibered manifolds with boundary. To state the theorem it is convenient to define a *fibered pair* which is a generalization of a pair of type (F, ϕ). We also enlarge our attention to orbifolds. An n-orbifold is a space that is locally modeled on a quotient of an open ball in $\mathbb{R}^n$ by a finite group. See [Walsh 2011] and Chapter 13 of [Thurston 1979] for more details.

Definition 2.1 [Calegari et al. 2011]. A *fibered pair* is a pair $(M, \mathscr{F})$, where M is a compact 3-manifold with boundary a union of tori and Klein bottles, and $\mathscr{F}$ is a foliation by compact surfaces. More generally, an orbifold fibered pair is a pair $(O, \mathcal{G})$, where O is a compact 3-orbifold, and $\mathcal{G}$ is a foliation of O by compact 2-orbifolds.

Definition 2.2 [Calegari et al. 2011]. A fibered pair $(\widetilde{M}, \widetilde{\mathcal{F}})$ *covers* $(M, \mathcal{F})$ if there is a finite covering of manifolds $\pi : \widetilde{M} \to M$ such that $\pi^{-1}(\mathcal{F})$ is isotopic to $\widetilde{\mathcal{F}}$. Two fibered pairs $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ are (*fibered*) *commensurable* if there is a third fibered pair $(\widetilde{M}, \widetilde{\mathcal{F}})$ that covers both.

For a given pair (F, ϕ), the mapping torus $[F, \phi]$ has a foliation $\mathcal{F}$ by surface leaves, which are homeomorphic to F and hence there is a corresponding fibered pair $([F, \phi], \mathcal{F})$.

Unlike the case of commensurability in Definition 1.2, it is easy to see that commensurability of fibered pairs is transitive. Suppose $(M_i, \mathcal{F}_i)$ and $(M_{i+1}, \mathcal{F}_{i+1})$ are commensurable for $i = 1, 2$ and $(\widetilde{M}_{12}, \widetilde{\mathcal{F}}_{12})$ (resp. $(\widetilde{M}_{23}, \widetilde{\mathcal{F}}_{23})$) is a common covering pair of $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ (resp. $(M_2, \mathcal{F}_2)$ and $(M_3, \mathcal{F}_3)$). Then there is a covering $p : \widetilde{N} \to M_2$ that corresponds to $p^{12}_* \pi_1(\widetilde{M}_{12}) \cap p^{23}_* \pi_1(\widetilde{M}_{23}) < \pi_1(M_2)$, where $p^{12} : \widetilde{M}_{12} \to M_2$ and $p^{23} : \widetilde{M}_{23} \to M_2$ are the covering maps. Then $(\widetilde{N}, p^{-1}(\mathcal{F}_2))$ covers both $(M_1, \mathcal{F}_1)$ and $(M_3, \mathcal{F}_3)$. Thus we see that fibered commensurability is a transitive relation.

We define another equivalence relation on fibered pairs so that the covering relation will be a partial order.

Definition 2.3 [Calegari et al. 2011]. We say that two fibered pairs $(M, \mathcal{F})$ and $(N, \mathcal{G})$ are *covering equivalent* if each covers the other. We call a covering equivalence class *minimal* if no representative covers any element of another covering equivalence class.

Remark 2.4 (see also Remark 2.9 of [Calegari et al. 2011]). Each covering equivalence class of the fibered pair associated to (F, ϕ) contains exactly one fibered pair unless ϕ is periodic. Therefore, when we consider pseudo-Anosov automorphisms, by abusing notation, we use the word “element” for each covering equivalent class.

Pseudo-Anosov automorphisms. The automorphisms on a compact surface are classified into three types: periodic, reducible, and pseudo-Anosov [Thurston 1988; Casson and Bleiler 1988]. By a result of Thurston, the (interior of the) mapping torus $[F, \phi]$ admits a hyperbolic metric of finite volume if and only if the automorphism ϕ is pseudo-Anosov (see [Thurston 1988], and compare [Otal 1996]).

Definition 2.5. A homeomorphism $f : F \to F$ is a *pseudo-Anosov homeomorphism* if there is a pair of transverse measured singular foliations $(\mathcal{F}^s, \mu^s)$ and $(\mathcal{F}^u, \mu^u)$ on F and a positive real number λ so that $f(\mathcal{F}^u) = \mathcal{F}^u$, $f(\mu^u) = \lambda \mu^u$ and $f(\mathcal{F}^s) = \mathcal{F}^s$, $f(\mu^s) = (1/\lambda)\mu^s$. We call $(\mathcal{F}^s, \mu^s)$ and $(\mathcal{F}^u, \mu^u)$ the stable and unstable measured singular foliations associated to f.

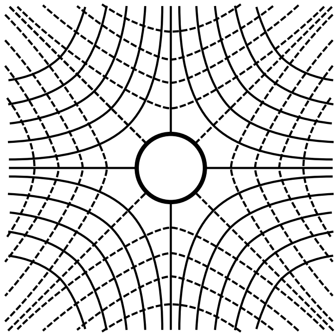

Figure 1. A shape of a singularity of degree 4 at the boundary.

See Figure 1 for a shape of the singularities of $(\mathcal{F}^s, \mu^s)$ and $(\mathcal{F}^u, \mu^u)$. An automorphism ϕ is said to be *pseudo-Anosov* if it has a pseudo-Anosov homeomorphism as a representative. We call the positive real number λ the *dilatation* of pseudo-Anosov automorphism ϕ and denote it by $\lambda(\phi)$.

In some cases, it is convenient to consider the restriction of automorphisms on the interior $\mathrm{Int}(F)$ of F. By considering $\phi|_{\mathrm{Int}(F)}$, we get a pseudo-Anosov automorphism on $\mathrm{Int}(F)$ and by abusing the notation we also denote it by ϕ. Note that $\mathrm{Int}(F)$ can be regarded as a surface with finitely many punctures, each corresponding to a boundary component of F. Then the singularities of $\mathcal{F}^s$ and $\mathcal{F}^u$ lie on $\mathrm{Int}(F)$ or the punctures. We denote the set of points and punctures that correspond to the singular points of associated singular foliations by $\mathrm{Sing}(\phi)$.

Uniqueness of the minimal element. In this subsection, we give a detailed discussion of Theorem 1.5 for the case where manifolds have boundary. By passing to a finite covering we may assume $\mathcal{F}$ to be co-orientable and hence M fibers over the circle; that is, M is the mapping torus $[F, \phi]$ of some surface F and pseudo-Anosov map ϕ. Since we are dealing with commensurability classes, it suffices to discuss the case where the foliations are co-orientable. The proof in [Calegari et al. 2011] assumes that all singular points of the singular foliations associated to ϕ lie on the interior of F. We prove this result for the case where some of the singular points lie on the boundary. This corresponds to the case where $\mathrm{Sing}(\phi)$ contains some punctures, by restricting the automorphism on the interior $\mathrm{Int}(F)$ of F.

Theorem 2.6 (see also [Calegari et al. 2011]). *Let $(M, \mathcal{F})$ be a hyperbolic co-orientable fibered pair and let (F, ϕ) be the pair associated to $(M, \mathcal{F})$. Then the commensurability class of $(M, \mathcal{F})$ contains a unique minimal (orbifold) element. Moreover, if $\mathrm{Int}(F) \cap \mathrm{Sing}(\phi) = \varnothing$, then the minimal element is a manifold.*

Proof. First, we recall the argument in [Calegari et al. 2011], since we will need it here. The stable and unstable singular foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ associated to ϕ

determine a unique singular Sol metric on $\mathrm{Int}(M)$. Pulling back this metric to the universal cover $\pi : \widetilde{M} \to \mathrm{Int}(M)$, $\widetilde{M}$ becomes a simply connected singular Sol manifold. Each fiber of $\widetilde{M}$ is a singular Euclidean plane. Let Λ be the full isometry group of the singular Sol metric. By appealing to the local Sol metric of $\widetilde{M}$, it can be verified that each element of Λ preserves the foliation by the singular Euclidean planes. Since $\pi_1(M) < \Lambda$, we have the covering $(\widetilde{M}, \widetilde{\mathscr{F}})/\pi_1(M) \to (\widetilde{M}, \widetilde{\mathscr{F}})/\Lambda$. We see that for any pair $(M', \mathscr{F}')$ commensurable with (M, F) the group $\pi_1(M')$ embeds into Λ and hence $(M', \mathscr{F}')$ covers $(\widetilde{M}, \widetilde{\mathscr{F}})/\Lambda$. Thus the theorem will be proved if we establish the following claim.

Claim 2.7. *Λ is discrete with respect to the compact open topology.*

For the proof of this claim, the condition $\mathrm{Sing}(\phi) \subset \mathrm{Int}(F)$ is assumed in [Calegari et al. 2011]. We prove this claim without the assumption. Note that if $\mathrm{Sing}(\phi) \not\subset \mathrm{Int}(F)$, the singular Sol metric is not necessarily complete. Let $\Lambda' < \Lambda$ be the subgroup consisting of isometries that preserve each fiber of $\widetilde{M}$ setwise. We first prove that the subgroup Λ' is discrete. Let S be a fiber of $\widetilde{M}$ and $\bar{S}$ be its completion with respect to the singular Euclidean metric. We will extend $p = \pi|_S : S \to \mathrm{Int}(F)$ to a local isometry $\bar{p} : \bar{S} \to \mathrm{Int}(F) \cup \mathrm{Sing}(\phi)$. Let $\{x_i\}$ be a Cauchy sequence in S. Then $\{p(x_i)\}$ is a Cauchy sequence in $\mathrm{Int}(F)$ and it converges to either an interior point of F or a point in $\mathrm{Sing}(\phi)$. Since $\bar{S}$ consists of the equivalence classes of Cauchy sequences in S, we can define $\bar{p} : [(x_i)] \mapsto \lim p(x_i)$. Since p is a local isometry, $\bar{p}$ is well defined and a local isometry. Therefore we get $E := \bar{S} \setminus S = \bar{p}^{-1}(\mathrm{Sing}(\phi))$ for the natural extension $\bar{p}$ of p. Any isometry $\varphi : S \to S$ extends to an isometry $\bar{\varphi} : \bar{S} \to \bar{S}$ and by construction we get $\bar{\varphi}(E) = E$. Suppose there is a sequence $\{\bar{\varphi}_i\}$ of isometries such that $\bar{\varphi}_i \to \mathrm{id}$. Since the distances between two distinct points in E are bounded from below by a positive constant, for large enough i, $\bar{\varphi}_i$ must fix E pointwise. Suppose that $\bar{\varphi} : \bar{S} \to \bar{S}$ is an isometry which preserves E pointwise. Since $\bar{S}$ is a singular Euclidean plane, we may find two points e_1, e_2 in E which can be joined by a unique geodesic γ. By appealing to the distance from e_1 and e_2, it follows that $\bar{\varphi}$ preserves γ pointwise. Note that every isometry on $\bar{S}$ leaves the set of leaves of $p^{-1}(\mathscr{F}^s)$ and $p^{-1}(\mathscr{F}^u)$ invariant. This implies that every leaf that intersects with γ is preserved by $\bar{\varphi}$. Let l be one of such leaves. Since $\bar{\varphi}$ is a local isometry of Sol metric, it locally acts as a translation on $\bar{S}$. Therefore $\bar{\varphi}$ fixes l pointwise. Since each leaf of $\mathscr{F}^s$ or $\mathscr{F}^u$ is dense in $\mathrm{Int}(F)$, the orbit of l under the action of the deck transformation group associated to p is also dense in $\bar{S}$. Hence $\bar{\varphi}$ is identity on a dense subset of $\bar{S}$ and since it is an isometry, we get $\bar{\varphi} = \mathrm{id}$. Therefore for large enough i, we get $\bar{\varphi}_i = \mathrm{id}$. This proves the discreteness of Λ'.

The discreteness of the dynamical direction of Λ follows from exactly the same argument in [Calegari et al. 2011]. We include the proof for completeness. Note that each isometry $\varphi \in \Lambda$ extends to the metric completion $\bar{M}$ of $\widetilde{M}$. We

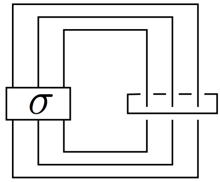

Figure 2. The fibered link associated to a braid $\sigma \in B_3$.

may parametrize each fiber by real numbers t in such a way that for any two fixed flow lines $a(t), b(t) \in E(t) \subset \bar{S}(t)$, the distance between $a(t)$ and $b(t)$ is $\sqrt{e^{2t}x^2 + e^{-2t}y^2}$ for some fixed x and y when $|t|$ is small enough. For small $|t|$ the distance between any two points in $E(t)$ are bounded from below by a constant which does not depend on t. Therefore since $\sqrt{e^{2t}x^2 + e^{-2t}y^2}$ is not a locally constant function, an isometry $\varphi \in \Lambda$ close enough to the identity must fix each fiber of the foliation by the singular Euclidean planes. Thus we see that Λ is discrete.

Since isometries may fix only singular points, if $\mathrm{Int}(F) \cap \mathrm{Sing}(\phi) = \varnothing$, then Λ has no fixed point in $\widetilde{M}$ and the last assertion holds. □

Corollary 2.8. *All the fibrations of $M_1 = S^3 \setminus 6_2^2$ and the magic 3-manifold M_2 are minimal elements.*

Proof. M_1 (resp. M_2) is homeomorphic to the complement of the fibered link associated to $\sigma_1\sigma_2^{-1} \in B_3$ (resp. $\sigma_1\sigma_2^{-1}\sigma_1 \in B_3$), where B_3 is the braid group on 3 strands (see Figure 2). It is well known that for every pseudo-Anosov element of B_3, all singularities are on the punctures. Therefore it suffices to prove that M_1 and M_2 are minimal *manifolds* (not orbifolds) with respect to usual covering relation. M_1 has volume $4V_0$, where $V_0 \approx 1.01\ldots$ is the volume of the ideal regular tetrahedron (see for example [Gehring et al. 1998]). By [Cao and Meyerhoff 2001], M_1 can only cover the figure-eight knot complements or its sister (m004 or m003 in SnapPea notation). However, SnapPy [Culler et al. 2013] can enumerate all double covers of m003 and m004 and none of them are homeomorphic to M_1. Similarly, the magic 3-manifold M_2 has volume $\approx 5.33\ldots$ and if it covers a manifold with degree 2, then its volume is $\approx 2.66\ldots$, which is less than the volume of the ideal regular octahedron ($\approx 3.66\ldots$). By [Agol 2010], such a manifold has only one cusp and cannot be doubly covered by M_2, which has 3 cusps. Moreover, since $\mathrm{Vol}(M_2)/3 \approx 1.77\cdots < 2V_0$, again by [Cao and Meyerhoff 2001], M_2 cannot cover any manifold with degree greater than 2. Now the result follows from the last assertion of Theorem 2.6. □

Remark 2.9. For a fixed surface, there exists a pseudo-Anosov automorphism with the smallest dilatation [Ivanov 1988]. It is interesting to compute the smallest dilatation for a given surface. Hironaka [2010] and Kin and Takasawa [2011] computed dilatations of the monodromy of each fiber of $S^3 \setminus 6_2^2$ and the magic 3-manifold respectively. It turns out that many small dilatation pseudo-Anosov automorphisms appear as the monodromies of fibrations of those manifolds. Corollary 2.8 shows that all such fibrations are minimal and hence their monodromies can be candidates for the smallest dilatation pseudo-Anosov maps.

Transitivity of commensurability in Definition 1.2. In this subsection, we discuss the subtle difference between fibered commensurability and commensurability in the sense of Definition 1.2. Here, two pairs of type (F, ϕ) are said to be *fibered commensurable* if associated fibered pairs are commensurable. It is easy to see that if two pairs (F_1, ϕ_1) and (F_2, ϕ_2) are fibered commensurable, they are commensurable in the sense of following definition.

Definition 2.10 [Carlson 2010]. Two pairs (F_1, ϕ_1) and (F_2, ϕ_2) are *commensurable* if there is a surface $\widetilde{F}$, an automorphism $\widetilde{\phi}$, and nonzero integers k_1 and k_2, so that $(\widetilde{F}, \widetilde{\phi})$ covers $(F_i, \phi_i^{k_i})$ for $i = 1, 2$.

In [Calegari et al. 2011], it is claimed without proof that two pairs (F_1, ϕ_1) and (F_2, ϕ_2) are fibered commensurable if and only if they are commensurable in the sense of Definition 1.2. Since a map cannot always be lifted even if a power of it can be lifted, the claim is not trivial. The claim would follow from the transitivity of commensurability in the sense of Definition 1.2, because taking powers of an automorphism is tantamount to a covering. In this subsection, we will prove that the transitivity of commensurability in Definition 1.2 is valid if the automorphisms are pseudo-Anosov.

Proposition 2.11. *Suppose that (F_i, ϕ_i) and (F_{i+1}, ϕ_{i+1}) are commensurable in the sense of Definition 1.2 for $i = 1, 2$. Suppose further that ϕ_i are pseudo-Anosov for $i = 1, 2, 3$. Then there exists a pair $(F_{123}, \widetilde{\phi}_i)$ that covers (F_i, ϕ_i) for each $i = 1, 2, 3$ such that $\widetilde{\phi}_1^{k_1} = \widetilde{\phi}_2^{k_2} = \widetilde{\phi}_3^{k_3}$ for some $k_1, k_2, k_3 \in \mathbb{Z} \setminus \{0\}$. In particular, commensurability in the sense of Definition 1.2 is transitive.*

Proof. In Theorem 2.6 we proved that each hyperbolic fibered commensurability class contains a unique minimal element. Let $M = [F_1, \phi_1]$. Recall that Λ is the group of isometries of the singular Sol metric on the universal cover $\widetilde{M}$ (see the proof of Theorem 2.6). By considering the subgroup Λ^+ that consists of isometries which preserve the orientation of $\widetilde{M}$ and the orientation of the leaf space of $\widetilde{M}$. By taking $M_{\min}^+ := \widetilde{M} / \Lambda^+$, we get a unique minimal element among all commensurable fibered pairs both orientable and co-orientable. Although there is a natural extension of this proof in the case where $\widetilde{M} / \Gamma^+$ is an orbifold, such a proof would require more terminology and could obfuscate the key ideas of the proof. Therefore, we only present the case where $\widetilde{M} / \Gamma^+$ is a manifold. In this case we get an associated pair $(F_{\min}, \phi_{\min})$ since $M_{\min}^+$ is orientable and co-orientable. Each

(F_i, ϕ_i) covers $(F_{\min}, \phi_{\min}^{l_i})$ for some $l_i \in \mathbb{Z}\setminus\{0\}$ $(i = 1, 2, 3)$. Note that $\phi_{\min}$ is not always lifted to F_i. Let $H_i < \pi_1(F_{\min})$ be a subgroup which is the image of $\pi_1(F_i)$ by the covering map for each $i = 1, 2, 3$. Further let $d = [\pi_1(F_{\min}) : H_1 \cap H_2 \cap H_3]$, and take $H_{123} := \bigcap\{H < \pi_1(F_{\min}) \mid [\pi_1(F_{\min}) : H] = d\}$. Recall that for a group G, a subgroup $H < G$ is called *characteristic* if for every isomorphism $f : G \to G$, we get $f(H) = H$. H_{123} is a characteristic subgroup and hence every homeomorphism on $F_{\min}$ lifts to the covering F_{123} that corresponds to $H_{123} < \pi_1(F_{\min})$. Since each $\phi_i : F_i \to F_i$ is a lift of $\phi_{\min}^{l_i}$, it can be lifted to $\widetilde{\phi}_i : F_{123} \to F_{123}$. Let l be the least common multiple of l_i's, then by putting $k_i = l/l_i$, we get $\widetilde{\phi}_1^{k_1} = \widetilde{\phi}_2^{k_2} = \widetilde{\phi}_3^{k_3}$ on F_{123}. □

Remark 2.12. We do not know if the transitivity or the equivalence of fibered commensurability and commensurability in the sense of Definition 1.2 holds for the case where the automorphisms are periodic or reducible.

3. Thurston norm and normalized entropy

Thurston norm. Let M be a fibered hyperbolic 3-manifold. In this subsection we recall briefly the Thurston norm on $H^1(M;\mathbb{R})$ and discuss the relationship between fibered commensurability of fibrations on a fixed manifold M and the normalized entropy. For more details about the Thurston norm, see [Thurston 1986; Kapovich 2001; Kin and Takasawa 2011]. For any (possibly disconnected) compact surface $F = F_1 \sqcup F_2 \sqcup \cdots \sqcup F_n$, let $\chi_-(F)$ be the sum of the absolute values of Euler characteristics $|\chi(F_i)|$ of components with negative Euler characteristics. For a given $\omega \in H^1(M;\mathbb{Z}) \subset H^1(M;\mathbb{R})$, we define $\|\omega\|$ to be

$$\min\bigl\{\chi_-(F) \mid F \text{ is an embedded orientable surface } (F,\partial F) \subset (M,\partial M), \text{ and } [F] \in H_2(M,\partial M;\mathbb{Z}) \text{ is the Poincaré dual of } \omega \in H^1(M;\mathbb{Z})\bigr\}.$$

If F realizes the minimum, we call F a minimal representative of ω. We can extend this norm to $H^1(M;\mathbb{Q})$ by $\|\omega\| = \|r\omega\|/r$. It turns out that $\|\cdot\|$ extends continuously to $H^1(M;\mathbb{R})$. Further, this $\|\cdot\|$ turns out to be seminorm on $H^1(M;\mathbb{R})$ and the unit ball $U = \{\omega \in H^1(M;\mathbb{R}) \mid \|\omega\| \leq 1\}$ is a compact convex polygon [Thurston 1986]. The seminorm $\|\cdot\|$ is called the Thurston norm on $H^1(M;\mathbb{R})$. We need some more terminologies to explain the relationship between $\|\cdot\|$ and fibrations on M. We denote

- the cone over a top-dimensional face Δ of the unit ball U by C_Δ,
- the set of integral classes on $\mathrm{Int}(C_\Delta)$ by $\mathrm{Int}(C_\Delta(\mathbb{Z}))$, and
- the set of rational classes on a top-dimensional face Δ by $\Delta(\mathbb{Q})$.

Theorem 3.1 [Thurston 1986]. *Let M be a fibered hyperbolic 3-manifold and F the fiber. Then there is a top-dimensional face Δ of U such that*

- *the dual of* $[F] \in H_2(M, \partial M; \mathbb{Z})$ *belongs to* $\operatorname{Int}(C_\Delta(\mathbb{Z}))$, *and*
- *for every primitive class* ω *in* $\operatorname{Int}(C_\Delta(\mathbb{Z}))$, *a minimal representative of* ω *is the fiber of a fibration on* M.

We call the face Δ in Theorem 3.1 a *fibered face* and the cone over a fibered face a *fibered cone*.

As a corollary, we see that if the first Betti number $b_1(M) > 1$ and M is fibered, then M has infinitely many distinct fibrations. We will discuss fibered commensurability of fibrations of a hyperbolic fibered 3-manifold.

Normalized entropy. The normalized entropy is shared by commensurable fibrations on a fixed hyperbolic 3-manifold.

Proposition 3.2. *Suppose that* $[F_1, \phi_1] = [F_2, \phi_2]$ *and their interior admit hyperbolic metrics. If* (F_1, ϕ_1) *is commensurable to* (F_2, ϕ_2), *then*

$$\chi(F_1) \log \lambda(\phi_1) = \chi(F_2) \log \lambda(\phi_2).$$

Proof. There are pairs $(\widetilde{F}, \widetilde{\phi}_i)$ that cover (F_i, ϕ_i) and $k_i \in \mathbb{Z} \setminus \{0\}$ for $i = 1, 2$ such that $\widetilde{\phi}_1^{k_1} = \widetilde{\phi}_2^{k_2}$. Then the mapping torus $[\widetilde{F}, \widetilde{\phi}_i^{k_i}]$ covers $[F_i, \phi_i]$ and the degree of this cover is $k_i \chi(\widetilde{F})/\chi(F_i)$. Since $[F_1, \phi_1] = [F_2, \phi_2]$, we get $k_1/\chi(F_1) = k_2/\chi(F_2)$. Since $\lambda(\phi) = \lambda(\widetilde{\phi})$,

$$\chi(\widetilde{F}) \log \lambda(\widetilde{\phi}_i^{k_i}) = \frac{\chi(\widetilde{F})}{\chi(F_1)} \chi(F_1) k_1 \log \lambda(\phi_1) = \frac{\chi(\widetilde{F})}{\chi(F_2)} \chi(F_2) k_2 \log \lambda(\phi_2).$$

Putting them all together, we get $\chi(F_1) \log \lambda(\phi_1) = \chi(F_2) \log \lambda(\phi_2)$. □

Each primitive integral class in $C_\Delta(\mathbb{Z})$ corresponds to a rational class in $\operatorname{Int}(\Delta)$. The normalized entropy defines a function $\operatorname{ent} : \Delta(\mathbb{Q}) \to \mathbb{R}$. In [Fried 1982], the function $1/\operatorname{ent}$ is shown to be concave and therefore it extends to $\operatorname{Int}(\Delta)$. Moreover:

Theorem 3.3 [McMullen 2000]. *The function* $1/\operatorname{ent} : \operatorname{Int}(\Delta) \to \mathbb{R}$ *is strictly concave.*

In Example 3.12 of [Calegari et al. 2011], it is remarked that some fibrations on $S^3 \setminus 6_2^2$ are not commensurable. In Corollary 2.8, it is proved that all fibrations on $S^3 \setminus 6_2^2$ are minimal elements and since each minimal element is unique, we see that two fibrations of $S^3 \setminus 6_2^2$ are either symmetric or noncommensurable. Here, we give an alternative proof of this fact in terms of the normalized entropy. In [Hironaka 2010; McMullen 2000], the unit ball of the Thurston norm on $H^1(S^3 \setminus 6_2^2)$ is computed to be a square. Further, the symmetries of the square all come from the symmetries of the manifold (see Example 4.5 for more details about the symmetries of $S^3 \setminus 6_2^2$). Therefore the function $1/\operatorname{ent}$ is invariant under the action of the symmetries of the unit ball. Since $1/\operatorname{ent}$ is *strictly* concave, this proves that any two fibrations that correspond to distinct elements in $H^1(M; \mathbb{Z})$ are either symmetric or noncommensurable. In other words, the normalized entropy determines the commensurability class of a fibration on $S^3 \setminus 6_2^2$ up to symmetry.

On the other hand, in [Kin et al. 2012, §2], it is observed that for the magic 3-manifold N there are rational points on a fibered face which share the same normalized entropy but which are not symmetric to each other. However, again by Corollary 2.8, we also see that any two distinct fibrations of N are either symmetric or noncommensurable. Hence for the magic 3-manifold, the commensurability classes of fibrations are not determined by the normalized entropies. We do not know for what kind of hyperbolic 3-manifolds the commensurability classes of fibrations on the same hyperbolic 3-manifold are determined by the normalized entropy up to symmetry.

4. Commensurability of fibrations on a hyperbolic 3-manifold

In this section we prove Theorems 1.4 and 1.3.

Manifolds without hidden symmetries. We start with some definitions. A *Kleinian group* is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$. Two Kleinian groups Γ_1 and Γ_2 are said to be *commensurable* if $\Gamma_1 \cap \Gamma_2$ is a finite-index subgroup of both Γ_1 and Γ_2. Let Γ be a Kleinian group. The *commensurator* $C^+(\Gamma)$ of Γ is

$$C^+(\Gamma) = \{h \in \mathrm{PSL}(2, \mathbb{C}) \mid \Gamma \text{ and } h\Gamma h^{-1} \text{ are commensurable}\},$$

and the *normalizer* $N^+(\Gamma)$ is

$$N^+(\Gamma) = \{h \in \mathrm{PSL}(2, \mathbb{C}) \mid \Gamma = h\Gamma h^{-1}\}.$$

Note that $N^+(\Gamma) < C^+(\Gamma)$.

Let M be a hyperbolic 3-manifold and $\rho : \pi_1(M) \to \Gamma < \mathrm{PSL}(2, \mathbb{C})$ a holonomy representation of $\pi_1(M)$. By the Mostow–Prasad rigidity theorem, any self-homeomorphism $\varphi : M \to M$ corresponds to a conjugation of Γ. Therefore we get $N(\Gamma)/\Gamma \cong \mathrm{Isom}(M)$, where $\mathrm{Isom}(M)$ is the group of self-homeomorphisms of M. If $C^+(\Gamma) \setminus N^+(\Gamma) \neq \varnothing$, each nontrivial element $h \in C^+(\Gamma) \setminus N^+(\Gamma)$ is said to be a *hidden symmetry*. Then M is said to have no hidden symmetries if Γ has no hidden symmetries. Note that by the Mostow–Prasad rigidity theorem, the holonomy representations of $\pi_1(M)$ are related by a conjugation. Hence the definition does not depend on the choice of a holonomy representation.

Proof of Theorem 1.4. Let $(M, \mathcal{F}_1)$ and $(M, \mathcal{F}_2)$ be commensurable fibered pairs that correspond to two distinct fibrations on M. By Theorem 2.6 we have a unique minimal element $(N, \mathcal{G})$ in the commensurability class. Let $\rho : \pi_1(N) \to \mathrm{PSL}(2, \mathbb{C})$ be a holonomy representation and $\Gamma := \rho(\pi_1(N))$. Since $(M, \mathcal{F}_1)$ and $(M, \mathcal{F}_2)$ cover $(N, \mathcal{G})$, there are two corresponding coverings $p_1, p_2 : M \to N$.

Let $\Gamma_i = \rho p_{i*}(\pi_1(M))$ for $i = 1, 2$. By the Mostow–Prasad rigidity theorem, there is $h \in \mathrm{PSL}(2, \mathbb{C})$ such that $h\Gamma_1 h^{-1} = \Gamma_2$. Further, since $\Gamma_2 < \Gamma \cap h\Gamma h^{-1}$, $h \in C^+(\Gamma) = C^+(\Gamma_1) = N^+(\Gamma_1)$. The last equality holds since M has no hidden symmetries. It follows that $\Gamma_1 = \Gamma_2$ and hence there exists a homeomorphism $\varphi : M \to M$ such that $p_1\varphi = p_2$. Therefore ω_1 and ω_2 are symmetric. □

Remark 4.1. Hyperbolic 3-manifolds with hidden symmetries are "rare" among all nonarithmetic hyperbolic 3-manifolds (see for example, [Goodman et al. 2008]). Hence we may expect that "most" hyperbolic 3-manifolds have no hidden symmetries and therefore have no nonsymmetric but commensurable fibration.

Remark 4.2. As mentioned above, there are no nonsymmetric but commensurable fibrations on $S^3 \setminus 6_2^2$ and the magic 3-manifold. However, $S^3 \setminus 6_2^2$ and the magic 3-manifold are arithmetic and by a result of Margulis [1991], they have lots of hidden symmetries. Therefore even though a manifold has hidden symmetries, it might not have any nonsymmetric but commensurable fibrations.

Nonsymmetric and commensurable fibrations. We now prove Theorem 1.3 by constructing examples of manifolds that have nonsymmetric but commensurable fibrations.

Lemma 4.3. *Let M be a fibered hyperbolic 3-manifold. Suppose two primitive elements $\omega_1 \neq \pm\omega_2 \in H^1(M; \mathbb{Z})$ correspond to fibrations with the fibers and the monodromies (F_1, ϕ_1) and (F_2, ϕ_2) respectively. We suppose further $(F_1, \phi_1) = (F_2, \phi_2)$ (that is, conjugate to each other). Then, for all large enough $n \in \mathbb{N}$, there exists a degree n covering space $p_n : M_n \to M$ such that $p_n^*(\omega_1)$ and $p_n^*(\omega_2)$ correspond to commensurable but nonsymmetric fibrations.*

Proof. Note that by the universal coefficient theorem, we have

$$H^1(M; \mathbb{Z}) \cong \mathrm{Hom}(H_1(M)/\mathrm{Tor}, \mathbb{Z}),$$

where Tor is the torsion part. This isomorphism is determined by a choice of a basis of $H_1(M; \mathbb{Z})/\mathrm{Tor}$. Let $A_i = \mathrm{ab}(\pi_1(F_i))/\mathrm{Tor}$, where $\mathrm{ab} : \pi_1(M) \to H_1(M)$ is the abelianization and $\pi_1(F_i) \hookrightarrow \pi_1(M)$ is an injection induced by the fiber bundle structure of M associated to (F_i, ϕ_i) for $i = 1, 2$. The fiber bundle structure of M gives the exact sequence

$$0 \to \pi_1(F_i) \to \pi_1(M) \xrightarrow{\rho_i} \pi_1(S^1) \cong \mathbb{Z} \to 0.$$

The map ρ_i factors through the abelianization since $\pi_1(S^1) \cong \mathbb{Z}$ is abelian. Hence we get $A_i = \mathrm{Ker}(\omega_i) \cong \mathbb{Z}^{b-1}$, where b is the first Betti number of M. We consider the dynamical covering $p_n : M_n \to M$ of degree n with respect to ω_1 (that is, the covering corresponding to (F_1, ϕ_1^n)). This is the covering corresponding to the

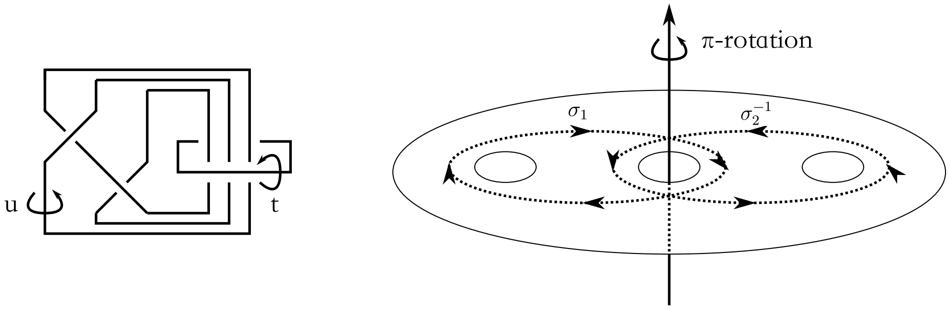

Figure 3. An involution map h on the 4-holed sphere.

surjective map

$$\pi_1(M) \xrightarrow{\text{ab}} H_1(M) \xrightarrow{\omega_1} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}.$$

For sufficiently large n, there exists $a \in A_2$ such that a maps to a nonzero element by the above surjective map. This means that each component of $p_n^{-1}(F_2)$ is not homeomorphic to F_2. $\square$

Example 4.4. The 3-manifold $S^3 \setminus 6_2^2$ and the magic manifold have symmetries that permute cusps, and therefore they do have two distinct elements in their first cohomology with homeomorphic fibers and conjugate monodromies.

Example 4.5. In this example we observe that $M := S^3 \setminus 6_2^2$ has two symmetric fibrations in the same fibered cone in $H^1(M)$. Although this fact can be checked by computing the symmetry group by SnapPy [Culler et al. 2013], we give a geometric proof. The first half of the following argument is due to Eriko Hironaka, see also [Hironaka 2010].

Let u, t be the generators of $H_1(M, \mathbb{Z})$ that correspond to the meridians of 6_2^2 (see the left picture of Figure 3). Let $U, T \in H^1(M; \mathbb{Z})$ be the dual of u, t respectively. Then U corresponds to the fibration of M with monodromy f that corresponds to $\sigma_1\sigma_2^{-1} \in B_3$. Let h be a π-rotation, which is depicted in Figure 3. We can see that $f^{-1} = \sigma_2\sigma_1^{-1} = hfh$, that is f and f^{-1} are conjugate to each other. Then we take the mirror image of 6_2^2. By isotopy and above conjugacy, we see that 6_2^2 is amphicheiral. The induced map on $H^1(M; \mathbb{Z})$ of the symmetry on M that gives amphicheirality satisfies $U \mapsto -U$ and $T \mapsto T$. This symmetry preserves the fibered face $\Delta := \{aU + bT \mid -1 < a < 1, b = 1\}$. By this symmetry, we see that fibrations on the cone C_Δ over Δ of the form $nU + mT$ and $-nU + mT$ $(n, m \in \mathbb{Z})$ are symmetric.

Proof of Theorem 1.3. Putting Lemma 4.3 and Example 4.5 together, we have a proof. $\square$

Acknowledgments

I would like to thank Sadayoshi Kojima and members of his research group for helpful conversations and advice. I would also like to thank Eriko Hironaka for her inspiring series of lectures about pseudo-Anosov maps at Tokyo Tech and for telling me the argument in Example 4.5. I would also like to thank Neil Hoffman, Tamás Kálmán, and Hongbin Sun for catching some errors in an earlier version of this paper. Finally I would like to thank the anonymous referee for helpful suggestions.

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Received October 24, 2012. Revised July 2, 2013.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 266, No. 2, 2013
dx.doi.org/10.2140/pjm.2013.266.329

MULTIPLICATIVE DIRAC STRUCTURES

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We introduce multiplicative Dirac structures on Lie groupoids, providing a unified framework to study both multiplicative Poisson bivectors (Poisson groupoids) and multiplicative closed 2-forms such as symplectic groupoids. We prove that for every source simply connected Lie groupoid G with Lie algebroid AG, there exists a one-to-one correspondence between multiplicative Dirac structures on G and Dirac structures on AG that are compatible with both the linear and algebroid structures of AG. We explain in what sense this extends the integration of Lie bialgebroids to Poisson groupoids and the integration of Dirac manifolds. We explain the connection between multiplicative Dirac structures and higher geometric structures such as $\mathscr{LA}$-groupoids and $\mathscr{CA}$-groupoids.

1. Introduction

Dirac structures were introduced by Courant and Weinstein [1988] as a common generalization of Poisson bivectors, closed 2-forms and regular foliations. A *Dirac structure* on a smooth manifold M consists of a vector subbundle $L \subseteq \mathbb{T}M := TM \oplus T^*M$ that is maximal isotropic with respect to the nondegenerate symmetric pairing on $\mathbb{T}M$,

$$\langle (X,\alpha), (Y,\beta)\rangle = \alpha(Y) + \beta(X),$$

and that satisfies the integrability condition

$$[\![\Gamma(L), \Gamma(L)]\!] \subseteq \Gamma(L),$$

MSC2010: 53D17, 53D18.
Keywords: Lie groupoids, Lie algebroids, multiplicative Dirac structures.

with respect to the Courant bracket $[\![\cdot,\cdot]\!]:\Gamma(\mathbb{T}M)\times\Gamma(\mathbb{T}M)\to\Gamma(\mathbb{T}M)$,

$$[\![(X,\alpha),(Y,\beta)]\!]=([X,Y],\mathscr{L}_X\beta-i_Y d\alpha).$$

The integrability in the sense of Courant unifies different integrability conditions, including closed 2-forms, Poisson bivectors and regular foliations (see [Courant 1990b; Courant and Weinstein 1988]). More precisely, a 2-form ω on a smooth manifold M induces a bundle map $\omega^\sharp : TM\to T^*M$, $X\mapsto\omega(X,\cdot)$ whose graph $L_\omega=\{(X,\omega^\sharp(X)\mid X\in TM)\}$ is a Lagrangian subbundle of $\mathbb{T}M$. In this case, the Courant integrability of L_ω is equivalent to ω being a closed 2-form. Similarly, any bivector π on M defines a bundle map $\pi^\sharp : T^*M\to TM$, $\alpha\mapsto\pi(\alpha,\cdot)$ whose graph $L_\pi=\{(\pi^\sharp(\alpha),\alpha)\}$ is a Lagrangian subbundle of $\mathbb{T}M$. One checks that L_π satisfies the Courant integrability condition if and only if π is a Poisson bivector. Also, if $F\subseteq TM$ is a regular subbundle we denote by $F^\circ\subseteq T^*M$ its annihilator. Then the Lagrangian subbundle $F\oplus F^\circ\subseteq\mathbb{T}M$ is integrable in the sense of Courant if and only if $F\subseteq TM$ is involutive with respect to the Lie bracket of vector fields.

The main objective of this paper is to study Dirac structures defined on Lie groupoids, satisfying a suitable compatibility condition with the groupoid multiplication. Our study is motivated by a variety of geometrical structures compatible with group or groupoid structures, including:

(i) *Poisson–Lie groups*: A Poisson–Lie group is a Lie group G with a Poisson structure π that is compatible in the sense that the multiplication map $m:G\times G\to G$ is a Poisson map. Equivalently, the Poisson bivector π is *multiplicative*, that is,

$$\pi_{gh}=(l_g)_*\pi_h+(r_h)_*\pi_g,$$

for every $g,h\in G$. Here l_g and r_h denote the left and right multiplication by g and h, respectively. Poisson–Lie groups arise as semiclassical limit of quantum groups, and they are infinitesimally described by *Lie bialgebras*. See for example, [Drinfel'd 1983].

(ii) *Symplectic groupoids*: A symplectic groupoid is a Lie groupoid G with a symplectic structure ω that is compatible with the groupoid multiplication in the sense that the graph

$$\operatorname{Graph}(m)\subseteq G\times G\times\overline{G}$$

is a Lagrangian submanifold with respect to the symplectic structure $\omega\oplus\omega\ominus\omega$. This compatibility condition is equivalent to saying that ω is *multiplicative*, that is,

$$m^*\omega=pr_1^*\omega+pr_2^*\omega,$$

where $pr_1, pr_2 : G_{(2)}\to G$ are the canonical projections and $G_{(2)}\subseteq G\times G$ is the set of composable groupoid pairs. Symplectic groupoids arise in the context of quantization of Poisson manifolds [Weinstein 1987; Weinstein and Xu 1991], connecting Poisson geometry to noncommutative geometry. In [Cattaneo and Felder 2001], symplectic groupoids appeared as phase spaces of certain sigma models. The infinitesimal description of symplectic groupoids is given by *Poisson structures*, see for example, [Weinstein 1987; Coste et al. 1987].

(iii) *Poisson groupoids*: These objects were introduced by A. Weinstein [1988] unifying Poisson–Lie groups and symplectic groupoids. A Poisson groupoid is a Lie groupoid G equipped with a Poisson structure π that is compatible with the groupoid multiplication in the sense that

$$\mathrm{Graph}(m) \subseteq G \times G \times \overline{G}$$

is a coisotropic submanifold. These structures are related to the geometry of the classical dynamical Yang–Baxter equation, see for instance [Etingof and Varchenko 1998]. At the infinitesimal level, Poisson groupoids are described by *Lie bialgebroids* [Mackenzie and Xu 1994].

(iv) *Presymplectic groupoids*: Lie groupoids equipped with a multiplicative closed 2-form were studied in [Bursztyn et al. 2004]. A presymplectic groupoid is a Lie groupoid G with a multiplicative closed 2-form ω satisfying suitable nondegeneracy conditions. These objects arise in connection with equivariant cohomology and generalized moment maps [Bursztyn and Crainic 2009]. The infinitesimal description of presymplectic groupoids is given by *Dirac structures*, extending the infinitesimal description of symplectic groupoids. More generally, Lie groupoids endowed with arbitrary multiplicative closed 2-forms are infinitesimally described by bundle maps $\sigma : AG \to T^*M$ called *IM-2-forms*. Here AG denotes the Lie algebroid of G, and T^*M is the cotangent bundle of the base of G.

The first goal of this work is to find a suitable definition of multiplicative Dirac structure that includes both multiplicative Poisson bivectors and multiplicative closed 2-forms, and hence encompasses all examples above. This is obtained by observing that given a Lie groupoid G over M with Lie algebroid AG, the tangent bundle TG and the cotangent bundle T^*G inherit natural Lie groupoid structures over TM and A^*G, respectively. One observes that a bivector π_G is multiplicative if and only if the bundle map $\pi_G^\sharp : T^*G \to TG$ is a groupoid morphism [Mackenzie and Xu 1994]. Similarly, a 2-form ω_G is multiplicative if and only if the bundle map $\omega_G^\sharp : TG \to T^*G$ is a morphism of Lie groupoids. It turns out that the direct sum vector bundle $TG \oplus T^*G$ is a Lie groupoid over $TM \oplus A^*G$, and graphs of both multiplicative Poisson bivectors and multiplicative closed 2-forms define Lie subgroupoids of $TG \oplus T^*G$. We say that a Dirac structure L_G on a Lie groupoid G is *multiplicative* if $L_G \subseteq TG \oplus T^*G$ is a Lie subgroupoid. A Lie groupoid G equipped with a multiplicative Dirac structure is referred to as a *Dirac groupoid*.

Our main purpose is to describe multiplicative Dirac structures infinitesimally, that is, in terms of Lie algebroid data. This work can be considered as a first step toward such a description. The main result of the present work says that for every source simply connected Lie groupoid G with Lie algebroid AG, multiplicative Dirac structures on G correspond to Dirac structures on AG suitably compatible with both the linear and Lie algebroid structures on AG. In the particular case of multiplicative Poisson bivectors and multiplicative 2-forms, we explain how this is equivalent to the known infinitesimal descriptions carried out in [Mackenzie and Xu 2000] and [Bursztyn et al. 2004], respectively. Along the way, we develop techniques that can treat all multiplicative structures above in a unified manner, often simplifying existing results and proofs.

The present paper is organized as follows. In Section 2 we recall the definition of tangent and cotangent structures including tangent and cotangent groupoids and their algebroids, that is, tangent and cotangent algebroids. We also give an intrinsic construction of the tangent lift of a Dirac structure, providing an alternative proof of the results shown in [Courant 1990a]. In Section 3 we define the main objects of our study, multiplicative Dirac structures. We discuss a variety of examples arising in nature, including foliated groupoids, Dirac Lie groups, tangent lifts of multiplicative Dirac structures, symmetries of multiplicative Dirac structures (for example, reduction of Poisson groupoids), B-field transformations of multiplicative Dirac structures and generalized complex groupoids. In Section 4 we introduce the notion of Dirac algebroid and also several examples are discussed, including foliated algebroids, Dirac Lie algebras, tangent lifts of Dirac algebroids, symmetries of Dirac algebroids (for example, reduction of Lie bialgebroids), B-field transformations of Dirac algebroids and generalized complex algebroids. In Section 5 we explain how the multiplicativity of a Dirac structure is reflected at the Lie algebroid level, proving the main result of this work, which says that if G is a source simply connected Lie groupoid with Lie algebroid AG, then there is a one-to-one correspondence between Dirac groupoid structures on G and Dirac algebroid structures on AG. Along the way, we explain the relation between multiplicative Dirac structures and higher structures such as $\mathscr{C}\mathscr{A}$-groupoids and $\mathscr{L}\mathscr{A}$-groupoids. We also relate the examples of Section 3 with the examples of Section 4, in the spirit of the correspondence established by the main result of the paper. In Section 6, we discuss conclusions and work in progress.

1A. ***Notation and conventions.*** For a Lie groupoid G over M we denote by $s, t : G \to M$ the source and target maps, respectively. The multiplication map is denoted by $m : G_{(2)} \to G$, where $G_{(2)} = \{(g, h) \in G \times G \mid s(g) = t(h)\}$ is the set of composable pairs. The Lie algebroid of G is defined by $AG := \operatorname{Ker}(Ts)|_M$, with Lie bracket given by identifying sections of AG with right-invariant vector fields on G and anchor map $\rho_{AG} := Tt|_{AG} : AG \to TM$. Given a Lie groupoid morphism $\Psi : G_1 \to G_2$, the corresponding Lie algebroid morphism is denoted by $A(\Psi) : AG_1 \to AG_2$. Arbitrary Lie algebroids are denoted by $A \to M$ with Lie bracket $[\cdot, \cdot]_A$ and anchor map ρ_A. Also, given a smooth manifold M, the tangent bundle is denoted by $p_M : TM \to M$ and the cotangent bundle is denoted by $c_M : T^*M \to M$.

2. Tangent and cotangent structures

2A. ***Tangent and cotangent groupoids.*** Let G be a Lie groupoid over M with Lie algebroid AG. The tangent bundle TG has a natural Lie groupoid structure over TM. This structure is obtained by applying the tangent functor to each of the structure maps defining G (source, target, multiplication, inversion and identity section). We refer to TG with this groupoid structure over TM as the *tangent groupoid* of G. The set of composable pairs of TG is $(TG)_{(2)} = T(G_{(2)})$, and for $(g, h) \in G_{(2)}$ and a tangent groupoid pair $(X_g, Y_h) \in (TG)_{(2)}$ the multiplication map on TG is

$$X_g \bullet Y_h := Tm(X_g, Y_h).$$

Now consider the cotangent bundle T^*G. It was shown in [Coste et al. 1987] that T^*G is a Lie groupoid over A^*G. The source and target maps are defined by

$$\tilde{s}(\alpha_g)u = \alpha_g\big(Tl_g(u - Tt(u))\big) \quad \text{and} \quad \tilde{t}(\beta_g)v = \beta_g(Tr_g(v)),$$

where $\alpha_g \in T^*_gG$, $u \in A_{s(g)}G$ and $\beta_g \in T^*_gG$, $v \in A_{t(g)}G$. The multiplication on T^*G is defined by

$$(\alpha_g \circ \beta_h)(X_g \bullet Y_h) = \alpha_g(X_g) + \beta_h(Y_h)$$

for $(X_g, Y_h) \in T_{(g,h)}G_{(2)}$.

We refer to T^*G with the groupoid structure over A^*G as the *cotangent groupoid* of G.

2B. ***Tangent and cotangent algebroids.*** Let $q_A : A \to M$ be a *vector bundle* over M. The tangent bundle TA has a natural structure of a *double vector bundle* [Pradines 1974], given by the diagram below.

(1)
$$\begin{tikzcd} TA \arrow[r, "Tq_A"] \arrow[d, "p_A"'] & TM \arrow[d, "p_M"] \\ A \arrow[r, "q_A"] & M \end{tikzcd}$$

Assume now that $q_A : A \to M$ has a Lie algebroid structure with anchor map $\rho_A : A \to TM$ and Lie bracket $[\cdot, \cdot]$ on $\Gamma_M(A)$.

As explained in [Mackenzie 2005], there is a canonical Lie algebroid structure on the vector bundle $Tq_A : TA \to TM$. Recall that there exists a *canonical involution* $J_M : TTM \to TTM$, which is a morphism of double vector bundles. In a local coordinates system $(x^i, \dot{x}^i, \delta x^i, \delta\dot{x}^i)$ on TTM this map is given by

$$J_M((x^i, \dot{x}^i, \delta x^i, \delta\dot{x}^i)) = (x^i, \delta x^i, \dot{x}^i, \delta\dot{x}^i).$$

Notice that the map J_M yields a vector bundle isomorphism as below.

$$\begin{tikzcd} TTM \arrow[r, "J_M"] \arrow[d, "Tp_M"'] & TTM \arrow[d, "p_{TM}"] \\ TM \arrow[r, "\mathrm{Id}"] & TM \end{tikzcd} \tag{2}$$

Now we can apply the tangent functor to the anchor map $\rho_A : A \to TM$, yielding a bundle map $T\rho_A : TA \to TTM$, where TTM is equipped with bundle projection $Tp_M : TTM \to TM$. Therefore, composing $T\rho_A$ with the canonical involution J_M we obtain the bundle map $\rho_{TA} : TA \to TTM$, defined by

$$\rho_{TA} := J_M \circ T\rho_A,$$

which is a vector bundle morphism from $TA \to TTM$, where the target bundle is the one corresponding to the usual bundle projection $p_{TM} : TTM \to TM$. The map $\rho_{TA} : TA \to TTM$, as above, defines the tangent anchor map. In order to define the tangent Lie bracket, we observe that every section $u \in \Gamma_M(A)$ induces two types of sections of $Tq_A : TA \to TM$. The first type corresponds to the *linear section* $Tu : TM \to TA$, which is given by applying the tangent functor to the section $u : M \to A$. The second type of section is the *core* section $\hat{u} : TM \to TA$, which is defined by

$$\hat{u}(X) = T(0^A)(X) +_A \overline{u(p_M(X))},$$

where $0^A : M \to A$ denotes the zero section, and $\overline{u(p_M(X))} = d/dt(tu(p_M(X)))|_{t=0}$. As observed in [Mackenzie and Xu 1994], sections of the form Tu and $\hat{u}$ generate the module of sections $\Gamma_{TM}(TA)$. Therefore, the tangent Lie bracket is determined by

$$[Tu, Tv] = T[u, v], \quad [Tu, \hat{v}] = \widehat{[u, v]}, \quad [\hat{u}, \hat{v}] = 0,$$

and we extend to other sections by requiring the Leibniz rule with respect to the tangent anchor ρ_{TA}. This defines the natural Lie algebroid structure on $Tq_A : TA \to TM$.

Example 2.1. Assume that $A = \mathfrak{g}$ is a Lie *algebra*, that is, a Lie algebroid over a point. In this case, the tangent Lie *algebra* structure on $T\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}$ is given by the semidirect product Lie algebra determined by the adjoint representation of $\mathfrak{g}$ on itself.

Following [Mackenzie 2005], the cotangent bundle of a Lie algebroid inherits a Lie algebroid structure. For that, let us explain the vector bundle structure $T^*A \to A^*$. If (x^i, u^a) are local coordinates on A, we induce a local coordinates system $(x^i, u^a, p_i, \lambda_a)$ on T^*A, where (p_i) determines a cotangent element in T_x^*M and $(\lambda_a) \in A_x^*$ is a cotangent element with respect to the tangent direction to the fibers of A. Now the bundle projection $r : T^*A \to A^*$ is described locally by $r(x^i, u^a, p_i, \lambda_a) = (x^i, \lambda_a)$. These vector bundle structures define a commutative diagram

$$
\begin{tikzcd}
T^*A \arrow[r, "r"] \arrow[d, "c_A"'] & A^* \arrow[d, "q_{A^*}"] \\
A \arrow[r, "q_A"] & M.
\end{tikzcd}
\tag{3}
$$

This endows T^*A with a double vector bundle structure. Suppose that $q_A : A \to M$ carries a Lie algebroid structure. Then we can consider the dual bundle A^* endowed with the linear Poisson structure induced by A. The cotangent bundle $T^*A^* \to A^*$ has the Lie algebroid structure determined by the linear Poisson bivector on A^*. There exists a Legendre type map $R : T^*A^* \to T^*A$ that is an antisymplectomorphism with respect to the canonical symplectic structures, and it is locally defined by $R(x^i, \xi_a, p_i, u^a) = (x^i, u^a, -p_i, \xi_a)$. For an intrinsic definition see [Mackenzie and Xu 1994; Tulczyjew 1977].

Definition 2.2. The *cotangent algebroid* of A is the vector bundle $T^*A \to A^*$ equipped with the unique Lie algebroid structure that makes the Legendre type transform $R : T^*(A^*) \to T^*A$ into an isomorphism of Lie algebroids.

Example 2.3. Suppose that $A = \mathfrak{g}$ is a Lie *algebra*, that is, a Lie algebroid over a point. Then, the cotangent algebroid $T^*\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ is given by the transformation Lie algebroid with respect to the coadjoint representation of $\mathfrak{g}$ on its dual vector space $\mathfrak{g}^*$.

Finally, recall also that the *Tulczyjew map* $\Theta_M : TT^*M \to T^*TM$ is the isomorphism which, in a local coordinates system $(x^i, p_i, \dot{x}^i, \dot{p}_i)$, is given by

$$\Theta_M(x^i, p_i, \dot{x}^i, \dot{p}_i) = (x^i, \dot{x}^i, \dot{p}_i, p_i).$$

See [Mackenzie and Xu 1994; Tulczyjew 1977] for an intrinsic definition. Consider now a Lie groupoid G over M with Lie algebroid $AG = \ker Ts|_M$. There exists a natural injective bundle map

$$i_{AG} : AG \to TG. \tag{4}$$

The canonical involution $J_G : TTG \to TTG$ restricts to an isomorphism of Lie algebroids $j_G : T(AG) \to A(TG)$. More precisely, there exists a commutative diagram

$$\begin{tikzcd} T(AG) \arrow[r, "j_G"] \arrow[d, "T(i_{AG})"'] & A(TG) \arrow[d, "i_{A(TG)}"] \\ TTG \arrow[r, "J_G"] & TTG \end{tikzcd} \tag{5}$$

In particular, the Lie algebroid $A(TG)$ of the tangent groupoid is canonically isomorphic to the tangent Lie algebroid $T(AG)$ of AG. Similarly, the Lie algebroid of the cotangent groupoid T^*G is isomorphic to the cotangent Lie algebroid $T^*(AG)$. For that, notice that the natural pairing $T^*G \oplus TG \to \mathbb{R}$ defines a groupoid morphism, and the application of the Lie functor yields a symmetric pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle : A(T^*G) \oplus A(TG) \to \mathbb{R}$, which is nondegenerate. See for example, [Mackenzie and Xu 1994; 2000]. In particular, we obtain an isomorphism $K_G : A(T^*G) \to A(TG)^*$, where the target dual is with respect to the fibration $A(TG) \xrightarrow{A(p_G)} AG$. Now we define a Lie algebroid isomorphism

$$j'_G : A(T^*G) \to T^*(AG), \tag{6}$$

determined by the composition $j'_G = j^*_G \circ K_G$, where $j^*_G : A(TG)^* \to T^*(AG)$ is the bundle map dual to the isomorphism $j_G : T(AG) \to A(TG)$. As $j_G : T(AG) \to A(TG)$ is a suitable restriction of the canonical involution $J_G : TTG \to TTG$, the isomorphism j'_G is related to the Tulczyjew map $\Theta_G : TT^*G \to T^*TG$, via

$$j'_G = (Ti_{AG})^* \circ \Theta_G \circ i_{A(T^*G)}.$$

2C. *Tangent lift of a Dirac structure.* The tangent lift of Dirac structures was originally studied by T. Courant [1990a], who described tangent Dirac structures locally. I. Vaisman [2005] gives an intrinsic construction of tangent Dirac structures, where the tangent lift of a Dirac structure is described via the sheaf of local sections defining a Dirac subbundle of $TTM \oplus T^*TM$. Here, we provide an alternative description of the tangent lift of a Dirac structure relied on the tangent lift of Lie algebroid structures described in the previous section.

In order to fix our notation, we begin by summarizing some of the main properties of tangent lifts of vector fields and differential forms, see [Grabowski and Urbański 1997; Yano and Ishihara 1973]. Let $f \in C^\infty(M)$ be a smooth function. Then we have a pair of smooth functions on TM defined by

$$f^v = f \circ p_M, \quad f^T = df.$$

We refer to f^v and f^T as the *vertical* and *tangent* lifts of f. One can see easily that the algebra of functions $C^\infty(TM)$ is generated by functions of the form f^v and f^T. Now, given a vector field X on M we define the *vertical* lift of X as the vector field X^v on TM that acts on vertical and tangent lifts of functions as

$$X^v(f^v) = 0, \quad X^v(f^T) = (Xf)^v.$$

The *tangent* lift of X is the vector field X^T on TM that acts on vertical and tangent lifts of functions in the following manner:

$$X^T(f^v) = (Xf)^v, \quad X^T(f^T) = (Xf)^T.$$

It is easy to see that vertical and tangent lifts of vector fields generate the space of all vector fields on TM. Now let us consider a 1-form α on a smooth manifold M. We define the *vertical* lift of α as the 1-form α^v on TM, which is determined by its value at vertical and tangent lifts of vector fields,

$$\alpha^v(X^v) = 0, \quad \alpha^v(X^T) = (\alpha(X))^v.$$

The *tangent* lift of α is the 1-form α^T on TM defined by

$$\alpha^T(X^v) = (\alpha(X))^v, \quad \alpha^T(X^T) = (\alpha(X))^T.$$

It is important to emphasize that vertical and tangent lifts of vector fields (resp. of 1-forms) are sections of the usual vector bundle structure $T(TM) \xrightarrow{p_{TM}} TM$ (resp. sections of $T^*(TM) \xrightarrow{c_{TM}} TM$), and they do not define sections of the tangent prolongation vector bundle $T(TM) \xrightarrow{Tp_M} TM$ (resp. of the tangent prolongation $T(T^*M) \xrightarrow{Tc_M} TM$). However, there exists a canonical relation between vector fields (resp. 1-forms) on TM and sections of the tangent prolongation vector bundle $T(TM) \to TM$ (resp. $T(T^*M) \to TM$). Given a vector field X and a 1-form α on M, we consider the linear sections $TX, T\alpha$ and the core sections $\hat{X}, \hat{\alpha}$ of the corresponding tangent prolongation vector bundles. It follows from the definition that

$$J_M(TX) = X^T, \quad J_M(\hat{X}) = X^v. \tag{7}$$

$$\Theta_M(T\alpha) = \alpha^T, \quad \Theta_M(\hat{\alpha}) = \alpha^v. \tag{8}$$

It turns out that many geometric properties of the direct sum vector bundle $T(TM) \oplus T^*(TM)$ can be understood in terms of tangent geometric properties of $T(TM) \oplus T(T^*M)$, using the canonical identification

$$J_M \oplus \Theta_M : T(TM) \oplus T(T^*M) \to T(TM) \oplus T^*(TM).$$

Now consider a Dirac structure L_M on M. Equivalently, we may think of L_M as a Lie algebroid over M with Lie bracket given by the Courant bracket on sections of L_M, and the anchor map ρ_M is the natural projection from $L_M \subseteq TM \oplus T^*M$ onto TM. According to a construction of K. Mackenzie and P. Xu [1994], we can consider the tangent prolongation Lie algebroid $TL_M \to TM$, with anchor map

$$\rho_{TM} = J_M \circ T\rho_M,$$

and Lie bracket defined by

$$[\hat{a}_1, \hat{a}_2]_{TL_M} = 0, \quad [Ta_1, \hat{a}_2]_{TL_M} = \widehat{[a_1, a_2]}, \quad [Ta_1, Ta_2]_{TL_M} = T[a_1, a_2],$$

where a_1, a_2 are sections of $L_M \to M$. We denote by L_{TM} the image of TL_M under the natural bundle map $J_M \oplus \Theta_M : TTM \oplus TT^*M \to TTM \oplus T^*TM$.

Proposition 2.4. *The subbundle $L_{TM} \subseteq TTM \oplus T^*TM$ is isotropic with respect to the nondegenerate symmetric pairing $\langle \cdot, \cdot \rangle_{TM}$ defined on $TTM \oplus T^*TM$.*

Proof. Let (X, α), (Y, β) be sections of L_M. Then, the tangent lifts (X^T, α^T) and (Y^T, β^T) define sections of L_{TM}. Notice that

$$\langle (X^T, \alpha^T), (Y^T, \beta^T) \rangle = (\beta(X))^T + (\alpha(Y))^T = (\langle (X, \alpha), (Y, \beta) \rangle)^T = 0.$$

This implies that L_{TM} is isotropic. □

The tangent Lie algebroid $TL_M \to TM$ induces a unique Lie algebroid structure on $L_{TM} \to TM$ characterized by the property that $J_M \oplus \Theta_M : TL_M \to L_{TM}$ is a Lie algebroid isomorphism. The space of sections $\Gamma(L_{TM})$ is generated by sections of the form $a^T := (J_M \oplus \Theta_M)(Ta)$ and $a^v := (J_M \oplus \Theta_M)\hat{a}$, where a is a section of $L_M \to M$. In particular the induced Lie bracket on sections of L_{TM} is completely determined by identities

$$[a_1^v, a_2^v] = 0, \quad [a_1^T, a_2^v] = [\![a_1, a_2]\!]^v, \quad [a_1^T, a_2^T] = [\![a_1, a_2]\!]^T,$$

and the Leibniz rule with respect to the induced anchor map $pr_{TTM} : L_{TM} \to TTM$.

Proposition 2.5. *The induced Lie bracket on sections $\Gamma(L_{TM})$ is a restriction of the Courant bracket $[\![\cdot, \cdot]\!]_{TM}$ on sections of $TTM \oplus T^*TM$.*

Proof. Due to the identities (7) and (8), we only need to check that the Courant bracket on sections of L_{TM}, naturally induced by $J_M \oplus \Theta_M$, satisfies the bracket identities that determine the induced Lie bracket on $\Gamma(L_{TM})$. One observes that vertical and tangent lifts are compatible with Lie derivatives in the sense that

(1) $\mathscr{L}_{X^v} \alpha^v = 0$,

(2) $\mathscr{L}_{X^T} \alpha^v = (\mathscr{L}_X \alpha)^v$,

(3) $\mathscr{L}_{X^T} \alpha^T = (\mathscr{L}_X \alpha)^T$,

and we conclude that

(1) $[\![X^v \oplus \alpha^v, Y^v \oplus \beta^v]\!] = 0$,

(2) $[\![X^T \oplus \alpha^T, Y^v \oplus \beta^v]\!] = [X, Y]^v \oplus (\mathscr{L}_X\beta - i_Y d\alpha)^v$,

(3) $[\![X^T \oplus \alpha^T, Y^T \oplus \beta^T]\!] = [X, Y]^T \oplus (\mathscr{L}_X\beta - i_Y d\alpha)^T$.

Thus the Lie bracket on $\Gamma_{TM}(L_{TM})$ induced by the tangent Lie bracket on $\Gamma_{TM}(TL_M)$ coincides with the Courant bracket. □

We have shown the following:

Proposition 2.6. *Let M be a smooth manifold. There exists a natural map*

$$\mathrm{Dir}(M) \to \mathrm{Dir}(TM)$$
$$L_M \mapsto L_{TM},$$

where $L_{TM} := (J_M \oplus \Theta_M)(TL_M)$.

The Dirac structure $L_{TM} \in \mathrm{Dir}(TM)$ given by the proposition above is referred to as the *tangent Dirac structure* induced by $L_M \in \mathrm{Dir}(M)$. It is straightforward to check that this construction unifies the tangent lift of both closed 2-forms and Poisson bivectors. Additionally, the presymplectic foliation of L_{TM} corresponds to taking the tangent bundle of each leaf endowed with the tangent lift of the leafwise presymplectic forms defined by L_M. See also [Boumaiza and Zaalani 2009] for a general construction of tangent lifts of Dirac structures on arbitrary Courant algebroids.

3. Multiplicative Dirac structures

This section introduces the main objects of study of this work, that is, Lie groupoids equipped with Dirac structures compatible with the groupoid multiplication, including both multiplicative Poisson and closed 2-forms as particular cases.

3A. ***Definition and main examples.*** Let G be a Lie groupoid over M, with Lie algebroid AG. Consider the direct sum Lie groupoid $\mathbb{T}G = TG \oplus T^*G$ with base manifold $TM \oplus A^*G$.

Definition 3.1. Let G be a Lie groupoid over M. A Dirac structure L_G on G is said to be *multiplicative* if $L_G \subseteq TG \oplus T^*G$ is a subgroupoid over some subbundle $E \subseteq TM \oplus A^*G$.

We refer to a pair (G, L_G), made up of a Lie groupoid G and a multiplicative Dirac structure L_G on G, as a *Dirac groupoid*. We use the notation $\mathrm{Dir}_{\mathrm{mult}}(G)$ to indicate the set consisting of all multiplicative Dirac structures on G.

It follows from the multiplicativity of L_G that $E \subseteq TM \oplus A^*G$ is a vector subbundle. In particular, a multiplicative Dirac structure L_G on a Lie groupoid G defines a $\mathscr{V}\mathscr{B}$-subgroupoid $L_G \subseteq \mathbb{T}G$.

Example 3.2. Let ω_G be a closed multiplicative 2-form on a Lie groupoid G. The multiplicativity property of ω_G is equivalent to saying that the bundle map $\omega_G^\sharp : TG \to T^*G$ is a morphism of Lie groupoids. Hence, the corresponding Dirac structure $L_{\omega_G} = \mathrm{Graph}(\omega_G) \subseteq \mathbb{T}G$ is a multiplicative Dirac structure. In this case we have a groupoid $L_{\omega_G} \rightrightarrows E$, where $E \subseteq TM \oplus A^*G$ is the subbundle given by the graph of the bundle map $-\sigma^t$ determined by the *IM-2-form* (see [Bursztyn et al. 2004]) σ associated to ω_G.

Example 3.3. Let (G, π_G) be a Poisson groupoid. The multiplicativity of π_G is equivalent to saying that $\pi_G^\sharp : T^*G \to TG$ is a morphism of Lie groupoids. Therefore, the associated Dirac structure $L_{\pi_G} = \mathrm{Graph}(\pi_G) \subseteq \mathbb{T}G$ defines a multiplicative Dirac structure. In this case we have a groupoid $L_{\pi_G} \rightrightarrows E$, where $E \subseteq TM \oplus A^*G$ is the subbundle given by the graph of dual anchor map $\rho_{A^*G} : A^*G \to TM$.

The examples discussed previously show that Dirac groupoids lead to a natural generalization of Poisson groupoids and presymplectic groupoids. Our main aim is to describe Dirac groupoids infinitesimally, establishing in particular, a connection between such an infinitesimal description and Lie bialgebroids and IM-2-forms.

3B. ***More examples of multiplicative Dirac structures.*** In addition to multiplicative closed 2-forms and multiplicative Poisson bivectors, there are several interesting multiplicative Dirac structures, which will be discussed throughout this subsection.

3B1. *Foliated groupoids.* A regular distribution $F_G \subseteq TG$ is called *multiplicative* if it defines a Lie subgroupoid of the tangent groupoid TG. A *foliated groupoid* is a pair (G, F_G), where G is a Lie groupoid and F_G is an involutive multiplicative regular distribution. In this case, the Dirac structure $F_G \oplus F_G^\circ \subseteq \mathbb{T}G$ is easily seen to be a multiplicative Dirac structure on G. The foliation tangent to an involutive multiplicative distribution is called a *multiplicative foliation*. Multiplicative foliations that are simultaneously transversal to the s-fibration and to the t-fibration were studied in [Tang 2006], providing interesting examples of noncommutative Poisson algebras. Also, multiplicative foliations arise in the context of geometric quantization of symplectic groupoids, namely, as polarizations compatible with a symplectic groupoid structure (see [Hawkins 2008]). In addition, the notion of multiplicative foliation has appeared in connection with exterior differential systems. For more details see [Crainic et al. 2012] and the references therein.

3B2. *Dirac Lie groups.* Dirac Lie groups, that is, Lie groups equipped with multiplicative Dirac structures, were first studied by the author in [Ortiz 2008], providing a generalization of Poisson Lie groups within the category of Lie groups. In that work, it is shown that, modulo regularity issues, Dirac Lie groups are given by the pull-back (in the sense of Dirac structures) of Poisson Lie groups via a surjective submersion which is also a Lie group morphism. Notice that whenever a Lie groupoid G over M is equipped with a multiplicative Dirac structure, then for every $x \in M$, the isotropy Lie group $G_x := s^{-1}(x) \cap t^{-1}(x)$ inherits a Dirac structure L_{G_x} making the pair (G_x, L_{G_x}) into a Dirac Lie group.

We emphasize that different notions of Dirac Lie groups exist in the literature. For instance, Li-Bland and Meinrenken [2011] have proposed a notion of multiplicativity that includes interesting examples of *twisted* Dirac structures on Lie groups such as the Cartan–Dirac structure on a compact Lie group.

3B3. *Tangent lift of a multiplicative Dirac structure.* In [Grabowski and Urbański 1995] it was proved that whenever a Lie group G carries a multiplicative Poisson bivector π_G, then the tangent Lie group TG equipped with the tangent Poisson structure π_{TG} becomes a Poisson Lie group. It is easy to extend the multiplicative Poisson case to abstract multiplicative Dirac structures on Lie groupoids. Assume that G is a Lie groupoid over M and consider the tangent groupoid TG over TM explained in Section 2A. Then, the tangent Dirac structure $L_{TG} \subseteq TTG \oplus T^*TG$ induced by a multiplicative Dirac structure $L_G \subseteq TG \oplus T^*G$ is also a multiplicative Dirac structure. Indeed, first observe that the bundle map $J_G : TTG \to TTG$ is a groupoid isomorphism over $J_M : TTM \to TTM$. Similarly, the bundle map $\Theta_G : TT^*G \to T^*TG$ is a groupoid isomorphism over the canonical identification $I : T(A^*G) \to (T(AG))^*$. Since L_G is a Lie subgroupoid of $TG \oplus T^*G$, then the tangent functor yields a Lie subgroupoid TL_G of $TTG \oplus TT^*G$. Due to the fact that L_{TG} is the image of TL_G via the groupoid isomorphism $J_G \oplus \Theta_G$, we see that L_{TG} inherits a natural structure of Lie subgroupoid of $TTG \oplus T^*TG$. Hence we conclude that L_{TG} defines a multiplicative Dirac structure on TG.

3B4. *Symmetries of multiplicative Dirac structures.* Let L_G be a multiplicative Dirac structure on a Lie groupoid $G \rightrightarrows M$, and let H be a Lie group acting on G by groupoid automorphisms. Assume that the H-action is free and proper and that the H-orbits coincide with the characteristic leaves of L_G. In this case the quotient space G/H inherits the structure of a Lie groupoid over M/H. Moreover, since G/H is the space of characteristic leaves of L_G, we conclude that there exists a Poisson structure π_{red} on G/H, making the quotient map $G \to G/H$ into both a backward and forward Dirac map. This fact together with the multiplicativity of L_G imply that π_{red} is a multiplicative Poisson bivector. In other words, the quotient space G/H is a Poisson groupoid. In the case where L_G is the graph of a multiplicative Poisson bivector and the action is Hamiltonian in the sense of Fernandes and Iglesias [≥ 2013], this recovers some of the results about reduction of Poisson groupoids carried out by those authors.

3B5. *Multiplicative B-field transformations.* Let $L \subseteq \mathbb{TM}$ be a Lagrangian subbundle. Given a 2-form $B \in \Omega^2(M)$ one can construct the Lagrangian subbundle

$\tau_B(L) \subseteq \mathbb{T}M$ defined by

$$\tau_B(L) = \{X \oplus \alpha + i_X B \mid X \oplus \alpha \in L\}.$$

A straightforward computation shows that $\tau_B(L)$ defines a Dirac structure on M if and only if B is a closed 2-form. See for instance [Bursztyn 2005; Gualtieri 2003]. In this case, we say that the Dirac structure $\tau_B(L)$ is obtained out of L by a *B-field transformation*.

Assume now that L_G is a multiplicative Dirac structure on a Lie groupoid G. Given a multiplicative closed 2-form B_G on G, one can consider the bundle map $\tau_{B_G} : \mathbb{T}G \to \mathbb{T}G$, $X \oplus \alpha \mapsto X \oplus \alpha + i_X(B_G)$. It follows from the multiplicativity of B_G that τ_{B_G} is a Lie groupoid isomorphism. As a result, the Dirac structure $\tau_{B_G}(L_G)$ on G is multiplicative. Our interest in B-field transformations of multiplicative Dirac structures is motivated by the work carried out in [Bursztyn 2005; Bursztyn and Radko 2003], where the authors study the connection between certain B-field transformations of symplectic and Poisson groupoids and the notion of Morita equivalence of Poisson manifolds.

3B6. *Generalized complex groupoids.* Generalized complex structures were introduced in [Hitchin 2003] and further developed in [Gualtieri 2003]. Given a smooth manifold M, one can consider the complexified vector bundle $\mathbb{T}_{\mathbb{C}}M := \mathbb{T}M \otimes \mathbb{C}$ endowed with the complex Courant bracket and the complex pairing $\langle \cdot, \cdot \rangle$. A generalized complex structure on M is a complex Dirac structure $L \subseteq \mathbb{T}_{\mathbb{C}}M$ such that $L \cap \bar{L} = \{0\}$, where $\bar{L}$ denotes the conjugate of L. Complexified versions of multiplicative Dirac structures give rise to generalized complex groupoids. More concretely, let G be a Lie groupoid equipped with a generalized complex structure L_G. We say that (G, L_G) is a *generalized complex groupoid* if $L_G \subseteq \mathbb{T}_{\mathbb{C}}G$ is a Lie subgroupoid. Generalized complex groupoids were introduced in [Jotz et al. 2012] under the name of Glanon groupoids. Structures such as symplectic groupoids and holomorphic Poisson groupoids are special instances of generalized complex groupoids.

4. Dirac algebroids

In this section we study Lie algebroids equipped with Dirac structures compatible with both the linear and Lie algebroid structure.

4A. Definition and main examples. Let $A \to M$ be a vector bundle. A Poisson bivector π_A on A is *linear* if the map $\pi_A^\sharp : T^*A \to TA$ is a morphism of double vector bundles. Similarly, a 2-form ω_A on A is *linear* if the map $\omega_A^\sharp : TA \to T^*A$ is a morphism of double vector bundles. The bundle map $\omega_A^\sharp$ in this case covers a bundle morphism $\lambda : TM \to A^*$. As shown in [Konieczna and Urbański 1999], a linear 2-form ω_A on a vector bundle $A \to M$ is closed if and only if $\omega_A = -(\lambda^t)^*\omega_{can}$, where ω_{can} is the canonical symplectic form on T^*M and $\lambda^t : A \to T^*M$ is a fiberwise dual map of $\lambda : TM \to A^*$. The definition below includes both linear Poisson bivectors and linear closed 2-forms as special instances.

Definition 4.1. A Dirac structure L_A on A is called *linear* if $L_A \subseteq \mathbb{T}A$ is a double vector subbundle of $\mathbb{T}A$.

A linear Dirac structure $L_A \subseteq \mathbb{T}A$ is not only a vector bundle over A, but also a vector bundle over a subbundle $E \subseteq TM \oplus A^*$. It follows directly from the definition that graphs of linear Poisson bivector and linear closed 2-forms define linear Dirac structures. Linear Dirac structures arise also in connection with Lagrangian and Hamiltonian mechanics, see for example, [Grabowska and Grabowski 2011].

Assume now that $A \to M$ carries also a Lie algebroid structure. Consider the direct sum Lie algebroid $\mathbb{T}A = TA \oplus T^*A$, whose base manifold is $TM \oplus A^*$.

Definition 4.2. A Dirac structure L_A on A is called *morphic* if L_A is a linear Dirac structure that is also a Lie subalgebroid of $\mathbb{T}A$.

We denote by $\mathrm{Dir}_{\mathrm{morph}}(A)$ the space of morphic Dirac structures on the Lie algebroid A.

A pair (A, L_A), where A is a Lie algebroid endowed with a morphic Dirac structure L_A, will be referred to as a *Dirac algebroid*.

Example 4.3. Let π_A be a linear Poisson bivector on a Lie algebroid $A \to M$. Then, the Dirac structure given by the graph of π_A is morphic if and only if $\pi_A^\sharp : T^*A \to TA$ is a Lie algebroid morphism. As shown in [Mackenzie and Xu 1994], this is equivalent to the pair (A, A^*) being a Lie bialgebroid.

Example 4.4. Let ω_A be a linear closed 2-form on a Lie algebroid $A \to M$, that is, $\omega_A = -\sigma^*\omega_{can}$, for some bundle map $\sigma : A \to T^*M$. The Dirac structure defined by the graph of ω_A is morphic if and only if $\omega_A^\sharp : TA \to T^*A$ is a Lie algebroid morphism. Equivalently, as shown in [Bursztyn et al. 2009a], the bundle map $\sigma : A \to T^*M$ is an IM-2-form on A. The notion of IM-2-form was introduced in [Bursztyn et al. 2004] motivated by the problem of the integration of Dirac structures. See also [Arias Abad and Crainic 2011], where IM-2-forms arise in connection with the Weil algebra and the Van Est isomorphism.

4B. ***More examples of Dirac algebroids.*** In addition to both morphic Poisson structures and morphic closed 2-forms, there are more examples of morphic Dirac structures, which we proceed to explain below.

4B1. *Foliated algebroids.* Let A be a Lie algebroid and $F_A \subseteq TA$ an involutive subbundle which is also a Lie subalgebroid of $TA \to TM$. In this case we say that (A, F_A) is a *foliated algebroid*. One can easily check that the Dirac structure $F_A \oplus F_A^\circ \subseteq \mathbb{T}A$ is a morphic Dirac structure. Foliated algebroids were studied in [Hawkins 2008] as a way to promote the notion of polarization in geometric quantization to the category of Lie algebroids. Also, a detailed discussion about foliated algebroids can be found in [Jotz and Ortiz 2012].

4B2. *Dirac Lie algebras.* Let $\mathfrak{g}$ be a Lie algebra. In this case, morphic Dirac structures are Lie subalgebroids of $T\mathfrak{g} \oplus T^*\mathfrak{g} \to \mathfrak{g}^*$. It follows from [Ortiz 2008] that Dirac Lie algebras are suitable pull-backs of Lie bialgebras.

4B3. *Tangent lifts of Dirac algebroids.* Let (A, L_A) be a Dirac algebroid. Consider the tangent Dirac structure L_{TA} on TA. By definition, the tangent Dirac structure is given by $L_{TA} := (J_A \oplus \Theta_A)(TL_A)$, where $TL_A \to TM$ is the tangent algebroid associated to the Dirac structure L_A viewed as a Lie algebroid over A. Since the bundle map $J_A \oplus \Theta_A : TTA \oplus TT^*A \to TTA \oplus T^*TA$ is a Lie algebroid isomorphism, we conclude that $L_{TA} \subseteq \mathbb{T}TA$ is a Lie subalgebroid. Therefore, the pair (TA, L_{TA}) is a Dirac algebroid.

4B4. *Symmetries of Dirac algebroids.* Let (A, L_A) be a Dirac algebroid. Consider a Lie group H acting on A by Lie algebroid automorphisms. Assume that the action is free and proper and that the H-orbits coincide with the characteristic leaves of L_A. One can check that the orbit space A/H inherits a Lie algebroid structure over M/H, making the quotient map $A \to A/H$ into a Lie algebroid morphism. Since the H-orbits are exactly the characteristic leaves of L_A, one concludes that A/H is equipped with a unique Poisson bivector π_{red} determined by the fact that $A \to A/H$ is a forward and backward Dirac map. Since L_A is morphic, we conclude that π_{red} is a morphic Poisson structure on A/H. In particular, due to [Mackenzie and Xu 1994], the pair $(A/H, (A/H)^*)$ is a Lie bialgebroid. In the special case where L_A is the graph of a morphic Poisson structure on A and the action is Hamiltonian in the sense of [Fernandes and Iglesias $\geq$ 2013], this recovers the reduction of Lie bialgebroids carried out in [Fernandes and Iglesias $\geq$ 2013].

4B5. *Morphic B-field transformations.* Let (A, L_A) be a Dirac algebroid. Associated to a morphic closed 2-form B_A on A is the Lie algebroid automorphism $\tau_{B_A} : \mathbb{T}A \to \mathbb{T}A$, $(X, \alpha) \mapsto (X, \alpha + i_X B_A)$. The Dirac structure $\tau_{B_A}(L_A) \subseteq \mathbb{T}A$ obtained out of L_A by applying the B-field transformation τ_{B_A} is morphic. Therefore, the pair (A, τ_{B_A}) is a Dirac algebroid. In particular, B-field transformations of morphic Poisson structures (that is, Lie bialgebroid structures on (A, A^*)) by morphic closed 2-forms are always morphic Dirac structures. If the B-field transformation is admissible, that is, the resulting Dirac structure is the graph of a Poisson bivector, such a bivector is necessarily morphic as well. In particular, we get a new bialgebroid structure on (A, A^*) referred to as a *gauge transformation* of the Lie bialgebroid (A, A^*). Gauge transformations of Lie bialgebroids were introduced in [Bursztyn 2005] motivated by the study of gauge transformations of Poisson groupoids and Morita equivalence of Poisson manifolds.

4B6. *Generalized complex algebroids.* Let $A \to M$ be a Lie algebroid. Consider the complexified Lie algebroid $\mathbb{T}_{\mathbb{C}}A = (TA \oplus T^*A) \otimes \mathbb{C}$ whose base manifold is $(TM \oplus A^*) \otimes \mathbb{C}$. A generalized complex structure L_A on A is *morphic* if $L_A \subseteq \mathbb{T}_{\mathbb{C}}A$ is a Lie subalgebroid. In this case, we say that (A, L_A) is a *generalized complex algebroid*. The notion of generalized complex algebroid was introduced in [Jotz et al. 2012] under the name of Glanon algebroids. Generalized complex algebroids include holomorphic Poisson structures and holomorphic Lie bialgebroids as particular cases.

5. Infinitesimal description of multiplicative Dirac structures

This section is the main part of the present work. Here we show that Dirac algebroids correspond to the infinitesimal counterpart of Dirac groupoids.

5A. ***The canonical $\mathcal{CA}$-groupoid.*** The main idea for studying multiplicative Dirac structures infinitesimally is based on the following observation. Given a Lie groupoid G over M, the canonical geometric objects associated to $\mathbb{T}G$ that are used to define Dirac structures (symmetric pairing and Courant bracket) are suitably compatible with the groupoid structure of $\mathbb{T}G$. This compatibility makes $\mathbb{T}G$ into a $\mathcal{CA}$-groupoid. The notion of $\mathcal{CA}$-groupoid was suggested by Mehta [2009] and further studied by Li-Bland and Ševera [2011]. More precisely, let $\langle\cdot,\cdot\rangle_G$ be the nondegenerate symmetric pairing on the direct sum Lie groupoid $\mathbb{T}G$.

Proposition 5.1. *The canonical pairing defines a morphism of Lie groupoids*

$$\langle\cdot,\cdot\rangle_G : \mathbb{T}G \oplus \mathbb{T}G \to \mathbb{R},$$

where $\mathbb{R}$ is equipped with the usual abelian group structure.

Proof. Since $\mathbb{R}$ is a groupoid over a point, we only need to check the compatibility of $\langle\cdot,\cdot\rangle_G$ with the corresponding groupoid multiplications. For that, consider elements $(X_g \oplus \alpha_g), (Y_g \oplus \beta_g) \in \mathbb{T}_gG$ and $(X'_h \oplus \alpha'_h), (Y'_h \oplus \beta'_h) \in \mathbb{T}_hG$. Then by definition of the groupoid structure on $\mathbb{T}G \oplus \mathbb{T}G$, we have

$$\big((X_g \oplus \alpha_g) \oplus (Y_g \oplus \beta_g)\big) * \big((X'_h \oplus \alpha'_h) \oplus (Y'_h \oplus \beta'_h)\big) = (X_g \bullet X'_h \oplus \alpha_g \circ \alpha'_h) \oplus (Y_g \bullet Y'_h \oplus \beta_g \circ \beta'_h),$$

therefore one gets

$$\begin{aligned}\big\langle(X_g \bullet X'_h \oplus \alpha_g \circ \alpha'_h), (Y_g \bullet Y'_h \oplus \beta_g \circ \beta'_h)\big\rangle_G &= (\alpha_g \circ \alpha'_h)(Y_g \bullet Y'_h) + (\beta_g \circ \beta'_h)(X_g \bullet X'_h)\\ &= \alpha_g(Y_g) + \alpha'_h(Y'_h) + \beta_g(X_g) + \beta'_h(X'_h)\\ &= \langle(X_g \oplus \alpha_g), (Y_g, \beta_g)\rangle_G + \langle(X'_h \oplus \alpha'_h), (Y'_h \oplus \beta'_h)\rangle_G. \qquad \square\end{aligned}$$

In order to explain the relation between the Courant bracket and the Lie groupoid structure on the direct sum vector bundle $\mathbb{T}G = TG \oplus T^*G$, we consider the direct product Courant algebroid $\mathbb{T}G \times \mathbb{T}G \to G \times G$. Every section $a^{(2)}$ of $\mathbb{T}G \times \mathbb{T}G$ can be written as

$$a^{(2)} = a_1 \circ pr_1 \oplus a_2 \circ pr_2,$$

where a_1, a_2 are sections of $\mathbb{T}G$, and $pr_1, pr_2 : \mathbb{T}G \times \mathbb{T}G \to \mathbb{T}G$ denote the natural projections. The direct product bracket on sections of $\mathbb{T}G \times \mathbb{T}G$ is defined as usual;

$$[a^{(2)}, \bar{a}^{(2)}] = [\![a_1, \bar{a}_1]\!] \circ pr_1 \oplus [\![a_2, \bar{a}_2]\!] \circ pr_2,$$

and the anchor map $\rho_{(\mathbb{T}G)_{(2)}} : \mathbb{T}G \times \mathbb{T}G \to TG \times TG$ is given by the canonical componentwise projection.

Proposition 5.2. *Let $m_{\mathbb{T}} : (\mathbb{T}G)_{(2)} \to \mathbb{T}G$ denote the groupoid multiplication of $\mathbb{T}G = TG \oplus T^*G$. If $a, b, a_i, b_i \in \Gamma(\mathbb{T}G)$, $i = 1, 2$ are sections such that*

$$m_{\mathbb{T}} \circ (a_1, a_2) = a \circ m_G, \quad m_{\mathbb{T}} \circ (b_1, b_2) = b \circ m_G,$$

then the following identities hold:

(i) $Tm_G\big(\rho_{(\mathbb{T}G)_{(2)}}(X_g^1 \oplus \alpha_g^1, X_h^2 \oplus \alpha_h^2)\big) = X_g^1 \bullet X_h^2$;

(ii) $m_{\mathbb{T}} \circ ([\![a_1, b_1]\!], [\![a_2, b_2]\!]) = [\![a, b]\!] \circ m_G$.

Proof. We begin by checking (i). For that, consider a section $a^{(2)} = a_1 \circ pr_1 \oplus a_2 \circ pr_2$ of $(\mathbb{T}G)_{(2)}$, where $a_1 = X^1 \oplus \alpha^1$ and $a_2 = X^2 \oplus \alpha^2$ are sections of $\mathbb{T}G$. The multiplication on the Lie groupoid $\mathbb{T}G$ maps the section $a^{(2)}$ into

$$m_{\mathbb{T}}(a_1 \circ pr_1 \oplus a_2 \circ pr_2)(g, h) = X_g^1 \bullet X_h^2 \oplus \alpha_g^1 \circ \alpha_h^2.$$

Applying the anchor map of $\mathbb{T}G$ we obtain

$$\rho_{\mathbb{T}G}(X_g^1 \bullet X_h^2 \oplus \alpha_g^1 \circ \alpha_h^2) = X_g^1 \bullet X_h^2.$$

On the other hand, the componentwise anchor map of $(\mathbb{T}G)_{(2)}$ applied to the section $a^{(2)}$ gives rise to

$$\rho_{(\mathbb{T}G)_{(2)}}(a_1 \circ pr_1 \oplus a_2 \circ pr_2)(g, h) = (X_g^1, X_h^2),$$

which followed by the derivative of $m_G : G_{(2)} \to G$ yields

$$Tm_G\big(\rho_{(\mathbb{T}G)_{(2)}}(X_g^1 \oplus \alpha_g^1, X_h^2 \oplus \alpha_h^2)\big) = X_g^1 \bullet X_h^2,$$

as required. In order to prove identity (ii), one considers

$$m_{\mathbb{T}} \circ a^{(2)} = a \circ m_G, \tag{9}$$

$$m_{\mathbb{T}} \circ \bar{a}^{(2)} = \bar{a} \circ m_G, \tag{10}$$

where $a^{(2)}, \bar{a}^{(2)} \in \Gamma_{G_{(2)}}((\mathbb{T}G)_{(2)})$ and $a, \bar{a} \in \Gamma_G(\mathbb{T}G)$. More concretely, write down sections as

$$\begin{aligned} a^{(2)} &= (X^1 \oplus \alpha^1) \circ pr_1 \oplus (X^2 \oplus \alpha^2) \circ pr_2, \\ \bar{a}^{(2)} &= (\bar{X}^1 \oplus \bar{\alpha}^1) \circ pr_1 \oplus (\bar{X}^2 \oplus \bar{\alpha}^2) \circ pr_2, \\ a &= Y \oplus \beta, \\ \bar{a} &= \bar{Y} \oplus \bar{\beta}. \end{aligned}$$

The identities (9), (10) then become

$$X^1_g \bullet X^2_h \oplus \alpha^1_g \circ \alpha^2_h = Y_{gh} \oplus \beta_{gh}, \tag{11}$$

$$\bar{X}^1_g \bullet \bar{X}^2_h \oplus \bar{\alpha}^1_g \circ \bar{\alpha}^2_h = \bar{Y}_{gh} \oplus \bar{\beta}_{gh}, \tag{12}$$

for any composable pair $(g, h) \in G \times G$. Now it follows directly from the definition of the direct product bracket that

$$\begin{aligned} &[a^{(2)}, \bar{a}^{(2)}] \\ &\quad = \big([X^1, \bar{X}^1] \oplus \mathscr{L}_{X^1}\bar{\alpha}^1 - i_{\bar{X}^1} d\alpha^1\big) \circ pr_1 \oplus \big([X^2, \bar{X}^2] \oplus \mathscr{L}_{X^2}\bar{\alpha}^2 - i_{\bar{X}^2} d\alpha^2\big) \circ pr_2. \end{aligned}$$

Then, composing with the groupoid multiplication of $\mathbb{T}G$, we have

$$\begin{aligned} &m_{\mathbb{T}} \circ [a^{(2)}, \bar{a}^{(2)}]_{(g,h)} \\ &\qquad = [X^1, \bar{X}^1]_g \bullet [X^2, \bar{X}^2]_h \oplus (\mathscr{L}_{X^1}\bar{\alpha}^1 - i_{\bar{X}^1} d\alpha^1)_g \circ (\mathscr{L}_{X^2}\bar{\alpha}^2 - i_{\bar{X}^2} d\alpha^2)_h. \end{aligned}$$

On the other hand,

$$[\![a, \bar{a}]\!] \circ m_G(g, h) = [Y, \bar{Y}]_{gh} \oplus (\mathscr{L}_Y \bar{\beta} - i_{\bar{Y}} d\beta)_{gh},$$

and using the identities (11) and (12) one concludes that

$$[Y, \bar{Y}]_{gh} = [X^1, \bar{X}^1]_g \bullet [X^2, \bar{X}^2]_h.$$

Thus, the tangent component of $[\![a, \bar{a}]\!]_{gh}$ coincides with the tangent component of $m_{\mathbb{T}} \circ [a^{(2)}, \bar{a}^{(2)}]_{(g,h)}$. It remains to show that we also have the equality of the corresponding cotangent parts. This is equivalent to showing that

$$\begin{aligned} &(\mathscr{L}_Y \bar{\beta} - \mathscr{L}_{\bar{Y}} \beta - d\langle \beta, \bar{Y}\rangle)_{gh} \\ &\qquad = \big(\mathscr{L}_{X^1}\bar{\alpha}^1 - \mathscr{L}_{\bar{X}^1}\alpha^1 - d\langle \alpha^1, \bar{X}^1\rangle\big)_g \circ \big(\mathscr{L}_{X^2}\bar{\alpha}^2 - \mathscr{L}_{\bar{X}^2}\alpha^2 - d\langle \alpha^2, \bar{X}^2\rangle\big)_h, \end{aligned}$$

for every composable pair $(g, h) \in G_{(2)}$. In order to prove this identity, we need to check that the left hand side (*LHS*), and the right hand side (*RHS*) above coincide at elements of the form $U_g \bullet V_h$. For that consider the 1-form on G defined by $\gamma := \mathscr{L}_Y \bar{\beta} - \mathscr{L}_{\bar{Y}} \beta - d\langle \beta, \bar{Y}\rangle$. We can look at the pull-back 1-form $m_G^*\gamma \in \Omega^1(G_{(2)})$, which at every tangent vector $(U_g, V_h) \in T_{(g,h)}G_{(2)}$ is given by

$$(m_G^*\gamma)_{(g,h)}(U_g, V_h) = \gamma_{gh}(U_g \bullet V_h) = (LHS)(U_g \bullet V_h).$$

The pull-back form $m_G^*\gamma$ involves three terms. Let us analyze the first term $m_G^*(\mathscr{L}_Y\bar\beta)$ of this pull-back form. It follows from the relation $Y=(m_G)_*(X^1,X^2)$ that

$$m_G^*(\mathscr{L}_Y\bar\beta)=\mathscr{L}_{(X^1,X^2)}m_G^*\bar\beta.$$

Notice that (12) implies that

$$\begin{aligned}(m_G^*\bar\beta)_{(g,h)}(U_g,V_H)&=\bar\beta_{gh}(U_g\bullet V_h)=(\bar\alpha_g^1\circ\bar\alpha_h^2)(U_g\bullet V_h)\\&=\bar\alpha_g^1(U_g)+\bar\alpha_h^2(V_h)=(\bar\alpha^1,\bar\alpha^2)_{(g,h)}(U_g,V_h).\end{aligned}$$

That is, $m_G^*(\mathscr{L}_Y\bar\beta)=\mathscr{L}_{X^1}\bar\alpha^1\oplus\mathscr{L}_{X^2}\bar\alpha^2$. A similar argument can be applied to the other terms of the pull-back form $m_G^*\gamma$, yielding

$$\begin{aligned}(LHS)(U_g\bullet V_h)&=(m_G^*\gamma)_{(g,h)}(U_g,V_h)\\&=(\mathscr{L}_{X^1}\bar\alpha^1)_g(U_g)+(\mathscr{L}_{X^2}\bar\alpha^2)_h(V_h)-(\mathscr{L}_{\bar X^1}\alpha^1)_g(U_g)\\&\quad-(\mathscr{L}_{\bar X^2}\alpha^2)_h(V_h)-d\langle\alpha^1,\bar X^1\rangle_g(U_g)-d\langle\alpha^2,\bar X^2\rangle_h(V_h)\\&=(RHS)(U_g\bullet V_h).\end{aligned}$$

Thus *RHS* and *LHS* coincide at elements of the form $U_g\bullet V_h$, and we conclude that $(m_{\mathbb{T}},m_G)$ is bracket preserving. □

Recall that, given a Courant algebroid $(\mathbb{E},\rho,[\![\cdot,\cdot]\!])$ over smooth manifold M and a submanifold $Q\subseteq M$, a *Dirac structure supported* on Q (see [Alekseev and Xu 2011; Bursztyn et al. 2009b]) is a subbundle $K\subset\mathbb{E}|_Q$ such that $K_x\subseteq\mathbb{E}_x$ is Lagrangian for all $x\in Q$ and the following conditions are fulfilled:

(1) $\rho(K)\subseteq TQ$;

(2) whenever $a_1,a_2\in\Gamma(\mathbb{E})$ satisfy $a_1|_Q,a_2|_Q\in\Gamma(K)$, then $[\![a_1,a_2]\!]|_Q\in\Gamma(K)$.

Dirac structures with support were used in [Bursztyn et al. 2009b] to introduce a natural notion of morphism between Courant algebroids. Let $\mathbb{E}_1,\mathbb{E}_2$ be Courant algebroids over M,N, respectively. A *Courant algebroid morphism* from $\mathbb{E}_1$ to $\mathbb{E}_2$ is a Dirac structure in $\mathbb{E}_2\times\overline{\mathbb{E}_1}$ supported on Graph(f), where $f:M\to N$ is a smooth map. Here $\overline{\mathbb{E}_1}$ denotes the Courant algebroid structure on the vector bundle $\mathbb{E}_1$ with the same bracket on $\Gamma(\mathbb{E}_1)$, anchor map and minus the usual symmetric pairing.

Combining Propositions 5.1 and 5.2, we obtain:

Proposition 5.3. *Let G be a Lie groupoid over M with multiplication map $m_G:G_{(2)}\to G$. Let $m_{\mathbb{T}}:(\mathbb{T}G)_{(2)}\to\mathbb{T}G$ denote the groupoid multiplication on $\mathbb{T}G$. Then* Graph$(m_{\mathbb{T}})\subseteq\mathbb{T}G\times\overline{\mathbb{T}G\times\mathbb{T}G}$ *is a Dirac structure supported on* Graph$(m_G)\subseteq G\times G\times G$. *That is,* Graph$(m_{\mathbb{T}})$ *is a Courant algebroid morphism from $\mathbb{T}G\times\mathbb{T}G$ to $\mathbb{T}G$.*

Using the terminology of [Li-Bland and Ševera 2011], Proposition 5.3 says that $\mathbb{T}G$ with its canonical Courant algebroid structure and groupoid multiplication is an example of $\mathscr{C}\mathscr{A}$-groupoid. See also Example 2.3.1 of [Li-bland 2012].

5B. *The $\mathscr{L}\mathscr{A}$-groupoid of a multiplicative Dirac structure.*

5B1. *Review of $\mathscr{L}\mathscr{A}$-groupoids.* An $\mathscr{L}\mathscr{A}$-groupoid is a Lie groupoid object in the category of Lie algebroids. More precisely, an *$\mathscr{L}\mathscr{A}$-groupoid* [Mackenzie 1992] is a square

$$
\begin{tikzcd}
H \arrow[r, "q_H"] \arrow[d, shift left] \arrow[d, shift right] & G \arrow[d, shift left] \arrow[d, shift right] \\
E \arrow[r, "q_E"] & M,
\end{tikzcd}
\tag{13}
$$

where the single arrows denote Lie algebroids and the double arrows denote Lie groupoids. These structures are compatible in the sense that all the structure mappings (that is, source, target, unit section, inversion and multiplication) defining the Lie groupoid H are Lie algebroid morphisms over the corresponding structure mappings which define the Lie groupoid G. We also require that the anchor map $\rho_H : H \to TG$ be a groupoid morphism over the anchor map $\rho_E : E \to TM$. Here TG is endowed with the tangent groupoid structure over TM. For describing the square given by an $\mathscr{L}\mathscr{A}$-groupoid we use the notation (H, G, E, M). It is worthwhile to explain how the groupoid multiplication defines a morphism of Lie algebroids. For that, let $m_H : H_{(2)} \subseteq H \times H \to H$ denote the groupoid multiplication of H, and similarly let $m_G : G_{(2)} \subseteq G \times G \to G$ denote the multiplication of G. The direct product vector bundle $H \times H \to G \times G$ inherits a natural Lie algebroid structure, and we have a Lie subalgebroid $H_{(2)}$ over $G_{(2)}$ which is just a pull-back algebroid, see for example, [Higgins and Mackenzie 1990] for details about the pull-back operation in the category of Lie algebroids. With respect to this Lie algebroid structure, the multiplication map m_H is required to be a Lie algebroid morphism covering m_G.

The Lie functor applied to an $\mathscr{L}\mathscr{A}$-groupoid (13) determines a double vector bundle

$$
\begin{tikzcd}
AH \arrow[r, "A(q_H)"] \arrow[d] & AG \arrow[d] \\
E \arrow[r, "q_E"] & M,
\end{tikzcd}
\tag{14}
$$

where each of the arrows define Lie algebroids. The top Lie algebroid structure is nontrivial, and it deserves a detailed explanation. The Lie algebroid structure $AH \to AG$ was constructed in [Mackenzie 2000] as a prolongation procedure similar to the tangent prolongation of a Lie algebroid, except that we replace the tangent functor by the Lie functor.

Definition 5.4. The *prolonged anchor map* $AH \to T(AG)$ is defined by

$$\tilde{\rho} := j_G^{-1} \circ A(\rho_H),$$

where $j_G : T(AG) \to A(TG)$ is the canonical identification defined in Equation (5).

Now we study the space of sections $\Gamma_{AG}(AH)$.

Definition 5.5. A section $u \in \Gamma_G(H)$ is called a *star section* if there exists a section $u_0 \in \Gamma_M(E)$ such that

(1) $\epsilon_E \circ u_0 = u \circ \epsilon_M$,

(2) $s_H \circ u = u_0 \circ s_G$.

Notice that since every star section $u : G \to H$ preserves the units and the source fibrations, we are allowed to apply the Lie functor to u, yielding a section $A(u)$ of the vector bundle $AH \xrightarrow{A(q_H)} AG$.

Definition 5.6. Let (H, G, E, M) be an $\mathcal{LA}$-groupoid. The *core* of H is the vector bundle over M defined by

$$K := \epsilon_M^* \ker(s_H).$$

Every section $k \in \Gamma(K)$ induces a section $k_H \in \Gamma_G(H)$ in the following way:

$$k_H(g) := k(t_G(g))0_g^H,$$

where 0_g^H is the zero element in the fiber H_g above $g \in G$. Notice that for every section $k \in \Gamma(K)$ the induced section $k_H \in \Gamma_G(H)$ satisfies

$$k_H \circ \epsilon_M = k.$$

It was proved in [Mackenzie 2000] that the core of the double vector bundle (AH, AG, E, M) is the vector bundle $K \to M$. Notice that a core element $k \in K$ induces a Lie algebroid element $\bar{k} \in AH$. Indeed, we observe that every element in AH has the form

$$W = \frac{d}{dt}(h_t)|_{t=0},$$

where h_t is a curve in H sitting in a fixed source fiber $s_H^{-1}(e)$ with $h_0 = \epsilon_E(e)$. Thus, for every core element $k \in K$ above $x \in M$ — that is, $s_H(k) = 0_x^E$ and $q_H(k) = \epsilon_M(x)$ — there exists a natural element $\bar{k} \in AH$, defined by

$$\bar{k} := \frac{d}{dt}(tk)|_{t=0}.$$

Definition 5.7. Given a section $k \in \Gamma(K)$, the *core* section induced by k is the section $k^{\text{core}} \in \Gamma_{AG}(AH)$ defined by

$$k^{\text{core}}(u_x) := A(0^H)u_x + \overline{k(x)}.$$

Proposition 5.8 [Mackenzie 2000]. *The space of sections $\Gamma_{AG}(AH)$ is generated by sections of the form $A(u)$, where $u : G \to H$ is a star section, and by sections of the form k^{core}, where $k : M \to K$ is a section of the core of H.*

The Lie bracket on $\Gamma_{AG}(AH)$ is defined in terms of star sections and core sections. First we observe that whenever $u, v \in \Gamma_G(H)$ are star sections, then the Lie bracket $[u, v] \in \Gamma_G(H)$ is also a star section. Thus the Lie bracket between sections of the form $A(u)$, $A(v)$ is defined by

$$[A(u), A(v)] = A([u, v]).$$

The bracket of a pair of core sections is defined by

$$[k_1^{\text{core}}, k_2^{\text{core}}] = 0.$$

In order to define the bracket of a star section and a core section we notice that every star section $u : G \to H$ induces a covariant differential operator

$$D_u : \Gamma(K) \to \Gamma(K), \quad k \mapsto [u, k_H] \circ \epsilon_M.$$

Now we define $[A(u), k^{\text{core}}] = (D_u(k))^{\text{core}}$.

The Lie bracket of other sections of $\Gamma_{AG}(AH)$ is defined by requiring the Leibniz rule

$$[w, fw'] = f[w, w'] + (\mathscr{L}_{\tilde{\rho}(w)} f)w'.$$

The vector bundle $AH \xrightarrow{A(q_H)} AG$ endowed with the anchor map

$$\tilde{\rho} = j_G^{-1} \circ A(\rho)$$

and the Lie bracket $[\,\cdot\,,\cdot\,]$ on $\Gamma_{AG}(AH)$ becomes a Lie algebroid called the *prolonged Lie algebroid* induced by $H \to G$, see [Mackenzie 2000].

Although the following remark is not mentioned in [Mackenzie 2000], it is important to notice that Mackenzie's construction of the prolonged Lie algebroid is natural in the following sense.

Proposition 5.9. *Let (H, G, E, M) be an $\mathscr{LA}$-groupoid. Consider the canonical embeddings $i_{AH} : AH \to TH$ and $i_{AG} : AG \to TG$. Endow $TH \to TG$ with the tangent algebroid structure and $AH \to AG$ with the prolonged algebroid structure. Then i_{AH} is a Lie algebroid morphism covering i_{AG}.*

Recall that (see for example, [Mackenzie 2005]) a vector bundle map $\Psi : A \to B$, covering $\psi : M \to N$, is a *Lie algebroid morphism* if

$$\rho_B \circ \Psi = T\psi \circ \rho_A,$$

and the following compatibility with brackets holds: For sections $u, v \in \Gamma(A)$ such that $\Psi(u) = \sum_j f_j \psi^* u_j$ and $\Psi(v) = \sum_i g_i \psi^* v_i$, where $f_j, g_i \in C^\infty(M)$ and $u_j, v_i \in \Gamma(B)$, we have

$$\Psi([u, v]_A) = \sum_{i,j} f_j g_i \psi^*[u_j, v_i]_B + \sum_i \mathcal{L}_{\rho_A(u)} g_i \psi^* v_i - \sum_j \mathcal{L}_{\rho_A(v)} f_j \psi^* u_j. \tag{15}$$

Proof. The compatibility with the anchor maps reads

$$\rho_{TH} \circ i_{AH} = T i_{AG} \circ \tilde{\rho},$$

which is exactly the definition of the prolonged anchor map.

Let us check now the compatibility with the Lie brackets. For that, consider a star section $u : G \to H$. Then, there are sections $Tu : TG \to TH$ and $A(u) : AG \to AH$. Both are related by $A(u) = Tu|_{AG}$. In particular, $i_{AH} \circ A(u) = Tu \circ i_{AG}$ holds. Similarly, every section $k \in \Gamma(K)$ of the core of H induces a section of the tangent prolongation $TH \to TG$. Indeed, first consider the induced section $k_H \in \Gamma_G(H)$ and then construct the core section $\hat{k}_H \in \Gamma_{TG}(TH)$ defined in the usual way:

$$\hat{k}_H(X_g) = T(0^H)X_g + \overline{k_H(g)}.$$

For every $x \in \epsilon_M(M) \subseteq G$ one has $k_H(x) = k(x)$, and thus at any $u_x \in (AG)_x \subseteq T_x G$ we get

$$\hat{k}_H(u_x) = A(0^H)u_x + \overline{k(x)}.$$

Hence we conclude that $i_{AH} \circ k^{\text{core}} = \hat{k}_H \circ i_{AG}$. Let us show that (15) holds for a pair of sections $A(u), A(v)$, where $u, v : G \to H$ are star sections. Indeed,

$$i_{AH} \circ [A(u), A(v)] = i_{AH} \circ A[u, v] = T[u, v] \circ i_{AG} = [Tu, Tv] \circ i_{AG},$$

as desired. It remains to show the bracket condition (15) for sections of the form $A(u), k^{\text{core}}$, where $u : G \to H$ is a star section and $k : M \to K$ is a section of the core. On the one hand, one has that

$$i_{AH} \circ [A(u), k^{\text{core}}] = i_{AH} \circ (D_u k)^{\text{core}} = (\widehat{D_u k})_H \circ i_{AG}.$$

On the other hand,

$$[Tu, \hat{k}_H] \circ i_{AG} = \widehat{[u, k_H]} \circ i_{AG}.$$

Notice that to conclude that (15) holds in this case it suffices to show that $(\widehat{D_u k})_H \circ i_{AG} = \widehat{[u, k_H]} \circ i_{AG}$. Indeed, using the fact that $k = k_H \circ \epsilon_M$ for every section $k : M \to K$, we conclude that if $v_x \in A_x G$, then

$$\widehat{[u, k_H]}(u_x) = T0_G^H(u_x) + \frac{d}{dt}(t[u, k_H](x))|_{t=0}$$
$$= T0_G^H(u_x) + \frac{d}{dt}(t(D_u k)_H(x))|_{t=0} = \widehat{(D_u k)}_H(v_x). \qquad \square$$

5B2. *Dirac groupoids as $\mathscr{LA}$-groupoids.* Let L_G be a multiplicative Dirac structure on a Lie groupoid $G \rightrightarrows M$. This means that we have a $\mathscr{VB}$-subgroupoid $L_G \rightrightarrows E$ of $\mathbb{T}G \rightrightarrows TM \oplus A^*G$, such that $L_G \subseteq \mathbb{T}G$ is also a Dirac subbundle. In particular there is a canonical Lie algebroid structure on $L_G \to G$ with anchor map $L_G \to TG$ the natural projection and Lie bracket $[\![\cdot , \cdot]\!]$ on $\Gamma_G(L_G)$. Given sections e_1, e_2 of E, there exist star sections a_1, a_2 of L_G covering e_1 and e_2, respectively. Since L_G is involutive with respect to the Courant bracket, we conclude that $[\![a_1, a_2]\!]$ is a star section of L_G covering a section e of E. We define $[e_1, e_2] := e$. A straightforward computation shows that with respect to this Lie bracket and the natural projection $E \to TM$, the vector bundle $E \to M$ becomes a Lie algebroid.

Proposition 5.10. *A multiplicative Dirac structure L_G on G gives rise to an $\mathscr{LA}$-groupoid*

$$\begin{tikzcd} L_G \arrow[r, "p_G \oplus c_G"] \arrow[d, shift left] \arrow[d, shift right] & G \arrow[d, shift left] \arrow[d, shift right] \\ E \arrow[r, "q_E"] & M, \end{tikzcd} \tag{16}$$

where p_G and c_G denote the tangent projection and the cotangent projection, respectively.

Proof. Since the structure mappings defining the Lie groupoid $L_G \rightrightarrows E$ are restrictions of the structure mappings of the tangent and cotangent groupoids, a straightforward computation shows that these structure mappings are Lie algebroid morphisms over the structure mapping of G. The fact that the multiplication on L_G is a Lie algebroid morphism over the multiplication on G follows from Proposition 5.2. An argument similar to the one used in the proof of Proposition 5.2 shows that the inversion map on L_G is a Lie algebroid morphism. $\square$

5C. ***The Lie algebroid of a multiplicative Dirac structure.*** We let G be a Lie groupoid over M with Lie algebroid AG. Let L_G be a multiplicative Dirac structure on a Lie groupoid G. According to Proposition 5.1, the canonical pairing $\langle \cdot , \cdot \rangle_G$: $\mathbb{T}G \oplus \mathbb{T}G \to \mathbb{R}$ is a Lie groupoid morphism. Applying the Lie functor yields a nondegenerate symmetric pairing

$$A(\langle \cdot , \cdot \rangle_G) : (A(TG) \oplus A(T^*G)) \times_{AG} (A(TG) \oplus A(T^*G)) \to \mathbb{R}.$$

Let $\langle \cdot , \cdot \rangle_{AG}$ denote the canonical nondegenerate symmetric pairing on $\mathbb{T}(AG)$. Recall that there exist canonical isomorphisms of Lie algebroids $j_G : T(AG) \to A(TG)$ and $j'_G : A(T^*G) \to T^*(AG)$ (see (5) and (6)). Since $\langle \cdot , \cdot \rangle_{AG}$ is just a suitable restriction of $T\langle \cdot , \cdot \rangle_G$, one concludes that the canonical map

$$j_G^{-1} \oplus j'_G : A(TG) \oplus A(T^*G) \to T(AG) \oplus T^*(AG)$$

is a fiberwise isometry with respect to $A(\langle \cdot , \cdot \rangle_G)$ and $\langle \cdot , \cdot \rangle_{AG}$. This is a useful tool for transporting Lagrangian subbundles of $TG \oplus T^*G$ to Lagrangian subbundles of $T(AG) \oplus T^*(AG)$. For instance, given a $\mathcal{VB}$-subgroupoid L_G of $TG \oplus T^*G$, we can apply the Lie functor to obtain a $\mathcal{VB}$-subalgebroid $A(L_G) \subseteq A(TG) \oplus A(T^*G)$. We mimic the construction of tangent Dirac structures, giving rise to a $\mathcal{VB}$-subalgebroid of $T(AG) \oplus T^*(AG)$ defined by

$$L_{AG} := (j_G^{-1} \oplus j'_G)(A(L_G)).$$

The following result is a straightforward consequence of the previous discussion.

Proposition 5.11. *Let $L_G \subseteq TG \oplus T^*G$ be a $\mathcal{VB}$-subgroupoid. Consider the associated $\mathcal{VB}$-subalgebroid $L_{AG} \subseteq T(AG) \oplus T^*(AG)$. Then L_G is isotropic with respect to $\langle \cdot , \cdot \rangle_G$ if and only if L_{AG} is isotropic with respect to $\langle \cdot , \cdot \rangle_{AG}$.*

In particular, if $L_G \subseteq TG \oplus T^*G$ is a $\mathcal{VB}$-subgroupoid with associated $\mathcal{VB}$-subalgebroid $L_{AG} \subseteq T(AG) \oplus T^*(AG)$ then L_G is an almost Dirac structure on G if and only if L_{AG} is an almost Dirac structure on AG.

Now we want to deal with integrability issues. For that, consider a multiplicative Dirac structure $L_G \subseteq \mathbb{T}G$ and let (L_G, G, E, M) be the associated $\mathcal{LA}$-groupoid. Applying the Lie functor we obtain the prolonged Lie algebroid structure on $A(L_G) \to AG$, and we use the canonical map

$$j_G^{-1} \oplus j'_G : A(TG) \oplus A(T^*G) \to T(AG) \oplus T^*(AG)$$

to define a Lie algebroid $L_{AG} = (j_G^{-1} \oplus j'_G)(A(L_G))$ over AG, characterized by the fact that $j_G^{-1} \oplus j'_G : A(L_G) \to L_{AG}$ is a Lie algebroid isomorphism. We have seen that $L_{AG} \subseteq \mathbb{T}(AG)$ is a Lagrangian subbundle with respect to the canonical pairing $\langle \cdot , \cdot \rangle_{AG}$ on $\mathbb{T}(AG)$. We claim that the Lie bracket on $\Gamma_{AG}(L_{AG})$ induced by the prolonged Lie bracket on $\Gamma_{AG}(A(L_G))$ coincides with the Courant bracket. Indeed, since the tangent Lie algebroid $TL_G \to TG$ is isomorphic to $L_{TG} \to TG$, where the latter is equipped with the algebroid structure induced by the tangent Dirac structure $L_{TG} \subset TTG \oplus T^*TG$, and $A(L_G)$ is a Lie subalgebroid of TL_G (Proposition 5.9), then the bracket on sections of L_{AG} induced by the identification $A(L_G) = L_{AG}$ is exactly the restriction of the Courant bracket on $\Gamma(T(AG) \oplus T^*(AG))$. We have proved:

Theorem 5.12. *Given a Lie groupoid G with Lie algebroid AG, there is a canonical map*

$$\mathrm{Dir}_{\mathrm{mult}}(G) \to \mathrm{Dir}_{\mathrm{morph}}(AG),\ L_G \mapsto L_{AG} := (j_G^{-1} \oplus j_G')(A(L_G)).$$

That is, up to a canonical identification, the Lie algebroid of a multiplicative Dirac structure $L_G \subset \mathbb{T}G$ defines a Dirac structure L_{AG} on AG which is also a Lie subalgebroid of $\mathbb{T}(AG)$.

It is interesting to observe that since L_{AG} is the Lie algebroid of the $\mathscr{LA}$-groupoid L_G, in particular L_{AG} inherits the structure of a *double Lie algebroid* [Mackenzie 2000]. Double Lie algebroids were introduced by Mackenzie [2011] as a way to understand Drinfeld's doubles of Lie bialgebroids. As a result, multiplicative Dirac structures provide interesting examples of double Lie algebroids.

5D. ***Dirac groupoids vs. Dirac algebroids.*** This section is concerned with the statement and proof of the main result of this work. We will prove that, whenever G is a source simply connected Lie groupoid with Lie algebroid AG, then the map in Theorem 5.12 is a bijection.

For that, recall that if M is a smooth manifold and $L \subset \mathbb{T}M$ is a Lagrangian subbundle, then there is a well-defined element $\mu_L \in \Gamma(\bigwedge^3 L^*)$ given by

$$\mu_L(a_1, a_2, a_2) := \langle [\![a_1, a_2]\!], a_3\rangle. \tag{17}$$

The element $\mu_L \in \Gamma(\bigwedge^3 L^*)$ is referred to as the *Courant* 3*-tensor* of L. Notice that a Lagrangian subbundle $L \subset \mathbb{T}M$ is a Dirac structure if and only if μ_L vanishes.

Proposition 5.13. *Let G be a Lie groupoid over M. Consider a Lagrangian subbundle $L_G \subset TG \oplus T^*G$, which is also a Lie subgroupoid. Then, the Courant* 3*-tensor of L_G is multiplicative*; *that is*,

$$\mu_{L_G} : \prod_{p_G \oplus c_G}^{3} L_G \to \mathbb{R}$$

is a groupoid morphism.

Proof. Let us consider composable pairs $a_g^i, \bar{a}_h^i$ in L_G with $i = 1, 2, 3$. Set $c_{gh}^i = m_{\mathbb{T}}(a_g^i, \bar{a}_h^i) \in (L_G)_{gh}$, for $i = 1, 2, 3$. Choose a section $c^i \in \Gamma(L_G)$ such that $c^i(gh) = c_{gh}^i$. Since L_G is a $\mathscr{VB}$-groupoid, the multiplication on L_G is fiberwise surjective. In particular, there exist sections $a^i, \bar{a}^i \in \Gamma(L_G)$ such that $m_{\mathbb{T}} \circ (a^i, \bar{a}^i) = c^i \circ m_G$, for every $i = 1, 2, 3$. Clearly $a^i(g) = a_g^i$ and $\bar{a}^i(h) = \bar{a}_h^i$, for $i = 1, 2, 3$.

Then,

$$\begin{aligned}\mu_{L_G}\big((a_g^1, a_g^2, a_g^3) * (\bar{a}_h^1, \bar{a}_h^2, \bar{a}_h^3)\big) &= \mu_{L_G}(c_{gh}^1, c_{gh}^2, c_{gh}^3) = \langle [\![c^1, c^2]\!](gh), c^3(gh)\rangle \\ &= \big\langle m_{\mathbb{T}}([\![a^2, a^2]\!], [\![\bar{a}^1, \bar{a}^2]\!])(g,h), m_{\mathbb{T}}(a^3, \bar{a}^3)(g,h)\big\rangle\end{aligned}$$

The last identity follows from the fact that $(m_{\mathbb{T}}, m_G)$ is a Courant morphism (see Proposition 5.2). Now we use the fact that $\langle\cdot,\cdot\rangle_G$ is a groupoid morphism to conclude that

$$\mu_{L_G}\big((a_g^1, a_g^2, a_g^3) * (\bar{a}_h^1, \bar{a}_h^2, \bar{a}_h^3)\big) = \mu_{L_G}(a_g^1, a_g^2, a_g^3) + \mu_{L_G}(\bar{a}_h^1, \bar{a}_h^2, \bar{a}_h^3).$$

This proves that the function μ_{L_G} is multiplicative. $\square$

We would like to describe explicitly the Lie algebroid morphism induced by the multiplicative tensor $\mu_{L_G} : \prod_{p_G \oplus c_G}^3 L_G \to \mathbb{R}$. For that, we need the next lemma.

Lemma 5.14. *Let M be a smooth manifold. Consider a Lagrangian subbundle $L_M \subset \mathbb{T}M$. Then, for every $(\dot{a}_1, \dot{a}_2, \dot{a}_3) \in TL_M$ the following identity holds*:

$$T\mu_{L_M}(\dot{a}_1, \dot{a}_2, \dot{a}_3) = \mu_{L_{TM}}\big((J_M \oplus \Theta_M)\dot{a}_1, (J_M \oplus \Theta_M)\dot{a}_2, (J_M \oplus \Theta_M)\dot{a}_3\big),$$

where $L_{TM} \subset \mathbb{T}(TM)$ is the tangent lift of L_M.

Proof. One has $T\mu_{L_M}(Ta_1, Ta_2, Ta_3) = T(\mu_{L_M}(a_1, a_2, a_3))$ for every $a_1, a_2, a_3 \in \Gamma_M(L_M)$. On the other hand, the canonical map $J_M \oplus \Theta_M$ applied to each of the sections Ta_1, Ta_2, Ta_3 gives $a_1^T, a_2^T, a_3^T \in \Gamma_{TM}(L_{TM})$. Thus we conclude that

$$\mu_{L_{TM}}(a_1^T, a_2^T, a_3^T) = \langle [\![a_1^T, a_2^T]\!], a_3^T\rangle_{TM} = (\langle [\![a_1, a_2]\!], a_3\rangle_M)^T,$$

which is exactly the tangent functor applied to the function $\mu_{L_M}(a_1, a_2, a_3)$. Therefore, for every triple of sections a_1, a_2, a_3 of L_M we get

$$T\mu_{L_M}(Ta_1, Ta_2, Ta_3) = \mu_{L_{TM}}(a_1^T, a_2^T, a_3^T). \tag{18}$$

Now we notice, using local coordinates, that for every point $\dot{a} \in TL_M$ above $\dot{x} \in TM$ there exists a section $a \in \Gamma_M(L_M)$ such that $Ta(\dot{x}) = \dot{a}$, where $Ta \in \Gamma_{TM}(TL_M)$ is the section obtained by applying the tangent functor to the section a of L_M. This fact together with identity (18) prove the statement. $\square$

As a consequence we obtain a direct proof of the Courant integrability of the tangent lift of a Dirac structure L_M on M.

Corollary 5.15. *Let L_M be an almost Dirac structure on M, and consider the induced almost Dirac structure L_{TM} on TM. Then L_{TM} is Courant integrable if L_M is Courant integrable.*

Now consider a multiplicative Dirac structure L_G on G. The application of the Lie functor to the groupoid morphism μ_{L_G} of Proposition 5.13 yields a Lie algebroid morphism

$$A(\mu_{L_G}) : \prod_{A(p_G \oplus c_G)}^{3} A(L_G) \to \mathbb{R}.$$

Since $A(\mu_{L_G}) = T\mu_{L_G}|_{A(L_G)}$, we conclude:

Proposition 5.16. *For the Lagrangian subbundle $L_{AG} = (j_G^{-1} \oplus j'_G)A(L_G) \subseteq \mathbb{T}(AG)$, we have*

$$A(\mu_{L_G}) = \mu_{L_{AG}} \circ (j_G^{-1} \oplus j'_G)^{(3)},$$

where $(j_G^{-1} \oplus j'_G)^{(3)} : \prod^3_{A(p_G \oplus c_G)} A(L_G) \to \prod^3_{p_{AG} \oplus c_{AG}} L_{AG}$ denotes the natural extension of $(j_G^{-1} \oplus j'_G)$.

Proof. This follows directly from Lemma 5.14 and the fact that j_G and j'_G are suitable restrictions of J_G and Θ_G, respectively. □

Now we are ready to state the main theorem of this work.

Theorem 5.17. *Let G be a source simply connected Lie groupoid with Lie algebroid AG. There is a one-to-one correspondence between multiplicative Dirac structures on G and morphic Dirac structures on AG. The correspondence is given by the map in Theorem 5.12.*

Proof. Let L_G be a multiplicative Dirac structure on G. Consider the Lagrangian subbundle $L_{AG} := (j_G^{-1} \oplus j'_G)(A(L_G)) \subset \mathbb{T}AG$. Since $\mu_{L_G} \equiv 0$, then Proposition 5.16 implies that $\mu_{L_{AG}} \equiv 0$. Thus, L_{AG} is a Dirac structure on AG which is clearly morphic. Notice that the integrability of L_{AG} is also a consequence of Theorem 5.12. Conversely, consider an element $L_A \in \mathrm{Dir}_{\mathrm{morph}}(AG)$; thus L_A is a linear Dirac structure on AG such that $L_A \subseteq \mathbb{T}AG$ is a $\mathcal{VB}$-subalgebroid. Since G is source simply connected, $\mathbb{T}G$ is the source simply connected Lie groupoid which integrates the Lie algebroid $\mathbb{T}AG$. As explained in [Bursztyn et al. ≥ 2013], the $\mathcal{VB}$-subalgebroid $L_A \subseteq \mathbb{T}A$ integrates to a source simply connected $\mathcal{VB}$-subgroupoid $L_G \subseteq \mathbb{T}G$. We will prove that L_G is a multiplicative Dirac structure on G. Since $L_{AG} \subseteq \mathbb{T}AG$ is Lagrangian with respect to the canonical symmetric pairing $\langle \cdot, \cdot \rangle_{AG}$ on $\mathbb{T}AG$, we conclude from Proposition 5.11 that L_G is Lagrangian with respect to the canonical symmetric pairing $\langle \cdot, \cdot \rangle_G$ on $\mathbb{T}G$. It remains to show that $L_G \subseteq \mathbb{T}G$ is integrable with respect to the Courant bracket. Equivalently, we have to prove that the Courant 3-tensor $\mu_{L_G} \in \Gamma(\bigwedge^3 L_G^*)$ is zero. Since $L_A \subseteq \mathbb{T}AG$ is a Dirac structure, the induced Courant 3-tensor $\mu_{L_A} \in \Gamma(\bigwedge^3 L_A^*)$ vanishes. Therefore, combining Proposition 5.16 (applied to the zero Lie algebroid morphism) with Lie's second theorem we conclude that $\mu_{L_G} \equiv 0$, as desired. This shows that L_G is a Dirac structure on G, which by definition is multiplicative. □

Remark 5.18. Theorem 5.17 provides a direct proof of the Notice that integrability of the Lagrangian subbundle $L_{AG} \subset \mathbb{T}(AG)$ associated to a multiplicative Dirac structure $L_G \subset \mathbb{T}G$, without using the theory of $\mathcal{LA}$-groupoids. In spite of this, we believe that the fact that L_{AG} inherits the structure of a double Lie algebroid is interesting in itself. This relies on the observation that L_G is an $\mathcal{LA}$-groupoid.

5E. ***Main examples revisited.*** We have shown several examples of Dirac groupoids and Dirac algebroids. See Sections 3 and 4, respectively. Here we will see that both classes of examples are related by the construction explained in Section 5C. Throughout this subsection G denotes a Lie groupoid over M with Lie algebroid AG.

5E1. *Poisson groupoids and Lie bialgebroids.* Consider a multiplicative Poisson bivector π_G on G. It is well known that in this case $M \subseteq G$ is a coisotropic submanifold and, in particular, the conormal bundle $N^*M \cong A^*G$ inherits a Lie algebroid structure. The Dirac structure L_G on G defined by the graph of π_G is a multiplicative Dirac structure. The multiplicativity of this Dirac structure is equivalent to $\pi_G^\sharp : T^*G \to TG$ being a morphism of Lie groupoids, and the associated Lie algebroid morphism coincides, up to identifications, with $\pi_{AG}^\sharp : T^*(AG) \to T(AG)$, where π_{AG} denotes the linear Poisson bivector on AG dual to the Lie algebroid A^*G. One concludes that the corresponding Dirac structure L_{AG} on AG is exactly the graph of π_{AG}. Since L_{AG} is a Lie subalgebroid of $\mathbb{T}AG$, the bundle map $\pi_{AG}^\sharp : T^*(AG) \to T(AG)$ is a Lie algebroid morphism. This is equivalent to saying that (AG, A^*G) is a Lie bialgebroid. As a corollary of Theorem 5.17 we obtain:

Corollary 5.19 [Mackenzie and Xu 2000]. *Let G be a source simply connected Lie groupoid with Lie algebroid AG. There is a one-to-one correspondence between multiplicative Poisson bivectors on G and Lie bialgebroid structures on (AG, A^*G).*

5E2. *Multiplicative 2-forms and IM-2-forms.* Assume that $\omega_G \in \Omega^2(G)$ is a multiplicative closed 2-form on G. The Dirac structure L_G given by the graph of $\omega_G^\sharp : TG \to T^*G$ is multiplicative. In this case, the corresponding Dirac structure L_{AG} on AG is given by the graph of the closed 2-form $\omega_{AG} := -\sigma^*\omega_{can}$, where $\sigma : AG \to T^*M$ is defined by $\sigma(u) = i_u\omega_G|_{TM}$. Since the Dirac structure L_{AG} is a Lie subalgebroid of $\mathbb{T}(AG)$, we conclude that the bundle map $\omega_{AG}^\sharp : T(AG) \to T^*(AG)$ is a Lie algebroid morphism. As shown in [Bursztyn et al. 2009a], this is equivalent to the bundle map $\sigma : AG \to T^*M$ being an *IM-2-form* on AG; that is, for every $u, v \in \Gamma(AG)$, the following conditions hold:

- $\langle\sigma(u), \rho_{AG}(v)\rangle = -\langle\sigma(v), \rho_{AG}(u)\rangle$;
- $\sigma[u, v] = \mathscr{L}_{\rho_{AG}(u)}\sigma(v) - \mathscr{L}_{\rho_{AG}(v)}\sigma(u) + d\langle\sigma(u), \rho_{AG}(v)\rangle$.

As a corollary of Theorem 5.17, we get:

Corollary 5.20 [Bursztyn et al. 2004]. *Let G be a source simply connected Lie groupoid with Lie algebroid AG. There is a one-to-one correspondence between multiplicative closed 2-forms on G and IM-2-forms on AG.*

5E3. *Foliated groupoids and foliated algebroids.* Let $F_G \subseteq TG$ be a multiplicative involutive subbundle. Then, the Dirac structure $L_G = F_G \oplus F_G^\circ$ is multiplicative.

The corresponding Dirac structure L_{AG} on AG associated to L_G is given by $L_{AG} = F_{AG} \oplus F_{AG}^\circ \subset \mathbb{T}(AG)$, where $F_{AG} := j_G^{-1}(A(F_G)) \subseteq T(AG)$. Since L_{AG} is a Dirac structure which is also a Lie subalgebroid of $\mathbb{T}(AG)$, we conclude that $F_{AG} \subseteq T(AG)$ is an involutive subbundle which is also a Lie subalgebroid of $T(AG) \to TM$. We refer to such a subbundle as a *morphic foliation* on AG. As a corollary of Theorem 5.17, we obtain the next result.

Corollary 5.21 [Hawkins 2008]. *Let G be a source simply connected Lie groupoid with Lie algebroid AG. There exists a one-to-one correspondence between multiplicative foliations on G and morphic foliations on AG.*

As shown in [Hawkins 2008; Jotz and Ortiz 2012], having a morphic foliation on AG is equivalent to AG be equipped with an *IM-foliation*, that is, a triple (F_M, K, ∇) where $F_M \subseteq TM$ is an involutive subbundle, $K \subseteq AG$ is a Lie subalgebroid with $\rho_{AG}(K) \subseteq F_M$, and ∇ is an F_M-connection on AG/K satisfying the following conditions:

- ∇ is flat.
- If $u \in \Gamma(AG)$ satisfies $\nabla_{\Gamma(F_M)}(u + K) \in \Gamma(K)$, then $[u, \Gamma(K)] \subseteq \Gamma(K)$.
- If $u, v \in \Gamma(AG)$ are such that $\nabla_{\Gamma(F_M)}(u + K), \nabla_{\Gamma(F_M)}(v + K) \in \Gamma(K)$, it follows that $\nabla_{\Gamma(F_M)}([u, v] + K) \in \Gamma(K)$.
- If $u \in \Gamma(AG)$ satisfies $\nabla_{\Gamma(F_M)}(u + K) \in \Gamma(K)$, then $[\rho_{AG}(u), \Gamma(F_M)] \subseteq \Gamma(F_M)$.

The properties as above determine completely the morphic foliation F_{AG} on AG. In particular, Dirac structures of the form $L_{AG} = F_{AG} \oplus F_{AG}^\circ$ are in one-to-one correspondence with IM-foliations. Additionally, there exists a conceptually clear interpretation of IM-foliations in terms of representations up to homotopy. See [Drummond et al. 2013] for more details.

5E4. *Dirac Lie groups and Dirac Lie algebras.* Let G be a Lie *group* with Lie algebra $\mathfrak{g}$ and let $L_G \in \mathrm{Dir}_{\mathrm{mult}}(G)$ be a multiplicative Dirac structure. Consider the Dirac structure $L_{\mathfrak{g}}$ on $\mathfrak{g}$ associated to L_G. It was shown in [Ortiz 2008] that $\ker(L_G) := L_G \cap TG$ is a regular involutive subbundle of TG, in particular $\ker(L_{\mathfrak{g}}) = j_G^{-1}(A(\ker(L_G)))$ is an involutive subbundle of $T\mathfrak{g}$. Since $\ker(L_{\mathfrak{g}})$ is a linear foliation on $\mathfrak{g}$, that is, multiplicative with respect to the abelian group structure on $\mathfrak{g}$, then the leaf through $0 \in \mathfrak{g}$ is a vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$. The other leaves are affine subspaces of $\mathfrak{g}$ modeled on $\mathfrak{h}$. In particular, the space of characteristic leaves of $L_{\mathfrak{g}}$ coincides with the quotient space $\mathfrak{g}/\mathfrak{h}$. The fact that $L_g \subseteq \mathbb{T}\mathfrak{g}$ is a Lie subalgebroid implies that $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal. Therefore, the space of characteristic leaves $\mathfrak{g}/\mathfrak{h}$ of $L_{\mathfrak{g}}$ inherits a unique Lie algebra structure making the quotient map $\phi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ into a surjective Lie algebra morphism. Since $\mathfrak{g}/\mathfrak{h}$ is the space of characteristic leaves of $L_{\mathfrak{g}}$, there is a unique Poisson structure π on $\mathfrak{g}/\mathfrak{h}$ making the quotient map $\phi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ into a forward and backward Dirac map. Since $L_{\mathfrak{g}}$ is a morphic Dirac structure, we conclude that π is a morphic bivector on $\mathfrak{g}/\mathfrak{h}$. In particular, the pair $(\mathfrak{g}/\mathfrak{h}, (\mathfrak{g}/\mathfrak{h})^*)$ is a Lie bialgebra. Conversely, given a Lie algebra $\mathfrak{g}$ and an ideal $\mathfrak{h} \subseteq \mathfrak{g}$ such that $(\mathfrak{g}/\mathfrak{h}, (\mathfrak{g}/\mathfrak{h})^*)$ is a Lie bialgebra, then the linear Poisson bivector π on $\mathfrak{g}/\mathfrak{h}$ is morphic. The surjective Lie algebra morphism $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ induces a Dirac structure $L_{\mathfrak{g}}$ on $\mathfrak{g}$ (the pull-back of π) which is morphic as well. We have proved the following result.

Proposition 5.22. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. There is a one-to-one correspondence between*

(1) *morphic Dirac structures on $\mathfrak{g}$, and*

(2) *ideals $\mathfrak{h} \subseteq \mathfrak{g}$ such that $(\mathfrak{g}/\mathfrak{h}, (\mathfrak{g}/\mathfrak{h})^*)$ is a Lie bialgebra.*

The proposition above recovers the results of [Ortiz 2008].

5E5. *Tangent lifts of Dirac structures.* Let L_G be a multiplicative Dirac structure on G. Consider the associated morphic Dirac structure L_{AG} on the Lie algebroid of G. We can lift L_G to a multiplicative Dirac structure on the tangent groupoid TG. Similarly, as explained in Section 4B3, the morphic Dirac structure L_{AG} can be lifted to a morphic Dirac structure $L_{T(AG)}$ on the tangent Lie algebroid $T(AG) \to TM$. It is straightforward to check that the morphic Dirac structure on $T(AG)$ associated to L_{TG} as in Theorem 5.17 coincides with the tangent lift $L_{T(AG)}$ of L_{AG}. That is, the tangent functor commutes with the Lie functor.

5E6. *Symmetries of Dirac groupoids.* Let L_G be a multiplicative Dirac structure on G. Consider the associated morphic Dirac structure L_{AG} on AG as in Theorem 5.17. Let H be a Lie group acting freely and properly on G by groupoid automorphisms $\Phi_h : G \to G$, $h \in H$. Applying the Lie functor to each $\Phi_h : G \to G$ yields a free and proper H-action on AG by Lie algebroid automorphisms $A(\Phi_h) : AG \to AH$, $h \in H$. Assume that the H-orbits of G coincide with the characteristic leaves of L_G. Then, the H-orbits of AG coincide with the characteristic leaves of L_{AG}. We have shown that in this situation we can endow the space of characteristic leaves G/H of L_G with a unique multiplicative Poisson bivector $\pi_{G/H}$ making the quotient map $G \to G/H$ into a forward and backward Dirac map. Similarly, the space of characteristic leaves AG/H of L_{AG} inherits a unique morphic Poisson structure $\pi_{AG/H}$ making the quotient map $AG \to AG/H$ into a forward and backward Dirac map. One can easily see that the morphic Dirac structure $L_{AG/H}$ associated to $\pi_{G/H}$ as in Section 5E1 coincides with the morphic Dirac structure on AG/H given by the graph of $\pi_{AG/H}$. As a consequence, the Lie bialgebroid of $(G/H, \pi_{G/H})$ is exactly $(AG/H, (AG/H)^*)$.

5E7. *B-field transformations.* Let L_G be a multiplicative Dirac structure on G. Assume that B_G is a multiplicative closed 2-form on G. Consider the Dirac structure L_G^B on G, obtained out of L_G by applying the B-field transformation with respect to B_G. As observed in [Bursztyn et al. 2009a], every multiplicative closed 2-form on G induces a morphic closed 2-form B_{AG} on AG. A direct computation shows that the morphic Dirac structure L_{AG}^B corresponding to L_G^B (as in Theorem 5.17) is given by the B-field transformation of L_{AG} with respect to B_{AG}, in agreement with [Ortiz 2012].

5E8. *Generalized complex groupoids.* Let $L_G \subseteq \mathbb{T}_{\mathbb{C}}G$ be a multiplicative generalized complex structure on G. The construction explained in Theorem 5.12 applies also to the case of multiplicative generalized complex structures. As a result, there is a morphic Dirac structure $L_{AG} \subseteq \mathbb{T}_{\mathbb{C}}AG$ given by $L_{AG} := (j_G^{-1} \oplus j'_G)_{\mathbb{C}}(A(L_G))$, where $(j_G^{-1} \oplus j'_G)_{\mathbb{C}} : A(\mathbb{T}_{\mathbb{C}}G) \to \mathbb{T}_{\mathbb{C}}(AG)$ denotes the complexification of the canonical isomorphism $(j_G^{-1} \oplus j'_G) : A(\mathbb{T}G) \to \mathbb{T}(AG)$. Observe that $L_{AG} \subseteq \mathbb{T}_{\mathbb{C}}AG$ is in fact a generalized complex structure making the pair (AG, L_{AG}) into a generalized Lie algebroid. For that, we only need to check that $L_{AG} \cap \overline{L_{AG}} = \{0\}$. Indeed, one easily checks that the conjugation map $\overline{(\,\cdot\,)}_G : \mathbb{T}_{\mathbb{C}}G \to \mathbb{T}_{\mathbb{C}}G$ is a Lie groupoid isomorphism. Therefore, the generalized complex structure $\bar{L}_G$ on G is also multiplicative. Since $\mathbb{T}_{\mathbb{C}}G = L_G \oplus \bar{L}_G$, the application of the Lie functor yields a decomposition

$$A(\mathbb{T}_{\mathbb{C}}G) = A(L_G) \oplus A(\bar{L}_G). \tag{19}$$

Straightforward computation shows that the Lie algebroid isomorphism $A(\overline{(\,\cdot\,)}_G) : A(\mathbb{T}_{\mathbb{C}}G) \to A(\mathbb{T}_{\mathbb{C}}G)$ satisfies

$$(j_G^{-1} \oplus j'_G)_{\mathbb{C}} \circ A(\overline{(\,\cdot\,)}_G) = \overline{(\,\cdot\,)}_{AG},$$

where the map of the right hand side of the identity above is the conjugation map $\mathbb{T}_{\mathbb{C}}(AG) \to \mathbb{T}_{\mathbb{C}}(AG)$. Hence, applying the canonical isomorphism $(j_G^{-1} \oplus j'_G)_{\mathbb{C}} : A(\mathbb{T}_{\mathbb{C}}G) \to \mathbb{T}_{\mathbb{C}}(AG)$ on both sides of (19), gives rise to

$$\mathbb{T}_{\mathbb{C}}AG = L_{AG} \oplus \bar{L}_{AG}.$$

Therefore, L_{AG} is transversal to $\bar{L}_{AG}$ and we conclude that L_{AG} is a morphic generalized complex structure. In this situation, Theorem 5.17 gives rise to the following result.

Proposition 5.23 [Jotz et al. 2012]. *Let G be a source simply connected Lie groupoid with Lie algebroid AG. There is a one-to-one correspondence between multiplicative generalized complex structures on G and morphic generalized complex structures on AG.*

6. Conclusions and final remarks

This work can be considered as the first step toward describing multiplicative Dirac structures infinitesimally. We have seen that every multiplicative Dirac structure L_G on a Lie groupoid G induces a Dirac structure L_{AG} on its Lie algebroid AG which is compatible with the algebroid structure in the sense that $L_{AG} \subseteq \mathbb{T}(AG)$ is a Lie subalgebroid. Notice that in the special situation of Poisson groupoids (resp. multiplicative closed 2-forms, multiplicative foliations) the induced Dirac structure on AG is equivalent to endowing (AG, A^*G) with a Lie bialgebroid structure (resp. IM-2-form, IM-foliation). Therefore, it would be interesting to introduce a suitable notion of *IM-Dirac structure*, providing a more explicit description of Dirac structures compatible with a Lie algebroid, unifying different infinitesimal structures such as Lie bialgebroids, IM-2-forms and IM-foliations. This study will be part of a future work.

Acknowledgements

The content of this work is based on the author's doctoral thesis [Ortiz 2009] carried out at IMPA under the supervision of Henrique Bursztyn. The author would like to thank Henrique Bursztyn for suggesting the problem of studying multiplicative Dirac structures as well as for several conversations and suggestions that have improved this work. While the author was at IMPA, he was supported by a PEC-PG doctoral scholarship from CAPES (Brazil). Finally, the author would like to thank the referee for all comments and suggestions that have improved the presentation of this paper.

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Received December 12, 2012. Revised May 14, 2013.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 266, No. 2, 2013
dx.doi.org/10.2140/pjm.2013.266.367

ON THE FINITE GENERATION OF A FAMILY OF EXT MODULES

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Dedicated to Professor L. L. Avramov on the occasion of his sixtieth birthday.

Let $(A, \mathfrak{m})$ be a local complete intersection ring. Let M, N be finitely generated A-modules and let I be an ideal in A. We show that

$$\bigcup_{n\geq 0}\bigcup_{i\geq 0} \operatorname{Ass}\operatorname{Ext}^i_A(M, I^n N)$$

is a finite set. We also show that there exist i_0, n_0 such that for all $i \geq i_0$ and $n \geq n_0$ we have

$$\operatorname{Ass}\operatorname{Ext}^{2i}_A(M, I^n N) = \operatorname{Ass}\operatorname{Ext}^{2i_0}_A(M, I^{n_0} N),$$
$$\operatorname{Ass}\operatorname{Ext}^{2i+1}_A(M, I^n N) = \operatorname{Ass}\operatorname{Ext}^{2i_0+1}_A(M, I^{n_0} N).$$

We prove analogous results for complete intersection rings which arise in algebraic geometry. We also prove that the complexity, $\operatorname{cx}(M, I^n N)$, is constant for all $n \gg 0$.

1. Introduction

Let A be a Noetherian ring. Let I be an ideal in A and let M be a finitely generated A-module. M. Brodmann [1979] proved that the set $\operatorname{Ass}_A M/I^n M$ is independent of n for all large n. This result is usually deduced by proving that $\operatorname{Ass}_A I^n M/I^{n+1} M$ is independent of n for all large n.

We state some generalizations of Brodmann's result. Fix $i \geq 0$. L. Melkersson and P. Schenzel [1993, Theorem 1] showed that

$$\operatorname{Ass}_A \operatorname{Tor}^A_i(M, I^n/I^{n+1}) \quad \text{and} \quad \operatorname{Ass}_A \operatorname{Tor}^A_i(M, A/I^n)$$

are independent of n for all large n. By the same argument,

$$\operatorname{Ass}_A \operatorname{Ext}^i_A(M, I^n/I^{n+1})$$

The work for this paper was done while the author was visiting University of Kentucky by a fellowship from Department of Science and Technology, India. The author is deeply grateful to DST for its financial support and University of Kentucky for its hospitality.
MSC2010: primary 13D07, 13H10; secondary 13A15, 13A02.
Keywords: local complete intersection, asymptotic associate primes, cohomological operators.

and, by [Katz and West 2004, 3.5],

$$\operatorname{Ass}_A \operatorname{Ext}^i_A(M, A/I^n)$$

are similarly independent of n. An example of A. Singh [2000] shows that

$$\operatorname{Ass}_A \varinjlim \operatorname{Ext}^i_A(A/I^n, M) \text{ need not be finite.}$$

So in this example

$$\bigcup_{n\geq 1} \operatorname{Ass}_A \operatorname{Ext}^i_A(A/I^n, M) \text{ is not even finite.}$$

I state some questions in this area that motivated me. They were raised respectively by W. Vasconcelos [1998, 3.5] and Melkersson and Schenzel [1993, page 936].

$$\text{Is the set } \bigcup_{i\geq 0} \operatorname{Ass}_A \operatorname{Ext}^i_A(M, A) \text{ finite?} \tag{1}$$

$$\text{Is the set } \bigcup_{i\geq 0}\bigcup_{n\geq 0} \operatorname{Ass}_A \operatorname{Tor}^A_i(M, A/I^n) \text{ finite?} \tag{2}$$

The motivation for the main result of this paper came from (1). I do not believe that the question has a positive answer in this generality, but I am unable to give a counterexample. Note that if A is a Gorenstein local ring then Vasconcelos's question has, trivially, a positive answer. If we change the question a little then we may ask: If M, D are two finitely generated A-modules,

$$\text{is the set } \bigcup_{i\geq 0} \operatorname{Ass}_A \operatorname{Ext}^i_A(M, D) \text{ finite?}$$

This is not known for Gorenstein rings in general. However, if $A = Q/(\boldsymbol{f})$, where $\boldsymbol{f} = f_1, \ldots, f_c$ is a regular sequence, and if $\operatorname{projdim}_Q M$ is finite, then the above question has a positive answer. This can be seen by using the theory of cohomology operators over such rings. This turns $\bigoplus_{i\geq 0} \operatorname{Ext}^i_A(M, D)$ into a finitely generated module over $A[t_1, \ldots, t_c]$, where t_i has degree 2 for each i.

Using Melkerson and Schenzel's question as a guidepost, I was interested to solve the the following questions: Let $(A, \mathfrak{m})$ be a local complete intersection of codimension c.

$$\text{Is the set } \bigcup_{i\geq 0}\bigcup_{j\geq 0} \operatorname{Ass}_A \operatorname{Ext}^i_A(M, D/I^jD) \text{ finite?} \tag{a}$$

$$\text{Is the set } \bigcup_{i\geq 0}\bigcup_{j\geq 0} \operatorname{Ass}_A \operatorname{Ext}^i_A(M, I^jD) \text{ finite?} \tag{b}$$

In Theorem 5.1 I prove that (b) holds. I have been unable to verify whether (a) holds.

Let $\mathscr{R}(I) = \bigoplus_{n \geq 0} I^n t^n$ be the Rees algebra of I. The main result in this paper concerns finite generation of a family of Ext modules:

Theorem 1.1. *Let Q be a Noetherian ring with finite Krull dimension and let $\boldsymbol{f} = f_1, \dots f_c$ be a regular sequence in Q. Set $A = Q/(\boldsymbol{f})$. Let M be a finitely generated A-module with* $\operatorname{projdim}_Q M$ *finite. Let I be an ideal in A and let $N = \bigoplus_{n \geq 0} N_n$ be a finitely generated $\mathscr{R}(I)$-module. Then*

$$\mathscr{E}(N) = \bigoplus_{i \geq 0} \bigoplus_{n \geq 0} \operatorname{Ext}^i_A(M, N_n)$$

is a finitely generated *bigraded $\mathscr{S} = \mathscr{R}(I)[t_1, \dots, t_c]$-module.*

Remark 1.2. See Section 2.3 for a description of $\mathscr{E}(N)$ as a $\mathscr{S} = \mathscr{R}(I)[t_1, \dots, t_c]$-module.

An easy consequence of this result is that (b) holds (by taking $N = \bigoplus_{n \geq 0} I^n D$); see Theorem 5.2. A complete, local complete intersection ring is a quotient of a regular local ring mod a regular sequence. So in this case (b) holds from Theorem 5.2. The proof of (b) for local complete intersections in general is a little technical; see Theorem 5.1. We also prove (b) for complete intersection rings which arise in algebraic geometry; see Section 6.

We next discuss a surprising consequence of Theorem 1.1. Let $(A, \mathfrak{m})$ be a local complete intersection of codimension c. Let M, N be two finitely generated A-modules. Let $\mu(X)$ denote the number of minimal generators of a finitely generated A-module X. Define

$$\operatorname{cx}_A(M, N) = \inf \left\{ b \in \mathbb{N} \;\middle|\; \varlimsup_{n \to \infty} \frac{\mu(\operatorname{Ext}^n_A(M, N))}{n^{b-1}} < \infty \right\}.$$

In Section 7 we prove (see Theorem 7.1) that

$$(\dagger) \qquad \operatorname{cx}_A(M, I^j N) \quad \text{is constant for all } j \gg 0.$$

We now describe in brief the contents of this paper. In Section 2 we give a module structure to $\mathscr{E}(N)$ over $\mathscr{S}$ (as in Theorem 1.1). We also discuss a few preliminaries. The local case of Theorem 1.1 is proved in Section 3 while the global case is proved in Section 4. In Section 5 we prove our results on asymptotic primes in the case of local complete intersections. In Section 6 we prove our result on asymptotic primes in complete intersection rings which arise in algebraic geometry. In Section 7 we prove $(\dagger)$.

2. Module structure

Let Q be a Noetherian ring and let $\boldsymbol{f} = f_1, \dots f_c$ be a regular sequence in Q. Set $A = Q/(\boldsymbol{f})$. Let M be a finitely generated A-module with $\operatorname{projdim}_Q M$ finite.

We will not change M throughout our discussion. Let I be an ideal in A. Let $\mathcal{R}(I) = \bigoplus_{n\geq 0} I^n X^n$ be the *Rees algebra* of I. We consider $\mathcal{R}(I)$ as a subring of the polynomial ring $A[X]$. Let $N = \bigoplus_{n\geq 0} N_n$ be a finitely generated $\mathcal{R}(I)$-module.

Set

$$\mathcal{E}(N) = \bigoplus_{i\geq 0} \bigoplus_{n\geq 0} \operatorname{Ext}^i_A(M, N_n).$$

In this section we show $\mathcal{E}(N)$ is a bigraded $\mathcal{S} = \mathcal{R}(I)[t_1, \dots, t_c]$-module. The grading on $\mathcal{S}$ is as follows: we set $\deg t_j = (0, 2)$ for $j = 1, \dots, c$, and for $a \in I^s$ we set $\deg aX^s = (s, 0)$. We also discuss two preliminary results that we will need later in this paper.

2.1. Let $\mathbb{F} : \cdots F_n \to \cdots \to F_1 \to F_0 \to 0$ be a free resolution of M as an A-module.

Let $t_1, \dots t_c : \mathbb{F}(+2) \to \mathbb{F}$ be the *Eisenbud operators*; see [Eisenbud 1980, Section 1]. Then:

(1) The t_i are uniquely determined up to homotopy.

(2) Any two of them commute up to homotopy.

Let $T = A[t_1, \dots, t_c]$ be a polynomial ring over A with variables $t_1, \dots, t_c$ of degree 2. Let D be an A-module. The operators t_j give well-defined maps

$$t_j : \operatorname{Ext}^i_A(M, D) \to \operatorname{Ext}^{i+2}_R(M, D) \quad \text{for } 1 \leq j \leq c \text{ and all } i,$$

which turn $\operatorname{Ext}^*_A(M, D) = \bigoplus_{i\geq 0} \operatorname{Ext}^i_A(M, D)$ into a module over T. Furthermore, these structures depend only on f, are natural in both module arguments and commute with the connecting maps induced by short exact sequences.

2.2. Gulliksen [1974, 3.1] proved that if $\operatorname{projdim}_Q M$ is finite then $\operatorname{Ext}^*_A(M, D)$ is a finitely generated T-module. If A is local and $D = k$, the residue field of A, Avramov [1989, 3.10] proved a converse; that is, if $\operatorname{Ext}^*_A(M, k)$ is a finitely generated T-module then $\operatorname{projdim}_Q M$ is finite. For a more general result, see [Avramov et al. 1997, 4.2].

2.3. Let $N = \bigoplus_{n\geq 0} N_n$ be a finitely generated module over $\mathcal{R}(I)$. Let $a \in I^s$. Consider $u = aX^s \in \mathcal{R}(I)_s$. The map

$$N_n \xrightarrow{u} N_{n+s}$$

yields a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}(\mathbb{F}, N_n) \arrow[r, "t_j"'] \arrow[d, "u"] & \operatorname{Hom}(\mathbb{F}, N_n)(+2) \arrow[d, "u"] \\
\operatorname{Hom}(\mathbb{F}, N_{n+s}) \arrow[r, "t_j"'] & \operatorname{Hom}(\mathbb{F}, N_{n+s})(+2).
\end{tikzcd}$$

Taking homology gives that $\mathscr{E}(N) = \bigoplus_{i\geq 0}\bigoplus_{n\geq 0} \operatorname{Ext}^i_A(M, N_n)$ is a bigraded $\mathscr{S}$-module, where $\mathscr{S} = \mathscr{R}(I)[t_1, \dots, t_c]$.

Remark 2.4. (1) For each i, the $\mathscr{R}(I)$-module $\bigoplus_{n\geq 0} \operatorname{Ext}^i_A(M, N_n)$ is finitely generated.

(2) For each n, the $A[t_1, \dots, t_c]$-module $\bigoplus_{i\geq 0} \operatorname{Ext}^i_A(M, N_n)$ is finitely generated.

2.5. ***Notation.*** (1) Let $N = \bigoplus_{n\geq 0} N_n$ be a graded $\mathscr{R}(I)$-module. Fix $j \geq 0$. Set

$$N_{\geq j} = \bigoplus_{n\geq j} N_n.$$

$\mathscr{E}(N_{\geq j})$ is naturally isomorphic to the submodule

$$\mathscr{E}(N)_{\geq j} = \bigoplus_{i\geq 0}\bigoplus_{n\geq j} \mathscr{E}(N)_{ij}$$

of $\mathscr{E}(N)$.

(2) If $A \to A'$ is a ring extension and if D is an A-module then set $D' = D \otimes_A A'$. Notice that if D is a finitely generated A-module then D' is a finitely generated A'-module.

(3) Set $\mathscr{S}' = \mathscr{S} \otimes_A A'$. Notice that $\mathscr{S}'$ is a finitely generated bigraded A'-algebra. Let $U = \bigoplus_{i\geq 0}\bigoplus_{n\geq 0} U_{i,n}$ be a graded $\mathscr{S}$-module. Then

$$U' = U \otimes_A A' = \bigoplus_{i\geq 0}\bigoplus_{n\geq 0} U'_{i,n}$$

is a graded $\mathscr{S}'$-module.

We state two lemmas that will help us in proving Theorem 1.1.

Lemma 2.6. *If $\mathscr{E}(N_{\geq j})$ is a finitely generated $\mathscr{S}$-module then $\mathscr{E}(N)$ is a finitely generated $\mathscr{S}$-module.*

Proof. Set $D = \mathscr{E}(N)/\mathscr{E}(N_{\geq j})$. We have the following exact sequence of $\mathscr{S}$-modules

$$0 \to \mathscr{E}(N_{\geq j}) \to \mathscr{E}(N) \to D \to 0.$$

Using Gulliksen's result it follows that D is a finitely generated $T = A[t_1, \dots, t_c]$-module. Since T is a subring of $\mathscr{S}$, we get that D is a finitely generated $\mathscr{S}$-module. Thus if $\mathscr{E}(N_{\geq j})$ is a finitely generated $\mathscr{S}$-module then $\mathscr{E}(N)$ is a finitely generated $\mathscr{S}$-module □

Lemma 2.7. (Keep the notation of 2.5(3).) *Let $A \to A'$ be a faithfully flat extension of rings and let $U = \bigoplus_{i\geq 0}\bigoplus_{n\geq 0} U_{i,n}$ be a graded $\mathscr{S}$-module. If U' is a finitely generated $\mathscr{S}'$-module then U is a finitely generated $\mathscr{S}$-module.*

Proof. The set

$$\mathfrak{D} = \{u_{in} \otimes 1 \mid u_{in} \in U_{in}, \text{ where } i, n \geq 0\}$$

generates U' as a $\mathcal{S}'$-module. As U' is a finitely generated $\mathcal{S}'$-module, we can choose a finite subset $\mathscr{C}$ of $\mathfrak{D}$ which generates U' as a $\mathcal{S}'$-module. Let

$$V = \langle u \mid u \otimes 1 \in \mathscr{C}\rangle.$$

Then V is a finitely generated submodule of U. Notice that $U' = V'$. Thus $(U/V) \otimes_A A' = 0$. Since A' is a faithfully flat A-algebra we get $U = V$. So U is a finitely generated $\mathcal{S}$-module. □

3. The local case

In this section we prove Theorem 1.1 when $(Q, \mathfrak{n})$ is local. Let $\mathfrak{m}$ be the maximal ideal of A. Set $k = A/\mathfrak{m}$. Let I be an ideal in A. Let

$$F(I) = \mathcal{R}(I) \otimes_A k = \bigoplus_{n \geq 0} I^n/\mathfrak{m}I^n$$

be the *fiber cone* of I.

3.1. Assume $N = \bigoplus_{n \geq 0} N_n$ is a finitely generated $\mathcal{R}(I)$-module. Notice that

$$F(N) = N \otimes_A k = \bigoplus_{n \geq 0} N_n/\mathfrak{m}N_n$$

is a finitely generated $F(I)$-module. Define

$$\operatorname{spread}(N) := \dim_{F(I)} N/\mathfrak{m}N.$$

Proof of Theorem 1.1 in the local case.

Case 1: *The residue field $k = A/\mathfrak{m}$ is infinite.* We induct on spread(N). First assume spread$(N) = 0$. This implies that $N_n/\mathfrak{m}N_n = 0$ for all $n \gg 0$. By Nakayama's lemma, $N_n = 0$ for all $n \gg 0$; say $N_n = 0$ for all $n \geq j$. Then $\mathscr{E}(N_{\geq j}) = 0$ and it is obviously a finitely generated $\mathcal{S}$-module. By Lemma 2.6 we get that $\mathscr{E}(N)$ is a finitely generated $\mathcal{S}$-module.

When spread$(N) > 0$ then there exists $u = xt \in \mathcal{R}(I)_1$ which is $(N \oplus F(N))$-filter-regular, that is, there exists j such that

$$(0 \colon {}_N u)_n = 0 \quad \text{and} \quad (0 \colon {}_{F(N)} u)_n = 0 \quad \text{for all } n \geq j.$$

Set $N_{\geq j} = \bigoplus_{n \geq j} N_n$ and $U = N_{\geq j}/uN_{\geq j}$. We have an exact sequence of $\mathcal{R}(I)$-modules

$$0 \to N_{\geq j}(-1) \xrightarrow{u} N_{\geq j} \to U \to 0.$$

For each $n \geq j$ the functor $\operatorname{Hom}_A(M, -)$ induces the long exact sequence of A-modules

$$\begin{array}{llllll} 0 \to \operatorname{Hom}_A(M, N_n) & \xrightarrow{u} & \operatorname{Hom}_A(M, N_{n+1}) & \to & \operatorname{Hom}_A(M, U_{n+1}) \\ \to \operatorname{Ext}^1_A(M, N_n) & \xrightarrow{u} & \operatorname{Ext}^1_A(M, N_{n+1}) & \to & \operatorname{Ext}^1_A(M, U_{n+1}) \\ \to \quad \cdots & \xrightarrow{u} & \cdots & \to & \cdots \\ \to \operatorname{Ext}^i_A(M, N_n) & \xrightarrow{u} & \operatorname{Ext}^i_A(M, N_{n+1}) & \to & \operatorname{Ext}^i_A(M, U_{n+1}) \\ \to \quad \cdots & \xrightarrow{u} & \cdots & \to & \cdots & . \end{array}$$

Using the naturality of Eisenbud operators we have the following exact sequence of $\mathcal{S}$-modules

$$\mathscr{E}(N_{\geq} j)(-1, 0) \xrightarrow{(u,0)} \mathscr{E}(N_{\geq} j) \to \mathscr{E}(U).$$

By construction,

$$\operatorname{spread}(U) = \operatorname{spread}(N_{\geq j}) - 1 = \operatorname{spread}(N) - 1.$$

By the induction hypothesis, $\mathscr{E}(U)$ is a finitely generated $\mathcal{S}$-module. Therefore by Lemma 3.2 we get $\mathscr{E}(N_{\geq j})$ is a finitely generated $\mathcal{S}$-module. Using Lemma 2.6 we get that $\mathscr{E}(N)$ is a finitely generated $\mathcal{S}$-module.

Case 2: *The residue field k is finite.*

In this case we do the standard trick. Let $Q' = Q[X]_{\mathfrak{n}Q[X]}$. Set $A' = A \otimes_Q Q'$. Notice that $A' = A[X]_{\mathfrak{m}A[X]}$ is a flat A-algebra with residue field $k(X)$ which is infinite. Notice that $f_1, \ldots, f_c$ is a Q'-regular sequence and $Q'/(\boldsymbol{f}) = A'$. Set $I' = IA'$ and $M' = M \otimes_Q Q' = M \otimes_A A'$. Notice that $\operatorname{projdim}_{Q'} M'$ is finite. Set $\mathcal{R}(I)' = \mathcal{R}(I')$, the Rees algebra of I'. Then $N' = N \otimes_A A'$ is a finitely generated $\mathcal{R}(I)'$-module. Also note that $\mathscr{E}(N') = \mathscr{E}(N) \otimes_A A'$.

By Case 1 we have that $\mathscr{E}(N')$ is a finitely generated $\mathcal{S}'$-module. So by Lemma 2.7 we get that $\mathscr{E}(N)$ is a finitely generated $\mathcal{S}$-module. $\square$

The next lemma is a bigraded version of Lemma 2.8(1) of [Puthenpurakal 2005].

Lemma 3.2. *Let R be a Noetherian ring* (*not necessarily local*) *and let $B = \bigoplus_{i,j\geq 0} B_{i,j}$ be a finitely generated bigraded R-algebra with $B_{0,0} = R$. Note that B need not be standard graded. Set*

$$B_y = \bigoplus_{j\geq 0} B_{(0,j)}.$$

Let $V = \bigoplus_{i,j\geq 0} V_{i,j}$ be a bigraded B-module satisfying these conditions:

(1) *For each $i \geq 0$, $V_i = \bigoplus_{j\geq 0} V_{i,j}$ is finitely generated as a B_y-module.*

(2) *There exists* $z \in B_{(r,0)}$ (*with* $r \geq 1$) *and a finitely generated bigraded* B*-module* D *such that we have an exact sequence of* B*-modules*

$$V(-r, 0) \xrightarrow{z} V \xrightarrow{\psi} D.$$

Then V *is a finitely generated* B*-module.*

Proof. Step 1. We begin by reducing to the case when ψ is *surjective*. Notice that $D' = \text{image}\,\psi$ is a finitely generated bigraded B-module. If $\psi' \colon V \to D'$ is the map induced by ψ then we have an exact sequence

$$V(-r, 0) \xrightarrow{z} V \xrightarrow{\psi'} D' \to 0.$$

Thus we may assume ψ is surjective.

Step 2. Choosing generators:

2.1. Choose a *finite* set W in V of homogeneous elements such that

$$\psi(W) = \{\psi(w) \mid w \in W\}$$

is a *generating set* for D.

2.2. Assume all the elements in W have x-coordinate $\leq c$.

2.3. For each $i \geq 0$, by hypothesis, V_i is a finitely generated B_y-module. So we may choose a *finite set* P_i of homogeneous elements in V_i which generates V_i as a B_y-module.

2.4. Set

$$G = W \cup \left(\bigcup_{i=0}^{c} P_i \right).$$

Clearly G is a *finite* set.

Claim. *G is a generating set for V.*

Let U be the B-submodule of V generated by G. It suffices to prove that $U_{i,j} = V_{i,j}$ for all $i, j \geq 0$. By construction we have that for $0 \leq i \leq c$

$$(*) \qquad U_{i,j} = V_{i,j} \quad \text{for each } j \geq 0.$$

We give $X := \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ the lex-order $\preceq$, making it well ordered. So we can prove our result by induction on X with respect to $\preceq$.

The base case is $(0, 0)$. In this case $U_{0,0} = V_{0,0}$ by (*). Let $(i, j) \in X \setminus \{(0, 0)\}$ and assume that for all $(r, s) \prec (i, j)$ we have $U_{r,s} = V_{r,s}$.

<u>Subcase 1</u>: $i \leq c$. By (*) we have $U_{i,j} = V_{i,j}$.

Subcase 2: $i > c$. Let $p \in V_{i,j}$. By construction, there exist $w_1, \dots, w_m \in W \subseteq G$ such that

$$\psi(p) = \sum_{l=1}^{m} h_l \psi(w_l), \quad \text{where } h_l \in B.$$

We may assume that $\deg h_l w_l = (i, j)$ for each l. Set $p' = \sum_{l=1}^{m} h_l w_l \in V_{i,j}$. Then $p' \in U_{i,j}$ and $p - p' \in \ker \psi$. So

$$p - p' = z \cdot q, \quad \text{where } q \in V_{(i-r,j)}.$$

If $q = 0$ then $p = p' \in U_{i,j}$. Otherwise, note that $(i-r, j) \prec (i, j)$. So by induction hypothesis, $q \in U_{(i-r,j)}$. It follows that $p \in U_{i,j}$. Thus $V_{i,j} \subseteq U_{i,j}$. Since $U_{i,j} \subseteq V_{i,j}$, by construction it follows that $U_{i,j} = V_{i,j}$. The result follows by induction on X. □

4. The global case

We need quite a few preliminaries to prove the global case of Theorem 1.1. See Section 4.2 for the difficulty in going from the local to the global case. Note that in the local case we proved the result by inducting on spread(N). This is unavailable to us in the global situation as there are usually infinitely many maximal ideals in a global ring. Most of this section will discuss two invariants of a graded $\mathscr{R}(I)$-module $N = \bigoplus_{n \geq 0} N_n$. We will use these invariants to prove Theorem 1.1 by induction.

4.1. ***Notation and conventions.*** We take the dimension of the zero-module to be -1. We also set the degree of the zero-polynomial to be -1.

Let $\mathfrak{P} \in \operatorname{Spec} Q$. If $\mathfrak{P} \supseteq \boldsymbol{f}$ then set $\mathfrak{p} = \mathfrak{P}/\boldsymbol{f}$. If $\mathfrak{P} \not\supseteq \boldsymbol{f}$ then any A-module localized at $\mathfrak{P}$ is zero. So assume $\mathfrak{P} \supseteq \boldsymbol{f}$.

(1) $\mathscr{R}(I)_\mathfrak{p} \cong \mathscr{R}(IA_\mathfrak{p})$ and $\mathscr{S}_\mathfrak{p} \cong \mathscr{R}(I)_\mathfrak{p}[t_1, \dots, t_c]$.

(2) $M_\mathfrak{p} = M_\mathfrak{P}$ has finite projective dimension as a $Q_\mathfrak{P}$-module.

(3) $\mathscr{E}(N)_\mathfrak{p} \cong \mathscr{E}(N_\mathfrak{p})$.

4.2. ***The difficulty in going from local to global.*** For each $\mathfrak{p} \in \operatorname{Spec} A$ it follows from Section 4.1 that $\mathscr{E}(N_\mathfrak{p})$ is a finitely generated $\mathscr{S}_\mathfrak{p}$-module. Usually $\operatorname{Supp}_A \mathscr{E}(N)$ will be an infinite set. So we cannot apply the local case and conclude.

The situation when $\operatorname{Supp}_A \mathscr{E}(N)$ is a finite set will help in the base step of our induction argument to prove Theorem 1.1. So we show it separately.

Lemma 4.3. *If* $\operatorname{Supp}_A \mathscr{E}(N)$ *is a finite set then* $\mathscr{E}(N)$ *is a finitely generated* $\mathscr{S}$-*module.*

Proof. We may choose a finite subset C of $\mathscr{E}(N)$ such that its image in $\mathscr{E}(N)_\mathfrak{p}$ generates $\mathscr{E}(N)_\mathfrak{p}$ for each $\mathfrak{p} \in \operatorname{Supp}_A \mathscr{E}(N)$. Set U to be the finitely generated submodule of $\mathscr{E}(N)$ generated by C.

Set $D = \mathscr{E}(N)/U$. Notice that $D_{\mathfrak{p}} = 0$ for each $\mathfrak{p} \in \operatorname{Spec} A$. So $D = 0$. Therefore $\mathscr{E}(N) = U$ is a finitely generated $\mathscr{S}$-module. $\square$

4.4. ***First inductive device.*** Since N is a finitely generated $\mathscr{R}(I)$-module we have $\operatorname{ann}_A N_n \subseteq \operatorname{ann}_A N_{n+1}$ for all $n \gg 0$. Since A is Noetherian it follows that $\operatorname{ann}_A N_n$ is constant for all $n \gg 0$. Call this stable value $\mathfrak{L}_N$. This enables us to define the *limit dimension* of N.

$$\operatorname{lim\,dim} N = \lim_{n\to\infty} \dim_A N_n = \dim A/\mathfrak{L}_N.$$

Since A has finite Krull dimension we get that $\operatorname{lim\,dim} N$ is finite.

4.5. Let $\mathfrak{P}$ be a prime ideal in A. If D is a finitely generated A-module then

$$\operatorname{ann}_{A_{\mathfrak{P}}} D_{\mathfrak{P}} = (\operatorname{ann}_A D)_{\mathfrak{P}} = (\operatorname{ann}_A D)A_{\mathfrak{P}}.$$

Therefore

$$(\mathfrak{L}_N)_{\mathfrak{P}} = \mathfrak{L}_{N_{\mathfrak{P}}}.$$

4.6. Note that if $\operatorname{lim\,dim}(N) = -1$ then $N_j = 0$, say for all $j \geq j_0$. So $\mathscr{E}(N_{\geq j_0}) = 0$. Using Lemma 2.6 it follows that $\mathscr{E}(N)$ is a finitely generated $\mathscr{S}$-module. The first nontrivial case is the following:

Proposition 4.7. *If* $\operatorname{lim\,dim}(N) = 0$ *then* $\mathscr{E}(N)$ *is a finitely generated* $\mathscr{S}$*-module.*

Proof. This implies that $A/\mathfrak{L}_N$ is Artinian. Say $\dim N_n = 0$ for $n \geq r$. Clearly,

$$\operatorname{Supp}_A \mathscr{E}(N_{\geq r}) \subseteq \operatorname{Supp}_A A/\mathfrak{L}_N,$$

a finite set of maximal ideals in A. It follows from Lemma 4.3 that $\mathscr{E}(N_{\geq r})$ is a finitely generated $\mathscr{S}$-module. Using Lemma 2.6 we get that $\mathscr{E}(N)$ is a finitely generated $\mathscr{S}$-module. $\square$

4.8. ***Higher-degree filter-regular element.*** We do not have filter-regular elements of degree 1 in the global situation. However we can do the following:

Set $E = N/H^0_{R_+}(N)$. Assume $E \neq 0$. As $H^0_{R_+}(E) = 0$ there exists homogeneous $u \in R_+$ such that u is E-regular [Bruns and Herzog 1993, 1.5.11]. Say $\deg u = s$. Since $E_n = N_n$ for all $n \gg 0$ it follows that the map $N_i \to N_{i+s}$ induced by multiplication by u is injective for all $i \gg 0$. We will say that u is an N filter-regular element of degree s.

4.9. ***The second inductive device.*** We now discuss a global invariant of N which patches well with local ones.

4.10. ***The local invariant.*** Let $(A, \mathfrak{m})$ be local and let $W = \bigoplus_{n\geq 0} W_n$ be a finitely generated $\mathcal{R}(I)$-module. Suppose $\mathfrak{L}_W = \operatorname{ann}_A W_n$ for all $n \geq c$. Let $\mathfrak{a} \subseteq \mathfrak{L}_W$ be an ideal. Fix $j \geq 0$. Set

$$d_{\mathfrak{a}}(W, j) = \begin{cases} 0 & \text{if } j < c, \\ 0 & \text{if } j \geq c \text{ and } \dim W_j < \dim A/\mathfrak{a}, \\ e(\mathfrak{m}, W_j) & \text{otherwise.} \end{cases}$$

Note that for $j \geq c$, W_j is an $A/\mathfrak{a}$-module. Furthermore $d_{\mathfrak{a}}(W, j)$ is the modified multiplicity function on the $A/\mathfrak{a}$-module W_j.

Remark 4.11. Notice if $\dim W_j = \dim A/\mathfrak{a}$ and $j \geq c$ then

$$d_{\mathfrak{a}}(W, j) = d_{\mathfrak{L}_W}(W, j).$$

Let $\mu(D)$ denote the minimal number of generators of an A-module D.

Lemma 4.12. *The function $d_{\mathfrak{a}}(W, -)$ is of polynomial type of degree $\leq \mu(I) - 1$.*

Proof. We may assume that the residue field of A is infinite. Set $T = \mathcal{R}(I)/\mathfrak{a}\mathcal{R}(I) = \bigoplus_{n\geq 0} T_n$. Notice $T_0 = A/\mathfrak{a}$. Let $\boldsymbol{x} = x_1, \dots, x_r$ be a minimal reduction of $\mathfrak{m}(A/\mathfrak{a})$. So $e(\mathfrak{m}, -) = e(\boldsymbol{x}, -)$ [Bruns and Herzog 1993, 4.6.5]. By a result due to Serre [Bruns and Herzog 1993, 4.7.6], we get that

$$e(\boldsymbol{x}, W_j) = \sum_{i=0}^{r} (-1)^i \ell\big(H_i(\boldsymbol{x}, W_j)\big).$$

Notice $H_i(\boldsymbol{x}, W) = \bigoplus_{j\geq c} H_i(\boldsymbol{x}, W_j)$ is a finitely generated $T/\boldsymbol{x}T$-module. Notice $(T/\boldsymbol{x}T)_0 = A/(\mathfrak{a}+\boldsymbol{x})$ is Artinian. Furthermore $(T/\boldsymbol{x}T)_1$ is a quotient of $\mathcal{R}(I)_1$ and so can be generated by $\mu(I)$ elements. Therefore the function $j \mapsto \ell\big(H_i(\boldsymbol{x}, W_j)\big)$ is of polynomial type of degree $\leq \mu(I) - 1$. The result follows. $\square$

Definition 4.13. $\theta(\mathfrak{a}, W)$ is the degree of the polynomial function $d_{\mathfrak{a}}(W, -)$.

Remark 4.14. Clearly $\theta(\mathfrak{a}, W)$ is nonnegative if and only if $\lim\dim W = \dim R/\mathfrak{a}$ and is -1 otherwise. Note that if $\dim A/\mathfrak{a} = \lim\dim W$ then $\theta(\mathfrak{a}, W) = \theta(\mathfrak{L}_W, W)$ is independent of $\mathfrak{a}$.

4.15. ***The global invariant.*** Let A be a Noetherian ring with finite Krull dimension. Let $I = (x_1, \dots, x_s)$ be an ideal in A. Let $W = \bigoplus_{n\geq 0} W_n$ be a finitely generated $\mathcal{R}(I)$-module. We assume that $\mathfrak{L}_W = \operatorname{ann}_A W_n$ for all $n \geq c$. Let $\mathfrak{a} \subseteq \mathfrak{L}_W$ be an ideal.

Set

$$\mathscr{C}(\mathfrak{a}) = \{\mathfrak{m} \mid \mathfrak{m} \in \text{m-Spec}(A), \mathfrak{m} \supseteq \mathfrak{a} \text{ and } \dim(A/\mathfrak{a})_{\mathfrak{m}} = \dim A/\mathfrak{a}\}.$$

Let $I = (x_1, \dots, x_s)$. If $\mathfrak{m} \in \mathscr{C}(\mathfrak{a})$ we have:

(a) $W_{\mathfrak{m}} = \bigoplus_{n\geq 0} (W_n)_{\mathfrak{m}}$.

(b) $\mathfrak{L}_{W_\mathfrak{m}} = (\mathfrak{L}_W)_\mathfrak{m}$. So $\mathfrak{a}_\mathfrak{m} \subseteq \mathfrak{L}_{W_\mathfrak{m}}$.

(c) $\theta(\mathfrak{a}_\mathfrak{m}, W_\mathfrak{m}) \leq s-1$.

Define

$$\theta(\mathfrak{a}, W) = \max\{\theta(\mathfrak{a}_\mathfrak{m}, W_\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{C}(\mathfrak{a})\}.$$

By (c) above we get that $\theta(\mathfrak{a}, W)$ is finite and is $\leq s-1$.

4.16. ***Properties of $\theta(\mathfrak{a}, W)$.*** We describe some properties of $\theta(\mathfrak{a}, W)$ we need for the proof of the global case of Theorem 1.1. Let $I = (x_1, \dots, x_s)$.

(i) $\theta(\mathfrak{a}, W) \leq s-1$. This is clear.

(ii) If $\mathfrak{L}_W \neq A$ then $\theta(\mathfrak{L}_W, W) \geq 0$. It suffices to consider the local case. Note that then $d_{\mathfrak{L}_W}(W, j) > 0$ for all $j \geq c$. It follows that $\theta(\mathfrak{L}_W, W) \geq 0$.

(iii) $\theta(\mathfrak{a}, W) = -1$ if and only if $\lim\dim W < \dim A/\mathfrak{a}$. If $\theta(\mathfrak{a}, W) = -1$ then $\theta(\mathfrak{a}_\mathfrak{m}, W_\mathfrak{m}) = -1$ for all $\mathfrak{m} \in \mathscr{C}(\mathfrak{a})$. This is equivalent to saying that $\lim\dim W_\mathfrak{m} < \dim(A/\mathfrak{a})_\mathfrak{m}$ for all $\mathfrak{m} \in \mathscr{C}(\mathfrak{a})$. By definition of $\mathscr{C}(\mathfrak{a})$ we have that

$$\dim A/\mathfrak{a} = \dim(A/\mathfrak{a})_\mathfrak{m} \quad \text{for each } \mathfrak{m} \in \mathscr{C}(\mathfrak{a}).$$

Also note that as $\mathfrak{a} \subseteq \mathfrak{L}_W$ we have

$$\lim\dim W = \max\{\lim\dim W_\mathfrak{m} \mid \mathfrak{m} \in \mathscr{C}(\mathfrak{a})\}.$$

So $\lim\dim W < \dim A/\mathfrak{a}$.

Conversely if $\lim\dim W < \dim A/\mathfrak{a}$ then for all $\mathfrak{m} \in \mathscr{C}(\mathfrak{a})$ we have

$$\lim\dim W_\mathfrak{m} \leq \lim\dim W < \dim A/\mathfrak{a} = \dim(A/\mathfrak{a})_\mathfrak{m}.$$

So $\theta(\mathfrak{a}_\mathfrak{m}, W_\mathfrak{m}) = -1$ for all $\mathfrak{m} \in \mathscr{C}(\mathfrak{a})$. Thus $\theta(\mathfrak{a}, W) = -1$.

(iv) If $\theta(\mathfrak{a}, W) \geq 0$ then $\theta(\mathfrak{L}_W, W) \leq \theta(\mathfrak{a}, W)$. By (iii) we get that $\lim\dim W = \dim A/\mathfrak{a}$. By hypothesis we also have $\mathfrak{a} \subseteq \mathfrak{L}_W$. Since $\dim A/\mathfrak{a} = \dim A/\mathfrak{L}_W$ it follows that $\mathscr{C}(\mathfrak{L}_W) \subseteq \mathscr{C}(\mathfrak{a})$. Using Remark 4.11 it follows that $\theta(\mathfrak{L}_W, W) \leq \theta(\mathfrak{a}, W)$.

(v) Let $u \in \mathscr{R}(I)_+$ be homogeneous of degree b. Assume u is W-filter regular and $W_n \neq 0$ for all $n \gg 0$. Set $E = W/uW$. Notice that $\mathfrak{L}_W \subseteq \mathfrak{L}_E$. Then

$$\theta(\mathfrak{L}_W, E) \leq \theta(\mathfrak{L}_W, W) - 1.$$

We have nothing to show if $\theta(\mathfrak{L}_W, E) = -1$. So assume $\theta(\mathfrak{L}_W, E) \geq 0$. Suppose $\theta(\mathfrak{L}_W, E) = \theta((\mathfrak{L}_W)_\mathfrak{p}, E_\mathfrak{p})$ for some $\mathfrak{p} \in \mathscr{C}(\mathfrak{L}_W)$. Since u is W-filter-regular, multiplication by u induces the exact sequence

$$0 \to W_{j-b} \to W_j \to E_j \to 0 \quad \text{for all } j \gg 0.$$

Localization at $\mathfrak{p}$ yields an exact sequence

$$0 \to (W_{j-b})_{\mathfrak{p}} \to (W_j)_{\mathfrak{p}} \to (E_j)_{\mathfrak{p}} \to 0 \quad \text{for all } j \gg 0.$$

Since $d_{\mathfrak{L}_{W_{\mathfrak{p}}}}(-,-)$ is an additive functor on $(A/\mathfrak{L}_W)_{\mathfrak{p}}$-modules we get that

$$\theta\big((\mathfrak{L}_W)_{\mathfrak{p}}, E_{\mathfrak{p}}\big) = \theta\big((\mathfrak{L}_W)_{\mathfrak{p}}, W_{\mathfrak{p}}\big) - 1.$$

The result follows since

$$\theta\big((\mathfrak{L}_W)_{\mathfrak{p}}, E_{\mathfrak{p}}\big) = \theta(\mathfrak{L}_W, E) \quad \text{and} \quad \theta\big((\mathfrak{L}_W)_{\mathfrak{p}}, W_{\mathfrak{p}}\big) \le \theta(\mathfrak{L}_W, W).$$

Proof of Theorem 1.1. We induct on $\lim\dim N$. If $\lim\dim N = -1, 0$ then the result follows from Section 4.6 and Proposition 4.7.

Assume $\lim\dim N \ge 1$ and assume the result holds for all $\mathscr{R}(I)$-modules E with $\lim\dim E \le \lim\dim N - 1$. Let $x \in \mathscr{R}(I)_+$ be homogeneous and an N-filter-regular element. Let $\deg x = r$. Set $D = N/xD$. By Lemma 2.6 it suffices to assume the case when x is N-regular.

We now induct on $\theta(\mathfrak{L}_N, N)$. If $\theta(\mathfrak{L}_N, N) = 0$ then $\theta(\mathfrak{L}_N, D) \le -1$, by Section 4.16(v). Using Section 4.16(iii) we get that

$$\lim\dim D < \dim A/\mathfrak{L}_N = \lim\dim N.$$

By the induction hypothesis (on lim dim) the module $\mathscr{E}(D)$ is a finitely generated $\mathscr{S}$-module. The short exact sequence of $\mathscr{R}(I)$-modules

$$0 \to N(-r) \xrightarrow{x} N \to D \to 0$$

induces an exact sequence of $\mathscr{S}$-modules

$$\mathscr{E}(N)(-r, 0) \xrightarrow{x} \mathscr{E}(N) \to \mathscr{E}(D).$$

By Lemma 3.2 we get that $\mathscr{E}(N)$ is a finitely generated $\mathscr{S}$-module.

We assume the result if $\theta(\mathfrak{L}_N, N) \le i$ and prove it when $\theta(\mathfrak{L}_N, N) = i + 1$. Let D be as above. So $\theta(\mathfrak{L}_N, D) \le i$, by Section 4.16(v). If $\theta(\mathfrak{L}_N, D) = -1$ then the argument as above yields $\mathscr{E}(N)$ to be a finitely generated $\mathscr{S}$-module.

If $\theta(\mathfrak{L}_N, D) \ge 0$ then by Section 4.16(iv) we get that $\theta(\mathfrak{L}_D, D) \le \theta(\mathfrak{L}_N, D) \le i$. So by induction hypothesis on $\theta(-,-)$ we get that $\mathscr{E}(D)$ is a finitely generated $\mathscr{S}$-module. By an argument similar to the one above we get that $\mathscr{E}(N)$ is a finitely generated $\mathscr{S}$-module. □

5. Application I: Asymptotic associated primes — the local case

In this section we give an answer to our main motivating question.

Theorem 5.1. *Let $(A, \mathfrak{m})$ be a local complete intersection. Let M be a finitely generated A-module. Let I be an ideal in A and let $N = \bigoplus_{n\ge 0} N_n$ be a finitely* *generated $\mathcal{R}(I)$-module. Then*

$$\bigcup_{n\geq 0}\bigcup_{i\geq 0}\operatorname{Ass}_A \operatorname{Ext}^i_A(M, N_n) \text{ is a finite set.}$$

Furthermore there exist i_0, n_0 such that for all $i \geq i_0$ and $n \geq n_0$ we have

$$\operatorname{Ass}_A \operatorname{Ext}^{2i}_A(M, N_n) = \operatorname{Ass}_A \operatorname{Ext}^{2i_0}_A(M, N_{n_0}),$$
$$\operatorname{Ass}_A \operatorname{Ext}^{2i+1}_A(M, N_n) = \operatorname{Ass}_A \operatorname{Ext}^{2i_0+1}_A(M, N_{n_0}).$$

Recall a local ring A is said to be a complete intersection if $\hat{A} = Q/(f_1, \dots, f_c)$, where $(Q, \mathfrak{n})$ is a complete regular local ring and $\boldsymbol{f}$ is a Q-regular sequence. If A is a complete intersection and a quotient of a regular local ring T then it can be shown that $A = T/(g_1, \dots, g_c)$, where $\boldsymbol{g}$ is a T-regular sequence (see [Matsumura 1980, 21.2]). In this case Theorem 5.1 holds by the following more general result:

Theorem 5.2. *Let Q be a Noetherian ring with finite Krull dimension and let $\boldsymbol{f} = f_1, \dots f_c$ be a regular sequence in Q. Set $A = Q/(\boldsymbol{f})$. Let M be a finitely generated A-module with* $\operatorname{projdim}_Q M$ *finite. Let I be an ideal in A and let $N = \bigoplus_{n\geq 0} N_n$ be a finitely generated $\mathcal{R}(I)$-module. Then*

$$\bigcup_{n\geq 0}\bigcup_{i\geq 0}\operatorname{Ass} \operatorname{Ext}^i_A(M, N_n) \text{ is a finite set.}$$

Furthermore there exist i_0, n_0 such that for all $i \geq i_0$ and $n \geq n_0$ we have

$$\operatorname{Ass}_A \operatorname{Ext}^{2i}_A(M, N_n) = \operatorname{Ass}_A \operatorname{Ext}^{2i_0}_A(M, N_{n_0}),$$
$$\operatorname{Ass}_A \operatorname{Ext}^{2i+1}_A(M, N_n) = \operatorname{Ass}_A \operatorname{Ext}^{2i_0+1}_A(M, N_{n_0}).$$

The following example shows that two sets of stable values of associate primes can occur.

Example 5.3. Let $Q = k[\![u, x]\!]$, $A = Q/(ux)$. Let $M = Q/(u)$, $I = A$ and $N = M[t]$ (so $N_n = M$ for all n).

For $i \geq 1$ one has (see [Avramov and Buchweitz 2000, 4.3])

$$\operatorname{Ext}^{2i-1}_A(M, M) = 0 \quad\text{and}\quad \operatorname{Ext}^{2i}_A(M, M) = k.$$

5.4. We now state a special case of a result due to E. West [2004, 3.2 and 5.1].

Let $R = A[x_1, \dots, x_r; y_1, \dots y_s]$ be a bigraded A-algebra with $\deg x_i = (2, 0)$ and $\deg y_j = (0, 1)$. Let $M = \bigoplus_{i,n\geq 0} M_{(i,n)}$ be a finitely generated R-module. Then:

(1) $\bigcup_{i\geq 0}\bigcup_{n\geq 0}\operatorname{Ass}_A M_{(i,n)}$ is a finite set.

(2) There exist i_0, n_0 such that for all $i \geq i_0$ and $n \geq n_0$ we have

$$\operatorname{Ass}_A M_{(2i,n)} = \operatorname{Ass}_A M_{(2i_0,n_0)}, \quad \operatorname{Ass}_A M_{(2i+1,n)} = \operatorname{Ass}_A M_{(2i_0+1,n_0)}.$$

Proof of Theorem 5.2. The result follows from our main theorem (1.1) and 5.4. □

We need the following exercise problem from [Matsumura 1980, 6.7, page 42].

Fact 5.5. *Let $f: A \to B$ be a ring homomorphism of Noetherian rings. Let U be a finitely generated B-module. Then*

$$\operatorname{Ass}_A U = \{\mathfrak{P} \cap A \mid \mathfrak{P} \in \operatorname{Ass}_B U\}.$$

In particular $\operatorname{Ass}_A U$ *is a finite set.*

There exist complete intersection rings which are not quotients of a regular local ring (see [Heitmann and Jorgensen 2012]). So Theorem 5.2 does not settle Theorem 5.1. To prove an analog of Theorem 5.2 for a local complete intersection we need the following result.

Lemma 5.6. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $\hat{A}$ be the completion of A with respect to $\mathfrak{m}$. Let B be a finitely generated $\hat{A}$-algebra containing $\hat{A}$. Let E be an A-module such that $E \otimes_A \hat{A}$ is a finitely generated B-module. Let D be any A-module. Then*:

(a) $\operatorname{Ass}_{\hat{A}} E \otimes_A \hat{A}$ *is a finite set.*

(b) $\operatorname{Ass}_A D = \{\mathfrak{P} \cap A \mid \mathfrak{P} \in \operatorname{Ass}_{\hat{A}}(D \otimes_A \hat{A})\}$.

(c) $\operatorname{Ass}_A E$ *is a finite set.*

To prove this result we need Theorem 23.3 from [Matsumura 1980]. Unfortunately, there is a typographical error there, so we state it here.

Theorem 5.7. *Let $\varphi: A \to B$ be a homomorphism of Noetherian rings*, *and let E be an A-module and G a B-module. Suppose that G is flat over A*; *then we have the following*:

(i) *If $\mathfrak{p} \in \operatorname{Spec} A$ and $G/\mathfrak{p}G \neq 0$ then*

$$^{a}\varphi(\operatorname{Ass}_B(G/\mathfrak{p}G)) = \operatorname{Ass}_A(G/\mathfrak{p}G) = \{\mathfrak{p}\}.$$

(ii) $\displaystyle \operatorname{Ass}_B(E \otimes_A G) = \bigcup_{\mathfrak{p} \in \operatorname{Ass}_A(E)} \operatorname{Ass}_B(G/\mathfrak{p}G).$

Remark 5.8. In [Matsumura 1980], $\operatorname{Ass}_A(E \otimes G)$ is written instead of $\operatorname{Ass}_B(E \otimes G)$. Also note that $^{a}\varphi(\mathfrak{P}) = \mathfrak{P} \cap A$ for $\mathfrak{P} \in \operatorname{Spec} B$.

Proof of Lemma 5.6. We consider the natural ring homomorphisms

$$\alpha: A \hookrightarrow \hat{A}, \quad \beta: \hat{A} \hookrightarrow B.$$

(a) We use the map β and Fact 5.5 to get our result.

(b) Set $X = \{\mathfrak{P} \cap A \mid \mathfrak{P} \in \operatorname{Ass}_{\hat{A}}(D \otimes_A \hat{A})\}$. We consider the flat map α.

Let $\mathfrak{q} \in X$. Say $\mathfrak{q} = \mathfrak{P} \cap A$, where $\mathfrak{P} \in \operatorname{Ass}_{\hat{A}} D \otimes \hat{A}$. By Theorem 5.7(ii), $\mathfrak{P} \in \operatorname{Ass}_{\hat{A}} \hat{A}/\mathfrak{p}\hat{A}$ for some $\mathfrak{p} \in \operatorname{Ass}_A D$. Notice $\hat{A}/\mathfrak{p}\hat{A} \neq 0$. By Theorem 5.7.(i) it follows that $\mathfrak{p} = \mathfrak{P} \cap A = \mathfrak{q}$. So $X \subseteq \operatorname{Ass}_A D$.

Conversely, if $\mathfrak{p} \in \operatorname{Ass}_A D$, then by Theorem 5.7(ii), $\operatorname{Ass}_{\hat{A}} \hat{A}/\mathfrak{p}\hat{A} \subseteq \operatorname{Ass}_{\hat{A}} D \otimes \hat{A}$. Notice $\hat{A}/\mathfrak{p}\hat{A} \neq 0$. Let $\mathfrak{P} \in \operatorname{Ass}_{\hat{A}} \hat{A}/\mathfrak{p}\hat{A}$. Then by Theorem 5.7(i) we have $\mathfrak{p} = \mathfrak{P} \cap A \in X$. Thus $\operatorname{Ass}_A D \subseteq X$. It follows that $\operatorname{Ass}_A D = X$.

(c) This follows from (a) and (b). □

Proof of Theorem 5.1. We consider the flat extension $\alpha \colon A \to \hat{A}$. Say $\hat{A} = Q/(\boldsymbol{f})$, where $(Q, \mathfrak{n})$ is a regular local ring and $\boldsymbol{f} = f_1, \dots, f_c \in \mathfrak{n}^2$ is a regular sequence.

(1) Consider $\mathscr{E}(N) = \bigoplus_{i \geq 0} \bigoplus_{n \geq 0} \operatorname{Ext}^i_A(M, N_n)$ as an A-module. By Theorem 1.1, $\mathscr{E}(N) \otimes \hat{A}$ is a finitely generated $B = \mathscr{R}(I\hat{A})[t_1, \dots t_c]$-algebra. By Lemma 5.6 we get that $\operatorname{Ass}_A \mathscr{E}(N)$ is a finite set. Notice that

$$\operatorname{Ass}_A \mathscr{E}(N) = \bigcup_{n \geq 0} \bigcup_{i \geq 0} \operatorname{Ass}_A \operatorname{Ext}^i_A(M, N_n).$$

(2) Set $\mathscr{E} = \mathscr{E}(N)$. By Theorem 1.1 there exist i_0 and n_0 such that for all $i \geq i_0$ and $n \geq n_0$ we have

$$\operatorname{Ass}_{\hat{A}} \mathscr{E}_{2i,n} \otimes \hat{A} = \operatorname{Ass}_{\hat{A}} \mathscr{E}_{2i_0,n_0} \otimes \hat{A}, \quad \operatorname{Ass}_{\hat{A}} \mathscr{E}_{2i+1,n} \otimes \hat{A} = \operatorname{Ass}_{\hat{A}} \mathscr{E}_{2i_0+1,n_0} \otimes \hat{A}.$$

By Lemma 5.6(b) it follows that for all $i \geq i_0$ and $n \geq n_0$ we have

$$\operatorname{Ass}_A \mathscr{E}_{2i,n} = \operatorname{Ass}_A \mathscr{E}_{2i_0,n_0}, \quad \operatorname{Ass}_A \mathscr{E}_{2i+1,n} = \operatorname{Ass}_A \mathscr{E}_{2i_0+1,n_0}.$$ □

6. Application II: Asymptotic associated primes — the geometric case

Let V be an affine or projective variety over an algebraically closed field K. Then V is said to be a local complete intersection if all of its local rings are complete intersections. Let A be the coordinate ring of V. In the affine case we have $A_\mathfrak{p}$ is a complete intersection for all $\mathfrak{p} \in \operatorname{Spec}(A)$. In the projective case we have $A_{(\mathfrak{p})}$ is a complete intersection for every $\mathfrak{p} \in \operatorname{Proj}(A)$. In this section we prove results analogous to Theorem 5.1 to coordinate rings of locally complete intersection varieties.

We first consider the affine case. In this case we prove the following general result. Recall a ring R is regular (a complete intersection) if $R_\mathfrak{p}$ is regular (a complete intersection) for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

Theorem 6.1. *Let Q be a regular ring of finite Krull dimension and let $\mathfrak{a}$ be an ideal in Q with $A = Q/\mathfrak{a}$ a complete intersection. Let M be a finitely generated A-module and let I be an ideal in A. Let $N = \bigoplus_{n \geq 0} N_n$ be a finitely generated $\mathscr{R}(I)$-module.*

Then

$$\bigcup_{n\geq 0}\bigcup_{i\geq 0}\operatorname{Ass}\operatorname{Ext}^i_A(M, N_n) \textit{ is a finite set.}$$

Furthermore there exist i_0, n_0 such that for all $i \geq i_0$ and $n \geq n_0$ we have

$$\operatorname{Ass}_A \operatorname{Ext}^{2i}_A(M, N_n) = \operatorname{Ass}_A \operatorname{Ext}^{2i_0}_A(M, N_{n_0}),$$
$$\operatorname{Ass}_A \operatorname{Ext}^{2i+1}_A(M, N_n) = \operatorname{Ass}_A \operatorname{Ext}^{2i_0+1}_A(M, N_{n_0}).$$

6.2. Before proving Theorem 6.1 we state the analogous result in the projective case. Let $\mathfrak{a}$ be a graded ideal in $Q = K[X_0, X_1, \dots, X_m]$, where $\deg X_i = 1$ for all i. Here K is not necessarily algebraically closed. Set $A = Q/\mathfrak{a}$. We assume $A_{(\mathfrak{p})}$ is a complete intersection for every $\mathfrak{p} \in \operatorname{Proj} A$. Recall that if U is the set of homogeneous elements in $A \setminus \mathfrak{p}$ then $A_{(\mathfrak{p})}$ is the degree-zero part of the graded ring $U^{-1}A$.

Let $\mathfrak{m}$ be the unique maximal homogeneous ideal of A. If E is a graded A-module then note that all its associate primes are homogeneous prime ideals of A. Set

$${}^*\operatorname{Ass}_A(E) = \operatorname{Ass}_A(E) \setminus \{\mathfrak{m}\},$$

the relevant associate primes of E. In the projective case our main theorem is this:

Theorem 6.3. (Keep the hypotheses of Section 6.2; note that $\mathcal{R}(I)$ is a bigraded ring.) *Let M be a finitely generated graded A-module and let I be a homogeneous ideal in A. Let $N = \bigoplus_{n\geq 0} N_n$ be a finitely generated bigraded $\mathcal{R}(I)$-module (so each N_n is a graded A-module). Then*

$$\bigcup_{n\geq 0}\bigcup_{i\geq 0}{}^*\operatorname{Ass}\operatorname{Ext}^i_A(M, N_n) \textit{ is a finite set.}$$

Furthermore there exist i_0, n_0 such that for all $i \geq i_0$ and $n \geq n_0$ we have

$${}^*\operatorname{Ass}_A \operatorname{Ext}^{2i}_A(M, N_n) = {}^*\operatorname{Ass}_A \operatorname{Ext}^{2i_0}_A(M, N_{n_0}),$$
$${}^*\operatorname{Ass}_A \operatorname{Ext}^{2i+1}_A(M, N_n) = {}^*\operatorname{Ass}_A \operatorname{Ext}^{2i_0+1}_A(M, N_{n_0}).$$

We now prove Theorems 6.1 and 6.3. We begin with the affine case. We need the following:

Lemma 6.4. *Suppose $A = Q/\mathfrak{a}$, where Q is a regular ring. Suppose for some $\mathfrak{p} \in \operatorname{Spec} A$ the ring $A_\mathfrak{p}$ is a complete intersection. Let $\mathfrak{q} \in \operatorname{Spec} Q$ with $\mathfrak{q}/\mathfrak{a} = \mathfrak{p}$. Then there exist $g \in Q \setminus \mathfrak{q}$ such that $\mathfrak{a}Q_g$ is generated by a Q_g-regular sequence.*

Proof. We have $A_{\mathfrak{p}} = Q_{\mathfrak{q}}/\mathfrak{a}Q_{\mathfrak{q}}$. Since $A_{\mathfrak{p}}$ is a complete intersection it follows from [Matsumura 1980, 21.2] that $\mathfrak{a}Q_{\mathfrak{q}}$ is generated by a regular sequence, say $f_1, \dots, f_c$. We may assume $f_i \in \mathfrak{a}$ for all i.

Set

$$E = \frac{\mathfrak{a}}{(f_1, \dots, f_c)} \quad \text{and} \quad D_i = \frac{(f_1, \dots, f_{i-1})\colon f_i}{(f_1, \dots, f_{i-1})} \quad \text{for } i = 1, \dots, c.$$

Let

$$L = E \oplus \left(\bigoplus_{i=1}^{c} D_i \right).$$

Then L is a finitely generated Q-module and $L_{\mathfrak{q}} = 0$. So there exists $g \in Q \setminus \mathfrak{q}$ such that $L_g = 0$. In Q_g note that $\mathfrak{a}Q_g = (f_1, \dots, f_c)Q_g$. Also as $(f_1, \dots, f_c)Q_{\mathfrak{q}} \neq Q_{\mathfrak{q}}$ we have that $(f_1, \dots, f_c)Q_g \neq Q_g$. Since $(D_i)_g = 0$ for $i = 1, \dots, c$ we get that $f_1, \dots, f_c$ is a Q_g-regular sequence. □

Proof of Theorem 6.1. Let $\mathfrak{p} \in \operatorname{Spec} A$. Then $A_{\mathfrak{p}}$ is a complete intersection. Let $\mathfrak{q} \in \operatorname{Spec} Q$ with $\mathfrak{q}/\mathfrak{a} = \mathfrak{p}$. Then by Lemma 6.4 there exist $g \in Q \setminus \mathfrak{q}$ such that $\mathfrak{a}Q_g$ is generated by a Q-regular sequence. Let $g_{\mathfrak{p}}$ be the image of g in A.

For $x \in A$ let $D(x) = \{\mathfrak{P} \in \operatorname{Spec} A \mid x \notin \mathfrak{P}\}$. Then $D(x)$ is a basic open set in $\operatorname{Spec}(A)$. Note that $\mathfrak{p} \in D(g_{\mathfrak{p}})$. Clearly

$$\operatorname{Spec} A = \bigcup_{\mathfrak{p} \in \operatorname{Spec} A} D(g_{\mathfrak{p}}).$$

As $\operatorname{Spec} A$ is quasicompact we have

$$\operatorname{Spec} A = D(g_{\mathfrak{p}_1}) \cup \dots \cup D(g_{\mathfrak{p}_m}) \quad \text{for some } m \geq 1.$$

Set $g_i = g_{\mathfrak{p}_i}$. Note that for any A-module E we have

$$\operatorname{Ass}_A E = \bigcup_{i=1}^{m} (\operatorname{Ass}_{A_{g_i}} E_{g_i}) \cap A,$$

and that $\mathscr{E}(N)_g = \mathscr{E}(N_g)$. Thus it suffices to prove the result for A_{g_i} for each i.

For each $i = 1, \dots, m$ we have that

$$A_{g_i} = \frac{Q_i \text{ a regular ring of finite Krull dimension}}{\text{regular sequence in } Q_i}.$$

As Q_i is a regular ring of finite Krull dimension we get that $\operatorname{projdim}_{Q_i} M_{g_i}$ is finite. So we can apply Theorem 5.2 to get the result. □

To prove Theorem 6.3 we need a few preliminaries. Recall that a $\mathbb{Z}$-graded ring $S = \bigoplus_{n \in \mathbb{Z}} S_n$ is said to be $*$-local if it has a unique proper maximal homogeneous ideal $\mathfrak{P}$. Note that $\mathfrak{P}$ is a prime ideal in S but not necessarily a maximal ideal in S. The functor $- \bigotimes S_{\mathfrak{P}}$ from the category of graded S-modules to the category of

$S_{\mathfrak{P}}$-modules is faithfully exact, by [Bruns and Herzog 1993, 1.5.15]. The following result is well known and can be easily proved using the same reference.

Lemma 6.5. *Let $S = \bigoplus_{n\in\mathbb{Z}} S_n$ be a $*$-local Cohen–Macaulay ring with unique maximal homogeneous ideal $\mathfrak{P}$. Let $\mathfrak{a}$ be a homogeneous ideal in S. If $\mathfrak{a}_{\mathfrak{P}}$ is generated by a regular sequence then $\mathfrak{a}$ is generated by a regular sequence of homogeneous elements. Furthermore if $\mathscr{C} = \{x_\alpha \mid \alpha \in \Delta\}$ is a generating set of $\mathfrak{a}$ consisting of homogeneous elements then we may choose $\boldsymbol{x} = x_1, \dots, x_c \in \mathscr{C}$ with $\mathfrak{a} = (\boldsymbol{x})$ and $\boldsymbol{x}$ is an S-regular sequence.*

To prove Theorem 6.3 we need the following analogue of Lemma 6.4.

Lemma 6.6. *Suppose $A = Q/\mathfrak{a}$, where $Q = K[X_0, \dots, X_n]$ is graded with* $\deg X_i =$ 1 *for all i and $\mathfrak{a}$ is a homogeneous ideal in Q. Suppose for some $\mathfrak{p} \in \operatorname{Proj} A$ the ring $A_{(\mathfrak{p})}$ is a complete intersection. Let $\mathfrak{q} \in \operatorname{Proj} Q$ with $\mathfrak{q}/\mathfrak{a} = \mathfrak{p}$. Then there exist homogeneous $g \in Q \setminus \mathfrak{q}$ such that $\mathfrak{a}Q_g$ is generated by a Q_g-regular sequence.*

Proof. Set

$$U = \{h \in A \mid h \text{ homogeneous and } h \notin \mathfrak{p}\},$$
$$W = \{h \in Q \mid h \text{ homogeneous and } h \notin \mathfrak{q}\}.$$

Then $U^{-1}A = W^{-1}Q/W^{-1}\mathfrak{a}$. Also note that some $X_i \notin \mathfrak{q}$. It follows that

$$U^{-1}A \cong A_{(\mathfrak{p})}[t, t^{-1}] \quad \text{and} \quad W^{-1}Q \cong Q_{(\mathfrak{q})}[t, t^{-1}].$$

Claim. *$U^{-1}A$ is a complete intersection.*

To see this, first observe that as $Q_{\mathfrak{q}}$ is a localization of $W^{-1}Q$ we have a flat map $Q_{(\mathfrak{q})} \to Q_{\mathfrak{q}}$ of local rings. As $Q_{\mathfrak{q}}$ is regular we have that $Q_{(\mathfrak{q})}$ is regular (see [Matsumura 1980, 23.7]). Notice that $A_{(\mathfrak{p})}$ is a quotient of a regular local ring $Q_{(\mathfrak{q})}$. So by [Bruns and Herzog 1993, 2.3.6], we have that $A_{(\mathfrak{p})}[t]$ is a complete intersection. As $U^{-1}A$ is a localization of $A_{(\mathfrak{p})}[t]$, it is also a complete intersection.

By Lemma 6.5 we have that $W^{-1}\mathfrak{a}$ is generated by a regular sequence $\boldsymbol{x} = x_1, \dots, x_c$ with $\boldsymbol{x} \in \mathfrak{a}$ homogeneous. Set

$$E = \frac{\mathfrak{a}}{(x_1, \dots, x_c)} \quad \text{and} \quad D_i = \frac{(x_1, \dots, x_{i-1}) : x_i}{(x_1, \dots, x_{i-1})} \quad \text{for } i = 1, \dots, c.$$

Set

$$L = E \oplus \left(\bigoplus_{i=1}^{c} D_i \right).$$

We have $W^{-1}L = 0$. Also, L is a finitely generated Q-module. So there exist $g \in W$ with $L_g = 0$. In Q_g note that $\mathfrak{a}Q_g = (x_1, \dots, x_c)Q_g$. Also, as $(x_1, \dots, x_c)W^{-1}Q \neq W^{-1}Q$ we have that $(x_1, \dots, x_c)Q_g \neq Q_g$. Since $(D_i)_g = 0$ for $i = 1, \dots, c$ we get that $x_1, \dots, x_c$ is a Q_g-regular sequence. $\square$

The proof of Theorem 6.3 is similar to that of Theorem 6.1, so we just sketch it.

Sketch of proof of Theorem 6.3. We use Lemma 6.6 and an argument analogous to the one used Theorem 6.1 to obtain

$$\operatorname{Proj} A = {}^*D(g_1) \cup \cdots \cup {}^*D(g_r) \quad \text{for some } r \geq 1,$$

for some homogeneous $g_i \in A$ and $A_{g_i} = Q_i/\mathfrak{a}_i$, where Q_i is regular of finite Krull dimension and $\mathfrak{a}_i$ is generated by a regular sequence. Note that for x homogeneous, ${}^*D(x) = \{\mathfrak{P} \in \operatorname{Proj} A \mid x \notin \mathfrak{P}\}$.

Let E be a graded A-module. Note that

$${}^*\operatorname{Ass}_A E = \bigcup_{i=1}^{r} (\operatorname{Ass}_{A_{g_i}} E_{g_i}) \cap A.$$

The result now follows by applying Theorem 5.2 to each $A_{g_i} = Q_i/\mathfrak{a}_i$. $\square$

7. Application III: Support varieties

Let $(A, \mathfrak{m})$ be a local complete intersection of codimension c. Let M, N be two finitely generated A-modules. Define

$$\operatorname{cx}_A(M, N) = \inf\left\{ b \in \mathbb{N} \;\middle|\; \varlimsup_{n\to\infty} \frac{\mu(\operatorname{Ext}^n_A(M, N))}{n^{b-1}} < \infty \right\}.$$

In this section we prove the following theorem:

Theorem 7.1. *Let $(A, \mathfrak{m})$ be a local complete intersection, M, N two finitely generated A-modules and let I be a proper ideal in A. Then*

$$\operatorname{cx}_A(M, I^j N) \quad \textit{is constant for all } j \gg 0.$$

7.2. ***Reduction to the case when A is complete and the residue field of A is algebraically closed.***

7.3. Suppose A' is a flat local extension of A such that $\mathfrak{m}' = \mathfrak{m}A'$ is the maximal ideal of A'. If E is an A-module then set $E' = E \otimes_A A'$. Notice that $I' \cong IA'$; we consider it as an ideal in A'. By [Avramov 1998, 7.4.3], A' is also a complete intersection. It can be easily checked that

$$\operatorname{cx}_{A'}(M', (I')^j N') = \operatorname{cx}_A(M, I^j N) \quad \text{for all } n \geq 0.$$

We now do our reduction in two steps.

By [Bourbaki 1983, Chapitre 9, appendice, corollaire du théoréme 1, p. IX.41], there exists a flat local extension $A \subseteq \widetilde{A}$ such that $\widetilde{\mathfrak{m}} = \mathfrak{m}\widetilde{A}$ is the maximal ideal of $\widetilde{A}$ and the residue field $\widetilde{k}$ of $\widetilde{A}$ is an algebraically closed extension of k. By

Section 7.3 it follows that we may assume k to be algebraically closed. We now complete A. Note that $\hat{A}$ is a flat extension of A which satisfies Section 7.3.

Thus we may assume that our local complete intersection A

(1) is complete. So $A = Q/(f_1, \dots, f_c)$, where $(Q, \mathfrak{n})$ is regular local and $f_1, \dots, f_c \in \mathfrak{n}^2$ is a regular sequence.

(2) has an algebraically closed residue field k.

Of course there exist many Q and $f_1, \dots, f_c$ of the type as indicated above. We simply fix one such representation of A.

7.4. Let U, V be two finitely generated A-modules.

Let $\operatorname{Ext}^*(U, V) = \bigoplus_{n \ge 0} \operatorname{Ext}^n_A(U, V)$ be the total ext module of U and V. We consider it as a (finitely generated) module over the ring of cohomological operators $A[t_1, \dots, t_c]$. Since $\operatorname{projdim}_Q U$ is finite $\operatorname{Ext}^*(U, V)$ is a finitely generated $A[t_1, \dots, t_c]$-module.

7.5. Let $\mathcal{C}(U, V) = \operatorname{Ext}^*(U, V) \otimes_A k$. Clearly $\mathcal{C}(U, V)$ is a finitely generated $T = k[t_1, \dots, t_c]$-module. (Here the degree of t_i is 2 for each $i = 1, \dots, c$). Set

$$\mathfrak{a}(U, V) = \operatorname{ann}_T \mathcal{C}(U, V).$$

Notice that $\mathfrak{a}(U, V)$ is a homogeneous ideal.

7.6. We now forget the grading of T and consider the affine space $\mathbb{A}^c(k)$. Let

$$\mathcal{V}(U, V) = \mathcal{V}(\mathfrak{a}(U, V)) \subseteq \mathbb{A}^c(k).$$

Since $\mathfrak{a}(U, V)$ is a graded ideal we get that $\mathcal{V}(U, V)$ is a cone.

7.7. By [Avramov and Buchweitz 2000, 2.4] we get that

$$\dim \mathcal{V}(U, V) = \operatorname{cx}_A(M, N).$$

Lemma 7.8. *If I is an ideal in A then there exists $j_0 \ge 0$ such that*

$$\mathcal{V}(U, I^j V) = \mathcal{V}(U, I^{j_0} V) \quad \textit{for all } j \ge j_0.$$

Proof of Theorem 7.1 assuming the lemma. By 7.3 we may assume that A is complete and has an algebraically closed residue field. The result now follows from 7.7 and Lemma 7.8. □

7.9. Let $N = \bigoplus_{n \ge 0} I^n V$. Set $\mathcal{E}(N) = \bigoplus_{n \ge 0} \bigoplus_{i \ge 0} \operatorname{Ext}^i_A(U, I^n V)$. Set $\mathcal{C}(N) = \mathcal{E}(N) \otimes_A k$. By Theorem 1.1, $\mathcal{E}(N)$ is a finitely generated $\mathcal{S} = \mathcal{R}(I)[t_1, \dots, t_c]$-module. It follows that $\mathcal{C}(N)$ is a finitely generated, bigraded, $G = F(I)[t_1, \dots, t_c]$-module. Recall that $F(I)$, the fiber cone of I, is a finitely generated k-algebra.

So we may as well consider $\mathcal{C}(N)$ as a bigraded $R = k[X_1, \dots, X_m, t_1, \dots t_c]$-module (of course here $X_1, \dots, X_m$ are variables). Furthermore $\deg X_l = (1, 0)$ for $l = 1, \dots m$ and $\deg t_s = (0, 2)$ for $s = 1, \dots, c$. Set $T = k[t_1, \dots, t_c]$.

7.10. ***Advantages of coarsening the grading on $\mathcal{C}(N)$.*** By forgetting the degree on the t_i we may consider $R = T[X_1, \dots, X_m]$. Notice that

$$\mathcal{C}(N) = \bigoplus_{n \geq 0} \mathcal{C}(U, I^n V)$$

as a graded R-module.

Proof of Lemma 7.8. We make the constructions as in Section 7.10. So $\mathcal{C}(N)$ is a finitely generated graded $R = T[X_1, \dots, X_m]$-module. Notice that R is $\mathbb{N}$-standard graded. So there exists j_0 such that

$$\operatorname{ann}_T \mathcal{C}(N)_j = \operatorname{ann}_T \mathcal{C}(N)_{j_0} \quad \text{for all } j \geq 0.$$

The results follows. □

Question 7.11 (With hypotheses as in Theorem 7.1).

Is $\operatorname{cx}_A(M, N/I^j N)$ *constant for all* $j \gg 0$?

Acknowledgements

I thank Professor L. L. Avramov and Professor J. Herzog for many discussions regarding this paper. I also thank the referee for many pertinent comments.

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Received September 18, 2008. Revised April 11, 2013.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 266, No. 2, 2013
dx.doi.org/10.2140/pjm.2013.266.391

INDEX FORMULAE FOR STARK UNITS AND THEIR SOLUTIONS

XAVIER-FRANÇOIS ROBLOT

Let K/k be an abelian extension of number fields with a distinguished place of k that splits totally in K. In that situation, the abelian rank-one Stark conjecture predicts the existence of a unit in K, called the Stark unit, constructed from the values of the L-functions attached to the extension. In this paper, assuming the Stark unit exists, we prove index formulae for it. In a second part, we study the solutions of the index formulae and prove that they admit solutions unconditionally for quadratic, quartic and sextic (with some additional conditions) cyclic extensions. As a result we deduce a weak version of the conjecture ("up to absolute values") in these cases and precise results on when the Stark unit, if it exists, is a square.

1. Introduction

Let K/k be an abelian extension of number fields. Denote by G its Galois group. Let S_∞ and S_{ram} denote respectively the set of infinite places of k and the set of finite places of k ramified in K/k. Let $S(K/k) := S_\infty \cup S_{\text{ram}}$. Fix a finite set S of places of k containing $S(K/k)$ and of cardinality at least 2. Assume that there exists at least one place in S, say v, that splits totally in K/k and fix a place w of K dividing v. Let e be the order of the group of roots of unity in K. In this setting Stark [1980] made the following conjecture.

Conjecture (abelian rank-one Stark conjecture). There exists an S-unit $\varepsilon_{K/k,S}$ in K such that

(1) For all characters χ of G,

$$L'_{K/k,S}(0,\chi) = \frac{1}{e}\sum_{g\in G}\chi(g)\log|\varepsilon^{g}_{K/k,S}|_w,$$

Supported by the JSPS Global COE *CompView*. This research was conducted while the author was a member of the Department of Mathematics of the Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-ku, Tokyo 152-8550, Japan .
MSC2010: 11R27, 11R29, 11R42.
Keywords: Stark conjectures, index formula, sextic extensions.

where $L_{K/k,S}(s,\chi)$ denotes the L-function associated to χ with Euler factors at prime ideals in S deleted.

(2) The extension $K(\varepsilon_{K/k,S}^{1/e})/k$ is abelian.

(3) If furthermore $|S| \geq 3$ then ε is a unit of K.

The unit $\varepsilon_{K/k,S}$ is called the Stark unit associated to the extension K/k, the set of places S and the place v.[1] It is unique up to multiplication by a root of unity in K. A good reference for this conjecture is [Tate 1984, Chap. IV].

The starting point of this research is the conjectural method used in [Cohen and Roblot 2000; Roblot 2000] (and inspired by [Stark 1977]) to construct totally real abelian extensions of totally real fields. Let L/k be such an extension. The idea is to construct a quadratic extension K/L, abelian over k, satisfying some additional conditions similar to the assumptions (A1), (A2) and (A3) below. Assuming the Stark conjecture for K/k, $S(K/k)$ and a fixed real place v of k, one can prove that $K = k(\varepsilon)$ and $L = k(\alpha)$, where $\alpha := \varepsilon + \varepsilon^{-1}$ and $\varepsilon := \varepsilon_{K/k,S(K/k)}$ is the corresponding Stark unit. Using part (1) of the conjecture, one computes the minimal polynomial $A(X)$ of α over k. The final step is to check unconditionally that the polynomial $A(X)$ does indeed define the extension L.

One notices in that setting that the rank of the units of K is equal to the rank of units of L plus the rank of the module generated by the Stark unit and its conjugates over k. A natural question to ask is whether the index of the group generated by the units of L and the conjugates of the Stark unit has finite index inside the group of units of K and, if so, if this index can be computed. A positive answer to the first question is given by Stark in [1976, Theorem 1]. In [Arakawa 1985], Arakawa gives a formula for this index when k is a quadratic field. Using similar methods, we obtain a general result (Theorem 2.2) in the next section. Then we derive a "relative" index formula (Theorem 2.3) that relates the index of the subgroup generated over $\mathbb{Z}[G]$ by the Stark unit inside the "minus-part" of the group of units of K to the cardinality of the "minus-part" of the class group of K.[2] In the third section, we use results of Rubin [1992] on a form of the Gras conjecture for Stark units to show that the relative index formula implies local relative index formulae (Theorem 3.2). Starting with the fourth section, we stop assuming the abelian rank-one Stark conjecture and study directly the solutions to the index formulae. In section 4, we look at how much these index formulae characterize the Stark unit (Proposition 4.1 and Corollary 4.5). In the next section, we introduce the algebraic tools that will be needed to prove the existence of solutions in some cases in the following sections. We also reprove in that section the abelian rank-one Stark conjecture for quadratic

[1] In fact the place w but changing the place w just amounts to replace the Stark unit by one of its conjugate.

[2] Similar in some way to the index formulae for cyclotomic units; see [Washington 1997, Chap. 8].

extensions (Theorem 5.5). Finally, sections 6 and 7 are devoted to a proof that solutions to the index formulae always exist for quartic extensions (Theorem 6.1) and sextic extensions (Theorem 7.1) with some additional conditions in that case. We show that the existence of solutions in those cases imply a weak version of the conjecture, where part (1) is satisfied only up to absolute values.[3] We also obtain results on when the Stark unit, if it exists, is a square (Corollary 2.4, Theorem 5.5, Corollary 6.2 and Corollary 7.2).

2. The index formulae

We assume from now on that the place v is infinite[4] and that k has at least two infinite places. Therefore we can always apply the conjecture for any finite set S containing $S(K/k)$. The cases that we are excluding are $k = \mathbb{Q}$ and k a complex quadratic field. In both cases the conjecture is proved and the Stark unit is strongly related to cyclotomic units and elliptic units respectively.

Fix a finite set S of places of k containing $S(K/k)$. We make the following additional assumptions.

(A1) k is totally real and the infinite places of K above v are real, the infinite places of K not above v are complex.

(A2) The maximal totally real subfield K^+ of K satisfies $[K : K^+] = 2$.

(A3) All the finite primes in S are either ramified or inert in K/K^+.

If S contains more than one place that splits totally in K/k then the conjecture is trivially true with the Stark unit being equal to 1. Therefore the only non trivial case excluded by (A1) is the case when k has exactly one complex place and K is totally complex. It is likely that most of the methods and results in this paper can be adapted to cover also that case. Assumptions (A2) and (A3) are necessary to ensure that the rank of the group generated by the units of K^+ and the conjugate of the Stark unit has finite index inside the group of units of K. Without these assumptions, global index formulae for Stark units as they are stated in this article cannot exist although it is still possible to prove index formulae for some p-adic characters if one takes also into account Stark units coming from subextensions (see [Rubin 1992] or Section 3).

[3]Unfortunately, in most cases the values are complex and there does not appear to be any obvious way to remove these absolute values.

[4]For v a finite place, the abelian rank-one Stark conjecture is basically equivalent to the Brumer–Stark conjecture; see [Tate 1984, §IV.6]. Recent results of Greither and Popescu [Greither and Popescu 2011] imply the validity of the Brumer–Stark conjecture away from its 2-part and under the hypothesis that an appropriate Iwasawa μ-invariant vanishes.

We assume until further notice that the conjecture is true for the extension K/k, the set of places S and the distinguished place v.[5]

Denote by $\varepsilon := \varepsilon_{K/k,S}$ the corresponding Stark unit. From now on, all subfields of K (including K itself) are identified with their image in $\mathbb{R}$ by w. We make the Stark unit unique by imposing that $\varepsilon > 0$. It follows that $\varepsilon^g > 0$ for all $g \in G$; see [Tate 1984, §IV.3.7]. One can also prove under these hypothesis — see [Roblot 2000, Lemma 2.8] — that $|S(K/k)| \geq 3$ and therefore ε is a unit of K by part (3) of the Conjecture, and that $|\varepsilon|_{w'} = 1$ for any place w' of K not above v.

Let m be the degree of K^+/k and d be the degree of $k/\mathbb{Q}$. Thus we have $[K:k]=2m$ and $[K:\mathbb{Q}]=2md$. Let τ denote the non trivial element of $\mathrm{Gal}(K/K^+)$. It is the complex conjugation of the extension K and, by the above remark, we have $\varepsilon^\tau = \varepsilon^{-1}$. Let G^+ denote the Galois group of K^+/k, thus $G^+ \cong G/\langle\tau\rangle$. It follows from (A1) that the signatures of K^+ and K are respectively $(dm, 0)$ and $(2m, m(d-1))$. Therefore the rank of U_{K^+} and U_K, the group of units of K^+ and K, are respectively $dm-1$ and $2m+m(d-1)-1 = (dm-1)+m$. Let U_{Stark} be the multiplicative $\mathbb{Z}[G]$-module generated by ± 1, ε and U_{K^+}. Let $R := \{\rho_1, \ldots, \rho_m\}$ be a fixed set of representatives of G modulo $\langle\tau\rangle$. Set $\varepsilon_\ell := \rho_\ell^{-1}(\varepsilon)$ for $\ell = 1, \ldots, m$. Since $\tau(\varepsilon) = \varepsilon^{-1}$, the group U_{Stark} is generated over $\mathbb{Z}$ by $\{\pm 1, \eta_1, \ldots, \eta_{dm-1}, \varepsilon_1, \ldots, \varepsilon_m\}$, where $\eta_1, \ldots, \eta_{dm-1}$ is a system of fundamental units of K^+. Let $|\cdot|_j$, $1 \leq j \leq (d+1)m$ denote the infinite normalized absolute values of K ordered in the following way. The $2m$ real absolute values of K, corresponding to the places over v, are $|\cdot|_j := |\rho_j(\cdot)|$ and $|\cdot|_{j+m} := |\rho_j\tau(\cdot)|$ for $1 \leq j \leq m$. The complex absolute values, corresponding to the infinite places not above v, are $|\cdot|_j$ for $2m+1 \leq j \leq (d+1)m$. The regulator of U_{Stark} is the absolute value of the determinant of the matrix of size $(d+1)m-1$ whose j-th row has entries

$$\log|\eta_1|_j,\ \log|\eta_2|_j,\ \ldots,\ \log|\eta_{dm-1}|_j,\ \log|\varepsilon_1|_j,\ \ldots,\ \log|\varepsilon_m|_j.$$

(We discard the last absolute value, $|\cdot|_{(d+1)m}$.) For $1 \leq j \leq (d+1)m$, let $|\cdot|_j^+$ denote the restriction of the absolute value $|\cdot|_j$ to K^+. For $1 \leq j \leq m$, the places corresponding to $|\cdot|_j$ and $|\cdot|_j^+$ are real and $\log|\eta_i|_j^+ = \log|\eta_i|_j = \log|\eta_i|_{j+m}$. For $2m+1 \leq j \leq (d+1)m$, the places corresponding to $|\cdot|_j$ and $|\cdot|_j^+$ are respectively complex and real, thus $\log|\eta_i|_j^+ = 2\log|\eta_i|_j$. Note also that $|\varepsilon_\ell|_{j+m} = |\varepsilon_\ell|_m^{-1}$ for $1 \leq j \leq m$ and $|\varepsilon_\ell|_j = 1$ for $2m+1 \leq j \leq (d+1)m$. Therefore the matrix is equal to

$$\left(\begin{array}{c|c} \log|\eta_i|_j^+ & \log|\varepsilon_\ell|_j \\ \hline \log|\eta_i|_j^+ & -\log|\varepsilon_\ell|_j \\ \hline 2\log|\eta_i|_{j'}^+ & 0 \end{array}\right)_{(j,j'),(i,\ell)},$$

[5] Since v is the only real place of k that stays real in K, we will usually not specify it.

where $1 \leq j \leq m$, $2m+1 \leq j' \leq (d+1)m-1$, $1 \leq i \leq dm-1$ and $1 \leq \ell \leq m$. Now we add the j-th row to the $(m+j)$-th row for $1 \leq j \leq m$ and we obtain finally the following matrix with the same determinant

$$\left(\begin{array}{c|c} \log|\eta_i|_j^+ & \log|\varepsilon_\ell|_j \\ \hline 2\log|\eta_i|_j^+ & 0 \\ \hline 2\log|\eta_i|_{j'}^+ & 0 \end{array}\right)_{(j,j'),(i,\ell)}.$$

Therefore the regulator of U_{Stark} is

$$\operatorname{Reg}(U_{\text{Stark}}) = \left|\det(\log|\varepsilon_\ell|_j)_{j,\ell} \det\bigl(2\log|\eta_i|_{j'}^+\bigr)_{j',i}\right|, \tag{2.1}$$

where $1 \leq \ell, j \leq m$, $1 \leq i \leq dm-1$ and j' runs through the set $\{1, \dots, m, 2m+1, \dots, (d+1)m-1\}$. The absolute values $|\cdot|_1^+, \dots, |\cdot|_m^+, |\cdot|_{2m+1}^+, \dots, |\cdot|_{(d+1)m-1}^+$ are the absolute values corresponding to all the infinite places of K^+ but one. Thus the second term is $2^{dm-1}R_{K^+}$. For the first term, we have

$$\left|\det(\log|\varepsilon_\ell|_j)_{j,\ell}\right| = \left|\det(\log|\varepsilon^{\rho\lambda^{-1}}|)_{\rho,\lambda\in R}\right|.$$

We say that a character χ of G is even if $\chi(\tau) = 1$, otherwise χ is odd and $\chi(\tau) = -1$. The even characters of G are the inflations of characters of G^+. We have the following modification of the classical determinant group factorization.

Lemma 2.1. *Let $a_g \in \mathbb{C}$, for $g \in G$, be such that $a_{\tau g} = -a_g$ for all $g \in G$. Then*

$$\det(a_{\rho\lambda^{-1}})_{\rho,\lambda\in R} = \prod_{\chi \text{ odd}} \sum_{\rho\in R} \chi(\rho) a_\rho.$$

Proof. Let E be the $\mathbb{C}$-vector space of functions $f : G \to \mathbb{C}$ such that $f(\tau g) = -f(g)$ for all $g \in G$. Clearly it has dimension m and admits $(\chi)_{\chi \text{ odd}}$ has a basis. Another basis is given by the functions $(\delta_\rho)_{\rho\in R}$ defined by

$$\delta_\rho(\rho) = 1,\ \delta_\rho(\tau\rho) = -1 \text{ and } \delta_\rho(g) = 0 \text{ for all } g \in G \text{ with } g \neq \rho, \tau\rho.$$

The group G acts on E by $f^\sigma : g \mapsto f(g\sigma)$ for $f \in E$ and $\sigma \in G$. In particular, we have $f^\tau = -f$. We extend this action linearly to give E a structure of $\mathbb{C}[G]$-module. Now consider the endomorphism defined by

$$T := \sum_{g\in G} a_g g.$$

We have

$$T(\delta_\rho) = \sum_{\substack{g\in G\\ \rho g^{-1}\in R}} a_g \delta_\rho^g + \sum_{\substack{g\in G\\ \rho g^{-1}\notin R}} a_g \delta_\rho^g = \sum_{\substack{g\in G\\ \rho g^{-1}\in R}} a_g \delta_{\rho g^{-1}} - \sum_{\substack{g\in G\\ \rho g^{-1}\notin R}} a_g \delta_{\tau\rho g^{-1}}.$$

We write $\lambda = \rho g^{-1}$ in the first sum and $\lambda = \tau\rho g^{-1}$ in the second one. We get

$$T(\delta_\rho) = \sum_{\lambda\in R} a_{\rho\lambda^{-1}}\delta_\lambda - \sum_{\lambda\in R} a_{\tau\rho\lambda^{-1}}\delta_\lambda = 2\sum_{\lambda\in R} a_{\rho\lambda^{-1}}\delta_\lambda.$$

Therefore the determinant of T is $2^m \det(a_{\rho\lambda^{-1}})_{\rho,\lambda\in R}$. On the other hand, for χ odd, we compute

$$T(\chi) = \sum_{g\in G} a_g \chi^g = \sum_{g\in G} a_g \chi(g)\chi.$$

Thus χ is an eigenvector for T with eigenvalue $\sum_{g\in G} a_g\chi(g) = 2\sum_{\rho\in R}\chi(\rho)a_\rho$. Therefore $\det(T) = 2^m \prod_{\chi \text{ odd}} \sum_{\rho\in R} \chi(\rho)a_\rho$ and the result follows. □

By the lemma, we get

$$\text{(2.2)}\quad \det(\log|\varepsilon^{\rho\lambda^{-1}}|)_{\rho,\lambda\in R} = \prod_{\chi \text{ odd}}\sum_{\rho\in R}\chi(\rho)\log|\varepsilon^\rho| = \prod_{\chi \text{ odd}}\frac{1}{2}\sum_{g\in G}\chi(g)\log|\varepsilon^g| = \prod_{\chi \text{ odd}} L'_{K/k,S}(0,\chi),$$

using part (1) for the last equality and the fact that the number of roots of unity in K is 2 since K is not totally complex by (A1). On the other hand, we have

$$\text{(2.3)}\quad \prod_{\chi \text{ odd}} L_{K/k,S}(s,\chi) = \frac{\zeta_{S,K}(s)}{\zeta_{S,K^+}(s)},$$

where $\zeta_{S,K}(s) := \zeta_{S_K,K}(s)$ and $\zeta_{S,K^+}(s) := \zeta_{S_{K^+},K^+}(s)$ denote respectively the Dedekind zeta functions of K and K^+ with the Euler factors at primes in S_K and S_{K^+} removed. Here S_K and S_{K^+} denote respectively the set of places of K and of K^+ above the places in S. We will often use by abuse the subscript S instead of S_K or S_{K^+} to simplify the notation. Taking the limit when $s \to 0$ in (2.3) and using the expression for the Taylor development at $s = 0$ of Dedekind zeta functions — see [Tate 1984, Corollary I.1.2] — we get

$$\text{(2.4)}\quad \prod_{\chi \text{ odd}} L'_{K/k,S}(0,\chi) = 2^{t_S}\frac{h_K R_K}{h_K^+ R_K^+},$$

where t_S is the number of prime ideals in S_{K^+} that are inert in K/K^+ and h_K, R_K, h_{K^+} and R_{K^+} are respectively the class numbers and regulators of K and K^+. Putting together equations (2.1), (2.2) and (2.4), we get the following result.

Theorem 2.2. *The index of U_{Stark} in the group of units of K is*

$$(U_K : U_{\text{Stark}}) = 2^{t_S+dm-1} h_K/h_{K^+},$$

where t_S is the number of prime ideals in S_{K^+} that are inert in K/K^+. □

Let Cl_K and Cl_{K^+} denote respectively the class groups of K and K^+. Define Cl_K^- and U_K^- as the kernel of the following maps induced by the norm $\mathcal{N} := 1+\tau$ of the extension K/K^+

$$\mathrm{Cl}_K^- := \mathrm{Ker}(\mathcal{N} : \mathrm{Cl}_K \to \mathrm{Cl}_{K^+}) \quad \text{and} \quad U_K^- := \mathrm{Ker}(\mathcal{N} : \bar{U}_K \to \bar{U}_{K^+}),$$

where $\bar{U}_K$ and $\bar{U}_{K^+}$ are respectively $U_K/\{\pm 1\}$ and $U_{K^+}/\{\pm 1\}$. From now on, we use the additive notation to denote the action of $\mathbb{Z}[G]$, and other group rings, on $\bar{U}_K$ and its subgroups $U_K^-, \bar{U}_{K^+}, \ldots$. For $x \in U_K$, we denote by $\bar{x}$ its class in $\bar{U}_K$ and adopt the following convention: if $\bar{x} \in \bar{U}_K$, we let x denote the unique element in the class $\bar{x}$ such that $x > 0$. Note that $\mathcal{N}(x) = \mathcal{N}(-x) = 1$ since K/K^+ is ramified at least one real place.

Theorem 2.3. *We have*

$$\left(U_K^- : \mathbb{Z}[G] \cdot \bar{\varepsilon}\right) = 2^{e+t_S} |\mathrm{Cl}_K^-|,$$

where $2^e = (\bar{U}_{K^+} : \mathcal{N}(\bar{U}_K))$.

Proof. By class field theory the map $\mathcal{N} : \mathrm{Cl}_K \to \mathrm{Cl}_{K^+}$ is surjective. Therefore $|\mathrm{Cl}_K^-| = h_K/h_{K^+}$. On the other hand, if we let $\bar{U}_{\mathrm{Stark}} := U_{\mathrm{Stark}}/\{\pm 1\}$, we have

$$\mathrm{Ker}\left(\mathcal{N} : \bar{U}_{\mathrm{Stark}} \to \bar{U}_{K^+}\right) = \mathbb{Z}[G] \cdot \bar{\varepsilon} \quad \text{and} \quad \mathrm{Im}\left(\mathcal{N} : \bar{U}_{\mathrm{Stark}} \to \bar{U}_{K^+}\right) = 2 \cdot \bar{U}_{K^+}.$$

Therefore we get

$$(\bar{U}_K : \bar{U}_{\mathrm{Stark}}) = (\mathcal{N}(\bar{U}_K) : 2 \cdot \bar{U}_{K^+}) \left(U_K^- : \mathbb{Z}[G] \cdot \bar{\varepsilon}\right).$$

Since $(\bar{U}_K : \bar{U}_{\mathrm{Stark}}) = (U_K : U_{\mathrm{Stark}})$, it follows from Theorem 2.2 that

$$\left(U_K^- : \mathbb{Z}[G] \cdot \bar{\varepsilon}\right) = \frac{2^{t_S + dm - 1} |\mathrm{Cl}_K^-|}{\left(\mathcal{N}(\bar{U}_K) : 2 \cdot \bar{U}_{K^+}\right)}.$$

We conclude by noting that

$$\left(\mathcal{N}(\bar{U}_K) : 2 \cdot \bar{U}_{K^+}\right) = \frac{\left(\bar{U}_{K^+} : 2 \cdot \bar{U}_{K^+}\right)}{\left(\bar{U}_{K^+} : \mathcal{N}(\bar{U}_K)\right)} = \frac{2^{dm-1}}{\left(\bar{U}_{K^+} : \mathcal{N}(\bar{U}_K)\right)}. \qquad \square$$

It has been observed that the Stark unit is quite often a square. The theorem provides us with a necessary condition for that to happen.

Corollary 2.4. *Let c be the 2-valuation of the order of Cl_K^-. A necessary condition for the Stark unit ε to be a square in K is*

$$e + t_S + c \geq m.$$

Proof. Assume that $\varepsilon = \eta^2$ with $\eta \in K$. Then it is easy to see that $\eta \in U_K^-$ and therefore $\left(\mathbb{Z}[G] \cdot \bar{\eta} : \mathbb{Z}[G] \cdot \bar{\varepsilon}\right) = 2^m$ divides $2^{e+t_S} |\mathrm{Cl}_K^-|$. $\square$

We will see in (5.10) that $e \geq (d-1)m-2$. Therefore the inequality in the corollary is always satisfied for $d \geq 2+2/m$. However, this is not enough to ensure that the Stark unit is a square in general. Indeed at the end of the paper we give an example of a cyclic sextic extension K/k satisfying (A1), (A2) and (A3), and with k a totally real cubic field where the Stark unit, assuming it exists, is not a square even though $e > m$. But, in all the cases that we study, we can prove that for d sufficiently large the Stark unit is always a square. Of course these cases are quite specific and it is difficult to draw from them general conclusions, but still we are lead to ask the following question.

Question. Fix a relative degree m. Does there exist a constant $D(m)$, depending only on m, such that for any extensions K/k of degree $2m$ and any finite set of places S containing $S(K/k)$ satisfying (A1), (A2) and (A3), and with $d \geq D(m)$, the corresponding Stark unit, assuming that it exists, is always a square in K?

It follows from the result of the next sections that the answer is positive for $1 \leq m \leq 3$ and that $D(1) = D(3) = 4$ and $D(2) = 3$.

3. Rubin's index formula

In [Rubin 1992], Rubin proves Gras conjecture type results for Stark units using Euler systems. His results are generalized by Popescu [2004]. In this section, we use the results of Rubin to get a similar result in our setting. To be able to use Rubin's results we need to make the following additional assumption:

(A4) K contains the Hilbert Class Field H_k of k.

We assume in this section that the conjecture is true for the extensions and set of places as described in [Rubin 1992].

We first introduce the results of Rubin. Let $\mathfrak{f}$ be the conductor of K/k. For any modulus $\mathfrak{g}$ dividing $\mathfrak{f}$, let $K_\mathfrak{g} = K \cap k(\mathfrak{g})$ be the intersection of K with the ray class field of k of conductor $\mathfrak{g}$. Since v is totally split in K/k, one can apply the conjecture to the extension $K_\mathfrak{g}/k$, the set of places $S(K_\mathfrak{g}/k)$ and the place v, and get a Stark unit that we denote by $\varepsilon_\mathfrak{g}$. Let $G_\mathfrak{g}$ be the Galois group of $K_\mathfrak{g}/k$. Note that by (A1) the group of roots of unity in $K_\mathfrak{g}$ is $\{\pm 1\}$. Part (2) of the conjecture is equivalent to the fact that $\varepsilon_\mathfrak{g}^{g-1} \in U_{K_\mathfrak{g}}^2$ for all $g \in G_\mathfrak{g}$; see [Tate 1984, Proposition IV.1.2]. Define R_{Stark} as the following $\mathbb{Z}[G]$-module:

$$R_{\text{Stark}} = \langle \pm 1,\ (\varepsilon_\mathfrak{g}^{g-1})^{1/2} \text{ for } \mathfrak{g} \mid \mathfrak{f} \text{ and } g \in G_\mathfrak{g} \rangle_{\mathbb{Z}[G]}.$$

Let p be a prime number that does not divide the order of G. In particular, p is an odd prime. Denote by $\hat{G}_p$ the set of irreducible $\mathbb{Z}_p$-characters of G. For $\psi \in \hat{G}_p$ and M a $\mathbb{Z}[G]$-module, we set

$$M^\psi := M \otimes_{\mathbb{Z}[G]} \mathbb{Z}_p[\psi],$$

where $\mathbb{Z}_p[\psi]$ is the ring generated over $\mathbb{Z}_p$ by the values of ψ and G acts on $\mathbb{Z}_p[\psi]$ via the character ψ. The following result is a direct consequence of Theorem 4.6 of [Rubin 1992].

Theorem 3.1 (Rubin). *If $\psi \in \hat{G}_p$ is odd then*

$$\left|(U_K/R_{\text{Stark}})^{\psi}\right| = \left|\text{Cl}_K^{\psi}\right|.$$

From this we deduce an analogous statement for our case.

Theorem 3.2. *For all $\psi \in \hat{G}_p$, we have*

$$\left|(U_K^-/\mathbb{Z}[G]\cdot\bar{\varepsilon})^{\psi}\right| = \left|(\text{Cl}_K^-)^{\psi}\right|.$$

Proof. For M a $\mathbb{Z}[G]$-module and $\psi \in \hat{G}_p$, it is direct to see that $M^{\psi} = (M^{1+\tau})^{\psi}$ if ψ is even and $M^{\psi} = (M^{1-\tau})^{\psi}$ if ψ is odd. In particular, if ψ is even, we get $|(U_K^-/\mathbb{Z}[G]\cdot\bar{\varepsilon})^{\psi}| = |(\text{Cl}_K^-)^{\psi}| = 1$ and the result follows trivially in that case. Assume now that ψ is odd. Let ε_0 be the Stark unit corresponding to the extension K/k, the set of places $S(K/k)$ and the distinguished place v. Assume first that $S = S(K/k) \cup \{\mathfrak{p}\}$ for some finite prime ideal $\mathfrak{p}$ of k not in $S(K/k)$. It follows from [Tate 1984, Proposition IV.3.4] that $\bar{\varepsilon} = (1 - \text{F}_{\mathfrak{p}}(K/k))\cdot\bar{\varepsilon}_0$, where $\text{F}_{\mathfrak{p}}(K/k)$ is the Frobenius at $\mathfrak{p}$ for the extension K/k. By (A3), τ is a power of $\text{F}_{\mathfrak{p}}(K/k)$ and thus $\psi(\text{F}_{\mathfrak{p}}(K/k))$ is a non trivial root of unity of order dividing $|G|$. Then $1 - \psi(F_{\mathfrak{p}}(K/k))$ is a p-adic unit and therefore $(\mathbb{Z}[G]\cdot\bar{\varepsilon})^{\psi} = (\mathbb{Z}[G]\cdot\bar{\varepsilon}_0)^{\psi}$. By repeating this argument if necessary, we see that this last equality also holds in the general case. Now, by taking $\mathfrak{g} = \mathfrak{f}$ and $\sigma = \tau$ in the definition of R_{Stark}, we see that $\varepsilon_0^{(\tau-1)/2} = \varepsilon_0^{-1} \in R_{\text{Stark}}$. Therefore we have $\varepsilon_0^{\mathbb{Z}[G]} \subset R_{\text{Stark}} \subset U_K$, and thus

$$\varepsilon_0^{2\mathbb{Z}[G]} \subset R_{\text{Stark}}^{\tau-1} \subset U_K^{\tau-1}.$$

We take the ψ-component, by the above remarks and the theorem, we get

$$\begin{aligned}|(U_K^-/\mathbb{Z}[G]\cdot\bar{\varepsilon})^{\psi}| &= |(U_K^-/\mathbb{Z}[G]\cdot\bar{\varepsilon}_0)^{\psi}| = |(U_K^{\tau-1}/\varepsilon_0^{2\mathbb{Z}[G]})^{\psi}| \\ &\geq |(U_K^{\tau-1}/R_{\text{Stark}}^{\tau-1})^{\psi}| = |(U_K/R_{\text{Stark}})^{\psi}| = |\text{Cl}_K^{\psi}| = |(\text{Cl}_K^-)^{\psi}|.\end{aligned}$$

Assume there exists a character ψ for which this is a strict inequality. Multiplying over all characters in $\hat{G}_p$, we get $|(U_K^-/\mathbb{Z}[G]\cdot\bar{\varepsilon})\otimes\mathbb{Z}_p| > |\text{Cl}_K^-\otimes\mathbb{Z}_p|$, a contradiction with Theorem 2.3. Therefore the equality holds for all $\psi \in \hat{G}_p$ and the theorem is proved. □

4. The index property

From now on, we do not assume anymore that the conjecture is true.

From the results of the previous sections, we see that the conjecture implies that there exists a unit $\bar{\varepsilon} \in U_K^-$ such that[6]

(P1) $\big(U_K^- : \mathbb{Z}[G] \cdot \bar{\varepsilon}\big) = 2^{e+t_S} |\mathrm{Cl}_K^-|$,

(P2) $\big|(U_K^-/\mathbb{Z}[G] \cdot \bar{\varepsilon})^{\psi}\big| = \big|(\mathrm{Cl}_K^-)^{\psi}\big|$ for all $p \nmid [K:k]$ and $\psi \in \hat{G}_p$.

A priori the existence of a solution to (P1) and (P2) does not imply in return the conjecture (except for quadratic extensions; see Theorem 5.5 below). Indeed, in general, properties (P1) and (P2) do not even characterize the Stark unit ε. To see that assume that $\bar{\eta}$ is a solution to (P1) and (P2), and let $\bar{\eta}' := u \cdot \bar{\eta}$, where $u \in \mathbb{Z}[G]^{\times}$ is a unit of $\mathbb{Z}[G]$. Then $\bar{\eta}'$ also satisfies (P1) and (P2). If u belongs to $\{\pm\gamma : \gamma \in G\} \subset \mathbb{Z}[G]^{\times}$, the group of trivial units of $\mathbb{Z}[G]$, then $\bar{\eta}'$ is essentially the *same solution* since it is a conjugate of $\bar{\eta}$ or the inverse of a conjugate of $\bar{\eta}$. However there may be some non trivial units in $\mathbb{Z}[G]$ (see the end of this section) and thus solutions to (P1) and (P2) that are not related in any obvious way to the Stark unit. In any case, we have the following result that shows that solutions to (P1) satisfy a very weak version of part (1) of the conjecture.

Proposition 4.1. *Let $\bar{\eta}$ be an element of U_K^- satisfying* (P1)*. Then we have*

$$\prod_{\chi \text{ odd}} \frac{1}{2} \sum_{g \in G} \chi(g) \log|\eta^g| = \pm \prod_{\chi \text{ odd}} L'_{K/k,S}(0,\chi). \tag{4.5}$$

Proof. Let $\bar{x} \in U_K^-$. Using the notations of Section 2, we have $|x^{\tau}|_j = |x|_j$ for $2m+1 \le j \le (d+1)m$ since these absolute values are complex and τ is the complex conjugation. Since, by construction, we have $x^{\tau} = x^{-1}$, it follows that $|x|_j^2 = |x^{1+\tau}|_j = 1$ and $|x|_j = 1$ for $2m+1 \le j \le (d+1)m$. We can therefore reproduce the determinant computation done in Section 2 replacing ε by η and U_{Stark} by the subgroup U_0 of U_K generated by U_{K^+} and the conjugates of η. We get

$$(U_K : U_0) = \pm 2^{dm-1} \frac{R_{K^+}}{R_K} \prod_{\chi \text{ odd}} \frac{1}{2} \sum_{g \in G} \chi(g) \log|\eta^g|.$$

We then proceed as in Theorem 2.3 by looking at the kernel of the norm map acting on $U_0/\{\pm 1\}$. Since $\bar{\eta}$ satisfies (P1), it follows that

$$2^{dm-1} \frac{R_{K^+}}{R_K} \prod_{\chi \text{ odd}} \frac{1}{2} \sum_{g \in G} \chi(g) \log|\eta^g| = \pm 2^{dm-1+t_S} |\mathrm{Cl}_K^-|.$$

[6] Although assumptions (A1) to (A4) are necessary to prove that the Stark unit is a solution of (P2), it is not necessary to assume (A4) to prove that solutions exist in the cases that we study below. It is an interesting question whether or not one could prove that the Stark unit is a solution to (P2) without having first to assume (A4).

Then, by (2.4), we get the result

$$\prod_{\chi \text{ odd}} \frac{1}{2} \sum_{g\in G} \chi(g) \log|\eta^g| = \pm 2^{ts} \frac{h_K R_K}{h_{K^+} R_{K^+}} = \pm \prod_{\chi \text{ odd}} L'_{K/k,S}(0,\chi). \qquad \square$$

We now turn to the study of the structure of the $\mathbb{Q}[G]$-module $U_K^- \otimes \mathbb{Q}$. Since U_K^- is killed by $1+\tau$, it is a $\mathbb{Q}[G]^-$-module where $\mathbb{Q}[G]^- := e^-\mathbb{Q}[G]$ and $e^- := \frac{1}{2}(1-\tau)$ is the sum of the idempotents of odd characters of G.[7] Since U_K^- injects into $U_K^- \otimes \mathbb{Q}$, we will identify it with its image. The following result describes the structure of $U_K^- \otimes \mathbb{Q}$ as a Galois module.

Proposition 4.2. *The module $U_K^- \otimes \mathbb{Q}$ is a free $\mathbb{Q}[G]^-$-module of rank* 1.

Proof. Let $\mathcal{Y}_K$ be the $\mathbb{Q}$-vector space with basis the elements z in the set $S_\infty(K)$ of infinite places of K. The group G acts on $\mathcal{Y}_K$ in the following way: z^g for $g \in G$ and $z \in S_\infty(K)$ is the infinite place defined by $x \mapsto z(x^g)$ for all $x \in K$. Denote by $\mathcal{X}_K$ the subspace of elements $\sum_z a_z z \in \mathcal{Y}_K$ such that $\sum_z a_z = 0$. Then the two $\mathbb{Q}[G]$-modules $\mathcal{X}_K$ and $U_K \otimes \mathbb{Q}$ are isomorphic by a result of Herbrand and Artin; see [Artin 1932]. Fix a $\mathbb{Q}[G]$-isomorphism $f : U_K \otimes \mathbb{Q} \to \mathcal{X}_K$. A direct computation shows that $\mathcal{X}_K^- := f(U_K^- \otimes \mathbb{Q})$ is spanned by the vectors $\{w^\rho - w^{\rho\tau}\}_{\rho\in R}$, where w is the fixed place of K above v. In particular, $\mathcal{X}_K^-$ is generated as a $\mathbb{Q}[G]^-$-module by the vector $w - w^\tau$. This proves the result. $\square$

Corollary 4.3. *There exist $\bar{\theta} \in \bar{U}_K^-$ and $q \in \mathbb{Q}^\times$ such that*

$$\prod_{\chi \text{ odd}} \frac{1}{2} \sum_{g\in G} \chi(g) \log|\theta^g| = q \prod_{\chi \text{ odd}} L'_{K/k,S}(0,\chi).$$

Proof. From the proposition, there exists $u \in U_K^- \otimes \mathbb{Q}$ such that $U_K^- \otimes \mathbb{Q} = \mathbb{Q}[G]^- \cdot u$. We let $\bar{\theta} := n \cdot u$, where $n \in \mathbb{N}$ is large enough so that $\bar{\theta} \in U_K^-$. Then we set

$$q := \frac{(U_K^- : \mathbb{Z}[G] \cdot \bar{\theta})}{2^{e+t}|\mathrm{Cl}_K^-|}.$$

The result follows by the proof of Proposition 4.1 *mutatis mutandis* and replacing q by $-q$ if necessary. $\square$

Thanks to Proposition 4.2, it is enough to study the structure of $\mathbb{Q}[G]^-$ to understand that of $U_K^- \otimes \mathbb{Q}$. Let X be the set of irreducible $\mathbb{Z}$-characters of G. Each $\xi \in X$ is the sum of the irreducible characters in a conjugacy class C_ξ of $\hat{G}$ under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. For $\xi \in X$, we let

$$e_\xi := \sum_{\chi \in C_\xi} e_\chi \in \mathbb{Q}[G]$$

[7] Note that $\mathbb{Q}[G]^-$ is a ring with identity e^-.

be the corresponding rational idempotent, where e_χ denotes the idempotent associated to the character χ. We have

$$\mathbb{Q}[G] = \bigoplus_{\xi \in X} e_\xi \mathbb{Q}[G] \simeq \bigoplus_{\xi \in X} \mathbb{Q}(\xi),$$

where $\mathbb{Q}(\xi)$ is the cyclotomic field generated by the values of any character in C_ξ. Let X_{odd} be the set of $\mathbb{Z}$-characters $\xi \in X$ such that one, and thus all, characters in C_ξ are odd. We have $e^- = \sum_{\xi \in X_{\text{odd}}} e_\xi$ and from the above decomposition, we get

$$\mathbb{Q}[G]^- = \bigoplus_{\xi \in X_{\text{odd}}} e_\xi \mathbb{Q}[G] \simeq \bigoplus_{\xi \in X_{\text{odd}}} \mathbb{Q}(\xi). \tag{4.6}$$

We now define $\mathbb{Z}[G]^- := e^-\mathbb{Z}[G]$ and let $\mathbb{O}_G^-$ be the maximal order of $\mathbb{Q}[G]^-$. We have

$$\mathbb{O}_G^- = \bigoplus_{\xi \in X_{\text{odd}}} e_\xi \mathbb{Z}[G] \simeq \bigoplus_{\xi \in X_{\text{odd}}} \mathbb{Z}[\xi]. \tag{4.7}$$

Now let p be a prime number. By (4.6), we get

$$\mathbb{Q}_p[G]^- \simeq \bigoplus_{\xi \in X_{\text{odd}}} \mathbb{Q}(\xi) \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \bigoplus_{\xi \in X_{\text{odd}}} \bigoplus_{\mathfrak{p} \in S_{\xi,p}} \mathbb{Q}(\xi)_{\mathfrak{p}}, \tag{4.8}$$

where $S_{\xi,p}$ is the set of prime ideals of $\mathbb{Q}(\xi)$ above p and $\mathbb{Q}(\xi)_{\mathfrak{p}}$ is the completion of $\mathbb{Q}(\xi)$ at the prime ideal $\mathfrak{p}$. On the other hand, each rational character $\xi \in X$ is the sum of irreducible $\mathbb{Z}_p$-characters, say $\xi = \sum_{\psi \in C_{\xi,p}} \psi$, and we have

$$\mathbb{Q}_p[G]^- = \bigoplus_{\xi \in X_{\text{odd}}} \bigoplus_{\psi \in C_{\xi,p}} e_\psi \mathbb{Q}_p[G]^-.$$

Therefore there is a bijection between the prime ideals in $S_{\xi,p}$ and the characters in $C_{\xi,p}$. For $\mathfrak{p}$ a prime ideal in $S_{\xi,p}$, we denote by $\psi_{\xi,\mathfrak{p}}$ the corresponding irreducible $\mathbb{Z}_p$-character. Before stating the first result, we need one more notation. Let T be a set of primes. We say that an element $u \in \mathbb{Q}[G]$ is a T-unit if $u \in \mathbb{Z}_p[G]^{-,\times}$ for all $p \notin T$, where $\mathbb{Z}_p[G]^{-,\times}$ is the group of units of $\mathbb{Z}_p[G]^-$.

Proposition 4.4. *Let M be a sub-$\mathbb{Z}[G]^-$-module of $\mathbb{Q}[G]^-$ of finite index. Let x be an element of M such that $x\mathbb{Z}[G]^-$ has finite index inside M. Assume that y is another element of M such that*

$$(M : x\mathbb{Z}[G]^-) = (M : y\mathbb{Z}[G]^-) \quad \text{and} \quad (M^\psi : (x\mathbb{Z}_p[G]^-)^\psi) = (M^\psi : (y\mathbb{Z}_p[G]^-)^\psi)$$

for all $p \nmid |G|$ and all $\psi \in \hat{G}_p$ with ψ odd. Then there exists a unique B-unit $u \in \mathbb{Q}[G]^-$ such that $y = ux$, where B is the set of primes dividing both $|G|$ and $(M : x\mathbb{Z}[G]^-)$.

Proof. Since $\mathbb{Q}[G]^- = x\mathbb{Q}[G]^-$, there exists $u \in \mathbb{Q}[G]^-$ such that $y = ux$. Assume $y = vx$ for another $v \in \mathbb{Q}[G]^-$. Then, for all $\xi \in X_{\text{odd}}$, we have $\xi(u)\xi(x) = \xi(v)\xi(x)$. Since $\xi(x) \neq 0$, it follows that $\xi(u) = \xi(v)$ and thus by (4.6), we get $u = v$ which proves that u is unique.

Let p be a prime. Assume first that p does not divide $|G|$. Let $\xi \in X_{\text{odd}}$ and $\mathfrak{p} \in S_{\xi,p}$. Write $\psi := \psi_{\xi,\mathfrak{p}}$ and denote by $\mathbb{Z}[\xi]_{\mathfrak{p}} := \psi(\mathbb{Z}_p[G]^-)$ the ring of integers of $\mathbb{Q}(\xi)_{\mathfrak{p}}$. Then M^ψ is an ideal of $\mathbb{Z}[\xi]_{\mathfrak{p}}$ and we have

$$\frac{(M^\psi : (x\mathbb{Z}_p[G]^-)^\psi)}{(M^\psi : (y\mathbb{Z}_p[G]^-)^\psi)} = \frac{(M^\psi : \psi(y)\mathbb{Z}[\xi]_{\mathfrak{p}})}{(M^\psi : \psi(x)\mathbb{Z}[\xi]_{\mathfrak{p}})} = |\psi(u)|_{\mathfrak{p}}.$$

Thus $\psi_{\xi,\mathfrak{p}}(u)$ is a unit in $\mathbb{Z}[\xi]_{\mathfrak{p}}$ for all $\xi \in X_{\text{odd}}$ and $\mathfrak{p} \in S_{\xi,p}$ and thus u lies in $\mathbb{Z}_p[G]^{-,\times}$. Assume now that p does not divide the index $(M : x\mathbb{Z}[G]^-)$. We have

$$(M \otimes \mathbb{Z}_p : x\mathbb{Z}_p[G]^-) = (M \otimes \mathbb{Z}_p : y\mathbb{Z}_p[G]^-) = 1.$$

Therefore $x\mathbb{Z}_p[G]^- = M \otimes \mathbb{Z}_p = y\mathbb{Z}_p[G]^-$ and $u \in \mathbb{Z}_p[G]^{-,\times}$. □

By Propositions 4.2 and 4.4, we get the following result.

Corollary 4.5. *Let B be the set of primes that divide both $|G|$ and $|\mathrm{Cl}_K^-|$. Assume there exist $\bar\eta$ and $\bar\eta'$ two elements of U_K^- satisfying* (P1) *and* (P2). *Then there exists a unique B-unit $u \in \mathbb{Q}[G]^-$ such that $\bar\eta' = u \cdot \bar\eta$.* □

From this result and the discussion at the beginning of the section, one cannot expect the properties (P1) and (P2) to characterize the Stark unit if $\mathbb{Z}[G]^-$ has some non trivial B-units and a fortiori if $\mathbb{Z}[G]^-$ has some non trivial units.[8] It follows from the methods of Higman [1940] that $\mathbb{Z}[G]^-$ has some non trivial units if and only if $\mathcal{O}_G^-$ does. By (4.7), this is the case if and only if there exists an odd character of G whose order does not divide 6. In particular, for G a cyclic group of even order, $\mathbb{Z}[G]^-$ has only trivial units if and only if the order of G is at most 6. We will prove in the next sections that there exist solutions to (P1) and (P2) in these cases (with some additional conditions for sextic extensions). From this we will deduce another proof of the conjecture for quadratic extensions and a weak version of the conjecture for quartic and sextic extensions.

5. Algebraic tools

In this section we introduce some algebraic tools and results that will be useful in the next sections. We start with the properties of Fitting ideals. Let R be a commutative ring with an identity element. Let M be a finitely generated R-module. Therefore there exists a surjective homomorphism $f : R^a \to M$ for some $a \geq 1$. The

[8]If $\bar\eta$ is a solution to (P1) and (P2) and u is a B-unit then $u \cdot \bar\eta$ is not necessarily a solution to (P1) and (P2). A necessary and sufficient condition for that is that the linear map $x \mapsto ux$ of $\mathbb{Q}[G]^-$ has determinant ± 1. This is always true if u is a unit of $\mathbb{Z}[G]^-$.

Fitting ideal of M as an R-module, denoted $\mathrm{Fitt}_R(M)$, is the ideal of R generated by $\det(\vec{v}_1, \dots, \vec{v}_a)$, where $\vec{v}_1, \dots, \vec{v}_a$ run through the elements of the kernel of f. One can prove that it does not depend on the choice of f. We will use the following properties of Fitting ideals; see [Northcott 1976, Chapter 3] or [Eisenbud 1995, Chapter 20].

- If there exist ideals $A_1, \dots, A_t$ of R such that $M \simeq R/A_1 \oplus \cdots \oplus R/A_t$, then we have
$$\mathrm{Fitt}_R(M) = A_1 \cdots A_t.$$
- Let T be an R-algebra. We have
$$\mathrm{Fitt}_T(M \otimes_R T) = \mathrm{Fitt}_R(M)T.$$
- Let N be another finitely generated R-module. We have
$$\mathrm{Fitt}_R(M \oplus N) = \mathrm{Fitt}_R(M)\mathrm{Fitt}_R(N).$$

Lemma 5.1. *Let M be a finite $\mathbb{O}_G^-$-module. Then*
$$|M| = |(\mathbb{O}_G^- / \mathrm{Fitt}_{\mathbb{O}_G^-}(M))|.$$

Proof. We have
$$(\mathbb{O}_G^- : \mathrm{Fitt}_{\mathbb{O}_G^-}(M)) = \prod_{\xi \in X_{\mathrm{odd}}} (e_\xi \mathbb{Z}[G] : e_\xi \mathrm{Fitt}_{\mathbb{O}_G^-}(M)) = \prod_{\xi \in X_{\mathrm{odd}}} (\mathbb{Z}[\xi] : \mathrm{Fitt}_{\mathbb{Z}[\xi]}(e_\xi M)).$$
Fix $\xi \in X_{\mathrm{odd}}$. Since $e_\xi M$ is a finite $\mathbb{Z}[\xi]$-module, there exist ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ such that
$$e_\xi M = \mathbb{Z}[\xi]/\mathfrak{a}_1 \oplus \cdots \oplus \mathbb{Z}[\xi]/\mathfrak{a}_r.$$
Therefore $\mathrm{Fitt}_{\mathbb{Z}[\xi]}(e_\xi M) = \mathfrak{a}_1 \cdots \mathfrak{a}_r$ and
$$(\mathbb{Z}[\xi] : \mathrm{Fitt}_{\mathbb{Z}[\xi]}(e_\xi M)) = N_{\mathbb{Q}(\xi)/\mathbb{Q}}(\mathfrak{a}_1 \cdots \mathfrak{a}_r) = |e_\xi M|.$$
It follows that $(\mathbb{O}_G^- : \mathrm{Fitt}_{\mathbb{O}_G^-}(M)) = \prod_{\xi \in X_{\mathrm{odd}}} |e_\xi M| = |M|$. □

Lemma 5.2. *Let M be a finite $\mathbb{Z}[G]^-$-module. Let p be a prime number not dividing $|G|$ and let ψ be an odd irreducible $\mathbb{Z}_p$-character. Then*
$$|M^\psi| = |(\mathbb{Z}[G]^- / \mathrm{Fitt}_{\mathbb{Z}[G]^-}(M))^\psi| = |(\mathbb{O}_G^- / \mathrm{Fitt}_{\mathbb{O}_G^-}(M))^\psi|.$$

Proof. We have $(\mathrm{Fitt}_{\mathbb{Z}[G]^-}(M))^\psi = \mathrm{Fitt}_{\mathbb{Z}_p[\psi]}(M^\psi)$. Since M^ψ is a finite $\mathbb{Z}_p[\psi]$-module, there exist integers $c_1, \dots, c_r \geq 1$ such that
$$M^\psi \simeq \bigoplus_{i=1}^{r} \mathbb{Z}_p[\psi]/\mathfrak{p}^{c_i},$$
where $\mathfrak{p}$ is the prime ideal of $\mathbb{Z}_p[\psi]$. Then $\mathrm{Fitt}_{\mathbb{Z}[G]^-}(M)^\psi = \mathfrak{p}^c$ with $c := c_1 + \cdots + c_r$

and therefore $|(\mathbb{Z}[G]^-/\mathrm{Fitt}_{\mathbb{Z}[G]^-}(M))^\psi| = (\mathbb{Z}_p[\xi] : \mathfrak{p}^c) = |M^\psi|$. The last equality is clear since $(\mathbb{O}_G^-)^\psi = \mathbb{Z}_p[\psi]$. □

In what follows we will also use repeatedly the Tate cohomology of finite cyclic groups; see [Lang 1994, §IX.1]. Let A be a finite cyclic group with generator a and let M be a $\mathbb{Z}[A]$-module. The zeroth and first group of cohomology are defined by

$$\hat{H}^0(A, M) := M^A/N_A(M) \quad \text{and} \quad \hat{H}^1(A, M) := \mathrm{Ker}(N_A : M \to M)/(1-a)M,$$

where $N_A := \sum_{b\in A} b$ and M^A is the submodule of elements in M fixed by A. Let N and P be two other $\mathbb{Z}[A]$-modules such that the following short sequence is exact:

$$1 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 1.$$

Then the hexagon below is also exact.

$$\begin{tikzcd}[column sep=small]
 & \hat{H}^0(A, M) \arrow[dr] & \\
\hat{H}^1(A, P) \arrow[ur] & & \hat{H}^0(A, N) \arrow[d] \\
\hat{H}^1(A, N) \arrow[u] & & \hat{H}^0(A, P) \arrow[dl] \\
 & \hat{H}^1(A, M) \arrow[ul] &
\end{tikzcd} \tag{5.9}$$

The Herbrand quotient of M is defined by

$$Q(A, M) := \frac{|\hat{H}^0(A, M)|}{|\hat{H}^1(A, M)|}.$$

The Herbrand quotient is multiplicative, that is for an exact short sequence as above, we have $Q(A, N) = Q(A, M)\, Q(A, P)$. The following result plays a crucial rôle in the next sections. It is a direct consequence of [Lang 1994, Corollary IX.4.2].

Lemma 5.3. *Let E/F be a quadratic extension with Galois group T. Let $R \geq 0$ be the number of real places in F that becomes complex in E. Then we have*

$$Q(T, U_E) = 2^{R-1}.$$ □

We use this result in the following way. Assume that $R \geq 1$. Write $\bar{U}_F$ and $\bar{U}_E$ for the group of units of F and E respectively modulo $\{\pm 1\}$. Then we have

$$\hat{H}^0(T, U_E) = \frac{U_F}{\mathcal{N}_{E/F}(U_E)} = \{\pm 1\} \times \frac{\bar{U}_F}{\mathcal{N}_{E/F}(\bar{U}_E)},$$

since -1 cannot be a norm in E/F. It follows from the lemma that $|\hat{H}^0(T, U_E)|$ is divisible by 2^{R-1} and therefore

$$2^{R-2} \mid (\bar{U}_F : \mathcal{N}_{E/F}(\bar{U}_F)). \tag{5.10}$$

In some cases we will not be able to get non trivial lower bounds with that method, but still be able to deduce that $\hat{H}^1(T, U_E)$ is trivial. In this situation, we have the following lemma.

Lemma 5.4. *Let E/F be a quadratic extension with Galois group T. Assume that $\hat{H}^1(T, U_E)$ is trivial. Then either E/F is unramified at finite places or there exists an element of order* 2 *in the kernel of the norm map from* Cl_E *to* Cl_F.

Proof. Consider the submodules of elements fixed by T in the short exact sequence

$$1 \longrightarrow U_E \longrightarrow E^\times \longrightarrow P_E \longrightarrow 1.$$

We get

$$1 \longrightarrow U_F \longrightarrow F^\times \longrightarrow P_E^T \longrightarrow \hat{H}^1(T, U_E) \longrightarrow \cdots .$$

Since $\hat{H}^1(T, U_E) = 1$ by hypothesis, it follows that the groups P_F and P_E^T are isomorphic. The isomorphism is the natural map that sends $\mathfrak{a} \in P_F$ to $\mathfrak{a}\mathbb{Z}_E \in P_E^T$, where $\mathbb{Z}_E$ is the ring of integers of E. Assume that there is a prime ideal $\mathfrak{p}$ of F that ramifies in E/F. Let $\mathfrak{P}$ be the unique prime ideal of E above $\mathfrak{p}$ and let $h \geq 1$ be the order of $\mathfrak{P}$ in Cl_E. Since $\mathfrak{P}^h \in P_E^T$, there exists a principal ideal $\mathfrak{a} \in P_F$ such that $\mathfrak{P}^h = \mathfrak{a}\mathbb{Z}_E$. Clearly $\mathfrak{a}$ is a power of $\mathfrak{p}$. Looking at valuations at $\mathfrak{P}$, it follows that h is even. We set $\mathfrak{C} := \mathfrak{P}^{h/2}$. Its class is an element of order 2 in Cl_E. But $\mathcal{N}_{E/F}(\mathfrak{C}) = \mathfrak{p}^{h/2} = \mathfrak{a}$ is a principal ideal. This concludes the proof. $\square$

To conclude this section we prove the conjecture in our settings when K/k is a quadratic extension. This result is proved in full generality in [Tate 1984, Theorem IV.5.4].

Theorem 5.5. *Let K/k be a quadratic extension and $S \supset S(K/k)$ be a finite set of places of k satisfying* (A1), (A2) *and* (A3). *Then the abelian rank-one Stark conjecture is satisfied for the extension K/k and the set S with the Stark unit being the unique solution, up to trivial units, of* (P1) *and* (P2). *Moreover the Stark unit is a square in K if and only if $e + t_S + c \geq 1$, where c is the* 2*-valuation of the order of* Cl_K^-. *In particular, if $d \geq 4$ then it always a square and, in fact, it is a 2^{d-3}-th power. It is also a square if $d = 3$ and the extension K/k is ramified at some finite prime.*

Proof. The only non trivial element of G is τ. Let χ be the character that sends τ to -1. It is the only non trivial character of G and also the only odd character. We have $\mathbb{Z}[G]^- = \mathbb{O}_G^- = e^-\mathbb{Z} \simeq \mathbb{Z}$. In particular, using Proposition 4.2, it is direct to see that there exists $\bar{\theta} \in U_K^-$ such that $U_K^- = \mathbb{Z} \cdot \bar{\theta}$. Define

$$\bar{\eta} := 2^{e+t_S} |\mathrm{Cl}_K^-| \cdot \bar{\theta}.$$

From its construction, it is clear that $\bar{\eta}$ satisfies (P1) and (P2). It follows from Proposition 4.1, and replacing η by η^{-1} if necessary, that

$$\frac{1}{2} \sum_{g \in G} \chi(g) \log |\eta^g| = L'_{K/k}(0, \chi).$$

This proves part (1) of the conjecture. Part (3) is direct by construction. It remains to prove part (2). But $(\tau - 1) \cdot \bar{\eta} = -2 \cdot \bar{\eta}$ so part (2) follows and the conjecture is proved in this case. Finally, from its definition, it is clear that η is a 2^r-th power in $K^\times$ if and only if $e + t_S + c \geq r$. Now, by (5.10), we have $e \geq d - 3$ and therefore the Stark unit is always a square if $d \geq 4$. Assume that $d = 3$ and that η is not a square. Then $e = 0$ and $|\hat{H}^0(G, U_K)| = 2$. From Lemma 5.3, we get $\hat{H}^1(G, U_K) = 1$ and therefore, since $c = 0$, the extension K/k is unramified at finite places by Lemma 5.4 $\square$

When $d = 2$, there exist extensions for which the Stark unit is a square and extensions for which it is not a square. Using the PARI/GP system [PARI 2011], we find the following examples.[9] Let $k := \mathbb{Q}(\sqrt{5})$ and let v_1, v_2 denote the two infinite places of k with $v_1(\sqrt{5}) < 0$ and $v_2(\sqrt{5}) > 0$. Let K be the ray class field modulo $\mathfrak{p}_{11} v_2$, where $\mathfrak{p}_{11} := (1/2 + 3\sqrt{5}/2)$ is one of the two prime ideals above 11. Then K/k is a quadratic extension that satisfies (A1), (A2) and (A3) with $S := S(K/k)$, and one can prove that the corresponding Stark unit is not a square. Now, on the other hand, let K be the ray class field modulo $\sqrt{5}\mathfrak{q}_{11} v_1$, where $\mathfrak{q}_{11} := (1/2 - 3\sqrt{5}/2)$ is the other prime ideal above 11. Then K/k is a quadratic extension that satisfies (A1), (A2) and (A3) with $S := S(K/k)$ and, in this case, the Stark unit is a square. When $d = 3$ and K/k is unramified both cases are possible. Indeed, let $k := \mathbb{Q}(\alpha)$, where $\alpha^3 - \alpha^2 - 13\alpha + 1 = 0$. It is a totally real cubic field. Let v_1, v_2, v_3 be the three infinite places of k with $v_1(\alpha) \approx -3.1829$, $v_2(\alpha) \approx 0.0765$ and $v_3(\alpha) \approx 4.1064$. Let K be the ray class field of k of conductor $\mathbb{Z}_k v_2 v_3$. Then K/k is a quadratic extension that satisfies (A1), (A2) and (A3) with $S := S(K/k)$, and that is unramified at finite places. One can prove in this setting that the Stark unit is not a square. On the other hand, let $k := \mathbb{Q}(\beta)$ with $\beta^3 - \beta^2 - 24\beta - 35 = 0$. It is a totally real cubic field. Let v_1, v_2, v_3 be the three infinite places of k with $v_1(\alpha) \approx -3.0999$, $v_2(\alpha) \approx -1.8861$, and $v_3(\alpha) \approx 5.9860$. Let K be the unique quadratic extension of k of conductor $\mathbb{Z}_k v_2 v_3$. Then K/k satisfies (A1), (A2) and (A3) with $S := S(K/k)$ and is unramified at finite places. One can prove that k is principal and the class number of K is 2. Therefore the Stark unit in this case is a square.

[9] PARI/GP was also used to find the examples given in the next two sections.

6. Cyclic quartic extensions

Theorem 6.1. *Let K/k be a cyclic quartic extension and $S \supset S(K/k)$ be a finite set of places of k satisfying* (A1), (A2) *and* (A3)*. Then there exists $\bar{\eta} \in U_K^-$ satisfying* (P1) *and* (P2)*. Furthermore, $\bar{\eta}$ is unique up to the action of $\pm G$, it satisfies*

$$\left|L'_{K/k,S}(0,\chi)\right| = \frac{1}{2}\left|\sum_{g\in G} \chi(g)\log|\eta^g|\right| \quad \text{for all } \chi \in \hat{G},$$

and the extension $K(\sqrt{\eta})/k$ is abelian.

Proof. Denote by γ a generator of G, therefore $\tau = \gamma^2$. Let χ be the character of G such that $\chi(\gamma) = i$ and let $\xi := \chi + \chi^3$ be the only element in X_{odd}. From the results of Section 4, we have

$$\mathbb{Q}[G]^- = e^-\mathbb{Q}[G] \simeq \mathbb{Q}(i),$$

where the ring isomorphism sends any element of $x \in \mathbb{Q}[G]^-$, written uniquely as $x = e^-(a + b\gamma)$ for $a, b \in \mathbb{Q}$, to $\chi(x) = a + bi$. In particular, we have $\mathbb{Z}[G]^- = \mathcal{O}_G^- \simeq \mathbb{Z}[i]$ and $\mathbb{Z}[G]^-$ is a principal ring. By Proposition 4.2, this implies that there exists $\bar{\theta} \in U_K^-$ such that $U_K^- = \mathbb{Z}[G]^- \cdot \bar{\theta}$.

We now prove the uniqueness of the solution. Assume that $\bar{\eta}$ and $\bar{\eta}'$ are two solutions to (P1) and (P2). By Corollary 4.5, there exists a unique 2-unit u in $\mathbb{Q}[G]^-$ such that $\bar{\eta}' = u \cdot \bar{\eta}$. Let $\mathfrak{p}_2 := (i+1)\mathbb{Z}[i]$ be the unique prime ideal above 2 in $\mathbb{Z}[i]$. Let $n := v_{\mathfrak{p}_2}(\chi(u))$. Assume, without loss of generality, that $n \geq 0$ (otherwise, exchange $\bar{\eta}$ and $\bar{\eta}'$ and replace u by u^{-1}) and therefore $\bar{\eta}' \in \mathbb{Z}[G]^- \cdot \bar{\eta}$. Let $x \in \mathbb{Z}[G]^-$ be such that $\bar{\eta} = x \cdot \bar{\theta}$. We have

$$\begin{aligned}(\mathbb{Z}[G]^- \cdot \bar{\eta} : \mathbb{Z}[G]^- \cdot \bar{\eta}') &= (x\mathbb{Z}[G]^- : ux\mathbb{Z}[G]^-)\\ &= (\chi(x)\mathbb{Z}[i] : \chi(u)\chi(x)\mathbb{Z}[i])\\ &= |\chi(u)| = 2^n.\end{aligned}$$

Therefore $n = 0$ and u is a unit. Since the only units of $\mathbb{Z}[i]$ are ± 1 and $\pm i$, it follows that $u = \pm e^- g$ with $g \in G$. This proves the uniqueness statement.

Next we prove that there exist solutions to (P1) and (P2). Let $\mathcal{F} := \operatorname{Fitt}_{\mathbb{Z}[G]^-}(\mathrm{Cl}_K^-)$ be the Fitting ideal of Cl_K^- as a $\mathbb{Z}[G]^-$-module. Let f be a generator of $\mathcal{F}$. We set $\bar{\eta} := f\,(\gamma+1)^{e+ts} \cdot \bar{\theta}$. We have by Lemma 5.1

$$(U_K^- : \mathbb{Z}[G] \cdot \bar{\eta}) = 2^{e+ts}(\mathbb{Z}[G]^- : \mathcal{F}) = 2^{e+ts}|\mathrm{Cl}_K^-|.$$

Thus $\bar{\eta}$ is a solution to (P1). In the same way it follows directly from Lemma 5.2 that it is a solution to (P2).

Now, since $\bar{\eta} \in U_K^-$, we have for $\nu = \chi_0$, the trivial character, or $\nu = \chi^2$ that

$$\frac{1}{2}\sum_{g\in G} \nu(g)\log|\eta^g| = 0.$$

On the other hand, $L'_{K/k,S}(\nu, 0) = 0$ follows directly from [Tate 1984, Proposition I.3.4]. From Proposition 4.1, using the fact that $\chi^3 = \bar{\chi}$, we get[10]

$$\begin{aligned}\left|L'_{K/k,S}(0,\chi)\right|^2 &= L'_{K/k,S}(0,\chi)L'_{K/k,S}(0,\chi^3)\\ &= \left(\frac{1}{2}\sum_{g\in G}\chi(g)\log|\eta^g|\right)\left(\frac{1}{2}\sum_{g\in G}\chi^3(g)\log|\eta^g|\right)\\ &= \left|\frac{1}{2}\sum_{g\in G}\chi(g)\log|\eta^g|\right|^2,\end{aligned}$$

and the equality to be proved follows by taking square roots.

Finally, to prove that $K(\sqrt{\eta})/k$ is abelian, we need to prove that $(\gamma-1)\cdot\bar{\eta}\in 2\cdot U_K^-$ by [Tate 1984, Proposition IV.1.2]. This is equivalent to proving that

$$(i-1)(i+1)^{e+t_S}\chi(f) \subset 2\mathbb{Z}[i],$$

that is one of the following assertions is satisfied: $e \geq 1$, $t_S \geq 1$ or 2 divides $|\mathrm{Cl}_K^-|$. We have $e \geq 2d-4$ by (5.10) and therefore the result is proved if $d \geq 3$. Assume that $d = 2$ and $e = 0$. Then it follows by Lemma 5.3 that $\hat{H}^1(T, U_K) = 1$, where $T := \langle\tau\rangle$. By Lemma 5.4 this implies that either 2 divides $|\mathrm{Cl}_K^-|$ and the result is proved, or K/K^+ is unramified at finite places. Assume the latter. At least one prime ideal of k ramifies in K by the proof of [Roblot 2000, Lemma 2.8] since k is a quadratic field. By (A3) this prime ideal is inert in K/K^+, thus $t_S \geq 1$. This concludes the proof. □

A consequence of this result is that we can say quite precisely when the Stark unit, it exists, is a square in that case. The result is very similar to the situation in the quadratic case (see Theorem 5.5).

Corollary 6.2. *Under the hypothesis of the theorem and assuming that the Stark unit exists, then it is a square in K if and only if $e + t_S + c \geq 2$, where c is the 2-valuation of $|\mathrm{Cl}_K^-|$. In particular, if $d \geq 3$ then it is always a square and, in fact, it is a 2^{d-2}-th power.*

Proof. We prove the equivalence. The inequality is satisfied when the Stark unit ε is a square by Corollary 2.4. Now assume that the inequality is satisfied. By the uniqueness statement of the theorem, we have $\bar{\varepsilon} = \bar{\eta}$ (replacing η by one of its conjugate if necessary). From the proof of the theorem, we see that $\bar{\eta}$ belongs to

[10]Note that is easy to see that both sides of (4.5) are positive in this case.

$2^r \cdot U_K^-$ if and only if $(i+1)^{e+t_S}\chi(f) \in 2^r\mathbb{Z}[i]$. Taking valuation at $\mathfrak{p}_2$, the only prime ideal above 2, we see that it is equivalent to $e+t_S+c \geq 2r$. This proves the first assertion. Now, to prove the second assertion, we see that $e \geq 2d-4$ by (5.10). Therefore $\bar\eta$ lies in $2^{d-2}\cdot U_K^-$. This proves the result. □

When $d=2$ it is possible to find examples for which the Stark unit, if it exists, is a square and examples for which it is not a square. For example, let $k := \mathbb{Q}(\sqrt{5})$ and let v_1, v_2 denote the two infinite places of k with $v_1(\sqrt{5}) < 0$ and $v_2(\sqrt{5}) > 0$. Let K be the ray class field modulo $\mathfrak{p}_{29}v_1$, where $\mathfrak{p}_{29} := (11/2-\sqrt{5}/2)$ is one of the two prime ideals above 29. Then K/k is a cyclic quartic extension that satisfies (A1), (A2) and (A3) with $S := S(K/k)$ and one can prove that, if it exists, the Stark unit is not a square. Now, on the other hand, let K be the ray class field modulo $\sqrt{5}\mathfrak{p}_{41}v_1$, where $\mathfrak{p}_{41} := (13/2-\sqrt{5}/2)$ is one of the two prime ideals above 41. Then K/k is a cyclic quartic extension that satisfies (A1), (A2) and (A3) with $S := S(K/k)$, but one can prove that, in this case, the Stark unit, if it exists, is a square.

7. Cyclic sextic extensions

In this final section we study the case when K/k is a cyclic sextic extension. We will need some additional assumptions to be able to prove that there exists solutions to (P1) and (P2).

Theorem 7.1. *Let K/k be a cyclic sextic extension such that* (A1), (A2) *and* (A3) *are satisfied with $S := S(K/k)$. Assume also that* 3 *does not divide the order of* Cl_K *and that no prime ideal above* 3 *is wildly ramified in K/k. Let F be the quadratic extension of k contained in K. Then there exists $\bar\eta \in U_K^-$ satisfying* (P1) *and* (P2) *and such that $\mathcal{N}_{K/F}(\eta)$ is the Stark unit for the extension F/k and the set of places S. Furthermore, $\bar\eta$ is unique up to the action of an element of* $\mathrm{Gal}(K/F)$, *satisfies for all $\chi \in \hat{G}$*

$$\left|L'_{K/k,S}(0,\chi)\right| = \frac{1}{2}\left|\sum_{g\in G}\chi(g)\log|\eta^g|\right|,$$

and the extension $K(\sqrt{\eta})/k$ is abelian.

Proof. Let γ be a generator of the Galois group G, thus $\tau=\gamma^3$. Let χ be the character that sends γ to $-\omega$, where ω is a fixed primitive third root of unity. It is a generator of the group of characters of G. We have $X_{\mathrm{odd}} = \{\xi_2, \xi_6\}$, where $\xi_2 := \chi^3$ and $\xi_6 := \chi+\chi^5$. The corresponding idempotents are

$$e_{\xi_2} = \tfrac{1}{6}(1-\gamma^3)(1+\gamma^2+\gamma^4) \quad \text{and} \quad e_{\xi_6} = \tfrac{1}{6}(1-\gamma^3)(2-\gamma^2-\gamma^4).$$

We have the ring isomorphism

$$\mathbb{Q}[G]^- = e_{\xi_2}\mathbb{Q}[G] + e_{\xi_6}\mathbb{Q}[G] \cong \mathbb{Q} \oplus \mathbb{Q}(\omega). \tag{7.11}$$

Let $\sigma := \gamma^2$ and let H be the subgroup of order 3 generated by σ. Any element $g \in \mathbb{Q}[G]^-$ can be written uniquely as $g = e^- h$, where h is an element of $\mathbb{Q}[H]$. The map $g \mapsto h$ is a ring isomorphism between $\mathbb{Q}[G]^-$ and $\mathbb{Q}[H]$, that restricts to an isomorphism between $\mathbb{Z}[G]^-$ and $\mathbb{Z}[H]$. From now on, we will identify $\mathbb{Q}[G]^-$ and $\mathbb{Q}[H]$. Note that, with that identification, both act in the same way on U_K^-, $U_K^- \otimes \mathbb{Q}$, Cl_K^-, etc. Let e_0 and e_1 be the image by the projection map of e_{ξ_2} and e_{ξ_6}. Then $e_0 = \frac{1}{3}(1 + \sigma + \sigma^2)$ is the idempotent of the trivial character of H and $e_1 = \frac{1}{3}(2 - \sigma - \sigma^2)$ is the sum of the idempotents of the two non trivial characters of H. The main difference between this case and the quartic case is the fact that the isomorphism between $\mathbb{Q}[H]$ and $\mathbb{Q} \oplus \mathbb{Q}(\omega)$ does not restrict to an isomorphism between $\mathbb{Z}[G]^-$ and $\mathbb{Z} \oplus \mathbb{Z}[\omega]$. In particular, $\mathbb{Z}[G]^-$ is not a principal ring. Because of that the proof is somewhat more intricate than in the quartic case. We will therefore proceed by proving a series of different claims. First, we define

$$\mathbb{O} := e_0\mathbb{Z}[H] + e_1\mathbb{Z}[H] \simeq \mathbb{Z} \oplus \mathbb{Z}[\omega]. \tag{7.12}$$

Note that, by the above identification, we have $\mathbb{O}_G^- \cong \mathbb{O}$.

Claim 1. *The ring $\mathbb{O}$ is principal and contains $\mathbb{Z}[H]$ with index* 3.

Let $\mathcal{I}$ be an ideal of $\mathbb{O}$. Then $e_0\mathcal{I}$ is an ideal of $e_0\mathbb{Z} \simeq \mathbb{Z}$. Thus there exists $a \in \mathbb{Z}$ such that $e_0\mathcal{I} = ae_0\mathbb{Z}[H]$. In the same way, $e_1\mathcal{I}$ is an ideal of $e_1\mathbb{Z} \simeq \mathbb{Z}[\omega]$. Since $\mathbb{Z}[\omega]$ is a principal ring, there exists $b, c \in \mathbb{Z}$ such that $e_1\mathcal{I} = e_1(b + c\sigma)\mathbb{Z}[H]$. One verify readily that $e_0a + e_1(b + c\sigma)$ is a generator of $\mathcal{I}$. To conclude the proof of Claim 1, we note that $\mathbb{O}/\mathbb{Z}[H] = \langle e_0 + \mathbb{Z}[H]\rangle = \langle e_1 + \mathbb{Z}[H]\rangle$ clearly has order 3.

Claim 2. *Let $\mathcal{A}$ be an ideal of $\mathbb{Z}[H]$ of finite index. Then there exists $g \in \mathcal{A}$ such that*

$$\mathbb{O}/\mathcal{A}\mathbb{O} \simeq \mathbb{Z}[H]/g\mathbb{Z}[H]. \tag{7.13}$$

Furthermore, $\mathcal{A} = g\mathbb{Z}[H]$ if $\mathcal{A}$ is a principal ideal. Otherwise $(\mathcal{A} : g\mathbb{Z}[H]) = 3$.

We prove this claim by considering the two cases: $\mathcal{A}\mathbb{O} \neq \mathcal{A}$ and $\mathcal{A}\mathbb{O} = \mathcal{A}$.

Claim 2.1. *Assume that $\mathcal{A}\mathbb{O} \neq \mathcal{A}$. Then $\mathcal{A}$ is a principal ideal.*

Let $g' = e_0a + e_1(b + c\sigma)$ be a generator of the principal ideal $\mathcal{A}\mathbb{O}$ of $\mathbb{O}$. If $e_1(b + c\sigma) \in \mathcal{A}$, then $e_1\mathcal{A} = e_1(b + c\sigma)\mathbb{Z}[H] \subset \mathcal{A}$ and it follows that $\mathcal{A} = \mathcal{A}\mathbb{O}$, a contradiction. Therefore $\mathcal{A}\mathbb{O}/\mathcal{A} = \langle e_1(b + c\sigma) + \mathcal{A}\rangle$ has order 3. Thus one of the three elements: $e_0a + e_1(b + c\sigma)$, $e_0a - e_1(b + c\sigma)$ or e_0a belongs to $\mathcal{A}$. It cannot be e_0a since that would imply, as above, that $\mathcal{A} = \mathcal{A}\mathbb{O}$. Denote by g the one element between $e_0a \pm e_1(b + c\sigma)$ that lies in $\mathcal{A}$. Clearly we still have

$g\mathbb{O} = \mathscr{A}\mathbb{O}$. Now, g is not a zero divisor since $\mathscr{A}$ has finite index in $\mathbb{Z}[H]$, so we have $(g\mathbb{O} : g\mathbb{Z}[H]) = (\mathbb{O} : \mathbb{Z}[H]) = 3$. Therefore we get

$$(\mathscr{A} : g\mathbb{Z}[H]) = \frac{(g\mathbb{O} : g\mathbb{Z}[H])}{(\mathscr{A}\mathbb{O} : \mathscr{A})} = 1$$

and $\mathscr{A} = g\mathbb{Z}[H]$. Equation (7.13) follows in that case from the equality

$$(\mathbb{O} : \mathscr{A}\mathbb{O})(\mathscr{A}\mathbb{O} : \mathscr{A}) = (\mathbb{O} : \mathbb{Z}[H])(\mathbb{Z}[H] : \mathscr{A}) \tag{7.14}$$

and the fact that $(\mathscr{A}\mathbb{O} : \mathscr{A}) = (\mathbb{O} : \mathbb{Z}[H])$ by the above.

Claim 2.2. *Assume that $\mathscr{A}\mathbb{O} = \mathscr{A}$. Then $\mathscr{A}$ is not a principal ideal, but there exists $g \in \mathscr{A}$ such that $(\mathscr{A} : g\mathbb{Z}[H]) = 3$.*

Let g be a generator of the principal ideal $\mathscr{A}\mathbb{O}$ of $\mathbb{O}$. Since $\mathscr{A}\mathbb{O} = \mathscr{A}$, g lies in $\mathscr{A}$ and we compute as above

$$(\mathscr{A} : g\mathbb{Z}[H]) = (\mathscr{A}\mathbb{O} : g\mathbb{O})(g\mathbb{O} : g\mathbb{Z}[H]) = 3.$$

Since $(\mathbb{O} : \mathbb{Z}[H])(\mathbb{Z}[H] : \mathscr{A}) = 3(\mathbb{Z}[H] : \mathscr{A}) = (\mathscr{A} : g\mathbb{Z}[H])(\mathbb{Z}[H] : \mathscr{A}) = (\mathbb{Z}[H] : g\mathbb{Z}[H])$ and $(\mathscr{A}\mathbb{O} : \mathscr{A}) = 1$, Equation (7.13) follows from (7.14). It remains to prove that $\mathscr{A}$ cannot be principal in that case. In order to prove this, we need another result. Let $x \in \mathbb{O}$. By the isomorphism in (7.12), it corresponds to a pair (x_0, x_1) in $\mathbb{Z} \oplus \mathbb{Z}[\omega]$. We define the norm of x as the following quantity

$$\mathrm{Norm}(x) := |x_0|\, N_{\mathbb{Q}(\omega)/\mathbb{Q}}(x_1).$$

Note that we recover the usual definition of the norm of $\mathbb{Q}[H]$ as a $\mathbb{Q}$-algebra. The proof of the following claim is straightforward and is left to the reader.

Claim 3. *Let $x \in \mathbb{O}$ with $\mathrm{Norm}(x) \neq 0$. Then $(\mathbb{O} : x\mathbb{O}) = \mathrm{Norm}(x)$. If furthermore $x \in \mathbb{Z}[H]$ then $(\mathbb{Z}[H] : x\mathbb{Z}[H]) = \mathrm{Norm}(x)$.*

We now finish the proof of Claim 2.2. Assume that $\mathscr{A}$ is principal, say $\mathscr{A} = h\mathbb{Z}[H]$. Then there exists $z \in \mathbb{Z}[H]$ such that $g = hz$ and we have $(\mathbb{O} : z\mathbb{O}) = 3$. Thanks to the above claim, we can explicitly compute all the elements $z \in \mathbb{O}$ such that $(\mathbb{O} : z\mathbb{O}) = 3$. There are the elements $z = e_0 a + e_1(b + c\sigma)$ with $a = \pm 1$ and $b + c\sigma \in \{\pm(1+2\sigma), \pm(2+\sigma), \pm(1-\sigma)\}$, or $a = \pm 3$ and $b + c\sigma \in \{\pm 1, \pm\sigma, \pm(1+\sigma)\}$. One can compute all possibilities and check that none of those belong to $\mathbb{Z}[H]$. This gives a contradiction and concludes the proof of Claim 2.2 and of Claim 2.

We now turn to the $\mathbb{Z}[H]$-structure of U_K^-. The principal result is the following claim that we will prove in several steps.

Claim 4. *There exists $\bar{\theta} \in U_K^-$ such that $U_K^- = \mathbb{Z}[H] \cdot \bar{\theta}$.*

Let $\bar{\theta}' \in U_K^-$ be such that $U_K^- \otimes \mathbb{Q} = \mathbb{Q}[H] \cdot \bar{\theta}'$. Note that the existence of $\bar{\theta}'$ follows from Proposition 4.2. We define

$$\Lambda := \left\{ x \in \mathbb{Q}[H] : x \cdot \bar{\theta}' \in U_K^- \right\}.$$

It is a fractional ideal of $\mathbb{Z}[H]$. The above claim is satisfied if and only if it is a principal ideal. Assume that this is not the case. Then, by the above, we have[11] $\Lambda\mathcal{O} = \Lambda$. Recall that F denotes the subfield of K fixed by H. It is a quadratic extension of k and $\mathrm{Gal}(F/k) = \langle \tau \rangle$. We define U_F^- as the kernel of the norm map from $U_F/\langle \pm 1 \rangle$ to $U_k/\langle \pm 1 \rangle$. We have also $U_F^- = U_K^- \cap (F^\times/\langle \pm 1 \rangle)$. Let $N_H := 1 + \sigma + \sigma^2$. It is the group ring element corresponding to the norm of the extensions K/F and K^+/k.

Claim 4.1. *$\Lambda\mathcal{O} = \Lambda$ if and only if $N_H \cdot U_K^- = 3 \cdot U_F^-$. If $\Lambda\mathcal{O} \neq \Lambda$, then $N_H \cdot U_K^- = U_F^-$.*

We have $\Lambda\mathcal{O} = \Lambda$ if and only if $e_0\Lambda \subset \Lambda$, that is $N_H \cdot U_K^- \subset 3 \cdot U_K^-$. Assume that it is the case. Let $\bar{\delta} \in U_K^-$ and set $\kappa := N_{K/F}(\delta) \in U_F$. Then the polynomial $X^3 - \kappa$ has a root, say ν, in U_K. If ν does not belong to F then all the roots of $X^3 - \kappa$ belongs to K since K/F is a Galois extension. It follows that K contains the third roots of unity, a contradiction. Therefore $\bar{\nu} \in U_F^-$ and $N_H \cdot U_K^- \subset 3 \cdot U_F^-$. The other inclusion is trivial and the first assertion of the claim is proved. If $\Lambda\mathcal{O} \neq \Lambda$ then $3 \cdot U_F^- \subsetneq N_H \cdot U_K^- \subset U_F^-$. Since U_F^- is a $\mathbb{Z}$-module of rank 1, it follows that $N_H \cdot U_K^- = U_F^-$. The claim is proved.

Let $\mathcal{S}$ be the set of prime ideals of K that are totally split in K/k. Denote by $I_{K,\mathcal{S}}$ the subgroup of I_K, the group of ideals of K, generated by the prime ideals in $\mathcal{S}$. Then, by Chebotarev's theorem, the following short sequence is exact

$$1 \longrightarrow P_K \cap I_{K,\mathcal{S}}^{1-\tau} \longrightarrow I_{K,\mathcal{S}}^{1-\tau} \longrightarrow \mathrm{Cl}_K^{1-\tau} \longrightarrow 1$$

where P_K is the group of principal ideals of K. We take the Tate cohomology of this sequence for the action of H. Since 3 does not divide the order of Cl_K, it does not divide the order of $\mathrm{Cl}_K^{1-\tau}$ and $\hat{H}^0(\mathrm{Cl}_K^{1-\tau}) = \hat{H}^1(\mathrm{Cl}_K^{1-\tau}) = 1$. Note that here and in what follows, to simplify the presentation, we drop the group H in the notation of the cohomology groups and write $\hat{H}^i(M)$ instead of $\hat{H}^i(H, M)$ for M a $\mathbb{Z}[H]$-module. It follows from the exact hexagon (5.9) for the above exact sequence that $\hat{H}^i(P_K \cap I_{K,\mathcal{S}}^{1-\tau}) \simeq \hat{H}^i(I_{K,\mathcal{S}}^{1-\tau})$ for $i = 0, 1$. Let $\mathfrak{A} \in P_K \cap I_{K,\mathcal{S}}^{1-\tau}$. There exist $\alpha \in K_{\mathcal{S}}^\times$, the subgroup of elements of $K^\times$ supported only by prime ideals in $\mathcal{S}$, and $\mathfrak{B} \in I_{K,\mathcal{S}}$ such that

$$\mathfrak{A} = (\alpha) = \mathfrak{B}^{1-\tau}.$$

[11] Strictly speaking, the claims above are only for integral ideals of $\mathbb{Z}[H]$ but they admit obvious and direct generalizations to fractional ideals.

We apply $1-\tau$ to this equation

$$\mathfrak{A}^{1-\tau} = (\alpha)^{1-\tau} = \mathfrak{B}^{(1-\tau)^2} = (\mathfrak{B}^{1-\tau})^2 = \mathfrak{A}^2.$$

Therefore we have $(P_K \cap I_{K,\mathcal{S}}^{1-\tau})^2 \subset P_{K,\mathcal{S}}^{1-\tau}$, where $P_{K,\mathcal{S}}$ is the subgroup of principal ideals generated by the elements of $K_{\mathcal{S}}^\times$. It follows that the quotient $(P_K \cap I_{K,\mathcal{S}}^{1-\tau})/P_{K,\mathcal{S}}^{1-\tau}$ is killed by 2 and therefore $\hat{H}^i(P_K \cap I_{K,\mathcal{S}}^{1-\tau}) = \hat{H}^i(P_{K,\mathcal{S}}^{1-\tau})$ for $i = 0$ or 1. We have proved the following claim.

Claim 4.2. *$\hat{H}^0(P_{K,\mathcal{S}}^{1-\tau}) \simeq \hat{H}^0(I_{K,\mathcal{S}}^{1-\tau})$ and $\hat{H}^1(P_{K,\mathcal{S}}^{1-\tau}) \simeq \hat{H}^1(I_{K,\mathcal{S}}^{1-\tau})$.*

Let $u \in U_K \cap (K_{\mathcal{S}}^\times)^{1-\tau}$. There exists $\alpha \in K_{\mathcal{S}}^\times$ such that $u = \alpha^{1-\tau}$. Therefore

$$u^{1-\tau} = \alpha^{(1-\tau)^2} = (\alpha^{1-\tau})^2 = u^2.$$

Reasoning as above, this implies that $\hat{H}^i(U_K \cap (K_{\mathcal{S}}^\times)^{1-\tau}) = \hat{H}^i(U_K^{1-\tau}) = \hat{H}^i(U_K^-)$ for $i = 0, 1$. We now consider the short exact sequence

$$1 \longrightarrow U_K \cap (K_{\mathcal{S}}^\times)^{1-\tau} \longrightarrow (K_{\mathcal{S}}^\times)^{1-\tau} \longrightarrow P_{K,\mathcal{S}}^{1-\tau} \longrightarrow 1.$$

Taking the Tate cohomology and using the above equalities, we extract the following exact sequence from the exact hexagon (5.9) corresponding to this exact sequence

$$(7.15) \qquad \cdots \longrightarrow \hat{H}^1(P_{K,\mathcal{S}}^{1-\tau}) \longrightarrow \hat{H}^0(U_K^-) \longrightarrow \hat{H}^0((K_{\mathcal{S}}^\times)^{1-\tau}) \longrightarrow \cdots$$

The next claim is just a reformulation of the first part of Claim 4.1.

Claim 4.3. *$\Lambda\mathbb{O} = \Lambda$ if and only if $\hat{H}^0(U_K^-) \simeq \mathbb{Z}/3\mathbb{Z}$.*

Assume the two followings claims for the moment.

Claim 4.4. *$\hat{H}^1(P_{K,\mathcal{S}}^{1-\tau})$ is trivial.*

Claim 4.5. *$\hat{H}^0((K_{\mathcal{S}}^\times)^{1-\tau})$ is trivial.*

By (7.15) we get that $\hat{H}^0(U_K^-) = 1$. Thus $\Lambda\mathbb{O} \neq \mathbb{O}$ by Claim 4.3 and therefore Λ is principal by Claim 2.1, and Claim 4 follows. It remains to prove Claims 4.4 and 4.5. We start with the proof of Claim 4.4. By Claim 4.2, this is equivalent to prove that $\hat{H}^1(I_{K,\mathcal{S}}^{1-\tau})$ is trivial. We have as $\mathbb{Z}[H]$-modules

$$I_{K,\mathcal{S}}^{1-\tau} = \prod_{\mathfrak{p}_0 \in \mathcal{S}_0}{}' \Big(\prod_{\mathfrak{P} | \mathfrak{p}_0} \mathfrak{P}^{\mathbb{Z}} \Big)^{1-\tau} \simeq \prod_{\mathfrak{p}_0 \in \mathcal{S}_0}{}' (1-\tau)\mathbb{Z}[G],$$

where $\mathcal{S}_0$ is the set of prime ideals of k that splits completely in K/k, $\mathfrak{P}$ runs through a set of representative of the prime ideals of K dividing $\mathfrak{p}_0$ under the action of τ and the $'$ indicates that it is a restricted product, that is the exponent of $\mathfrak{P}$ is zero for all but finitely many prime ideals. The isomorphism comes from fixing a prime ideal above $\mathfrak{p}_0$ and the fact that $\mathfrak{p}_0$ is totally split in K/k. Therefore we have

$$\hat{H}^1(I_{K,\mathcal{S}}^{1-\tau}) = \prod_{\mathfrak{p}_0\in\mathcal{S}_0}{}' \hat{H}^1((1-\tau)\mathbb{Z}[G]) \simeq \prod_{\mathfrak{p}_0\in\mathcal{S}_0}{}' \hat{H}^1(\mathbb{Z}[H]).$$

It is well-known that $\hat{H}^1(\mathbb{Z}[H]) = 1$, thus Claim 4.4 is proved.

To prove Claim 4.5, we prove that the norm from $(K_{\mathcal{S}}^\times)^{1-\tau}$ to $(F_{\mathcal{S}}^\times)^{1-\tau}$ is surjective. Let $\alpha^{1-\tau} \in (F_{\mathcal{S}}^\times)^{1-\tau}$. By the Hasse Norm Principle, $\alpha^{1-\tau}$ is a norm in K/F if and only if it is a norm in $K_{\mathfrak{P}}/F_{\mathfrak{p}}$ for all prime ideals $\mathfrak{P}$ of K, where $\mathfrak{p}$ denotes the prime ideal of F below $\mathfrak{P}$. If $\mathfrak{p}$ splits in K/F, then $\alpha^{1-\tau}$ is trivially a norm in $K_{\mathfrak{P}}/F_{\mathfrak{p}}$. Assume now that $\mathfrak{p}$ is inert. It follows from the theory of local fields; see [Lang 1994, §XI.4], that the norm of $K_{\mathfrak{P}}/F_{\mathfrak{p}}$ is surjective on the group of units of $F_{\mathfrak{p}}$. But $\alpha^{1-\tau}$ is a unit at $\mathfrak{P}$ since $\mathfrak{P} \notin \mathcal{S}$, and therefore it is a norm also in this case. Finally we assume that $\mathfrak{P}$ is ramified in K/F. Let p be the rational prime below $\mathfrak{P}$. By hypothesis, $p \neq 3$ since 3 is not wildly ramified in K/k. Write $\mu_{\mathfrak{P}}$, $\mathbb{U}_{\mathfrak{P}}$, $\mu_{\mathfrak{p}}$ and $\mathbb{U}_{\mathfrak{p}}$ for the group of roots of unity of order prime to p and the group of principal units of $K_{\mathfrak{P}}$ and $F_{\mathfrak{p}}$ respectively. We have $\mu_{\mathfrak{P}} = \mu_{\mathfrak{p}}$ and therefore $\mathcal{N}_{K_{\mathfrak{P}}/F_{\mathfrak{p}}}(\mu_{\mathfrak{P}}) = \mu_{\mathfrak{p}}^3$. On the other hand $\mathcal{N}_{K_{\mathfrak{P}}/F_{\mathfrak{p}}}(\mathbb{U}_{\mathfrak{p}}) = \mathbb{U}_{\mathfrak{p}}^3 = \mathbb{U}_{\mathfrak{p}}$ and the norm is surjective on principal units. Since $\mathfrak{P} \notin \mathcal{S}$, $v_{\mathfrak{p}}(\alpha) = 0$ and $\alpha = \zeta u$ with $\zeta \in \mu_{\mathfrak{p}}$ and $u \in \mathbb{U}_{\mathfrak{p}}$. It follows from the above discussion that $\alpha^{1-\tau}$ is a norm in $K_{\mathfrak{P}}/F_{\mathfrak{p}}$ if and only if $\zeta^{1-\tau} \in \mu_{\mathfrak{p}}^3$. Let $\mathfrak{p}_0$ be the prime ideal of k below $\mathfrak{p}$. Assume first that $\mathfrak{p}_0$ is ramified in F/k. Then $\mu_{\mathfrak{p}} \subset k_{\mathfrak{p}_0}$ and $\zeta^{1-\tau} = 1$, thus $\alpha^{1-\tau}$ is a norm in $K_{\mathfrak{P}}/F_{\mathfrak{p}}$. Assume now that $\mathfrak{p}_0$ is inert[12] in F/k. Denote by f the residual degree of $\mathfrak{p}_0$. The group $\mu_{\mathfrak{p}_0}$ of roots of unity in $k_{\mathfrak{p}_0}$ of order prime to p has order $p^f - 1$. Let $\mathfrak{P}^+ := \mathfrak{P} \cap K^+$. The extension $K^+_{\mathfrak{P}^+}/k_{\mathfrak{p}_0}$ is a tamely ramified cyclic cubic extension. Therefore it is a Kummer extension by [Lang 1994, Proposition II.5.12] and $k_{\mathfrak{p}_0}$ contains the third roots of unity, that is 3 divides $p^f - 1$. Since τ is the Frobenius element at $\mathfrak{p}_0$ of the extension F/k, we have $\zeta^{1-\tau} = \zeta^{1-p^f} = (\zeta^{(1-p^f)/3})^3 \in \mu_{\mathfrak{p}}^3$ and therefore $\alpha^{1-\tau}$ is a norm in $K_{\mathfrak{P}}/F_{\mathfrak{p}}$. We have proved that $\alpha^{1-\tau}$ is a norm everywhere locally. It follows by the Hasse Norm Principle that there exists $\beta \in K^\times$ such that $\mathcal{N}_{K/F}(\beta) = \alpha^{1-\tau}$. Let $\mathfrak{P}$ be a prime ideal of K not in $\mathcal{S}$ and, as above, let $\mathfrak{p}$ be the prime ideal of F below $\mathfrak{P}$. Assume first that $\mathfrak{P}$ is ramified or inert in K/F, then $v_{\mathfrak{P}}(\beta) = v_{\mathfrak{p}}(\alpha^{1-\tau})$ or $\frac{1}{3}v_{\mathfrak{p}}(\alpha^{1-\tau})$ respectively. In both cases we get $v_{\mathfrak{P}}(\beta) = 0$ since $\alpha \in K_{\mathcal{S}}^\times$. If $\mathfrak{P}$ is split in K/F then it must be inert or ramified in K/K^+ by (A3). It follows that $v_{\mathfrak{P}}(\beta^{1-\tau}) = 0$. Therefore $\delta := \beta^{1-\tau} \in K_{\mathcal{S}}^\times$. We now compute

$$\mathcal{N}_{K/F}(\delta^{1-\tau}) = \mathcal{N}_{K/F}(\beta)^{(1-\tau)^2} = (\alpha^{1-\tau})^{2(1-\tau)} = (\alpha^{1-\tau})^4.$$

[12] By (A3), it cannot be split in F/k.

Thus $\alpha^{1-\tau}$ is the norm of $(\delta/\alpha)^{1-\tau} \in (K_{\mathcal{S}}^{\times})^{1-\tau}$. This concludes the proof of Claim 4.5 and therefore also the proof of Claim 4. The next claim follows from Claim 4.1 and the fact, seen in the proof of Claim 4, that $\Lambda\mathbb{O} \neq \Lambda$.

Claim 5. $$N_H \cdot U_K^- = U_F^-.$$

Let $\mathcal{F} := \mathrm{Fitt}_{\mathbb{Z}[H]}(\mathrm{Cl}_K^-)$ be the Fitting ideal of Cl_K^- as a $\mathbb{Z}[H]$-module. Apply Claim 2 to the ideal $\mathcal{F}$ and call f the element of $\mathcal{F}$ such that $\mathbb{O}/\mathcal{F}\mathbb{O} \simeq \mathbb{Z}[H]/f\mathbb{Z}[H]$. Set $\bar{\eta}' := f \cdot \bar{\theta}$. Thanks to Claim 4, we find that

$$(U_K^- : \mathbb{Z}[H] \cdot \bar{\eta}') = (\mathbb{Z}[H] : f\mathbb{Z}[H]) = (\mathbb{O} : \mathcal{F}\mathbb{O}) = |\mathrm{Cl}_K^-|. \tag{7.16}$$

For this last equality, we first use the fact that, since 3 does not divide the order of Cl_K^-, we can make e_0 and e_1 act on it and see it therefore as an $\mathbb{O}$-module. By the properties of the Fitting ideal, $\mathcal{F}\mathbb{O}$ is the Fitting ideal of Cl_K^- as an $\mathbb{O}$-module and the equality follows from Lemma 5.1.

Claim 6. *Let $n, m \geq 0$ be two integers. Then there exists $\kappa_{n,m} \in \mathbb{Z}[H]$, unique up to a trivial unit, such that*

$$\mathrm{Norm}(\kappa_{n,m}) = 2^{n+2m} \quad \textit{and} \quad e_0\kappa_{n,m} = e_0 2^n. \tag{7.17}$$

We define

$$\kappa_{n,m} := 2^n e_0 + (-1)^{n+m} 2^m e_1.$$

It is clear from its construction that $\kappa_{n,m}$ satisfies (7.17). One can see also directly that $\kappa_{n,m} \in \mathbb{Z}[H]$ since $2 \equiv -1 \pmod 3$. It remains to prove the uniqueness statement. Clearly $e_0\kappa_{n,m}$ is fixed by construction. On the other hand $e_1\kappa_{n,m}$ is an element of norm 2^{2m} in $e_1\mathbb{Z}[H] \simeq \mathbb{Z}[\omega]$. Since 2 is inert in $\mathbb{Z}[\omega]$, there exists only one element in $\mathbb{Z}[\omega]$ of norm 2^{2m} up to units. This concludes the proof of the claim.

Let $e' \in \mathbb{N}$ be such that $2^{e'} = (\bar{U}_k : \mathcal{N}(\bar{U}_F))$.

Claim 7. *The integer $e - e'$ is nonnegative and even.*

We consider the natural map $\bar{U}_k \to \bar{U}_{K^+}/\mathcal{N}(\bar{U}_K)$ that comes from the inclusion $U_k \subset U_{K^+}$. Let $\bar{u} \in \bar{U}_k$ be in the kernel of this map. Thus there exists $\bar{x} \in \bar{U}_K$ such that $\bar{u} = \mathcal{N}(\bar{x})$. Set $\bar{y} := N_H \cdot \bar{x} - \bar{u} \in \bar{U}_F$. We have

$$\mathcal{N}(\bar{y}) = N_H \cdot \mathcal{N}(\bar{x}) - \mathcal{N}(\bar{u}) = 3 \cdot \bar{u} - 2 \cdot \bar{u} = \bar{u}.$$

Therefore the kernel of the above map is $\mathcal{N}(\bar{U}_F)$ and there is a well-defined injective group homomorphism from $\bar{U}_k/\mathcal{N}(\bar{U}_F)$ to $\bar{U}_{K^+}/\mathcal{N}(\bar{U}_K)$. This proves that[13] $e \geq e'$. The cokernel of this map is

$$\frac{\bar{U}_{K^+}/\mathcal{N}(\bar{U}_K)}{\bar{U}_k/\mathcal{N}(\bar{U}_F)} \simeq \bar{U}_{K^+}/(\bar{U}_k + \mathcal{N}(\bar{U}_K)). \tag{7.18}$$

[13] This inequality follows also from Claim 10 below.

It is a finite $\mathbb{Z}[H]$-module of order $2^{e-e'}$. In particular, the idempotents e_0 and e_1 act on it. We have $e_0 \cdot \bar{U}_{K^+}/(\bar{U}_k + \mathcal{N}(\bar{U}_K)) = N_H \cdot \bar{U}_{K^+}/(\bar{U}_k + \mathcal{N}(\bar{U}_F)) = 1$. It follows that $\bar{U}_{K^+}/(\bar{U}_k + \mathcal{N}(\bar{U}_K)) = e_1 \cdot \bar{U}_{K^+}/(\bar{U}_k + \mathcal{N}(\bar{U}_K))$ is a $\mathbb{Z}[\omega]$-module. Since 2 is inert in $\mathbb{Z}[\omega]$, the order of $\bar{U}_{K^+}/(\bar{U}_k + \mathcal{N}(\bar{U}_K))$ is an even power of 2. This concludes the proof of the claim.

Let $\kappa := \kappa_{e'+t_S,(e-e')/2}$. We define

$$\bar{\eta} := \pm\kappa \cdot \bar{\eta}'. \tag{7.19}$$

The sign will be chosen in the proof of the next claim. We have

$$\begin{aligned}(U_K^- : \mathbb{Z}[H] \cdot \bar{\eta}) &= (U_K^- : \mathbb{Z}[H] \cdot \bar{\eta}')(\mathbb{Z}[H] \cdot \bar{\eta}' : \mathbb{Z}[H]\kappa \cdot \bar{\eta}') \\ &= |\mathrm{Cl}_K^-| \ \mathrm{Norm}(\kappa) = 2^{e+t_S}|\mathrm{Cl}_K^-|\end{aligned}$$

using (7.16), Claim 3 and the definition and properties of κ. Therefore, $\bar{\eta}$ satisfies (P1). Let p be a prime not dividing $[K : k]$ and let ψ be an odd irreducible $\mathbb{Z}_p$-character. By the construction of η' and the fact that p is odd and κ is a 2-unit, we find that

$$\left|(U_K^-/\mathbb{Z}[H] \cdot \bar{\eta})^{\psi}\right| = \left|(U_K^-/\mathbb{Z}[H] \cdot \bar{\eta}')^{\psi}\right| = \left|(\mathbb{O}/\mathcal{F})^{\psi}\right| = \left|(\mathrm{Cl}_K^-)^{\psi}\right|$$

(the last equality comes from Lemma 5.2). Hence $\bar{\eta}$ is also a solution to (P2).

Claim 8. *Up to the right choice of sign in (7.19), we have*

$$\frac{1}{2}\sum_{g\in G} \chi^3(g) \log |\eta^g|_w = L'_{K/k,S}(0, \chi^3).$$

The $\mathbb{Z}[H]$-module $U_K^-/(\mathbb{Z}[H] \cdot \bar{\eta})$ has order not divisible by 3 since $\bar{\eta}$ satisfies (P1). Thus it is a $\mathbb{O}$-module and we can split it into two parts corresponding to the two idempotents e_0 and e_1. On the one hand, using Claim 5, we have

$$e_0 \cdot \left(U_K^-/\mathbb{Z}[H] \cdot \bar{\eta}\right) = N_H \cdot \left(U_K^-/\mathbb{Z}[H] \cdot \bar{\eta}\right) \simeq U_F^-/\mathbb{Z} \cdot \bar{\eta}_F,$$

where $\bar{\eta}_F := N_H \cdot \bar{\eta} \in U_F^-$. On the other hand, we compute

$$e_1 \cdot \left(U_K^-/\mathbb{Z}[H] \cdot \bar{\eta}\right) \simeq e_1\left(\mathbb{Z}[H]/\kappa f \mathbb{Z}[H]\right) \simeq \mathbb{Z}[\omega]/2^{(e-e')/2}\mathcal{F}_1,$$

where $\mathcal{F}_1$ is the Fitting ideal of $(\mathrm{Cl}_K^-)^{e_1}$ viewed as an $\mathbb{Z}[\omega]$-module. Indeed, by construction, $e_1 f \mathbb{Z}[H] = e_1\mathcal{F}\mathbb{O} \simeq \mathrm{Fitt}_{\mathbb{Z}[\omega]}((\mathrm{Cl}_K^-)^{e_1})$. Since $\mathrm{Cl}_K^- = (\mathrm{Cl}_K^-)^{e_0} \oplus (\mathrm{Cl}_K^-)^{e_1}$ and $(\mathrm{Cl}_K^-)^{e_0} \simeq \mathcal{N}_{K/F}(\mathrm{Cl}_K^-)$, we have

$$(\mathbb{Z}[\omega] : \mathcal{F}_1) = |(\mathrm{Cl}_K^-)^{e_1}| = \frac{|\mathrm{Cl}_K^-|}{|\mathcal{N}_{K/F}(\mathrm{Cl}_K^-)|}.$$

Claim 8.1. $$\mathcal{N}_{K/F}(\mathrm{Cl}_K^-) = \mathrm{Cl}_F^-.$$

Consider the composition of maps $\mathrm{Cl}_F^- \to \mathrm{Cl}_K^- \to \mathrm{Cl}_F^-$, where the first map is the map induced by the lifting of ideals from F to K and the second map is the norm $\mathcal{N}_{K/F}$. This is the map of multiplication by 3 and therefore, if the order of Cl_F^- is not divisible by 3, it is a bijection and the claim is proved. Assume that $3 \mid |\mathrm{Cl}_F^-|$. Let h_E denote the class number of a number field E. Thus $h_K^- := |\mathrm{Cl}_K^-| = h_K/h_{K^+}$ and $h_F^- := |\mathrm{Cl}_F^-| = h_F/h_k$. If K/F is ramified at some finite prime then h_F divides h_K. As h_F^- divides h_F, it follows that $3 \mid h_K$, a contradiction. Assume now that K/F is unramified at finite primes. Therefore 3 divides h_F and $h_F/3$ divides h_K. In the same way, K^+/k is unramified and therefore 3 divides h_k. Since $3 \mid h_F^-$, this implies that 9 divides h_F and therefore 3 divides h_K, a contradiction. It follows that 3 does not divide $|\mathrm{Cl}_F^-|$ and the claim is proved.

Putting together the claim and the computation that precedes it, we find that

$$(U_F^- : \mathbb{Z} \cdot \bar{\eta}_F) = \frac{(U_K^- : \mathbb{Z} \cdot \bar{\eta})}{2^{e-e'}} \frac{|\mathrm{Cl}_F^-|}{|\mathrm{Cl}_K^-|} = 2^{e'+t_S} |\mathrm{Cl}_F^-|. \tag{7.20}$$

Let $\mathfrak{P}^+ \in S_{K^+}$ and denote by $\mathfrak{p}_0$ the prime ideal of k below $\mathfrak{P}^+$. Then $\mathfrak{P}^+$ is inert in K/K^+ if and only if $\mathfrak{p}_0$ is inert in F/k. Furthermore, if $\mathfrak{P}^+$ is inert in K/K^+, then it is ramified[14] in K^+/k and it is the only prime ideal in S_{K^+} above $\mathfrak{p}_0$. It follows that the number t_S of prime ideals in S_{K^+} that are inert in K/K^+ is equal to the number of prime ideals in S that are inert in F/k. Therefore $\bar{\eta}_F$ satisfy the properties (P1) and (P2) for the extension F/k and the set of primes S. As a consequence of Theorem 5.5, we see that either η_F or η_F^{-1} is the Stark unit for the extension K/F and the set of places S. By choosing the right sign in (7.19), we can assume that η_F is the Stark unit. Therefore we have

$$\tfrac{1}{2}(\log|\eta_F|_w + \nu(\tau)\log|\eta_F^\tau|_w) = L'_{F/k,S}(0,\nu),$$

where ν is the non trivial character of F/k. It follows from the functorial properties of L-functions that $L_{F/k,S}(s,\nu) = L_{K/k,S}(s,\chi^3)$, and from the definition of η_F that

$$\log|\eta_F|_w + \nu(\tau)\log|\eta_F^\tau|_w = \sum_{g\in G} \chi^3(g)\log|\eta^g|_w.$$

This completes the proof of the claim.

Now, by Proposition 4.1, we know that

$$\Big(\frac{1}{2}\sum_{g\in G}\chi(g)\log|\eta^g|\Big)\Big(\frac{1}{2}\sum_{g\in G}\chi^3(g)\log|\eta^g|\Big)\Big(\frac{1}{2}\sum_{g\in G}\chi^5(g)\log|\eta^g|\Big)$$
$$= \pm L'_{K/k,S}(0,\chi)L'_{K/k,S}(0,\chi^3)L'_{K/k,S}(0,\chi^5).$$

[14]Recall that $S = S(K/k)$.

We cancel the non zero terms corresponding to χ^3 using Claim 8 and, since χ and χ^5 are conjugate, we get

$$\left|\frac{1}{2}\sum_{g\in G}\chi(g)\log|\eta^g|\right|^2 = \left(\frac{1}{2}\sum_{g\in G}\chi(g)\log|\eta^g|\right)\left(\frac{1}{2}\sum_{g\in G}\chi^5(g)\log|\eta^g|\right)$$
$$= L'_{K/k,S}(0,\chi)L'_{K/k,S}(0,\chi^5) = |L'_{K/k,S}(0,\chi)|^2.$$

Taking square roots, we get

$$\left|\frac{1}{2}\sum_{g\in G}\chi(g)\log|\eta^g|\right| = \left|\frac{1}{2}\sum_{g\in G}\chi^5(g)\log|\eta^g|\right| = |L'_{K/k,S}(0,\chi)| = |L'_{K/k,S}(0,\chi^5)|.$$

Note that we have directly using [Tate 1984, Proposition I.3.4]

$$\frac{1}{2}\sum_{g\in G}\chi_0(g)\log|\eta^g| = \frac{1}{2}\sum_{g\in G}\chi^2(g)\log|\eta^g| = \frac{1}{2}\sum_{g\in G}\chi^4(g)\log|\eta^g|$$
$$= L'_{K/k,S}(0,\chi_0) = L'_{K/k,S}(0,\chi^2) = L'_{K/k,S}(0,\chi^4) = 0.$$

We now prove that $\bar{\eta}$ is unique up to multiplication by an element of H. Assume that $\bar{\eta}'$ is another element of U_K^- satisfying (P1), (P2) and such that $N_H\cdot\bar{\eta}'$ is the Stark unit for the extension F/k and the set of places S. Let $u\in\mathbb{Q}[H]$ be such that $\bar{\eta}' = u\cdot\bar{\eta}$. By Corollary 4.5, u is a 2-unit. Now, by hypothesis, $\bar{\eta}_F = N_H\cdot(u\cdot\bar{\eta}) = u\cdot(N_H\cdot\bar{\eta}) = u\cdot\bar{\eta}_F$ and thus $e_0u = e_0$. Write u_1 for the element of $\mathbb{Q}(\omega)$ such that $(1,u_1)$ corresponds to u by the isomorphism in (7.11). Since both $\bar{\eta}$ and $\bar{\eta}'$ satisfy (P1), we have $\mathrm{Norm}(u)=1$ and thus $N_{\mathbb{Q}(\omega)/\mathbb{Q}}(u_1)=1$. But u_1 is a 2-unit in $\mathbb{Q}(\omega)$ and there is only prime ideal above 2 in $\mathbb{Q}(\omega)$. Therefore u_1 is in fact a unit and $u\in H$.

Finally, it remains to prove that $K(\sqrt{\eta})/k$ is an abelian extension. As noted before, this is equivalent to proving that $(\gamma-1)\cdot\bar{\eta}\in 2\cdot\bar{U}_K^-$ by [Tate 1984, Proposition IV.1.2]. Now γ acts on U_K^- as $-\sigma^2$. Thus, by the definition of $\bar{\eta}$ and the isomorphism between U_K^- and $\mathbb{Z}[H]$, this is equivalent to proving that

$$(\sigma^2+1)\kappa f\in 2\mathbb{Z}[H]. \tag{7.21}$$

Claim 9. *Let $x\in\mathbb{Z}[H]$. Then $x\in 2\mathbb{Z}[H]$ if and only if $xe_0\in 2e_0\mathbb{Z}[H]$ and $xe_1\in 2e_1\mathbb{Z}[H]$.*

If $x\in 2\mathbb{Z}[H]$ then clearly $xe_0\in 2e_0\mathbb{Z}[H]$ and $xe_1\in 2e_1\mathbb{Z}[H]$. Reciprocally, assume that $xe_0 = 2e_0a_0$ and $xe_1 = 2e_1a_1$ with $a_0,a_1\in\mathbb{Z}[H]$. Let $a := e_0a_0+e_1a_1$. We have by construction $2a = x\in\mathbb{Z}[H]$ and $3a = (3e_0)a_0+(3e_1)a_1\in\mathbb{Z}[H]$. Therefore a belongs to $\mathbb{Z}[H]$ and the claim is proved.

We prove (7.21) using the claim. On one hand, we have

$$e_0(\sigma^2+1)\kappa f = 2^{e'+ts+1} e_0 f \in 2e_0\mathbb{Z}[H].$$

On the other hand, we have

$$e_1(\sigma^2+1)\kappa f = 2^{(e-e')/2} e_1(\sigma^2+1) f.$$

The proof will be complete if we prove that $e-e'>0$. For that we use the following claim.

Claim 10. $|\hat{H}^0(T, U_K/U_F)| = 2^{e-e'}$.

Let U_K° be the subgroup of elements $u \in U_K$ such that $u^{1-\tau} \in U_F$. We have

$$\begin{aligned}\hat{H}^0(T, U_K/U_F) &= \frac{(U_K/U_F)^T}{\mathcal{N}(U_K/U_F)} = \frac{U_K^\circ/U_F}{(\mathcal{N}(U_K)\,U_F)/U_F} \\ &\simeq \frac{U_K^\circ}{\mathcal{N}(U_K)\,U_F} \simeq \frac{\bar{U}_K^\circ}{\mathcal{N}(\bar{U}_K)+\bar{U}_F}.\end{aligned}$$

By (7.18), it is enough to prove the following group isomorphism

$$\bar{U}_{K^+}/(\mathcal{N}(\bar{U}_K)+\bar{U}_k) \simeq \bar{U}_K^\circ/(\mathcal{N}(\bar{U}_K)+\bar{U}_F). \tag{7.22}$$

Since $\bar{U}_{K^+} \cap (\mathcal{N}(\bar{U}_K)+\bar{U}_F) = \mathcal{N}(\bar{U}_K)+\bar{U}_k$, there is a natural injection of the left side of (7.22) in the right side, induced by the inclusion $\bar{U}_{K^+} \subset \bar{U}_K^\circ$. We prove now that this map is surjective. Let $\bar{u} \in \bar{U}_K^\circ$. Thus $\bar{x} := (1-\tau)\cdot\bar{u} \in \bar{U}_F$. Note that $(1-\tau)\cdot\bar{x} = 2\cdot\bar{x}$. Define $\bar{y} := N_H\cdot\bar{u}-\bar{x} \in \bar{U}_F$ and $\bar{z} := \bar{u}-\bar{y}$. We have

$$(1-\tau)\cdot\bar{z} = (1-\tau)\cdot\bar{u}-(1-\tau)N_H\cdot\bar{u}+(1-\tau)\cdot\bar{x} = \bar{x}-N_H\cdot\bar{x}+2\cdot\bar{x} = 0.$$

Thus $\bar{z} \in \bar{U}_{K^+}$. This proves that $\bar{u} = \bar{z}+\bar{y} \in \bar{U}_{K^+}+\bar{U}_F$. Equation (7.22) follows and the proof of the claim is finished.

Now by the multiplicativity of the Herbrand quotient and Lemma 5.3, we find that

$$Q(T, U_K/U_F) = \frac{Q(T, U_K)}{Q(T, U_F)} = 2^{2d-2}. \tag{7.23}$$

Therefore $e-e' \geq 2d-2 \geq 2$. This concludes the proof that $K(\sqrt{\eta})$ is abelian over k and the proof of the theorem. □

Corollary 7.2. *Under the hypothesis of the theorem and assuming that the Stark unit exists, then it is a square in K if and only if the Stark unit for the extension F/k and the set S is a square and $(e-e')/2+c-c' \geq 1$, where c is the 2-valuation of $|\mathrm{Cl}_K^-|$, c' is the 2-valuation of $|\mathrm{Cl}_F^-|$ and $(\bar{U}_k : \mathcal{N}(\bar{U}_F)) = 2^{e'}$. In particular, if $d \geq 4$ then it is always a square and, in fact, it is a 2^{d-3}-th power. It is also a square if $d=3$ and the extension K/k is ramified at some finite prime.*

Proof. We use the notations and results of the proof of the theorem. By the uniqueness statement, the Stark unit, if it exists, is equal to η or one of this conjugate over F. In particular, the Stark unit is a 2^r-th power in K if and only if $\bar{\eta} \in 2^r \cdot U_K^-$. By Claim 9 and the construction of $\bar{\eta}$, this is equivalent to $2^{e'+t_S} e_0 f \in 2^r e_0 \mathbb{Z}$ and $2^{(e-e')/2} e_1 f \in 2^r e_1 \mathbb{Z}[H]$.

Now $e_0 f \mathbb{Z} = e_0 |\mathrm{Cl}_F^-| \mathbb{Z}$ by the definition of f, Claim 8.1 and the discussion that precedes it. Thus the first condition is equivalent to $e' + t_S + c' \geq r$. For $r = 1$, this is equivalent to the fact that the Stark unit for F/k and the set S is a square by Theorem 5.5 and the discussion that follows (7.20) on the number of primes in S that are inert in F/k. For the second condition, recall that $e_1 f \mathbb{Z}[H] \simeq \mathrm{Fitt}_{\mathbb{Z}[\omega]}((\mathrm{Cl}_K^-)^{e_1})$ and therefore $e_1 f \in 2^v \mathbb{Z}[H]$, where v is the 2-valuation of the index $(\mathbb{Z}[\omega] : \mathrm{Fitt}_{\mathbb{Z}[\omega]}((\mathrm{Cl}_K^-)^{e_1}))$. By Claim 8.1 and the computation before it, this index is equal to $|\mathrm{Cl}_K^-|/|\mathrm{Cl}_F^-|$. Therefore the second condition is equivalent to $(e - e')/2 + c - c' \geq r$. This proves the first assertion: the Stark unit for K/k and $S = S(K/k)$ is a square if and only if the Stark unit for the extension F/k and the set S is a square and $(e - e')/2 + c - c' \geq 1$. For the second assertion, we have $e' \geq d - 3$ by (5.10) and $(e - e')/2 \geq d - 1$ by Claim 10 and (7.23). Thus $\bar{\eta} \in 2^{d-3} \cdot U_K^-$ for $d \geq 4$ and we have that the Stark unit is a 2^{d-3}-th power if $d \geq 4$. Finally, for $d = 3$, the condition $2^{(e-e')/2} e_1 f \in 2 e_1 \mathbb{Z}[H]$ is always satisfied. Assume that the extension K/k is ramified at some finite prime. If F/k is also ramified at some finite prime then the Stark unit for the extension F/k and the set S is a square by Theorem 5.5. If F/k is unramified at finite primes then any prime ideal that ramifies in K/k is inert in F/k by (A3). Therefore $t_S \geq 1$ and the Stark unit for the extension F/k and the set S is a also square by Theorem 5.5. It follows that the Stark unit for K/k is a square by the first part. This concludes the proof. □

Note that the condition in the case $d = 3$ is sharp. Indeed let $k := \mathbb{Q}(\alpha)$, where $\alpha^3 + \alpha^2 - 9\alpha - 8 = 0$, be the smallest totally real cubic field of class number 3. Let v_1, v_2, v_3 be the three infinite places of k with $v_1(\alpha) \approx -3.0791$, $v_2(\alpha) \approx -0.8785$ and $v_3(\alpha) \approx 2.9576$. Let K be the ray class field of k of modulus $\mathbb{Z}_k v_2 v_3$. The extension K/k is cyclic of order 6, satisfies (A1), (A2) and (A3) with $S := (S/k)$, and is unramified at finite places. One can check that, if it exists, the corresponding Stark unit is not a square in K.

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Received March 14, 2012. Revised June 11, 2012.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 266, No. 2, 2013
dx.doi.org/10.2140/pjm.2013.266.423

THE SHORT TIME ASYMPTOTICS OF NASH ENTROPY

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Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Rc} \geq -Kg$, $H(x, y, t)$ be the heat kernel on M^n, and $H = (4\pi t)^{-n/2}e^{-f}$. Nash entropy is defined as $N(H, t) = \int_{M^n}(fH)\,d\mu(x) - n/2$. We study the asymptotic behavior of $N(H, t)$ and $\partial N(H, t)/\partial t$ as $t \to 0^+$ and get the asymptotic formulas at $t = 0$. In the appendix, we get a Hamilton-type upper bound for the Laplacian of the positive solution to the heat equation on such manifolds, which is itself interesting.

1. Introduction

On a complete manifold (M^n, g) with $\mathrm{Rc} \geq -Kg$, where $K > 0$ is a constant, for fixed $y \in M^n$, it is well known that the heat kernel $H(x, y, t)$ on (M^n, g) is unique. We assume $H = (4\pi t)^{-n/2}e^{-f}$. As in [Ni 2004b], Nash entropy is defined as follows.

Definition 1.1.
$$N(H, t) = \int_{M^n} (fH)\,d\mu(x) - \frac{n}{2}.$$

Nash entropy is closely related to $\mathcal{W}$-entropy for the linear heat equation, and the large time asymptotics of this entropy reflects the volume growth rate of the manifold; see [Ni 2004a; 2004b; 2010].

In this paper, we study the asymptotic behavior of $N(H, t)$ and $\partial N(H, t)/\partial t$ as $t \to 0^+$, and solve Problem 23.36 of [Chow et al. 2010]. More precisely, we prove:

Theorem 1.2. *Let (M^n, g) be a complete Riemannian manifold with* $\mathrm{Rc} \geq -Kg$, *where $K > 0$ is a constant. Then*

$$N(H, t) = -\tfrac{1}{2}R(y) \cdot t + O(t^{3/2}) \tag{1-1}$$

and

$$\frac{\partial}{\partial t}[N(H, t)] = -\tfrac{1}{2}R(y) + o(1), \tag{1-2}$$

where $\limsup_{t\to 0} O(t^{3/2})t^{-3/2}$ *is bounded*, $\lim_{t\to 0} o(1) = 0$, *and t is small enough.*

MSC2010: 35K15, 53C44.
Keywords: Nash entropy, short time asymptotics.

One motivation to study the short time asymptotics of Nash entropy is Li–Yau–Perelman type estimates for the heat equation on manifolds with Ricci curvature bounded from below. Motivated by Perelman's differential Harnack estimate for Ricci flow on a closed manifold (M^n, g) with $\mathrm{Rc} \geq 0$, Ni [2004a] proved the following Li–Yau–Perelman type estimate for the heat equation when $t > 0$:

$$2\Delta f(x, y, t) - |\nabla f(x, y, t)|^2 + \frac{f(x, y, t) - n}{t} \leq 0, \tag{1-3}$$

where $H(x, y, t) = (4\pi t)^{-n/2} e^{-f}$ is the heat kernel. In fact, (1-3) is also true for the heat kernel on a complete manifold (M^n, g) with $\mathrm{Rc} \geq 0$; see [Chow et al. 2008].

Perelman made the following claim.

Claim 1.3 [Perelman 2002, Remark 9.6]. *If (M^n, g) is a compact Riemannian manifold, $g_{ij}(x, t)$ evolves according to $(g_{ij})_t = A_{ij}(t)$ and $g_{ij}(x, 0) = g_{ij}(x)$, $t \in (-T, 0]$. Define $\Box = \partial/(\partial t) - \Delta$ and its conjugate $\Box^* = -\partial/(\partial t) - \Delta - \frac{1}{2}A$ (where $A = g^{ij} A_{ij}$). Consider the fundamental solution $u = (-4\pi t)^{-n/2} e^{-f}$ for $\Box^*$, starting as a δ-function at some point $(p, 0)$. Then, for general A_{ij}, the function $(\Box \bar{f} + \bar{f}/t)(q, t)$, where $\bar{f} = f - \int_{M^n} f u$, is of order $O(1)$ for (q, t) near $(p, 0)$.*

We focus on the special case where the evolving metrics are the static metric. From Theorem 1.2, it is easy to show that Perelman's claim in the static metric case is equivalent to the following claim on compact manifolds:

$$2\Delta f(x, y, t) - |\nabla f(x, y, t)|^2 + \frac{f(x, y, t) - n}{t} = -R(y) + O(t + d^2(x, y)). \tag{1-4}$$

If (1-4) is true, it is an improvement of (1-3) when $t + d^2(x, y)$ is small enough and $R(y) > 0$. But, using the explicit formula (cf. [Grigor'yan 2009, Section 9.2])

$$H = (4\pi t)^{-3/2} \frac{d}{\sinh d} \exp\Bigl(-\frac{d^2}{4t} - t\Bigr)$$

for the heat kernel on a hyperbolic manifold $\mathbb{H}^3$, it is easy to check that (1-4) is not true generally. Hence Claim 1.3 is not generally true for the static metric case on complete manifolds.

As observed in [Ni 2004b], the integrand of $\partial N(H, t)/\partial t$ is simply the expression in Li and Yau's gradient estimate for the heat kernel multiplied with the heat kernel, which is $-(\Delta \ln H + n/(2t))H$. Because so far there is no sharp Li–Yau-type gradient estimate for the heat kernel or solutions to the heat equation on complete manifolds with Ricci curvature bounded from below by a negative constant, we hope that (1-2) will be helpful in better understanding this estimate.

On the other hand, in the case where (M^n, g) is a compact Riemannian manifold, the short time behavior of the logarithm of the heat kernel has been studied by many probabilists. Although the heat kernel $H(x, y, t)$ has an infinite sequence expansion at $t = 0$, generally there is no such expansion of $\ln H$ at $t = 0$, and the singularity of $\ln H$ at $t = 0$ can have many complicated situations. However, Varadhan [1967] proved

$$\lim_{t\to 0} t \ln H(x, y, t) = -\frac{d^2(x, y)}{4}. \tag{1-5}$$

Moreover, using stochastic processes methods, Malliavin and Stroock proved [1996] that the above equation is preserved while taking the first and second spatial derivatives on a domain outside of the cut locus. Using analytic methods, (1-5) was proved for complete Riemannian manifolds by Cheng, Li, and Yau [Cheng et al. 1981]. We hope that Theorem 1.2 will be useful in studying the short time behavior of the logarithm of the heat kernel on complete manifolds by analytic methods.

The strategy to prove (1-1) is to use the infinite sequence expansion $H_N(x, y, t)$ of $H(x, y, t)$ at $t = 0$, although generally $\ln H_N$ does not converge to $\ln H$ near $t = 0$ uniformly. In the integral sense of Definition 1.1, we show there is a uniform convergence in Lemma 3.1 by using an improved estimate of $H - H_N$ obtained in Theorem 2.2. The rest of the calculation of the integral of H_N is standard, but, for completeness, we give the details.

To prove (1-2), because the manifold M^n can be noncompact, we need to be more careful when switching the order of differentiation and integration. A detailed proof of the validity of the switch is given in the beginning of Section 4. We need an upper bound of H_t/H to verify the above switch. This type of bound is known for closed manifolds [Hamilton 1993], and in [Chow et al. 2008] (see also [Ni 2006]) the proof is sketched for complete manifolds with $\mathrm{Rc} \geq 0$ using a strategy similar to that in [Kotschwar 2007]. A detailed proof of this Hamilton-type upper bound for complete manifolds with $\mathrm{Rc} \geq -Kg$ is included in the appendix for completeness.

The paper is organized as follows. In Section 2, we state some preliminary results about the heat kernel and get some improved estimates of $H - H_N$. In Section 3, we prove (1-1). In Section 4, using (1-1) and results in the appendix, we prove (1-2). In the appendix, we prove Hamilton-type upper bound of H_t/H on complete manifolds with Ricci curvature bounded from below.

2. Preliminaries

We first define some notations and functions. In the rest of the paper, we fix $y \in M^n$ and define

$$\Omega_y = \{x \in M^n : d(x, y) < \mathrm{inj}_g(y)\},$$

where $\mathrm{inj}_g(y)$ denotes the injectivity radius of the metric g at y. Define

$$B(\rho) = \{x : d(x, y) \leq \rho\} \quad \text{and} \quad B_z(\rho) = \{x : d(x, z) \leq \rho\}.$$

Hence $B(\rho) = B_y(\rho)$. $V(B_z(\rho))$ is used to denote the volume of $B_z(\rho)$ and $V_{-K}(\rho)$ is the volume of the geodesic ball of radius ρ in the constant $(-K/(n-1))$ sectional curvature space form.

Fix $r \in (0, \frac{1}{4}\operatorname{inj}_g(y))$ and let $N_0 = n/2 + 3$. Define

$$E = (4\pi t)^{-n/2} \exp\left(-\frac{d^2(x, y)}{(4t)}\right) \quad \text{and} \quad \widetilde{E} = (4\pi t)^{-n/2} \exp\left(-\frac{d^2(x, y)}{(5t)}\right).$$

When the meaning is clear from context, we sometimes simplify notation by denoting $B(r/2)$ by B and $d(x, y)$ by d.

Assume $\eta : [0, \infty) \to [0, 1]$ is a C^∞ cut-off function with

$$\eta(s) = \begin{cases} 1 & \text{if } s \le r, \\ 0 & \text{if } s \ge 2r. \end{cases} \tag{2-1}$$

The following theorem collects several known results about the heat kernel on complete manifolds; see, for example, [Chow et al. 2010; Garofalo and Lanconelli 1989; Li 2012].

Theorem 2.1. *(M^n, g) is a complete Riemannian manifold with* $\mathrm{Rc} \ge -Kg$, *where $K > 0$ is a constant. Then there exists a unique positive fundamental solution $H(x, y, t)$ to the heat equation, which is called the heat kernel. Moreover,*

$$H(x, y, t) \in C^\infty(M^n \times M^n \times (0, \infty))$$

is symmetric in x and y, and

(i)

$$\int_{M^n} H(x, y, t)\, d\mu(x) \equiv 1; \tag{2-2}$$

(ii)

$$H(x, y, t) = P_{N_0}(x, y, t) + F_{N_0}(x, y, t) \tag{2-3}$$

$$P_{N_0}(x, y, t) = \eta(d(x, y)) H_{N_0}(x, y, t), \tag{2-4}$$

$$H_{N_0}(x, y, t) = (4\pi t)^{-n/2} \exp\left(-\frac{d^2(x, y)}{4t}\right) \cdot \sum_{k=0}^{N_0} \varphi_k(x, y) t^k, \tag{2-5}$$

where $\varphi_k(x, y) \in C^\infty(\Omega_y)$ and $k = 0, 1, \ldots, N_0$. Also, H_{N_0} satisfies

$$\left(\Delta - \frac{\partial}{\partial t}\right) H_{N_0}(x, y, t) = E \Delta \varphi_{N_0} t^{N_0}; \tag{2-6}$$

(iii) *let $\{x^k\}_{k=1}^n$ be exponential normal coordinates centered at $y \in M^n$. Then φ_0 and φ_1 have the asymptotic expansion*

$$\varphi_0(x, y) = 1 + \tfrac{1}{12} R_{pq}(y) x^p x^q + O(d^3(x, y)), \tag{2-7}$$

$$\varphi_1(x, y) = \frac{R(y)}{6} + O(d(x, y)). \tag{2-8}$$

We prove an estimate for F_{N_0}. This estimate is an improvement of the usual estimate of F_{N_0}, which only gives $t^{N_0+1-n/2}$ bound. The improved estimate (2-9) is the key to the proof of Lemma 3.1.

Theorem 2.2. *For $F_{N_0}(x, y, t)$ in Theorem 2.1, we have the following estimates*:

$$|F_{N_0}(x, y, t)| \le C t^4 \exp\Big(-\frac{d^2(x, y)}{5t}\Big), \tag{2-9}$$

$$\Big|\frac{\partial}{\partial t} F_{N_0}(x, y, t)\Big| \le C t^2 \exp\Big(-\frac{d^2(x, y)}{5t}\Big), \tag{2-10}$$

where t is small enough and C is a positive constant independent of x, t.

Remark 2.3. (2-9) was proved in [Garofalo and Lanconelli 1989] for uniformly parabolic operators. Our proof of (2-9) and (2-10) is motivated by an argument in [Li 2012] and is different from the proof in [Garofalo and Lanconelli 1989].

Proof. (Δ). We first prove (2-9). From the definition of $P_{N_0}(x, y, t)$, it is easy to see that $\lim_{t\to 0} P_{N_0}(x, y, t) = \delta_y(x)$. In particular,

$$\begin{aligned} F_{N_0}(x, y, t) &= H(x, y, t) - P_{N_0}(x, y, t) \\ &= -\int_0^t \frac{\partial}{\partial s} \int_{M^n} H(x, z, t-s) P_{N_0}(z, y, s)\, d\mu(z)\, ds \\ &= -\int_0^t \int_{M^n} \Big(\frac{\partial}{\partial s} - \Delta_z\Big) P_{N_0}(z, y, s) \cdot H(x, z, t-s)\, d\mu(z)\, ds, \end{aligned} \tag{2-11}$$

where Δ_z is the Laplacian with respect to the variable z.

From (2-6) and the definition of η, when $z \in B(r)$,

$$\Big|\Big(\frac{\partial}{\partial s} - \Delta_z\Big) P_{N_0}(z, y, s)\Big| \le C_1 s^3 \exp\Big(-\frac{d^2(z, y)}{4s}\Big), \tag{2-12}$$

and when $z \in B(2r) \backslash B(r)$,

$$\Big|\Big(\frac{\partial}{\partial s} - \Delta_z\Big) P_{N_0}(z, y, s)\Big| \le C_2 s^{-n/2-1} \exp\Big(-\frac{d^2(z, y)}{4s}\Big). \tag{2-13}$$

Hence

$$
\begin{aligned}
(2\text{-}14)\quad & |F_{N_0}(x,y,t)| \\
&\le C_1 \int_0^t s^3 \int_{B(r)} H(x,z,t-s) \exp\left(-\frac{d^2(z,y)}{4s}\right) d\mu(z)\, ds \\
&\qquad + C_2 \int_0^t s^{-n/2-1} \int_{B(2r)\setminus B(r)} H(x,z,t-s) \exp\left(-\frac{d^2(z,y)}{4s}\right) d\mu(z)\, ds \\
&\le (a)+(b)
\end{aligned}
$$

We can find $0 < t_1 \le 1$ and $k_0 > 0$ such that if $s \in (0, t_1)$,

$$V(B_p(\sqrt{s})) \ge k_0 s^{n/2} \quad \text{for any } p \in B_y(3r).$$

In the rest of the proof, assume $t \in (0, t_1]$. We have two cases.

Case I. If $x \in B_y(3r)$ and $z \in B_y(2r)$, then, from [Li and Yau 1986] and the above volume lower bound,

$$
\begin{aligned}
(2\text{-}15)\quad & H(x,z,t-s) \\
&\le C V^{-1/2}(B_x(\sqrt{t-s}))V^{-1/2}(B_z(\sqrt{t-s})) \cdot \exp\left[CK(t-s) - \frac{6d^2(z,x)}{25(t-s)}\right] \\
&\le C(K,k_0,n)(t-s)^{-n/2} \exp\left(-\frac{6d^2(z,x)}{25(t-s)}\right)
\end{aligned}
$$

Case II. If $x \notin B_y(3r)$ and $z \in B_y(2r)$, using (2-15), $d(x,z) \ge r$, and the volume comparison theorem,

$$
\begin{aligned}
(2\text{-}16)\quad H(x,z,t-s) &\le C V^{-1}(B_z(\sqrt{t-s})) \cdot \left[\frac{V_{-K}(\sqrt{t-s}+d(x,z))}{V_{-K}(\sqrt{t-s})}\right]^{1/2} \\
&\qquad \cdot \exp\left[CK(t-s) - \frac{6d^2(z,x)}{25(t-s)}\right] \\
&\le C(K,k_0,n,r) \exp\left(-\frac{23d^2(z,x)}{100(t-s)}\right)
\end{aligned}
$$

Note that, in Case I, $\mathrm{inj}_g(x)$ has a uniform lower bound. Hence it is easy to get

$$(2\text{-}17)\qquad \int_{B_y(r)} s^{-n/2} \exp\left(-\frac{d^2(z,x)}{100s}\right) d\mu(z) \le C$$

for any $s \in (0, t_1]$.

Now, using (2-15), (2-16), (2-17) and the classical inequality

$$\frac{d^2(x,z)}{t-s} + \frac{d^2(y,z)}{s} \ge \frac{d^2(x,y)}{t},$$

we can get

$$\int_{B_y(r)} H(x,z,t-s)\exp\Big(-\frac{d^2(z,y)}{4s}\Big)\,d\mu(z) \le C\exp\Big(-\frac{23d^2(x,y)}{100t}\Big).$$

Hence

$$(a) \le Ct^4\exp\Big(-\frac{23d^2(x,y)}{100t}\Big). \tag{2-18}$$

Similarly,

$$\begin{aligned}\int_{B_y(2r)\setminus B_y(r)} H(x,z,t-s)\exp\Big(-\frac{d^2(z,y)}{4s}\Big)\,d\mu(z)\\ \le C\exp\Big(-\frac{3r^2}{100s}\Big)\exp\Big(-\frac{d^2(x,y)}{5t}\Big).\end{aligned}$$

Hence

$$\begin{aligned}(b) &\le C_2\Big[\int_0^t s^{-\frac{n}{2}-1}\exp\Big(-\frac{3r^2}{100s}\Big)\,ds\Big]\exp\Big(-\frac{d^2(x,y)}{5t}\Big)\\ &\le Ct^4\exp\Big(-\frac{d^2(x,y)}{5t}\Big).\end{aligned} \tag{2-19}$$

By (2-18) and (2-19), (2-9) is proved.

(Θ). The strategy to prove (2-10) is similar.

$$\begin{aligned}\frac{\partial}{\partial t}F_{N_0}(x,y,t) &= \frac{\partial}{\partial t}\Big[-\int_0^t\int_{M^n}\Big(\frac{\partial}{\partial s}-\Delta_z\Big)P_{N_0}(z,y,s)\cdot H(x,z,t-s)\,d\mu(z)\,ds\Big]\\ &= -\int_0^t\int_{M^n}\Big(\frac{\partial}{\partial s}-\Delta_z\Big)P_{N_0}(z,y,s)\cdot\Big(\frac{\partial}{\partial t}H(x,z,t-s)\Big)\,d\mu(z)\,ds\\ &\qquad + \Big(\Delta_x-\frac{\partial}{\partial t}\Big)P_{N_0}(x,y,t)\\ &= (\gamma)+(\tau)\end{aligned}$$

From (2-12), (2-13), and $P_{N_0}(x,y,t)=0$, when $x\notin B(2r)$,

$$(\tau) \le Ct^4\exp\Big(-\frac{d^2(x,y)}{5t}\Big). \tag{2-20}$$

Now we estimate (γ).

$$\begin{aligned}(\gamma) &= -\int_0^t\int_{M^n}\Big(\frac{\partial}{\partial s}-\Delta_z\Big)P_{N_0}(z,y,s)\cdot(\Delta_z H(x,z,t-s))\,d\mu(z)\,ds\\ &= -\int_0^t\int_{M^n}\Big[\Delta_z\Big(\frac{\partial}{\partial s}-\Delta_z\Big)P_{N_0}(z,y,s)\Big]\cdot H(x,z,t-s)\,d\mu(z)\,ds.\end{aligned}$$

Similarly as with (2-12) and (2-13), from (2-6), when $z \in B(r)$,

$$\left|\Delta_z\Big(\frac{\partial}{\partial s} - \Delta_z\Big) P_{N_0}(z, y, s)\right| \le C_3 s \exp\Big(-\frac{d^2(z, y)}{4s}\Big), \tag{2-21}$$

and when $z \in B(2r)\backslash B(r)$,

$$\left|\Delta_z\Big(\frac{\partial}{\partial s} - \Delta_z\Big) P_{N_0}(z, y, s)\right| \le C_4 s^{-n/2-3} \exp\Big(-\frac{d^2(z, y)}{4s}\Big). \tag{2-22}$$

Following a similar argument as in the proof of (2-9), using (2-21) and (2-22) instead of (2-12) and (2-13),

$$(\gamma) \le Ct^2 \exp\Big(-\frac{d^2(x, y)}{5t}\Big). \tag{2-23}$$

From (2-20) and (2-23),

$$\left|\frac{\partial}{\partial t} F_{N_0}(x, y, t)\right| \le (\gamma) + (\tau) \le Ct^2 \exp\Big(-\frac{d^2(x, y)}{5t}\Big). \qquad \square$$

3. The short time asymptotics of $N(H, t)$

From (2-5) and (2-7) in Theorem 2.1, there exists $0 < t_0 \le 1$ such that

$$\tfrac{1}{2} \le (4\pi t)^{n/2} \exp\Big(\frac{d^2(x, y)}{4t}\Big) H_{N_0}(x, y, t) \le 2 \tag{3-1}$$

holds when $x \in B(r/2)$ and $0 < t \le t_0$. In Sections 3 and 4, we assume that $t \in (0, t_0]$ and (M^n, g) and H are from Theorem 2.1.

Lemma 3.1.

$$\int_{B(r/2)} \Big[\ln \frac{H(x, y, t)}{H_{N_0}(x, y, t)}\Big] \cdot H(x, y, t)\, d\mu(x) = O(t^2). \tag{3-2}$$

Proof. Assume $x \in B(r/2)$, $t \le t_0$. Then $P_{N_0}(x, y, t) = H_{N_0}(x, y, t)$. Hence

$$F_{N_0}(x, y, t) = H(x, y, t) - H_{N_0}(x, y, t).$$

From (2-9),

$$|F_{N_0}(x, y, t)| \le Ct^{N_0+1-n/2} \exp\Big(-\frac{d^2(x, y)}{5t}\Big). \tag{3-3}$$

If $F_{N_0}(x, y, t) > 0$,

$$\begin{aligned}\left|\ln \frac{H}{H_{N_0}} \cdot H\right|(x, y, t) &= \ln\Big(1 + \frac{F_{N_0}}{H_{N_0}}\Big) \cdot H \le \frac{F_{N_0}}{H_{N_0}} \cdot H \\ &\le Ct^{N_0+1} \exp\Big(\frac{d^2(x, y)}{20t}\Big) \cdot H(x, y, t).\end{aligned}$$

If $F_{N_0}(x, y, t) \le 0$, $H(x, y, t) \le H_{N_0}(x, y, t)$ and

$$\left|\ln \frac{H}{H_{N_0}} \cdot H\right|(x, y, t) = |\ln H(x, y, t) - \ln H_{N_0}(x, y, t)| \cdot H(x, y, t)$$
$$= \left|\frac{1}{\xi}[H(x, y, t) - H_{N_0}(x, y, t)]\right| \cdot H(x, y, t),$$

where $H(x, y, t) \le \xi \le H_{N_0}(x, y, t)$. Hence

$$\left|\ln \frac{H}{H_{N_0}} \cdot H\right|(x, y, t) \le \left|\frac{F_{N_0}}{H}\right| \cdot H = F_{N_0} \le Ct^{N_0+1-n/2} \exp\left(-\frac{d^2(x, y)}{5t}\right).$$

By the above,

$$\left|\ln \frac{H}{H_{N_0}} \cdot H\right|(x, y, t) \le Ct^4\left[t^{n/2} \exp\left(\frac{d^2(x, y)}{20t}\right) \cdot H + \exp\left(-\frac{d^2(x, y)}{5t}\right)\right]. \tag{3-4}$$

From (3-1) and (3-3),

$$\begin{aligned} H(x, y, t) &\le |H_{N_0}| + |F_{N_0}| \\ &\le 2(4\pi t)^{-n/2} \exp\left(-\frac{d^2(x, y)}{4t}\right) + Ct^4 \cdot \exp\left(-\frac{d^2(x, y)}{5t}\right). \end{aligned} \tag{3-5}$$

By (3-4) and (3-5),

$$\left|\ln \frac{H}{H_{N_0}} \cdot H\right|(x, y, t) \le Ct^4. \tag{3-6}$$

Hence

$$\int_{B(r/2)}\left[\ln \frac{H(x, y, t)}{H_{N_0}(x, y, t)}\right] \cdot H(x, y, t)\, d\mu(x) = O(t^2). \qquad \square$$

Proof of (1-1).

$$-\int_{M^n} fH\, d\mu = \int_{M^n \setminus B(r/2)} (-fH)\, d\mu + \int_{B(r/2)} (-fH)\, d\mu = \text{(I)} + \text{(II)}.$$

First we estimate (I). From [Li and Yau 1986], we have

$$H(x, y, t) \le CV^{-1/2}(B_x(\sqrt{t}))V^{-1/2}(B_y(\sqrt{t})) \cdot \exp\left[CKt - \frac{d^2(x, y)}{5t}\right].$$

If $x \in M^n \setminus B(r/2)$ and t is small enough, using the volume comparison theorem,

$$\begin{aligned} H(x, y, t) &\le CV^{-1}(B_y(\sqrt{t})) \cdot \frac{V_{-K}(\sqrt{t} + d)}{V_{-K}(\sqrt{t})} \cdot \exp\left[CKt - \frac{d}{5t}\right] \\ &\le Ct^2 \exp\left(-\frac{d^2}{6t}\right), \end{aligned} \tag{3-7}$$

where C depends on n, K, r, and the metric g near y. Choose t small enough such that $H \le Ct^2 \le e^{-1}$. Then, by the monotonicity of $h(x) = \ln x \cdot x$ on $(0, e^{-1}]$,

$$|\ln H(x, y, t) \cdot H(x, y, t)| \le \left|\ln\Big[Ct^2 \exp\Big(-\frac{d^2}{6t}\Big)\Big] \cdot \Big[Ct^2 \exp\Big(-\frac{d^2}{6t}\Big)\Big]\right|.$$

Hence

$$\begin{aligned}
|(\mathrm{I})| &= \left|\int_{M^n \setminus B(r/2)} \Big[\ln H + \frac{n}{2}\ln(4\pi t)\Big] \cdot H\, d\mu(x)\right| \\
&\le \int_{M^n \setminus B(r/2)} \left|\ln\Big[Ct^2 \exp\Big(-\frac{d^2}{6t}\Big)\Big] \cdot \Big[Ct^2 \exp\Big(-\frac{d^2}{6t}\Big)\Big]\right| d\mu(x) \\
&\qquad + \frac{n}{2}\int_{M^n \setminus B(r/2)} \left|\ln(4\pi t) \cdot \Big[Ct^2 \exp\Big(-\frac{d^2}{6t}\Big)\Big]\right| d\mu(x) \\
&\le O(t^{3/2}).
\end{aligned} \tag{3-8}$$

In the last inequality, we used $\mathrm{Rc} \ge -Kg$ and the volume comparison theorem.

$$\begin{aligned}
|(\mathrm{II})| &= \int_{B(r/2)} \Big[\ln H + \frac{n}{2}\ln(4\pi t)\Big] \cdot H\, d\mu \\
&= \int_{B(r/2)} \ln\frac{H}{H_{N_0}} \cdot H\, d\mu(x) + \int_{B(r/2)} \Big[\ln H_{N_0} + \frac{n}{2}\ln(4\pi t)\Big] \cdot H\, d\mu(x) \\
&= (\mathrm{III}) + (\mathrm{IV}).
\end{aligned}$$

By Lemma 3.1, $(\mathrm{III}) = O(t^2)$. From Lemma 3.2, which follows,

$$(\mathrm{IV}) = -\frac{n}{2} + \frac{1}{2}R(y) \cdot t + O(t^{3/2}). \qquad \square$$

Lemma 3.2.

$$\int_{B(r/2)} \Big[\ln H_{N_0} + \frac{n}{2}\ln(4\pi t)\Big] \cdot H d\mu(x) = -\frac{n}{2} + \tfrac{1}{2}R(y) \cdot t + O(t^{3/2}). \tag{3-9}$$

Proof. Set $(\mathrm{I}) := \int_{B(r/2)} [\ln H_{N_0} + (n/2)\ln(4\pi t)] \cdot H\, d\mu(x)$. From Theorem 2.1,

$$\ln H_{N_0} = -\frac{n}{2}\ln(4\pi t) - \frac{d^2(x, y)}{4t} + \ln\Big(\sum_{k=0}^{N_0} \varphi_k t^k\Big)$$

and

$$\ln\Big(\sum_{k=0}^{N_0} \varphi_k t^k\Big) = \ln\varphi_0 + \frac{\varphi_1}{\varphi_0} \cdot t + O(t^2).$$

Hence

$$(\mathrm{I}) = \int_{B(r/2)} \Big[-\frac{d^2(x, y)}{4t} + \ln\varphi_0 + \frac{\varphi_1}{\varphi_0} \cdot t + O(t^2)\Big] \cdot H\, d\mu(x).$$

Now using Theorem 2.1(iii),

$$
\begin{aligned}
\text{(3-10)} \qquad \text{(I)} &= \int_{B(r/2)} \left[-\frac{d^2}{4t} + \tfrac{1}{12} R_{pq}(y) x^p x^q + O(d^3) + \left(\frac{R(y)}{6} + O(d) \right) t \right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot H \, d\mu(x) + O(t^2) \\
&= \text{(II)} + \text{(III)} + \text{(IV)} + \text{(V)} + \text{(VI)} + O(t^2),
\end{aligned}
$$

where

$$
\begin{aligned}
\text{(II)} &= \int_{B(r/2)} \left(-\frac{d^2(x,y)}{4t} \right) \cdot H \, d\mu(x), \\
\text{(III)} &= \frac{1}{12} \int_{B(r/2)} \left(R_{pq}(y) x^p x^q \right) \cdot H \, d\mu(x), \\
\text{(IV)} &= C \int_{B(r/2)} d^3(x,y) \cdot H(x,y,t) \, d\mu(x), \\
\text{(V)} &= \frac{R(y)}{6} t \cdot \int_{B(r/2)} H(x,y,t) \, d\mu(x), \\
\text{(VI)} &= Ct \cdot \int_{B(r/2)} d(x,y) \cdot H(x,y,t) \, d\mu(x).
\end{aligned}
$$

From (3-7),

$$
\int_{B(r/2)} H = \int_{M^n} H - \int_{M^n \setminus B(r/2)} H = 1 + O(t^2).
$$

Hence

$$
\text{(V)} = \tfrac{1}{6} R(y) \cdot t + O(t^2).
$$

Using (3-5) and the fact that

$$
\int_{R^n} O(|x|^k) (4\pi t)^{-n/2} \exp\left(-\frac{|x|^2}{4t} \right) dx = O(t^{k/2}),
$$

where k is any nonnegative integer, we can get (IV) $= O(t^{3/2})$ and (VI) $= O(t^{3/2})$. Similarly,

$$
\text{(III)} = \tfrac{1}{6} R(y) \cdot t + O(t^2)
$$

Finally, from Lemma 3.3, which follows,

$$
\text{(II)} = -\frac{n}{2} + \tfrac{1}{6} R(y) \cdot t + O(t^{3/2}). \qquad \square
$$

Lemma 3.3.

$$
\text{(3-11)} \qquad -\frac{1}{4t} \int_{B(r/2)} d^2(x,y) \cdot H d\mu(x) = -\frac{n}{2} + \tfrac{1}{6} R(y) \cdot t + O(t^{3/2}).
$$

Proof. (II) $:= -(1/(4t))\int_{B(r/2)} d^2(x,y)\cdot H\,d\mu(x)$. Then

$$\text{(II)} = -\frac{1}{4t}\int_{B(r/2)} d^2(x,y)\cdot(H_{N_0}+F_{N_0})\cdot\alpha\,dx, \tag{3-12}$$

where dx in the integral of (3-12) is the volume element of Euclidean space $\mathbb{R}^n$, and

$$\alpha = \sqrt{\det(g)} = 1 - \tfrac{1}{6}R_{pq}(y)x^px^q + O(d^3(x,y)).$$

Then

$$\begin{aligned}
\text{(II)} &= -\frac{1}{4t}\int_{B(r/2)} d^2(x,y)\cdot(4\pi t)^{-n/2}\exp\Big(-\frac{d^2(x,y)}{4t}\Big)(\varphi_0+\varphi_1 t)\cdot\alpha\,dx + O(t^2)\\
&= \Big[-\frac{1}{4t}-\frac{1}{24}R(y)\Big]\cdot\int_{B(r/2)} d^2\cdot(4\pi t)^{-n/2}\exp\Big(-\frac{d^2}{4t}\Big)\,dx\\
&\qquad + \frac{1}{48t}\int_{B(r/2)}(4\pi t)^{-n/2}(R_{pq}(y)x^px^q)d^2\cdot\exp\Big(-\frac{d^2}{4t}\Big)\,dx + O(t^{3/2})\\
&= \Big[-1/4t - 1/24R(y)\Big]\cdot 2nt + \frac{1}{48t}I_n + O(t^{3/2}),
\end{aligned}$$

where

$$I_n = \int_{\mathbb{R}^n}(4\pi t)^{-n/2}\Big(\sum_{k=1}^n\lambda_k x_k^2\Big)\cdot\Big(\sum_{i=1}^n x_i^2\Big)\exp\Big(-\frac{1}{4t}\cdot\sum_{j=1}^n x_j^2\Big)\,dx.$$

In the above we diagonalize $R_{pq}(y)$ and let $\lambda_k = R_{kk}(y)$.

We can get $I_1 = 12\lambda_1 t^2$ and the induction formula

$$I_n = I_{n-1} + 4\Big(\sum_{i=1}^n\lambda_i\Big)t^2 + 4(n+1)\lambda_n t^2.$$

Then it is easy to get

$$I_n = 4(n+2)\Big(\sum_{i=1}^n\lambda_i\Big)t^2 = 4(n+2)R(y)t^2. \tag{3-13}$$

By all the above (II) $= -n/2 + (R(y)/6)t + O(t^{3/2})$. □

4. The short time asymptotics of $\partial N(H,t)/\partial t$

To study $\partial N(H,t)/\partial t$, we must first switch the order of differentiation with integration. Because the manifold M^n can be noncompact, we need to be more careful when doing this. The following lemma justifies this switch in our case.

Lemma 4.1.

$$\frac{\partial}{\partial t}\left[\int_{M^n} H(-f)\,d\mu(x)\right] = \int_{M^n} [H(-f)]_t\,d\mu(x). \tag{4-1}$$

Proof. Define $\varphi_\rho(x) = \phi(d(x, y)/\rho)$, where ϕ is defined in the appendix, $\rho > 1$ is a constant. Fix $t > 0$ and define $G(x, t) = [H(-f)](x, y, t)$. For any $\epsilon > 0$, assume $1 > l > 0$ (if $l < 0$, a similar argument works). Then

$$\begin{aligned}
&\left|\int_{M^n} \frac{G(x, t+l) - G(x, t)}{l}\,d\mu(x) - \int_{M^n} G_t\varphi_\rho\,d\mu(x)\right| \\
&\quad\le \int_{B(\rho)} |G_t(x, t+\xi_x l) - G_t(x, t)|\,d\mu(x) + 2\int_{M^n\setminus B(\rho)} \sup_{s\in[t,t+l]} |G_t(x, s)|\,d\mu(x) \\
&\quad\le \int_{B(\rho)} \left|\frac{\partial^2}{\partial^2 t} G(x, t+\zeta_x l)\right| d\mu(x)\cdot l + 2\int_{M^n\setminus B(\rho)} \Big(\sup_{s\in[t,t+l]} |G_t(x, s)|\Big)\,d\mu(x) \\
&\quad\le \text{(I)} + \text{(II)}.
\end{aligned}$$

We first estimate (II). From [Li and Yau 1986], for $s \in [t, t+l]$,

$$\frac{H_t}{H}(x, y, s) \ge \tfrac{1}{2}\left[\frac{|\nabla H|^2}{H^2} - \frac{2n}{s} - CK\right] \ge -\frac{C}{s}, \tag{4-2}$$

where $C = C(K, n)$. From Corollary A.10,

$$\begin{aligned}
\frac{H_t}{H}(x, s) &\le \frac{2}{s}\left\{n + (4+Ks)\ln\frac{C(K, t+1)}{H(x, y, s)V^{1/2}(B_x(\sqrt{s/2}))V^{1/2}(B_y(\sqrt{s/2}))}\right\} \\
&\le \frac{C}{s}\left(1 + |\ln H| + \left|\ln\left[V\left(B_x\left(\sqrt{\frac{s}{2}}\right)\right)\cdot V\left(B_y\left(\sqrt{\frac{s}{2}}\right)\right)\right]\right|\right).
\end{aligned} \tag{4-3}$$

When $x \in M^n\setminus B(\rho)$, using the volume comparison theorem,

$$\begin{aligned}
&\left|\ln\left[V\left(B_x\left(\sqrt{\frac{s}{2}}\right)\right)\cdot V\left(B_y\left(\sqrt{\frac{s}{2}}\right)\right)\right]\right| \\
&\qquad\le 2\left|\ln V\left(B_y\left(\sqrt{\frac{s}{2}}\right)\right)\right| + \left|\ln V_{-K}\left(\sqrt{\frac{s}{2}}\right)\right| + |\ln V_{-K}(s + d(x, y))| \\
&\qquad\le C(|\ln s| + s + d),
\end{aligned} \tag{4-4}$$

where C is independent of ρ. From (4-2), (4-3), and (4-4),

$$\left|\frac{H_t}{H}\right|(x, s) \le \frac{C}{s}(|\ln H| + |\ln s| + s + d) \tag{4-5}$$

when $x \in M^n\setminus B(\rho)$.

From (4-5), on $M^n \backslash B(\rho)$,

$$
\begin{aligned}
(4\text{-}6)\quad |(-f)H_t|(x,s) &\le \Big[|\ln H| + \frac{n}{2}|\ln(4\pi s)|\Big]\cdot |H_t|(x,s) \\
&\le \Big[|\ln H| + \frac{n}{2}|\ln(4\pi s)|\Big]\cdot C|H|\cdot s^{-1}(|\ln H| + |\ln s| + s + d) \\
&\le \frac{C}{s}\cdot H[|\ln H|^2 + |\ln s|^2 + s^2 + d^2].
\end{aligned}
$$

From (4-5) and (4-6), if $s \in [t, t+l]$ and $x \in M^n \backslash B(\rho)$,

$$
\begin{aligned}
(4\text{-}7)\quad & |G_t(x,s)| \\
&\le \Big[|H_t| + \frac{n}{2s}|H| + |(-f)H_t|\Big](x,s) \\
&\le \frac{C}{s}H\cdot(|\ln H| + |\ln s| + s + d) + \frac{n}{2s}|H| + \frac{C}{s}H\cdot(|\ln H|^2 + |\ln s|^2 + s^2 + d^2) \\
&\le \frac{C}{s}H\cdot(|\ln H|^2 + |\ln s|^2 + s^2 + d^2),
\end{aligned}
$$

where C is independent of ρ. We can choose l smooth enough such that $(t+l) \le 2t$. Then, using (3-7) and (4-7), on $x \in M^n \backslash B(\rho)$,

$$
\begin{aligned}
(4\text{-}8)\quad |G_t(x,s)| &\le Cs\exp\Big(-\frac{d^2}{6s}\Big)\cdot\Big[\Big|C + 2\ln s - \frac{d^2}{6s}\Big|^2 + |\ln s|^2 + s^2 + d^2\Big] \\
&\le C(t+l)\exp\Big(-\frac{d^2}{6(t+l)}\Big)\cdot\Big[t^2 + d^2 + |\ln t|^2 + \Big(\frac{d^2}{t}\Big)^2\Big] \\
&\le Ct\exp\Big(-\frac{d^2}{12t}\Big)\cdot\Big[t^2 + |\ln t|^2 + \Big(\frac{d^2}{t}\Big)^2\Big].
\end{aligned}
$$

Hence, for any $\epsilon > 0$, we can find $\rho_0 > 1$ such that if $\rho \ge \rho_0$,

$$
(4\text{-}9)\qquad \int_{M^n\backslash B(\rho)} \big(\sup_{s\in[t,t+l]} |G_t(x,s)|\big)\, d\mu(x) < \frac{\epsilon}{4}
$$

On the other hand, because $0 < l < 1$,

$$
(4\text{-}10)\qquad \int_{B(\rho)} |G_{tt}(x, t+\zeta_x l)|\, d\mu(x) \le \int_{B(\rho)} \sup_{s\in[t,t+1]} |G_{tt}(x,s)|\, d\mu(x) \le C(\rho).
$$

Choose $l \le \epsilon/(4C(\rho))$. From (4-9) and (4-10), if $\rho > \rho_0$,

$$
(4\text{-}11)\qquad \left|\int_{M^n} \frac{G(x,t+l) - G(x,t)}{l}\, d\mu(x) - \int_{M^n} G_t\varphi_\rho\, d\mu(x)\right| < \epsilon.
$$

It is easy to see from Lemma 4.3 and its proof that

$$
\lim_{\rho\to\infty}\int_{M^n} G_t\phi_\rho
$$

exists and

$$\lim_{\rho\to\infty}\int_{M^n} G_t\phi_\rho = \int_{M^n} G_t. \tag{4-12}$$

From (4-11) and (4-12), we get our conclusion. □

From results in [Cheng et al. 1981], $\lim_{t\to 0} t\ln H = -d^2/4$ and the limit is uniform for any x in $B(r)$. Hence we can assume

$$t\ln H(x,y,t) = -\frac{d^2(x,y)}{4} + \epsilon(t,x,y).$$

We sometimes simplify notation by denoting $\epsilon(t,x,y)$ by ϵ. Then

$$t(-f) = \frac{n}{2}t\ln(4\pi t) - \frac{d^2}{4} + \epsilon, \tag{4-13}$$

where $\lim_{t\to 0}\epsilon(t,x,y) = 0$, and the limit is uniform for any x in $B(r)$. Without loss of generality, we can assume that $\varphi_0(x,y) \geq 1/2$ when $x \in B(r/2)$.

Lemma 4.2.

$$\int_{B(r/2)} E(-f)\,d\mu(x) = -\frac{n}{2} + \tfrac{1}{3}R(y)\cdot t + o(t), \tag{4-14}$$

$$\int_{B(r/2)} E(-f)\,O(d(x,y))\,d\mu(x) = o(1), \tag{4-15}$$

where $\lim_{t\to 0} o(t)/t = 0$.

Proof.

$$\begin{aligned}\int_B E(-f)\,d\mu(x) &\\ &= \int_B \frac{H_{N_0}}{\sum_{k=0}^{N_0}\varphi_k t^k}\cdot(-f)\,d\mu(x)\\ &= \int_B \left(\frac{1}{\varphi_0} - \frac{\varphi_1}{\varphi_0^2}t\right)H(-f)\,d\mu(x) + o(t)\\ &= \int_B \left(1 + \tfrac{1}{12}R_{pq}(y)x^p x^q - \frac{R(y)}{6}t\right)H(-f) + o(t)\\ &= -\frac{n}{2} + \left(\frac{1}{2}+\frac{n}{12}\right)R(y)t + \frac{1}{12}\int_B R_{pq}(y)x^p x^q\cdot H(-f)\,d\mu(x) + o(t).\end{aligned} \tag{4-16}$$

In the last equation, we used (1-1).

We estimate the third term on the right side of (4-16).

$$
\begin{aligned}
\text{(I)} &:= \frac{1}{12}\int_B R_{pq}(y)x^p x^q \cdot H(-f)\,d\mu(x) \\
&= \frac{1}{12}\int_B R_{pq}(y)x^p x^q \cdot H\Big[\ln H_{N_0} + \frac{n}{2}\ln(4\pi t)\Big]d\mu(x) \\
&= \frac{1}{12}\int_B R_{pq}(y)x^p x^q \cdot H_{N_0}\Big[-\frac{d^2}{4t} + \ln\varphi_0\Big]\cdot\alpha\,dx + o(t) \\
&= -\frac{1}{48t}\int_B E\cdot d^2\cdot R_{pq}(y)x^p x^q\,dx + o(t) \\
&= -\frac{n+2}{12}R(y)t + o(t).
\end{aligned}
\tag{4-17}
$$

In the last equation above, we used (3-13). From (4-16) and (4-17), we get (4-14). To prove (4-15), we follow a similar strategy.

$$
\begin{aligned}
\int_B E(-f)O(d)\,d\mu(x) &= \int_B \Big(\frac{1}{\varphi_0} - \frac{\varphi_1}{\varphi_0^2}t\Big)H(-f)O(d)\,d\mu(x) + o(1) \\
&= \int_B H_{N_0}\Big[\ln H_{N_0} + \frac{n}{2}\ln(4\pi t)\Big]O(d)\,d\mu(x) + o(1) \\
&= \int_B E\Big(-\frac{d^2}{4t} + \ln\varphi_0\Big)O(d)\,d\mu(x) + o(1) = o(1). \qquad \square
\end{aligned}
$$

Lemma 4.3.

$$\int_{M^n\setminus B} |(-f)H_t|\,d\mu(x) = O(t^{1/2}),$$

where $t \ll 1$ is small enough.

Proof. Similarly as with (4-6), on $M^n\setminus B$,

$$|(-f)H_t| \le \frac{C}{t}\cdot H[|\ln H|^2 + |\ln t|^2 + t^2 + d^2]. \tag{4-18}$$

Hence

$$
\begin{aligned}
\int_{M^n\setminus B} |(-f)H_t| &\le \frac{C}{t}\int_{M^n\setminus B} H\cdot|\ln H|^2 + \frac{C}{t}\int_{M^n\setminus B} H(|\ln t|^2 + t^2 + d^2) \\
&= \text{(I)} + \text{(II)}.
\end{aligned}
$$

Similarly as in the proof of (3-8), using (3-7), the volume comparison theorem, and the monotonicity of $h(x) = x(\ln x)^2$, when $x \in (0, e^{-2}]$,

$$\text{(I)} \le O(t^{1/2}).$$

Using (2-9), when $x \in M^n \backslash B$,

$$H \le |\eta H_{N_0}| + |F_{N_0}| \le C\Big[t^{-n/2}\exp\Big(-\frac{d^2}{4t}\Big) + t^4 \cdot \exp\Big(-\frac{d^2}{5t}\Big)\Big] = O(t^2)\widetilde{E}. \tag{4-19}$$

From (4-19), it is easy to get

$$(\mathrm{II}) \le O(t). \qquad \square$$

Proof of (1-2).

$$\begin{aligned}\frac{\partial}{\partial t}\Big[\int_{M^n} H(-f)\,d\mu(x)\Big] &= \int_{M^n}\Big[H_t + \frac{n}{2t}H + (-f)H_t\Big]\,d\mu(x)\\ &= \frac{n}{2t} + \int_{M^n\backslash B(r/2)} (-f)H_t\,d\mu(x) + \int_{B(\frac{r}{2})}(-f)H_t\,d\mu(x)\\ &= \frac{n}{2t} + (\mathrm{I}) + (\mathrm{II}).\end{aligned}$$

From Lemma 4.3, we have

$$(\mathrm{I}) = O(t^{1/2}).$$

From Lemma 4.4, which follows, we get

$$(\mathrm{II}) = -\frac{n}{2t} + \frac{1}{2}R(y) + o(1). \qquad \square$$

Lemma 4.4.

$$\int_B (-f)H_t\,d\mu(x) = -\frac{n}{2t} + \frac{1}{2}R(y) + o(1).$$

Proof. From (2-10) and (4-13),

$$\int_B (-f)H_t d\mu(x) = \int_B (-f)\cdot(H_{N_0})_t + O(t)$$

and

$$\begin{aligned}&\int_B (-f)(H_{N_0})_t\,d\mu(x)\\ &= \int_B \Big(\frac{d^2}{4t^2} - \frac{n}{2t}\Big)H_{N_0}\cdot(-f)\,d\mu(x) + \int_B E\varphi_1(-f)\,d\mu(x) + o(1)\\ &= \frac{1}{4t^2}\int_B H_{N_0}(-f)d^2\,d\mu(x) - \frac{n}{2t}\int_B H_{N_0}(-f)\,d\mu(x) + \int_B E\varphi_1(-f)\,d\mu(x) + o(1)\\ &= (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + o(1).\end{aligned}$$

Using Lemma 4.2,

$$\begin{aligned}\text{(III)} &= \int_B E\varphi_1(-f)\,d\mu(x) = \tfrac{1}{6}R(y)\int_B E(-f)\,d\mu(x) + \int_B E(-f)\cdot O(d)\\ &= -\frac{n}{12}R(y) + o(1).\end{aligned}$$

From (2-9) and (1-1),

$$\begin{aligned}\text{(II)} &= -\frac{n}{2t}\int_B H_{N_0}(-f)\,d\mu(x)\\ &= -\frac{n}{2t}\int_B H(-f)\,d\mu(x) - \frac{n}{2t}\int_B O(t^{N_0+1})\widetilde{E}(-f)\,d\mu(x)\\ &= \frac{n^2}{4t} - \frac{n}{4}R(y) + o(1).\end{aligned}$$

Similarly, by Lemma 4.5, which follows,

$$\begin{aligned}\text{(I)} &= \frac{1}{4t^2}\int_B (H + O(t^{N_0+1})\widetilde{E})(-f)\cdot d^2\,d\mu(x) = \frac{1}{4t^2}\int_B H(-f)\cdot d^2\,d\mu(x) + o(1)\\ &= -\frac{n(n+2)}{4t} + \Big(\frac{n}{3}+\frac{1}{2}\Big)R(y) + o(1).\end{aligned}$$

From all the above,

$$\int_B (-f)H_t\,d\mu(x) = -\frac{n}{2t} + \frac{1}{2}R(y) + o(1) \qquad \square$$

Lemma 4.5.

$$\frac{1}{4t^2}\int_B H(-f)\cdot d^2 d\mu(x) = -\frac{n(n+2)}{4t} + \Big(\frac{n}{3}+\frac{1}{2}\Big)R(y) + o(1).$$

Proof. We use a strategy similar to that used in the proof of (1-1).

$$\begin{aligned}&\frac{1}{4t^2}\int_B H(-f)\cdot d^2 d\mu(x)\\ &\qquad = \frac{1}{4t^2}\int_B \Big[\ln H_{N_0} + \frac{n}{2}\ln(4\pi t)\Big] H d^2 d\mu(x) + \frac{1}{4t^2}\int_B \Big[\ln\frac{H}{H_{N_0}}\Big] H d^2 d\mu(x).\end{aligned}$$

From (3-6),

$$\Big[\ln\frac{H}{H_{N_0}}\Big]H = O(t^4).$$

Hence,

$$\begin{aligned}
&\frac{1}{4t^2}\int_B H(-f)\cdot d^2\,d\mu(x)\\
&=\frac{1}{4t^2}\int_B\left[-\frac{d^2}{4t}+\ln\varphi_0+\frac{\varphi_1}{\varphi_0}t+O(t^2)\right]Hd^2\cdot\alpha\,dx+o(1)\\
&=\frac{1}{4t^2}\int_B\left(-\frac{d^2}{4t}+\frac{1}{12}R_{pq}(y)x^px^q+\frac{1}{6}R(y)t-\frac{R(y)}{24}d^2+\frac{1}{48t}R_{pq}(y)x^px^q\cdot d^2\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\cdot Ed^2\,dx+o(1)\\
&=-\frac{n(n+2)}{4t}+\frac{-n^2+2n+4}{24}R(y)+\frac{1}{192t^3}\int_{\mathbb{R}^n}ER_{pq}(y)x^px^q\cdot d^4\,dx+o(1).
\end{aligned}$$

Define

$$Q_n=\int_{\mathbb{R}^n}ER_{pq}(y)x^px^q\cdot d^4\,dx=\int_{\mathbb{R}^n}E\cdot\left(\sum_{i=1}^n\lambda_ix_i^2\right)\cdot\left(\sum_{j=1}^n x_j^2\right)dx,$$

where we diagonalize $R_{pq}(y)$ and let $\lambda_i=R_{ii}(y)$. We can get $Q_1=120\lambda_1t^3$ and the induction formula

$$Q_n=Q_{n-1}+8(2n+5)\left(\sum_{i=1}^n\lambda_i\right)t^3+8(n^2+4n+3)\lambda_n\cdot t^3.$$

Then it is easy to get $Q_n=8(n^2+6n+8)R(y)\cdot t^3$. Hence

$$\frac{1}{4t^2}\int_B H(-f)\cdot d^2\,d\mu(x)=-\frac{n(n+2)}{4t}+\left(\frac{n}{3}+\frac{1}{2}\right)R(y)+o(1). \qquad\square$$

Appendix

Richard Hamilton [1993] established an upper bound of the Laplacian of the positive solution to the heat equation on closed manifolds. We generalize his theorem to complete manifolds with Ricci curvature bounded below. Our proof follows a strategy similar to that in [Kotschwar 2007]. We firstly establish a preliminary estimate on $t|\Delta u|$ so that the maximum principle of Ni and Tam [2004] may be applied to the quantity of interest in Hamilton's second derivative estimate.

We introduce a cut-off function ϕ defined on $\mathbb{R}$, which is a smooth nonnegative nonincreasing function which is 1 on $(-\infty, 1)$ and 0 on $[2, +\infty)$. We can further assume

$$|\phi'|\le 2,\quad |\phi''|+\frac{(\phi')^2}{\phi}\le 16. \tag{A-1}$$

To prove the following Bernstein-type local estimate, we employ a technique of W.-X. Shi [1989] from the estimation of derivatives of curvature under the Ricci flow (see also [Chow et al. 2008]). Define $F=(C+t|\nabla u|^2)t^2|\Delta u|^2$ and consider the evolution of F.

Lemma A.6. *Suppose (M^n, g) is a complete Riemannian manifold. If $|u(x,t)|\le \mathcal{M}$ is a solution to the heat equation on $B_p(4\rho)\times[0,T]$ for some $p\in M^n$, constants $\mathcal{M}$, ρ, T, $K>0$, and $\mathrm{Rc}\ge -Kg$ on $B_p(4\rho)$,*

$$t|\Delta u|\le C(n,K,\mathcal{M})\Big[1+T\Big(1+\frac{1}{\rho^2}\Big)\Big]\cdot\Big(\frac{1}{\rho}+1\Big)\cdot\Big[T+\coth\Big(\sqrt{\frac{K}{n-1}}\rho\Big)\Big] \tag{A-2}$$

holds on $B_p(\rho)\times[0,T]$.

Proof. From [Kotschwar 2007], we get that

$$t|\nabla u|^2\le C_1\Big[1+T\Big(1+\frac{1}{\rho^2}\Big)\Big]=: C_2 \tag{A-3}$$

holds on $B_p(2\rho)\times[0,T]$, where $C_1=C_1(K,\mathcal{M})$. Define $C_3=4C_2$, and

$$F(x,t)=(C_3+t|\nabla u(x,t)|^2)t^2|\Delta u(x,t)|^2.$$

A long but straightforward computation gives

$$\begin{aligned}\Big(\frac{\partial}{\partial t}-\Delta\Big)F&\\ =-2(C_3+t|\nabla u|^2)|\nabla\Delta u|^2&-8t^3\sum_{i,j}\nabla_i\nabla_j u\nabla_i\Delta u\nabla_j u\Delta u-2t^3|\nabla^2u|^2\cdot|\Delta u|^2\\ &+2t(C_3+t|\nabla u|^2)|\Delta u|^2+[|\nabla u|^2-2t\,\mathrm{Rc}(\nabla u,\nabla u)]t^2|\Delta u|^2.\end{aligned}$$

When $x\in B_p(4\rho)$, using $t|\nabla u|^2\le C_2=\frac14 C_3$ and $\mathrm{Rc}\ge -Kg$,

$$\begin{aligned}\Big(\frac{\partial}{\partial t}-\Delta\Big)F&\le -10t^3|\nabla u|^2\cdot|\nabla\Delta u|^2+8t^3|\nabla u|\cdot|\nabla\Delta u|\cdot|\nabla^2u|\cdot|\Delta u|\\ &\qquad -2t^3|\nabla^2 u|^2\cdot|\Delta u|^2+C_4t|\Delta u|^2\\ &\le -\tfrac25 t^3|\nabla^2u|^2\cdot|\Delta u|^2+C_4t|\Delta u|^2,\end{aligned}$$

where $C_4=(2KT+11)C_2$. The term with the coefficient $-\frac25$ arose from the inequality $-10x^2+8xy-2y^2\le-\frac25y^2$. On the other hand, we know that $|\nabla^2u|^2\ge(1/n)|\Delta u|^2$. Hence

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)F\le-\frac{2}{5n}t^3|\Delta u|^4+C_4t|\Delta u|^2\le-\frac{1}{5nt}[t^2|\Delta u|^2]^2+\frac{5n}{4t}C_4\le-\frac{C_6}{t}F^2+\frac{C_5}{t}.$$

In the last equality we used $F\le(C_3+C_2)t^2|\Delta u|^2=5C_2t^2|\Delta u|^2$, and

$$C_5=C(n,K,\mathcal{M})(1+T)\Big[1+T\Big(1+\frac{1}{\rho^2}\Big)\Big], \tag{A-4}$$

$$C_6=C(n,K,\mathcal{M})\Big[1+T\Big(1+\frac{1}{\rho^2}\Big)\Big]^{-2}. \tag{A-5}$$

Define $\gamma(x)=\phi(d(x,p)/\rho)$. Then $\gamma(x)F(x,t)$ attains its maximum at a point $(x_0,t_0)\in B_p(2\rho)\times[0,T]$. The rest of the computation is at (x_0,t_0);

$$0\le\left(\frac{\partial}{\partial t}-\Delta\right)(\gamma F)\le\gamma\left(-\frac{C_6}{t}F^2+\frac{C_5}{t}\right)-\Delta\gamma\cdot F-2\nabla\gamma\nabla F.$$

Note that at (x_0,t_0), $\nabla(\gamma F)=0$. Letting $G=(\gamma F)(x_0,t_0)$, we get

$$0\le-\frac{C_6}{t}G^2+\left(2\frac{|\nabla\gamma|^2}{\gamma}-\Delta\gamma\right)G+\frac{C_5}{t}\tag{A-6}$$

and

$$\begin{aligned}\left(2\frac{|\nabla\gamma|^2}{\gamma}-\Delta\gamma\right)&=\frac{2}{\rho^2}\cdot\frac{|\phi'|^2}{\phi}-\frac{\phi''}{\rho^2}-\frac{\phi'}{\rho}\Delta d(x,p)\\&\le\frac{32}{\rho^2}+\frac{2}{\rho}\cdot\coth\Bigl(\sqrt{\frac{K}{n-1}}\rho\Bigr).\end{aligned}\tag{A-7}$$

In the last inequality we used (A-1), $\mathrm{Rc}\ge-Kg$, and the Laplacian comparison theorem. From (A-4)–(A-7),

$$\begin{aligned}0\le-G^2+C(n,K,\mathcal{M})\Bigl[1+T\Bigl(1+\frac{1}{\rho^2}\Bigr)\Bigr]^2T\cdot\Bigl[\frac{1}{\rho^2}+\frac{1}{\rho}\coth\Bigl(\sqrt{\frac{K}{n-1}}\rho\Bigr)\Bigr]G&\\+C(n,K,\mathcal{M})\Bigl[1+T\Bigl(1+\frac{1}{\rho^2}\Bigr)\Bigr]^3(1+T).&\end{aligned}$$

Then it is easy to get

$$G\le C(n,K,\mathcal{M})\cdot\Bigl[1+T\Bigl(1+\frac{1}{\rho^2}\Bigr)\Bigr]^2(1+T)\cdot\Bigl[\Bigl(\frac{1}{\rho^2}+\frac{1}{\rho}\Bigr)\coth\Bigl(\sqrt{\frac{K}{n-1}}\rho\Bigr)+1+T\Bigl(1+\frac{1}{\rho^2}\Bigr)\Bigr].$$

Hence, on $B_p(\rho)$,

$$\begin{aligned}t^2|\Delta u|^2&\le C_3^{-1}F\le C_3^{-1}G\\&\le C(n,K,\mathcal{M})\cdot\Bigl[1+T\Bigl(1+\frac{1}{\rho^2}\Bigr)\Bigr]^2\cdot\Bigl[\Bigl(\frac{1}{\rho^2}+1\Bigr)\cdot\Bigl(T+\coth\Bigl(\sqrt{\frac{K}{n-1}}\rho\Bigr)\Bigr)+1\Bigr]\end{aligned}$$

Taking the square root in the above inequality, we get our conclusion. □

Letting $\rho\to\infty$, we get the following global estimate.

Corollary A.7. *Suppose (M^n,g) is a complete Riemannian manifold with* $\mathrm{Rc}\ge-Kg$, *and $|u(x,t)|\le\mathcal{M}$ is a solution to the heat equation on $M^n\times[0,T]$, where K, $\mathcal{M}$, T are positive constants. Then*

$$t|\Delta u|\le C(n,K,\mathcal{M})(1+T)^2\tag{A-8}$$

holds on $M^n\times[0,T]$.

We also need a maximum principle, due originally to Karp and Li [1982], which was stated more generally by Ni and Tam.

Theorem A.8 [Ni and Tam 2004, Theorem 1.2]. *Suppose (M^n, g) is a complete Riemannian manifold and $h(x,t)$ is a smooth function on $M^n \times [0, T]$ such that*

$$\Big(\frac{\partial}{\partial t} - \Delta\Big) f(x,t) \le 0$$

whenever $f(x,t) \ge 0$. Assume that

$$\int_0^T \int_{M^n} e^{-a \cdot d^2(x,p)} f_+^2(x,s)\, d\mu(x)\, ds < \infty$$

for some $a > 0$, where p is a fixed point on M^n and $f_+(x,t) := \max\{f(x,t), 0\}$. If $f(x,0) \le 0$ for all $x \in M^n$, $f(x,t) \le 0$ for all $(x,t) \in M^n \times [0,T]$.

Now we are ready to prove Hamilton's theorem in the complete case.

Theorem A.9. *Suppose (M^n, g) is a complete Riemannian manifold with $\mathrm{Rc} \ge -Kg$, and $0 < u(x,t) \le \mathcal{M}$ is a solution to the heat equation on $M^n \times [0,T]$, where $K, \mathcal{M}, T$ are positive constants. Then*

$$t\Big(\frac{\Delta u}{u} + \frac{|\nabla u|^2}{u^2}\Big) \le n + (4 + 2Kt) \ln \frac{\mathcal{M}}{u}. \tag{A-9}$$

Proof. Defining $u_\epsilon = u + \epsilon$ for $\epsilon > 0$, we obtain a solution satisfying $\epsilon < u_\epsilon \le \mathcal{M} + \epsilon =: \mathcal{M}_\epsilon$. Once the estimate has been proved for u_ϵ, the theorem follows by letting $\epsilon \to 0$. Consider the function

$$F(x,t) = t\Big(\Delta u_\epsilon + \frac{|\nabla u_\epsilon|^2}{u_\epsilon}\Big) - u_\epsilon\Big[n + (4 + 2Kt) \ln \frac{\mathcal{M}_\epsilon}{u_\epsilon}\Big].$$

A long but straightforward computation gives

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta\Big) F(x,t) = u_\epsilon\Big[&-2t|\nabla^2 \ln u_\epsilon|^2 + \Delta \ln u_\epsilon - (2+2Kt)|\nabla \ln u_\epsilon|^2 \\ &- 2t\,\mathrm{Rc}(\nabla \ln u_\epsilon, \nabla \ln u_\epsilon) - 2K \ln \frac{\mathcal{M}_\epsilon}{u_\epsilon}\Big] \\ \le u_\epsilon\Big[&-\frac{2t}{n}|\Delta \ln u_\epsilon|^2 + \Delta \ln u_\epsilon - 2|\nabla \ln u_\epsilon|^2\Big].\end{aligned} \tag{A-10}$$

If $F(x,t) \ge 0$ at (x,t),

$$-2|\nabla \ln u_\epsilon|^2 \le \Delta \ln u_\epsilon - \frac{n}{t}. \tag{A-11}$$

From (A-10) and (A-11),

$$\Big(\frac{\partial}{\partial t} - \Delta\Big) F(x,t) \le u_\epsilon\Big[-\frac{2t}{n}|\Delta \ln u_\epsilon|^2 + 2\Delta \ln u_\epsilon - \frac{n}{t}\Big] \le -\frac{n}{2t} < 0. \tag{A-12}$$

In (A-3) let $\rho \to \infty$. Then

$$t|\nabla u|^2 \le C(K, \mathcal{M}_\epsilon, T). \tag{A-13}$$

From (A-13) and (A-8),

$$F_+^2(x,t) \le \left[t\left(\Delta u_\epsilon + \frac{|\nabla u_\epsilon|^2}{u_\epsilon}\right)\right]^2 \le C(\epsilon, n, K, \mathcal{M}_\epsilon, T). \tag{A-14}$$

Using (A-14), for any $p \in M^n$ and $\rho > 0$,

$$\begin{aligned}\int_0^T \int_{B_p(\rho)} \exp\big(-d^2(x,p)\big) F_+^2(x,t)\, d\mu(x)\, dt \\ \le C(\epsilon, n, K, \mathcal{M}_\epsilon, T) \int_{M^n} \exp\big(-d^2(x,p)\big)\, d\mu(x) \le C.\end{aligned} \tag{A-15}$$

In the last equality we used the volume comparison theorem and $\mathrm{Rc} \ge -Kg$. Letting $\rho \to \infty$,

$$\int_0^T \int_{M^n} \exp -d^2(x,p) F_+^2(x,t)\, d\mu(x)\, dt \le C < \infty. \tag{A-16}$$

From (A-12) and (A-16), using Theorem A.8, we get $F(x,t) \le 0$ for all $0 \le t \le T$, completing the proof. □

We now give an the upper bound for the Laplacian of the heat kernel.

Corollary A.10. *Suppose (M^n, g) is a complete Riemannian manifold such that* $\mathrm{Rc} \ge -Kg$, *$H(x,y,t)$ is the heat kernel on M^n, and $0 < t \le T$, where K, T are positive constants. Then*

$$\begin{aligned}&\left(\Delta H + \frac{|\nabla H|^2}{H}\right)(x,y,t) \\ &\quad\le \frac{2H(x,y,t)}{t}\left\{n + (4+Kt)\ln \frac{C(K,T)}{H(x,y,t)V^{1/2}(B_x(\sqrt{t/2}))V^{1/2}(B_y(\sqrt{t/2}))}\right\}.\end{aligned}$$

Proof. Note that if $s \in [t/2, t]$, from [Li and Yau 1986],

$$H(x,y,t) \le C(K,T) \cdot V^{1/2}\big(B_x\big(\sqrt{t/2}\big)\big) V^{1/2}\big(B_y\big(\sqrt{t/2}\big)\big).$$

Then apply Theorem A.9 on $u(x,s) = H(x,y,s+t/2)$ and $M^n \times [0, t/2]$. The conclusion follows from (A-9). □

Acknowledgements

The author thanks Zhiqin Lu, Brett Kotschwar for interest and suggestions and Peter Li and Jiaping Wang for their interest. After this paper was posted on arXiv, Jia-yong Wu kindly informed us that he had independently proved a Hamilton-type upper bound for complete manifolds with $\mathrm{Rc} \ge 0$ in detail; see [Wu 2013].

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Received October 8, 2012. Revised November 16, 2012.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 266, No. 2, 2013
dx.doi.org/10.2140/pjm.2013.266.449

SEVERAL SPLITTING CRITERIA FOR VECTOR BUNDLES AND REFLEXIVE SHEAVES

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In this paper, we show two splitting criteria for vector bundles on complex projective spaces by analytic method. We also prove a splitting criterion for reflexive sheaves on Horrocks schemes by algebraic method.

1. Introduction

Vector bundles are favored objects studied in algebraic geometry and commutative algebra. We say that a vector bundle splits if it is isomorphic to a direct sum of line bundles. A very interesting problem is whether there are nonsplitting vector bundles of small ranks on complex projective spaces. Although such vector bundles exist in lower dimensions, they seem to be extremely rare as the dimension increases. In fact, Hartshorne conjectured:

Conjecture 1.1 [Hartshorne 1974]. *If $n \geq 7$, all rank-2 vector bundles on the projective space $\mathbb{P}^n$ split.*

Under some additional conditions, the conjecture was proved. However, the conjecture is still open. The most well-known condition is the vanishing of certain intermediate (local) cohomology groups. The first splitting criterion of this type is attributed to Horrocks.

Theorem 1.2 [Horrocks 1964]. *Let E be a vector bundle on the projective space $\mathbb{P}^n$ with $n \geq 2$. Then E splits if and only if $H^i(\mathbb{P}^n, E(k)) = 0$ for all $k \in \mathbb{Z}$ and $1 \leq i \leq n-1$.*

A standard proof is to apply induction to n and use the following so-called "restriction criterion".

Theorem 1.3. *Let E be a rank-r vector bundle over $\mathbb{P}^n$, with $n \geq 3$. E splits if and only if its restriction $E|_H$ to some hyperplane $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ splits.*

The first author gratefully acknowledges the support from the National Nature Science Foundation of China (Grant No. 31271408) and Tsinghua University.
MSC2010: 14F05, 14J60, 32L10.
Keywords: holomorphic vector bundles, connection, reflexive sheaves, Horrocks scheme, splitting Criterion.

Evans and Griffith improved Horrocks' criterion in the 1980's.

Theorem 1.4 [Evans and Griffith 1981]. *Let E be a vector bundle on the projective space $\mathbb{P}^n$ of rank $r < n$. Then E splits if and only if $H^i(\mathbb{P}^n, E(k)) = 0$ for all $k \in \mathbb{Z}$ and $1 \leq i \leq r-1$.*

Kumar, Peterson and Rao obtained another improvement of Horrocks' theorem.

Theorem 1.5 [Kumar et al. 2003]. *Let E be a vector bundle on $\mathbb{P}^n$. If* rank $E < 2[n/2]$, *then E splits if and only if $H^i(\mathbb{P}^n, E(k)) = 0$ for all $k \in \mathbb{Z}$ and $1 < i < n-1$.*

Another type of splitting criteria involves extensibility of vector bundles. Let X be an algebraic variety and Y be a subvariety of X. A vector bundle E on Y is said to extend to X if there exists a vector bundle F on X such that $F|_Y = E$. Barth and van de Ven [1974] showed that a rank-2 vector bundle E on $\mathbb{P}^n$ splits if and only if E extends to $\mathbb{P}^N$ for all $N > n$. Their result was generalized to vector bundles of arbitrary rank by Sato [1977].

For any coherent sheaf $\mathcal{F}$, we denote the dual by $\mathcal{F}^* = \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$. The next theorem combines extensibility and vanishing of cohomology groups.

Theorem 1.6 [Kempf 1990]. *Let E be a vector bundle on the projective space $\mathbb{P}^n$ with $n \geq 2$ and E^* be its dual. Then E splits if and only if the following two conditions are satisfied*:

(1) *E extends to $\mathbb{P}^{n+1}$.*

(2) *$H^1(\mathbb{P}^n, E \otimes E^*(-k)) = 0$ for all positive integer k.*

Proofs of the above mentioned theorems are all algebraic. However, the following remarkable criterion uses an analytic method.

Theorem 1.7 [Luk and Yau 1993]. *Let E be a holomorphic vector bundle on $\mathbb{CP}^n$ with $n \geq 2$ and E^* be its dual. Then E splits if and only if $H^1(\mathbb{CP}^n, E \otimes E^*(k))$ vanish for all $k \in \mathbb{Z}$.*

In this paper, we employ Luk and Yau's idea to provide analytic proofs of some splitting criteria of vector bundles on complex projective spaces.

Let $p : \mathbb{CP}^{n+1} \setminus \{\xi\} \to \mathbb{CP}^n$ be the projection from a point $\xi \in \mathbb{CP}^{n+1} \setminus \mathbb{CP}^n$. We prove the following theorems in this paper.

Theorem A. *A holomorphic vector bundle E on $\mathbb{CP}^n$ splits if and only if p^*E extends to a vector bundle on $\mathbb{CP}^{n+1}$.*

Theorem B. *Let E be a holomorphic vector bundle on $\mathbb{CP}^n$. If* rank $E < 2[n/2]$, *then E splits if and only if the local cohomology groups $H^i_{\{\xi\}}(\mathbb{P}^{n+1}, \tilde{E}(k)) = 0$ for all $k \in \mathbb{Z}$ and $1 < i < n$, where $\tilde{E}$ is the extension of p^*E on $\mathbb{CP}^{n+1}$.*

Kumar observed that condition (2) in Theorem 1.6 implies that p^*E extends to a vector bundle on $\mathbb{CP}^{n+1}$. Thus Theorem A implies the following theorem.

Theorem 1.8 [Kumar 2003]. *Let E be a holomorphic vector bundle on $\mathbb{CP}^n$ with $n \geq 2$ and E^* be its dual. Then E splits if and only if $H^1(\mathbb{CP}^n, E \otimes E^*(-k)) = 0$ for all positive integer k.*

On the other hand, generalizations of splitting criteria to reflexive sheaves and more general varieties have been obtained. Abe and Yoshinaga [2008] generalized the restriction criterion for reflexive sheaves on projective spaces. On the other hand, Bakhtary [2011] generalized the restriction criterion to Horrocks varieties.

Definition 1.9. A coherent sheaf $\mathscr{F}$ on X is reflexive if $\mathscr{F} \cong \mathscr{F}^{**}$. It is normal if for every open set $U \subset X$ and every closed subset $Y \subset U$ of codimension ≥ 2, the restriction map $\mathscr{F}(U) \to \mathscr{F}(U \setminus Y)$ is bijective. We define the singular locus of $\mathscr{F}$ as $\operatorname{Sing}(\mathscr{F}) := \{x \in X \mid \mathscr{F}_x \text{ is not locally free}\}$.

Definition 1.10. An algebraic variety X is called a splitting variety if $H^1(X, L) = 0$ for any line bundle L on X. A Horrocks variety is a splitting variety X with $H^2(X, L) = 0$ for any line bundle L on X.

In this paper, we prove a generalization of both the theorem of Bakhtary and the theorem of Abe and Yoshinaga.

Theorem C. *Let H be an effective ample divisor on a smooth projective variety X of dimension* $\dim X \geq 4$. *Assume that X is a Horrocks variety. Then a reflexive sheaf $\mathscr{F}$ on X is splitting if and only if the restriction $\mathscr{F}|_H$ is splitting.*

Theorems A and B will be proved in Section 2 and Theorem C will be proved in the last section.

2. Splitting criteria via connections

Let E be a rank-r complex vector bundle over a complex n-dimensional manifold M. Denote by $\mathscr{A}^q$ and $\mathscr{A}^q(E)$ the sheaves of smooth q-forms on M and smooth q-forms on M with coefficients in E respectively.

Definition 2.1. A connection on E is a $\mathbb{C}$-linear morphism

$$\mathrm{D} : \mathscr{A}^0(E) \to \mathscr{A}^1(E)$$

such that for any open subset U of M

$$\mathrm{D}(f\gamma) = \mathrm{d}f \otimes \gamma + f\mathrm{D}(\gamma),$$

for any $f \in \Gamma(U, \mathscr{A}^0)$ and $\gamma \in \Gamma(U, \mathscr{A}^0(E))$.

A connection D of a vector bundle E localized over any open subset U is determined by a matrix $\omega = (\omega_j^k)$ of smooth 1-forms, called a connection matrix of D over U. It is well known that if the complex vector bundle E is holomorphic, then we have a connection D which can be decomposed into $\mathrm{D}' + \bar{\partial}$ such that

$\mathrm{D}' : \mathscr{A}^{(0,0)}(E) \to \mathscr{A}^{(1,0)}(E)$ and $\bar\partial : \mathscr{A}^{(0,0)}(E) \to \mathscr{A}^{(0,1)}(E)$, where $\mathscr{A}^{(p,q)}(E)$ are the sheaves of smooth (p,q)-forms with coefficients in E.

Definition 2.2. Let E be a holomorphic vector bundle over $\mathbb{C}^{n+1} \setminus \{0\}$ with a connection $\mathrm{D} = \mathrm{D}' + \bar\partial$. Denote by $\left(\sum_{j=0}^{n} \theta_{ij}^{k}\, \mathrm{d}z_j\right)$ the connection matrix of D'. We say D is holomorphic in the direction z_l for some $0 \le l \le n$ if the θ_{il}^{k} are holomorphic for all $1 \le i, k \le r$.

Theorem 2.8 of [Luk and Yau 1993] says that the existence of such a connection will force the cohomology groups $H^i(\mathbb{C}^{n+1} \setminus \{0\}, E)$ to vanish for $0 < i < n$. To prove our theorems, we will look for such a connection.

Let L be the line in $\mathbb{C}^{n+2}$ defined by $z_0 = z_1 = \cdots = z_n = 0$ and $\xi = [0, \ldots, 0, 1]$ be a point in $\mathbb{CP}^{n+1} \setminus \mathbb{CP}^n$. Set $X = \mathbb{C}^{n+2} \setminus L$ and $U = \mathbb{CP}^{n+1} \setminus \{\xi\}$. Consider the projections $\pi : X \to U$, $\pi((z_0, \ldots, z_{n+1})) = [z_0, \ldots, z_{n+1}]$ and $p : U \to \mathbb{CP}^n$, $p([z_0, \ldots, z_{n+1}]) = [z_0, \ldots, z_n]$. It is well known that p defines a line bundle structure on U. In fact, U is the total space of the line bundle $\mathbb{O}_{\mathbb{CP}^n}(1)$.

Assume that E is a holomorphic vector bundle on $\mathbb{CP}^n$. We claim that $\pi^* p^* E$ has a connection which is holomorphic in the direction z_{n+1}. Let $D = D' + \bar\partial$ be a connection on E. Assume that D' has a connection matrix $\omega^i = (\omega_\beta^\alpha)$ over $U_i = \{z_i \neq 0\} \subset \mathbb{CP}^n$, and $\omega_\beta^\alpha = \sum_{j=0}^{n} \theta_{\beta j}^{\alpha}\, \mathrm{d}(z_j/z_i)$. Pulling back E and the connection D to X, we see that $\pi^* p^* E$ admits a connection whose D'-part is defined by the connection matrix

$$\left(\sum_{j=0}^{n} (\theta_{\beta j}^{\alpha} \circ p \circ \pi) \frac{1}{z_i}\, \mathrm{d}z_j - \sum_{j=0}^{n} (\theta_{\beta j}^{\alpha} \circ p \circ \pi) \frac{z_j}{(z_i)^2}\, \mathrm{d}z_i\right),$$

which is clearly holomorphic in the direction z_{n+1}.

Now we are ready to prove Theorems A and B. The next result is a special case of Lemma 2.2 in [Luk and Yau 1993].

Lemma 2.3. $$H^i(X, \pi^* p^* E) = \bigoplus_{k=-\infty}^{+\infty} H^i(U, p^* E(k)).$$

Proof. Notice that the projection $\pi : X \to U$ admits a bundle structure whose fibers are the punctured complex line $\mathbb{C}^*$. In fact, one can check that X is the total space of $\mathbb{O}_U(1)$ with the zero section U removed. Applying Lemma 2.2 of [Luk and Yau 1993], we get the equality. □

Denote by $\iota : \mathbb{CP}^{n+1} \setminus \{\xi\} \hookrightarrow \mathbb{CP}^{n+1}$ the inclusion and $\tilde{E} = \iota_* p^* E$ the extension of E. The following proposition is the key to prove Theorems A and B.

Proposition 2.4. *Assume that the local cohomology groups*

$$H^i_{\{\xi\}}(\mathbb{CP}^{n+1}, \tilde{E}(k)) = H^{i+1}_{\{\xi\}}(\mathbb{CP}^{n+1}, \tilde{E}(k)) = 0,$$

for $0 < i < n$ *and all* $k \in \mathbb{Z}$. *Then* $H^i(X, \pi^* p^* E) = 0$ *for* $0 < i < n$.

Proof. By the assumption and the exact sequence of local cohomology

$$H^i_{\{\xi\}}(\mathbb{CP}^{n+1}, \tilde{E}(k)) \to H^i(\mathbb{CP}^{n+1}, \tilde{E}(k)) \\ \to H^i(U, p^*E(k)) \to H^{i+1}_{\{\xi\}}(\mathbb{CP}^{n+1}, \tilde{E}(k)),$$

we see that

$$H^i(U, p^*E(k)) = H^i(\mathbb{CP}^{n+1}, \tilde{E}(k)).$$

Hence $H^i(U, p^*E(k))$ is of finite dimension. By Serre vanishing theorem, there exists an integer N such that $H^i(\mathbb{CP}^{n+1}, \tilde{E}(-k)) = 0$ for all integers i, k with $|k| \geq N$ and $1 \leq i \leq n$. Therefore there are only finitely many $H^i(U, p^*E(-k)) \neq 0$. Thus $H^i(X, \pi^*p^*E) = \bigoplus_{k=-N}^{N} H^i(U, p^*E(-k))$ is of finite dimension by Lemma 2.3.

We know that π^*p^*E admits a connection which is holomorphic in the direction z_{n+1}. Applying the same argument as in the proof of Theorem 2.8 in [Luk and Yau 1993], we conclude that $H^i(X, \pi^*p^*E) = 0$. $\square$

Theorem A follows very easily from Proposition 2.4.

Theorem A. *Let E be a holomorphic vector bundle on $\mathbb{CP}^n$. Then E splits if and only if p^*E extends to a vector bundle on $\mathbb{CP}^{n+1}$.*

Proof. Assume that $E = \bigoplus_a \mathbb{O}_{\mathbb{CP}^n}(a)$. Then $p^*\mathbb{O}_{\mathbb{CP}^n}(a)$ extends uniquely to $\mathbb{O}_{\mathbb{CP}^{n+1}}(a)$. Hence $\tilde{E}$ is a vector bundle, moreover $\tilde{E}$ splits. Conversely, assume that $\tilde{E}$ is a vector bundle, then $\operatorname{depth}_x \tilde{E} = \operatorname{depth} \tilde{E}_x = \dim \mathbb{O}_x = n+1$. By [Hartshorne 1967, Proposition 1.4 and Theorem 3.8], we know that $H^i_{\{\xi\}}(\mathbb{CP}^{n+1}, \tilde{E}(-k)) = 0$ for all $k \in \mathbb{Z}$ and $i \leq n$. By Proposition 2.4 and Lemma 2.3, we see that $H^i(U, p^*E(-k)) = 0$ for all $k \in \mathbb{Z}$ and $1 \leq i \leq n$. Hence $H^i(\mathbb{CP}^{n+1}, \tilde{E}(-k)) = 0$ for all $k \in \mathbb{Z}$ and $1 \leq i \leq n$. It follows from Theorem 1.2 that $\tilde{E}$ splits, and so does E. $\square$

Another consequence of Proposition 2.4 is the following local cohomology version of Theorem 1.5 of Kumar, Peterson and Rao.

Theorem B. *Let E be a vector bundle on $\mathbb{CP}^n$. If* $\operatorname{rank} E < 2[n/2]$, *then E splits if and only if for all $k \in \mathbb{Z}$ and $1 < i < n$ the local cohomology groups $H^i_{\{\xi\}}(\mathbb{CP}^{n+1}, \tilde{E}(k)) = 0$.*

Proof. It is clear that if E splits then $H^i_{\{\xi\}}(\mathbb{CP}^{n+1}, \tilde{E}(k)) = 0$ for all $k \in \mathbb{Z}$ and $1 < i < n$. Assume that $H^i_{\{\xi\}}(\mathbb{CP}^{n+1}, \tilde{E}(k)) = 0$ for all $k \in \mathbb{Z}$ and $1 < i < n$. By Proposition 2.4,

$$H^i(X, \pi^*p^*E) = \bigoplus_{k=-\infty}^{+\infty} H^i(U, p^*E(-k)) = 0$$

for $2 \leq i \leq n-2$. In particular, $H^i(U, p^*E) = \bigoplus_{k=-\infty}^{+\infty} H^i(\mathbb{CP}^n, E(-k)) = 0$ for $2 \leq i \leq n-2$. By Theorem 1.5, we see that E splits. $\square$

Another local version of Theorem 1.5 was obtained as one of the main theorems in [Majidi-Zolbanin 2005].

3. Splitting of reflexive sheaves

The proof of Theorem C relies on the following two propositions.

Proposition 3.1. *Let $\mathcal{F}$ be a reflexive sheaf on a smooth projective variety X and H be an effective ample divisor X. Assume that H is a splitting variety. If $\mathcal{F}|_H$ splits into a direct sum of line bundles, then*

$$H^1(X, \mathcal{F}(kH)) = 0 \quad \textit{for all } k \in \mathbb{Z}.$$

Proof. By assumption, we have a surjective morphism

$$(*) \qquad H^1\big(X, \mathcal{F}((k-1)H)\big) \to H^1(X, \mathcal{F}(kH)) \to 0.$$

By Serre duality, $H^1(X, \mathcal{F}(kH)) \cong \operatorname{Hom}\big(\operatorname{Ext}^{n-1}(\mathcal{F}(kH), \omega_X), k\big)$. It suffices to show that $\operatorname{Ext}^{n-1}(\mathcal{F}(kH), \omega_X) = 0$ for $k \ll 0$. Consider the spectral sequence of local and global Ext functors

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\mathcal{F}, \omega_X)) \Rightarrow E^{p+q} = \operatorname{Ext}^{p+q}(\mathcal{F}, \omega_X).$$

Since $\mathcal{F}|_H$ is free, then $\operatorname{Sing}(\mathcal{F}|_H) = \varnothing$, which implies that $\operatorname{Sing}(\mathcal{F}) \cap H = \varnothing$. Note that the singular locus $\operatorname{Sing}(\mathcal{F})$ is a closed subset of X and H is ample. If $\dim \operatorname{Sing}(\mathcal{F}) = d > 0$, then $\operatorname{Sing}(\mathcal{F}) \cdot H^d > 0$. In particular, $\operatorname{Sing}(\mathcal{F}) \cap H \neq \varnothing$. Therefore, $\dim \operatorname{Sing}(\mathcal{F}) = 0$. Since a coherent sheaf $\mathcal{F}$ is free at a point $p \in X$ if and only if the stalk $(\mathcal{E}xt^q(\mathcal{F}, \mathcal{G}))_p = \operatorname{Ext}^q(\mathcal{F}_p, \mathcal{G}_p) = 0$ for all $q > 0$ and any coherent sheaf $\mathcal{G}$, we see that $\dim \operatorname{Supp}(\mathcal{E}xt^q(\mathcal{F}, \omega_X)) = 0$ for $q > 0$. Hence, $H^p(X, \mathcal{E}xt^q(\mathcal{F}, \omega_X)) = 0$ for $p > 0$ and $q > 0$. Now there are only two E^2-terms $H^0(X, \mathcal{E}xt^{n-1}(\mathcal{F}, \omega_X))$ and $H^{n-1}(X, \mathcal{H}om(\mathcal{F}, \omega_X))$ which may contribute to $\operatorname{Ext}^{n-1}(\mathcal{F}, \omega_X)$. Since $\mathcal{F}$ is reflexive, then $\operatorname{depth}_x \mathcal{F} \geq 2$ for all $x \in X$ by [Hartshorne 1980, Proposition 1.3]. Thus $H^1_{\{x\}}(\mathcal{F}_x) = 0$. By local duality, we get $\mathcal{E}xt^{n-1}(\mathcal{F}_x, \omega_{X,x}) = H^1_{\{x\}}(\mathcal{F}_x) = 0$. Hence $\mathcal{E}xt^{n-1}(\mathcal{F}, \omega_X) = 0$. The spectral sequence then tells us that there is a surjective morphism

$$(**) \qquad H^{n-1}(X, \mathcal{H}om(\mathcal{F}, \omega_X)) \twoheadrightarrow \operatorname{Ext}^{n-1}(\mathcal{F}, \omega_X).$$

Since $H^{n-1}(X, \mathcal{F}^*(-kH) \otimes \omega_X) = 0$ for $-k \gg 0$ by Serre vanishing theorem, replacing $\mathcal{F}$ by $\mathcal{F}(kH)$ in $(**)$, we conclude that

$$\operatorname{Ext}^{n-1}(\mathcal{F}(kH), \omega_X) = 0 \quad \text{for all } k \ll 0. \qquad \square$$

Theorem 3.2. *Let $\mathcal{E}$ be a vector bundle and $\mathcal{F}$ be a reflexive sheaf over a smooth projective variety X of dimension* $\dim X \geq 4$. *Let H be an effective ample divisor on X. If $\mathcal{F}|_H \cong \mathcal{E}|_H$ and $H^1(X, \mathcal{E}^* \otimes \mathcal{F}(-H)) = 0$, then $\mathcal{F} \cong \mathcal{E}$.*

Proof. By the assumption, we see that $\mathcal{H}om(\mathcal{E}, \mathcal{F}) = \mathcal{E}^* \otimes \mathcal{F}$ is also reflexive. Since a reflexive sheaf is torsion free, then the following sequence is exact:

$$0 \to \mathcal{E}^* \otimes \mathcal{F}(-H) \to \mathcal{E}^* \otimes \mathcal{F} \to (\mathcal{E}^* \otimes \mathcal{F})|_H \to 0.$$

Since $H^1(X, \mathcal{E}^* \otimes \mathcal{F}(-H)) = 0$, then $H^0(X, \mathcal{E}^* \otimes \mathcal{F}) \to H^0(H, (\mathcal{E}^* \otimes \mathcal{F})|_H)$ is surjective. Note that $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) = H^0(X, \mathcal{H}om(\mathcal{E}, \mathcal{F}))$. Therefore, there is a morphism $\varphi : \mathcal{E} \to \mathcal{F}$ extending the isomorphism $\phi : \mathcal{E}|_H \to \mathcal{F}|_H$. We need to show that φ is an isomorphism. Consider $\det \varphi : \det \mathcal{E} \to \det \mathcal{F}$. Since $\mathcal{E}|_H \cong \mathcal{F}|_H$ and $\mathrm{Pic}(X) = \mathrm{Pic}(H)$ by Grothendieck–Lefschetz theorem, we conclude that $\det \mathcal{E} = \mathcal{O}_X(c_1(\mathcal{E})) = \mathcal{O}_X(c_1(\mathcal{F})) = \det \mathcal{F}$. Therefore $\det \varphi \in H^0(X, \det \mathcal{E}^* \otimes \det \mathcal{F}) = \mathbb{C}$ because a reflexive rank-1 sheaf is a line bundle (see [Okonek et al. 1980, Lemma 1.1.15]). Clearly, $\det \varphi$ is a nonzero constant, since $\det \phi$ is a nonzero constant. Thus at each $x \in X \setminus (\mathrm{Sing}(\mathcal{E}^* \otimes \mathcal{F}))$, the morphism φ_x is an isomorphism. Since $\mathcal{E}^* \otimes \mathcal{F}$ is reflexive, then $\mathrm{codim}(\mathrm{Sing}(\mathcal{E}^* \otimes \mathcal{F})) \geq 3$ and hence φ is an isomorphism by [Hartshorne 1980, Proposition 1.6]. □

Theorem C follows easily from Proposition 3.1 and Theorem 3.2.

Theorem C. *Let H be an effective ample divisor on a smooth projective variety X of dimension* $\dim X \geq 4$. *Assume that X is a Horrocks variety. Then a reflexive sheaf $\mathcal{F}$ on X is splitting if and only if the restriction $\mathcal{F}|_H$ is splitting.*

Proof. By [Bakhtary 2011, Proposition 4.13], X is Horrocks if and only if X and all effective ample divisors are splitting. Clearly, if $\mathcal{F}$ is splitting, then $\mathcal{F}|_H$ is splitting. Conversely, assume that $\mathcal{F}|_H$ is splitting. By Grothendieck–Lefschetz theorem, there is a splitting vector bundle $\mathcal{E}$ on X such that $\mathcal{E}|_H = \mathcal{F}|_H$. By Proposition 3.1, we know that $H^1(X, \mathcal{E}^* \otimes \mathcal{F}(-H)) = 0$. Therefore, $\mathcal{E} \cong \mathcal{F}$ by Theorem 3.2. □

In [Abe and Yoshinaga 2008], the authors also generalized Horrocks cohomology criterion for reflexive sheaves on projective spaces. However, Horrocks cohomology criterion may not hold on Horrocks varieties in general. There exist smooth hypersurfaces in $\mathbb{P}^5$ with nonsplit vector bundles satisfying the Horrocks cohomology condition (see Remarks in the introduction of [Kumar et al. 2007]). It will be very interesting to know under what cohomology conditions a vector bundle on a Horrocks variety splits.

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Received October 13, 2012.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 266, No. 2, 2013
dx.doi.org/10.2140/pjm.2013.266.457

THE MINIMAL VOLUME ORIENTABLE HYPERBOLIC 3-MANIFOLD WITH 4 CUSPS

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We prove that the 8^4_2 link complement is the minimal volume orientable hyperbolic manifold with 4 cusps. Its volume is twice the volume V_8 of the ideal regular octahedron; that is, $7.32\ldots = 2V_8$. The proof relies on Agol's argument used to determine the minimal volume hyperbolic 3-manifolds with 2 cusps. We also need to estimate the volume of a hyperbolic 3-manifold with totally geodesic boundary which contains an essential surface with non-separating boundary.

1. Introduction

The volumes of hyperbolic 3-manifolds are known to be topological invariants. The structure of the set of the volumes of hyperbolic 3-manifolds is known.

Theorem 1.1 (Jørgensen and Thurston's; see [Benedetti and Petronio 1992, Corollaries E.7.1 and E.7.5]). *The set of the volumes of orientable hyperbolic* 3*-manifolds is a well-ordered set of the type ω^ω with respect to the order of $\mathbb{R}$. The volume of an orientable hyperbolic* 3*-manifold with n cusps corresponds to an n-fold limit ordinal.*

This theorem gives rise to the problem of determining the minimal volume orientable hyperbolic 3-manifolds with n cusps. The answers are known in the cases where $0 \le n \le 2$.

- In the case where $n = 0$ (closed manifold),
  Gabai, Meyerhoff and Milley [2009] showed that the Weeks manifold has the minimal volume. Its volume is $0.94\ldots$.
- In the case where $n = 1$,
  Cao and Meyerhoff [2001] showed that the figure-eight knot complement and the manifold obtained by the (5, 1)-Dehn surgery from the Whitehead link

This research is supported by Global COE Program "New Development in Mathematics" from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan.
MSC2010: 57M27.
Keywords: hyperbolic 3-manifold, essential surface, geodesic boundary.

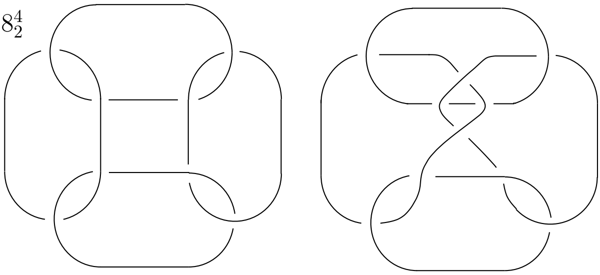

Figure 1. The 8^4_2 link and a link whose complement is homeomorphic to that of the 8^4_2 link.

complement have the minimal volume. Their volume is $2.02\ldots = 2V_3$, where V_3 is the volume of the ideal regular tetrahedron.

- In the case where $n = 2$,

  Agol [2010] showed that the $(-2, 3, 8)$-pretzel link complement and the Whitehead link complement have the minimal volume. Their volume is $3.66\ldots = 4\sum_{k=0}^{\infty}(-1)^k/(2k+1)^2 = V_8$, where V_8 is the volume of the ideal regular octahedron.

In the case where $n \geq 3$, Adams [1988] showed that the volume of an n-cusped hyperbolic 3-manifold is not less than nV_3. Agol [2010] conjectured the following:

- In the case where $3 \leq n \leq 10$,

  the minimally twisted hyperbolic chain link complement has the minimal volume.

- In the case where $n \geq 11$,

  the $(n-1)$-fold covering of Whitehead link complement has the minimal volume.

In this paper, we prove this conjecture in the case where $n = 4$.

Theorem 1.2. *The minimal volume orientable hyperbolic* 3*-manifold with* 4 *cusps is homeomorphic to the* 8^4_2 *link complement. Its volume is* $7.32\ldots = 2V_8$.

We remark that this link is not the unique one to determine the complement. For example, the complement of the link on the right of Figure 1 is homeomorphic to the 8^4_2 link complement.

We will prove Theorem 1.2 in Sections 4 and 5. The proof owes much to Agol [2010].

2. Review of Agol's argument

In this section, we set up some notation and review the argument used by Agol [2010] to determine the minimal volume of 2-cusped hyperbolic 3-manifolds. We treat compact smooth 3-manifolds with boundary and corners. We only consider surfaces in a compact 3-manifold which are properly embedded or contained in the boundary. Let $I = [0, 1]$.

Let M be a 3-manifold with boundary. For a properly embedded surface $X \subset M$, let $M \setminus\setminus X$ denote the path-metric closure of $M - X$. We will say that X is *essential* if X is incompressible and ∂-incompressible and has no component parallel to the boundary. Essential surfaces are not assumed to be connected.

A finite volume orientable hyperbolic 3-manifold can be the interior of a compact 3-manifold with the boundary which consists of tori. Its boundary component is called a *cusp*. When we say a hyperbolic manifold in what follows, it often means this compact manifold. We also consider hyperbolic manifolds with totally geodesic boundary. In this case there may be *annular cusps* which adjoin the totally geodesic boundary. The *double* of a hyperbolic manifold M with totally geodesic boundary is the manifold obtained from two copies of M by gluing along the totally geodesic boundary. Then two annular cusps form one torus cusp in its double.

We introduce the notion of pared manifolds. It was defined by Thurston to characterize a topological property of geometrically finite hyperbolic manifolds.

Definition 2.1 [Thurston 1986, Section 7; Morgan 1984, Definition 4.8]. A *pared manifold* is a pair (M, P) such that

- M is a compact orientable irreducible 3-manifold,
- $P \subset \partial M$ is a union of annuli and tori which are incompressible in M,
- every abelian, noncyclic subgroup of $\pi_1(M)$ is peripheral with respect to P (that is, conjugate to a subgroup of the fundamental group of a component of P), and
- every map $\phi\colon (S^1 \times I, S^1 \times \partial I) \to (M, P)$ which induces injective maps on the fundamental groups deforms, as maps of pairs, into P.

We call P the *parabolic locus* of the pared manifold (M, P), and an annular component of P is called a *pared annulus*. We denote the surface $\partial M - \operatorname{int}(P)$ by $\partial_0 M$.

Let (M, P) be a pared manifold. If every map $\psi\colon (S^1 \times I, S^1 \times \partial I) \to (M, \partial_0 M)$ that induces injective maps on the fundamental groups deforms either into $\partial_0 M$ or into P, then we call (M, P) *acylindrical*.

Since a finite volume orientable hyperbolic 3-manifold is atoroidal, it is a pared manifold by letting its parabolic locus be the cusp tori. Conversely:

Theorem 2.2. *Let (M, P) be an acylindrical Haken pared manifold, and assume that $\partial_0 M$ is incompressible. We assume that M is not a 3-ball, a $T^2 \times I$ or a solid torus. Then $M - P$ admits a finite volume hyperbolic structure with totally geodesic boundary $\partial_0 M$. This hyperbolic structure is unique up to isometry.*

Since the double DM of an acylindrical pared manifold (M, P) is atoroidal, DM admits a finite volume hyperbolic structure, where DM is obtained from two copies of M by gluing along $\partial_0 M$. Then the diffeomorphism swapping the two copies of M can be taken to be an isometry. The fixed point set $\partial_0 M$ is totally geodesic [Leininger 2006, Lemma 2.6].

When a hyperbolic manifold is cut along an essential surface, the obtained manifold is a pared manifold.

Lemma 2.3 [Agol 2010, Lemma 3.2]. *Let M be a finite volume orientable hyperbolic 3-manifold, and ∂M be the parabolic locus P of M. Let $X \subset M$ be an essential surface. Then $(M \setminus\setminus X, P \setminus\setminus \partial X)$ is a pared manifold.*

Theorem 2.4 (JSJ decomposition for a pared manifold; see [Jaco and Shalen 1979; Johannson 1979; Morgan 1984, Section 11]). *Let (M, P) be a pared manifold such that $\partial_0 M$ is incompressible. There is a canonical set of essential annuli $(A, \partial A) \subset (M, \partial_0 M)$ called the characteristic annuli. It is characterized up to isotopy by the property that they are the maximal collection of nonparallel essential annuli such that every other essential annulus $(B, \partial B) \subset (M, \partial_0 M)$ may be relatively isotoped to an annulus $(B', \partial B') \subset (M, \partial_0 M)$ so that $B' \cap A = \varnothing$. Then each complementary component $(L, \partial_0 L) \subset (M \setminus\setminus A, \partial_0 M \setminus\setminus \partial A)$ is one of the following types*:

1. $(T^2 \times I, (T^2 \times I) \cap \partial_0 M)$, *where one of the boundary components $T^2 \times \partial I$ is a torus component of P.*
2. $(S^1 \times D^2, (S^1 \times D^2) \cap \partial_0 M)$, *which is a solid torus with annuli in the boundary.*
3. (I*-bundle*, ∂I*-subbundle*), *which is an I-bundle over a surface whose Euler characteristic is negative*, *and the I-bundle over the boundary is contained in $A \cup P$.*
4. $(L, L \cap \partial_0 M)$, *where L has no essential annuli whose boundary is contained in $L \cap \partial_0 M$.*

A neighborhood of a torus component of P is either of type 1 or of type 4. One of the boundary components $T^2 \times \partial I$ of type 1 is a torus component of P, and the intersection of the other boundary component and $\partial_0 M$ is a union of essential annuli in the torus. The intersection $(S^1 \times D^2) \cap \partial_0 M$ in a component of type 2 is a union of essential annuli in $\partial(S^1 \times D^2)$. The union of components of type 3 is called the *window*. A component of type 4 is the acylindrical pared manifold $(L, L - \partial_0 M)$. The union of the components of type 4 is called the *guts* and denoted by $\mathrm{Guts}(M, P)$. A torus boundary component of the guts is a torus component of P.

The definition of guts in [Agol 2010] is a bit different from ours. There, the guts are defined to be the union of types 1, 2 and 4. The definition in [Agol et al. 2007] is same as ours, and it is appropriate for our purpose.

Let M be a finite volume orientable hyperbolic manifold. For an essential surface $X \subset M$, $(M \setminus\!\setminus X, P \setminus\!\setminus \partial X)$ is a pared manifold by Lemma 2.3. Therefore, we can define $\mathrm{Guts}(X) = \mathrm{Guts}(M \setminus\!\setminus X, P \setminus\!\setminus \partial X)$. Then the components of $\mathrm{Guts}(X)$ admit hyperbolic structures with geodesic boundary by Theorem 2.2. Hence the volume $\mathrm{vol}\,\mathrm{Guts}(X)$ is defined. This volume is not greater than the volume of M.

Theorem 2.5 [Agol et al. 2007, Theorem 9.1]. *Let M be a finite volume orientable hyperbolic manifold, and $X \subset M$ be an essential surface. Then*

$$\mathrm{vol}\, M \geq \mathrm{vol}\,\mathrm{Guts}(X) \geq \frac{V_8}{2}|\chi(\partial\,\mathrm{Guts}(X))|.$$

Moreover, the refinement by Calegari, Freedman and Walker (in [Calegari et al. 2010, Theorem 5.5]) *implies that M is obtained from ideal regular octahedra by gluing along the faces when the equality holds.*

The estimate of $\mathrm{vol}\,\mathrm{Guts}(X)$ from below in Theorem 2.5 follows from the following theorem.

Theorem 2.6 [Miyamoto 1994, Theorem 5.2]. *Let M be a hyperbolic manifold with totally geodesic boundary. Then* $\mathrm{vol}\, M \geq V_8/2|\chi(\partial M)|$. *Moreover, M is obtained from ideal regular octahedra by gluing along their faces when the equality holds.*

Lemma 2.7. *Let M be a finite volume orientable hyperbolic* 3*-manifold, and $X \subset M$ be a nonempty essential surface. Then each component of* $\mathrm{Guts}(X)$ *has negative Euler characteristic.*

Proof. Since the Euler characteristic of every closed 3-manifold is 0, we have $\chi(\mathrm{Guts}(X)) = \frac{1}{2}\chi(\partial\,\mathrm{Guts}(X))$. Assume that there is a component L of $\mathrm{Guts}(X)$ such that $\chi(L) \geq 0$. Since no component of $\partial\,\mathrm{Guts}(X)$ is a sphere, $\chi(L) = 0$ and ∂L consists of tori. Since M is atoroidal, $\partial L \subset \partial M$. This implies $L = M$ by connectedness of M. This contradicts the fact that X is not empty. □

This lemma implies that $\chi(\partial\,\mathrm{Guts}(X)) \leq -4$ if $\mathrm{Guts}(X)$ is not connected.

We will use annular compressions to obtain a surface whose guts are not empty.

Definition 2.8. Let $(X, \partial X) \subset (M, \partial M)$ be an essential surface in a 3-manifold. A *compressing annulus* is an embedding $i\colon (S^1 \times I, S^1 \times \{0\}, S^1 \times \{1\}) \hookrightarrow (M, X, \partial M)$ such that

- i_* induces injective maps on π_1,
- $i(S^1 \times I) \cap X = i(S^1 \times \{0\})$, and
- $i(S^1 \times \{0\})$ is not isotopic in X to ∂X.

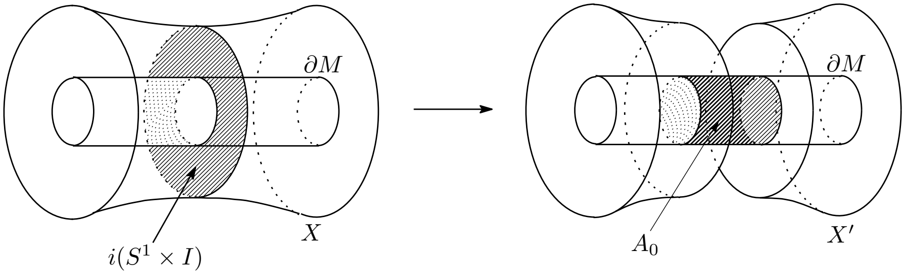

Figure 2. An annular compression.

An *annular compression* of $(X, \partial X) \subset (M, \partial M)$ is the surgery along a compressing annulus $i(S^1 \times I)$. Let U be a regular neighborhood of $i(S^1 \times I)$ in $M \setminus\!\setminus X$, and put $\partial_0 U = \partial U \cap (X \cup \partial M)$ and $\partial_1 U = \overline{\partial U - (X \cup \partial M)}$. Then the surface $X' = (X - \partial_0 U) \cup \partial_1 U$ is called the annular compression of X. If X is essential, X' is also essential. We will say that $A_0 = \partial U \cap \partial M$ is the annulus in the boundary created by the annular compression (Figure 2). This annulus is not contained in the window of $M \setminus\!\setminus X'$.

Lemma 2.9 [Agol 2010, Lemma 3.3]. *Let M be a finite volume orientable hyperbolic manifold. Let $X \subset M$ be an essential surface. If X has a compressing annulus, let X' be the annular compression of X. Then the annulus in the boundary created by this annular compression is not contained in the window of $M \setminus\!\setminus X'$.*

The following lemma is used in the proof of [Agol 2010, Theorem 3.4]. Lemmas 2.9 and 2.10 imply that a torus or an annulus in the boundary is contained in the boundary of the gut regions after we perform annular compressions as many times as possible.

Lemma 2.10. *Let M and X be as above. We assume that a $T^2 \times I$ component or an $S^1 \times D^2$ component intersects a component T of ∂M. Then we can perform an annular compression for X toward T.*

The following theorem is a result in [Culler and Shalen 1984, Theorem 3]. We will use it to find an essential surface to start the proof of Theorem 4.3.

Theorem 2.11. *Let M be a finite volume orientable hyperbolic manifold with n cusps, and let $\partial M = T_1 \cup \cdots \cup T_n$, where T_i is a torus for $1 \leq i \leq n$. Let k be an integer such that $1 \leq k \leq n$. Then there is an essential surface $X \subset M$ such that $\partial X \cap T_i \neq \varnothing$ for $1 \leq i \leq k$ and $\partial X \cap (T_{k+1} \cup \cdots \cup T_n) = \varnothing$.*

3. Essential surfaces in 3-manifold with boundary

In this section we find an essential surface in a hyperbolic 3-manifold with geodesic boundary. Using this we will estimate the volume of a hyperbolic 3-manifold with geodesic boundary with at least 4 cusps. Essential surfaces are found by a homological argument for 3-manifolds, and it is not necessary to assume the hyperbolic structure.

Lemma 3.1 [Hatcher 2007, Lemma 3.5]. *Let M be a compact orientable 3-manifold. Then the rank of the boundary homomorphism $\partial_*\colon H_2(M, \partial M; \mathbb{Q}) \to H_1(\partial M; \mathbb{Q})$ is half of the dimension of $H_1(\partial M; \mathbb{Q})$.*

Lemma 3.2. *Let L be an orientable hyperbolic 3-manifold with geodesic boundary S, with k annular cusps $A_1, \dots, A_k$ and with $n-k$ torus cusps $T_{k+1}, \dots, T_n$, where $1 \le k \le 3$ and $n \ge 4$. Assume that $\chi(S) = -2$. Then there is an essential surface $Y \subset L$ such that $Y \cap S = \varnothing$ and $[\partial Y] \neq 0 \in H_1(\partial L; \mathbb{Z})$.*

Proof. The union $S' = S \cup A_1 \cup \cdots \cup A_k$ is a closed surface of genus 2. We note that there are only two types of essential closed curves in S'; one separates S' and the other does not. There are no pairs of disjoint separating curves in S'.

We can take $k-1$ annuli of $\{A_1, \dots, A_k\}$ such that the complement of them is connected. The image of $\partial_*\colon H_2(L, \partial L; \mathbb{Q}) \to H_1(\partial L; \mathbb{Q})$ is a subspace of $H_1(\partial L; \mathbb{Q}) \cong \mathbb{Q}^{2(n-k)+4}$ of dimension $n-k+2$, by Lemma 3.1. We consider the subspace V of $H_1(\partial L; \mathbb{Q})$ spanned by all the elements represented by curves in $A_1, \dots, A_{k-1}, T_{k+1}, \dots, T_n$. Since the dimension of V is $2(n-k)+(k-1)$, V intersects $\mathrm{Im}(\partial_*)$ in a nontrivial subspace of $H_1(\partial L; \mathbb{Q})$. Hence there exists a nonzero element z in $H_2(L, \partial L; \mathbb{Q})$ such that $\partial_* z \neq 0$ and z belongs to V. By taking a multiple of z, there exists a nonzero element z' in $H_2(L, \partial L; \mathbb{Z})$ such that $\partial_* z' \neq 0$ and $\partial_* z'$ is represented by a union of closed curves in $A_1, \dots, A_{k-1}, T_{k+1}, \dots, T_n$. We can find an essential surface Y representing z' such that

$$\partial Y \subset A_1 \cup \cdots \cup A_{k-1} \cup T_{k+1} \cup \cdots \cup T_n. \qquad \square$$

4. Estimate of volume

Now we are going to estimate the volume of a hyperbolic manifold with geodesic boundary. Lemma 3.2 and Theorem 4.1 imply that the volume of an orientable hyperbolic 3-manifold with 4 cusps and with geodesic boundary is not less than $2V_8$.

Theorem 4.1. *Let L be an orientable hyperbolic 3-manifold with geodesic boundary S. Suppose that there is an essential surface $Y \subset L$ such that $Y \cap S = \varnothing$ and $[\partial Y] \neq 0 \in H_1(\partial L; \mathbb{Z})$. Then there is an essential surface Y' such that $\chi(\partial\,\mathrm{Guts}(L \setminus\!\!\setminus Y')) \le -4$ and $\mathrm{vol}\, L \ge 2V_8$.*

If $\chi(S) \le -4$, then $\mathrm{vol}\, L \ge 2V_8$ by Theorem 2.6. Hence we may assume that $\chi(S) = -2$. Let S' denote the surface which is the union of S and the annular cusps of L. ∂L consists of S' and the torus cusps of L.

We will find an essential surface Y' such that $\chi(\partial\,\mathrm{Guts}(L \setminus\!\!\setminus Y')) \le -4$. Then $\chi(\partial\,\mathrm{Guts}(DL \setminus\!\!\setminus (DY' \cup S)) \le -8$, where DL is the double of L (the hyperbolic manifold obtained from two copies of L by gluing along the geodesic boundary S) and DY' is the union of two copies of Y' in DL. Then Theorem 2.5 implies $\operatorname{vol} DL \geq 4V_8$, and so $\operatorname{vol} L \geq 2V_8$.

We will find a gut component intersecting S. For this we need to know how a window component intersects S.

Lemma 4.2. *Let L, S and Y be as above. Assume that S intersects a component $(J, \partial_0 J)$ of the window of $L \setminus\!\!\setminus Y$. Then $(J, \partial_0 J)$ is a product I-bundle and intersects S only on one component of the ∂I-bundle.*

Proof. Suppose that the base space of J is nonorientable. Since $\partial_0 J$ is connected, $\partial_0 J \subset S$. We take a simple closed curve α in J such that α is projected to an orientation-reversing loop in the base space of J. There is a simple closed curve β in $\partial_0 J$ such that $[\beta] = [\alpha]^2 \in \pi_1(DL) \subset \operatorname{Isom}^+(\mathbb{H}^3)$. If β is homotopic to the boundary of $\partial_0 J$, the base space of J is a Möbius band. It contradicts the definition of the window. Hence $[\beta] \in \pi_1(S) \subset \operatorname{Isom}^+(\mathbb{H}^2)$ is hyperbolic element. The simple closed curve β is homotopic to a simple closed geodesic β' in S [Ratcliffe 2006, Theorem 9.6.5]. But the fact that $[\beta'] = [\alpha]^2$ contradicts the fact that an element represented by a simple closed geodesic in a hyperbolic manifold has no roots [Ratcliffe 2006, Theorem 9.6.2]. Therefore no twisted I-bundle intersects S.

Suppose that the base space of J is orientable and both components Q_0 and Q_1 of $\partial_0 J$ are contained in S. Since $\chi(Q_0) = \chi(Q_1) < 0$, there are (not necessarily simple) closed curves $\gamma_i \subset Q_i$ $(i = 0, 1)$ such that γ_i is not homotopic to the boundary of Q_i and γ_0 and γ_1 are homotopic in L. Let γ_i' be the closed geodesic in Q_i homotopic to γ_i. Since L is totally geodesic, the two closed geodesics γ_0' and γ_1' are homotopic in L. It contradicts the uniqueness of the closed geodesic in a homotopy class. Therefore a product I-bundle intersects S on at most one side of the ∂I-bundle. □

Proof of Theorem 4.1. Let Y_0 be an essential surface in L such that $Y_0 \cap S = \varnothing$ and $[\partial Y_0] \neq 0 \in H_1(\partial L; \mathbb{Z})$. Moreover let $|\chi(Y_0)|$ be minimal among the surfaces satisfying these conditions. Since L has no essential sphere, disk, torus or annulus, $\chi(Y_0) < 0$. Let $p : L \setminus\!\!\setminus Y_0 \to L$ be the natural projection.

(i) First we consider the case where S intersects a component $(J, \partial_0 J)$ of the window of $L \setminus\!\!\setminus Y_0$. Then $\chi(J)$ is equal to -1 or -2. We will show that $\chi(J) = -1$. Assume that $\chi(J) = -2$. $S \cap p(J)$ is a 2-punctured torus or a 4-punctured sphere. (If it is a closed surface, $Y_0 \cap p(J)$ is a component of Y_0 which is parallel to S'. It contradicts that Y_0 is essential.) Let Y_0' be the surface which is the union of annuli (Figure 3) and $Y_0 - (Y_0 \cap p(J))$. If there is an annulus in $L - Y_0$ whose boundary is two components of the frontier of $Y_0 - (Y_0 \cap p(J))$, we glue $Y_0 - (Y_0 \cap p(J))$ and this annulus (the upper of Figure 3). Since $Y_0 \cap p(J)$ is connected, the orientation matches. Otherwise, there is an annular cusp which is homotopic to the frontier of

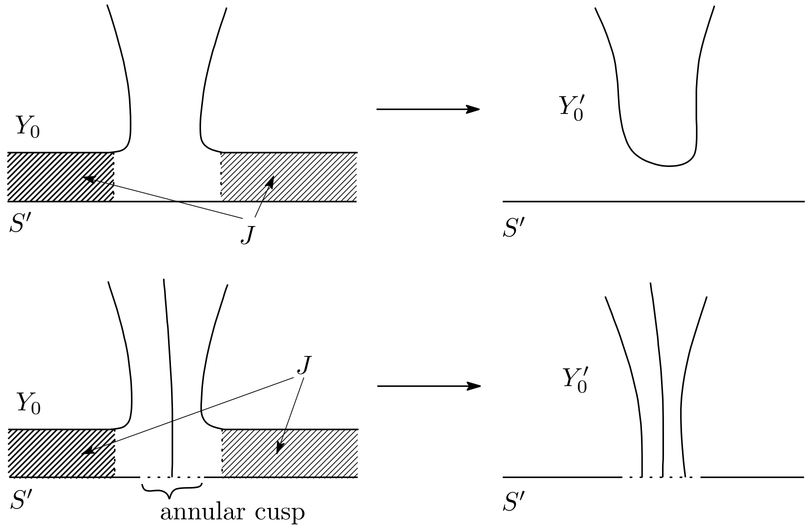

Figure 3. Constructions in the case where S intersects a component of the window whose Euler characteristic is -2.

$Y_0 - (Y_0 \cap p(J))$. Then we can glue $Y_0 - (Y_0 \cap p(J))$ and the two annuli, where one of the boundary components of each annulus is contained in this annular cusp (the lower of Figure 3). Then $[Y_0', \partial Y_0'] = [Y_0, \partial Y_0] \in H_2(L, \partial L; \mathbb{Z})$. We obtain an essential surface from Y_0' by compressing if necessary. Then $|\chi(Y_0')| < |\chi(Y_0)|$, contradicting the choice of Y_0. Therefore $\chi(J) = -1$.

We will find an essential surface Y_1 such that S intersects only one component of the window of $L \setminus\!\setminus Y_1$. If S intersects only one component of the window of $L \setminus\!\setminus Y_0$ already, put $Y_1 = Y_0$. Suppose that S intersects two components $(J, \partial_0 J)$ and $(J', \partial_0 J')$ of the window of $L \setminus\!\setminus Y_0$. Let Y_0' be the surface which is the union of $Y_0 - (Y_0 \cap p(J))$ and a surface in $p(J')$ (Figure 4). Then

$$[Y_0', \partial Y_0'] = [Y_0, \partial Y_0] \in H_2(L, \partial L; \mathbb{Z}).$$

Note that since the orientation may not match, we cannot construct a surface as in Figure 3. If Y_0' is not essential, we obtain an essential surface simpler than Y_0 by compressing Y_0'. Since this Y_0' is essential.

Suppose that S intersects two components of the window of $L \setminus\!\setminus Y_0'$ again. Then one of these two components is contained in $p(J')$. We can perform the above construction again and remove a part of Y_0' which is contained in the boundary of the window. Since no I-bundle can intersect S essentially along both components of the boundary by Lemma 4.2, the part of the obtained surface in $p(J')$ is not contained in the boundary of the component of the window which intersects S and

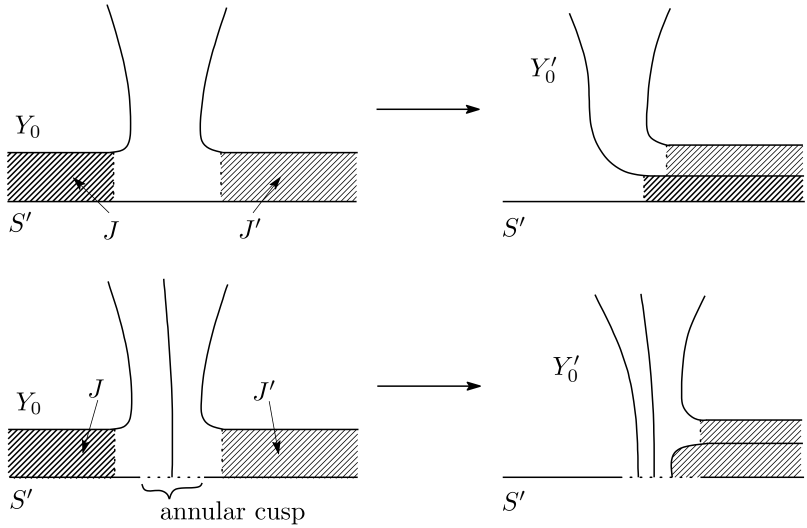

Figure 4. Constructions in the case where S intersects 2 components of the window whose Euler characteristics are -1.

lies on the same side as $p(J)$. Hence the above construction can be performed only finitely many times.

Let Y_1 be the essential surface obtained by performing the above construction as many times as possible. The Euler characteristic of the intersection of S and the window of $L \setminus\setminus Y_1$ equals -1. Therefore the Euler characteristic of the intersection of S and $\mathrm{Guts}(L \setminus\setminus Y_1)$ is equal to -1. In particular, $\mathrm{Guts}(L \setminus\setminus Y_1) \neq \varnothing$.

We will find an essential surface Y_2 such that $\chi(\partial\,\mathrm{Guts}(L \setminus\setminus Y_2)) \leq -4$. If $\chi(\partial\,\mathrm{Guts}(L \setminus\setminus Y_1)) \leq -4$, put $Y_2 = Y_1$. Suppose that $\chi(\partial\,\mathrm{Guts}(L \setminus\setminus Y_1)) = -2$. Since the Euler characteristic of $\partial\,\mathrm{Guts}(L \setminus\setminus Y_1) - S'$ is equal to -1, it is either a 1-punctured torus or a 3-punctured sphere.

Suppose that $\partial\,\mathrm{Guts}(L \setminus\setminus Y_1) - S'$ is a 1-punctured torus. Then $\partial\,\mathrm{Guts}(L \setminus\setminus Y_1) - S'$ can contain a pared annulus, and $Y_1 \cap \partial\,\mathrm{Guts}(L \setminus\setminus Y_1)$ is a 1-punctured torus or a 3-punctured sphere. If $Y_1 \cap \partial\,\mathrm{Guts}(L \setminus\setminus Y_1)$ is a 1-punctured torus, let Y_1' be the surface which is the union of $Y_1 - (Y_1 \cap \partial\,\mathrm{Guts}(L \setminus\setminus Y_1))$ and a surface in $p(J')$ (Figure 5). If $Y_1 \cap \partial\,\mathrm{Guts}(L \setminus\setminus Y_1)$ is a 3-punctured sphere, we obtain the surface $\tilde{Y}_1$ by modifying Y_1 around the pared annulus in $\partial\,\mathrm{Guts}(L \setminus\setminus Y_1) - S'$ (Figure 6). Here $\tilde{Y}_1 \cap \partial\,\mathrm{Guts}(L \setminus\setminus Y_1)$ is a 1-punctured torus. Thus we obtain an essential surface Y_1' as the union of $\tilde{Y}_1 - (\tilde{Y}_1 \cap \partial\,\mathrm{Guts}(L \setminus\setminus Y_1))$ and a surface in $p(J')$ (Figure 5).

Suppose that $\partial\,\mathrm{Guts}(L \setminus\setminus Y_1) - S'$ is a 3-punctured sphere. $\partial\,\mathrm{Guts}(L \setminus\setminus Y_1) - S'$ does not contain a pared annulus. Let Y_1' be the surface which is the union of $Y_1 - (Y_1 \cap \partial\,\mathrm{Guts}(L \setminus\setminus Y_1))$ and a surface in $p(J')$.

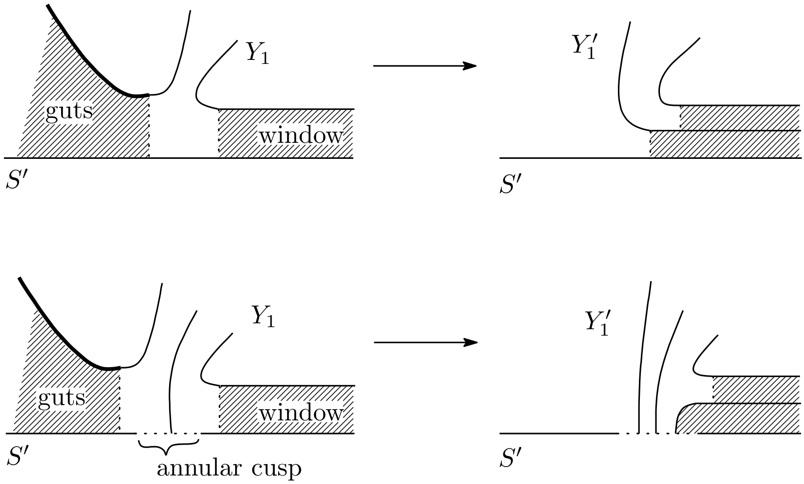

Figure 5. Constructions in the case where $\partial\,\mathrm{Guts}(L \setminus\!\setminus Y_1) = -2$.

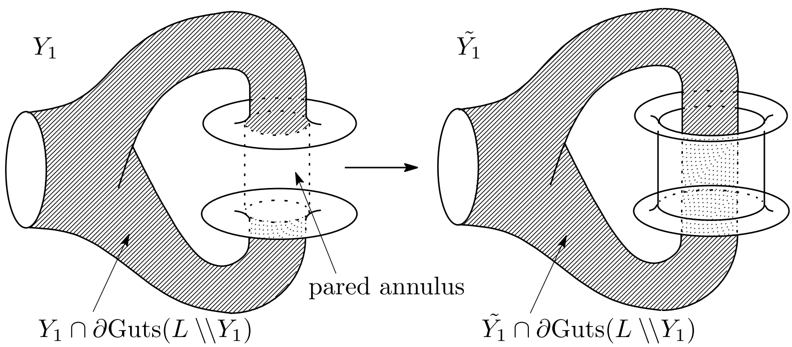

Figure 6. A construction around a pared annulus in the boundary of $\mathrm{Guts}(L \setminus\!\setminus Y_1) - S'$.

We have obtained a surface Y_1' in these ways. Then, in general, we have $[Y_1', \partial Y_1'] \neq [Y_1, \partial Y_1] \in H_2(L, \partial L; \mathbb{Z})$, but $[\partial Y_1'] = [\partial Y_1] \neq 0 \in H_1(\partial L; \mathbb{Z})$. Since $|\chi(Y_1)| = |\chi(Y_0)|$, Y_1 is essential.

Since $Y_1 \cap \partial\,\mathrm{Guts}(L \setminus\!\setminus Y_1)$ cannot be contained in the window of $L \setminus\!\setminus Y_1'$, it follows that $S \cap \partial\,\mathrm{Guts}(L \setminus\!\setminus Y_1)$ is not contained in the window of $L \setminus\!\setminus Y_1'$. Hence we can consider that $\mathrm{Guts}(L \setminus\!\setminus Y_1')$ contains $S \cap \partial\,\mathrm{Guts}(L \setminus\!\setminus Y_1)$. Since Y_1 is essential, the added surface in the window is not contained in $Y_1' \cap \partial\,\mathrm{Guts}(L \setminus\!\setminus Y_1')$. Hence the above construction can be performed only finitely many times.

Let Y_2 be the essential surface obtained by performing the above construction as many times as possible. Then $\chi(\partial\,\mathrm{Guts}(L \setminus\setminus Y_2))$ is no longer equal to -2, and $\chi(\partial\,\mathrm{Guts}(L \setminus\setminus Y_2)) \leq -4$.

(ii) Suppose that S intersects no component of the window of $L \setminus\setminus Y_0$. Then $\chi(\mathrm{Guts}(L \setminus\setminus Y_0) \cap S) = -2$. Assume that $\chi(\partial\,\mathrm{Guts}(L \setminus\setminus Y_0)) = -2$. $\partial\,\mathrm{Guts}(L \setminus\setminus Y_0)$ is a closed surface which is the union of a surface in S and annuli. Since L is atoroidal, $\partial\,\mathrm{Guts}(L \setminus\setminus Y_0)$ contains the closed surface S'. Hence $\partial\,\mathrm{Guts}(L \setminus\setminus Y_0)$ consists of S' and some torus cusps of L. The connectivity of L implies that $L = \mathrm{Guts}(L \setminus\setminus Y_0)$. It contradicts that Y_0 is nonempty. Therefore $\chi(\partial\,\mathrm{Guts}(L \setminus\setminus Y_0)) \leq -4$. □

We prove the essential part of Theorem 1.2.

Theorem 4.3. *Let M be an orientable hyperbolic manifold with* 4 *cusps. Then* $\mathrm{vol}\, M \geq 2V_8$. *Moreover, if* $\mathrm{vol}\, M = 2V_8$, *M is obtained from two ideal regular octahedra by gluing along the faces.*

Proof. It is sufficient to find an essential surface $X \subset M$ such that $\chi(\partial\,\mathrm{Guts}(X)) \leq -4$. Then Theorem 4.3 follows from Theorem 2.5.

Let $T_1, \dots, T_4$ be the cusps of M. We take an essential surface X_0 such that $X_0 \cap T_1 \neq \varnothing$ and $X_0 \cap T_i = \varnothing (2 \leq i \leq 4)$ by Theorem 2.11. We perform annular compressions for X_0 as many times as possible to obtain an essential surface X_1. When annular compression is performed, the number of boundary components of the surface increases and its Euler characteristic does not change. Since the Euler characteristic of each component of the obtained essential surface is negative, annular compressions can be performed only finitely many times.

We will show that $\mathrm{Guts}(X_1)$ intersects $T_2, \dots, T_4$. Let k be the number of cusps intersecting $X_1 (1 \leq k \leq 4)$. Let $T_1, \dots, T_k$ be the cusps intersecting X_1. Let $A_2, \dots, A_k$ be the annuli in $T_2 \setminus\setminus \partial X_1, \dots, T_k \setminus\setminus \partial X_1$ created by the last annular compressions to $T_2, \dots, T_k$. Since there are no compressing annuli any more, Lemma 2.10 implies that $A_2, \dots, A_k$ are not contained in a solid torus component of the JSJ decomposition of $M \setminus\setminus X_1$ and $T_{k+1}, \dots, T_4$ are not contained in a $T^2 \times I$ component of it. Since compressing annuli to different cusps can be taken disjointly, we may change the order of annular compressions to different cusps. By Lemma 2.9, $A_2, \dots, A_k$ are not contained in the window of $M \setminus\setminus X_1$. Therefore $A_2, \dots, A_k, T_{k+1}, \dots, T_4 \subset \partial\,\mathrm{Guts}(X_1)$.

If $\mathrm{Guts}(X_1)$ is not connected, then $\chi(\partial\,\mathrm{Guts}(X_1)) \leq -4$ as desired. Suppose that $\mathrm{Guts}(X_1)$ is connected. Then $A_2, \dots, A_k, T_{k+1}, \dots, T_4$ are contained in one component N of $M \setminus\setminus X_1$. We will find an essential surface X_2 such that $\partial\,\mathrm{Guts}(X_2)$ contains at least 4 pared components.

(i) Suppose that $(T_1 \setminus\setminus \partial X_1) \cap N \neq \varnothing$. If $N = \mathrm{Guts}(X_1)$, let A_1 be an annulus which is a component of $(T_1 \setminus\setminus \partial X_1) \cap N$. Otherwise let A_1 be an separating annulus of the JSJ decomposition intersecting $\mathrm{Guts}(X_1)$. In either case, A_1 is a pared annulus of $\mathrm{Guts}(X_1)$ different from $A_2, \dots, A_k$. Then it is sufficient to put $X_2 = X_1$.

(ii) Suppose that $(T_1 \setminus\setminus \partial X_1) \cap N = \varnothing$. Let $X_1' = X_1 \cap p(N)$, where $p\colon M \setminus\setminus X_1 \to M$ is the natural projection. Then X_1' is an essential surface in M and $T_1 \cap X_1' = \varnothing$. X_1' is the union of the components of X_1 intersecting N. If we cannot perform an annular compression for X_1' to T_1, $\mathrm{Guts}(X_1')$ contains a neighborhood of T_1 which is in the complement of N. Since $\mathrm{Guts}(X_1')$ is not connected, $\chi(\partial\, \mathrm{Guts}(X_1')) \le -4$. Then it is sufficient to put $X_2 = X_1'$.

If we can perform an annular compression for X_1' to T_1, we obtain X_2 by performing annular compressions to T_1 as many times as possible. Let A_1 be the innermost annulus in T_1. Since X_1 is obtained by performing annular compressions as many times as possible, there is no compressing annulus for X_1' to $A_2, \dots, A_k, T_{k+1}, \dots, T_4$ in $p(N)$. Hence there is no compressing annulus for X_2 to $A_2, \dots, A_k, T_{k+1}, \dots, T_4$ in $p(N)$. Since the surface which is obtained by filling X_1' with $A_2, \dots, A_k$ consists of components of a surface in the process of the annular compression from X_0 to X_1, it is essential. This implies that $A_1, \dots, A_k$ are not contained in the window of $M \setminus\setminus X_2$ by Lemma 2.9. Therefore $A_1, \dots, A_k, T_{k+1}, \dots, T_4 \subset \mathrm{Guts}(X_2)$.

Finally, we will find an essential surface X such that $\chi(\partial\, \mathrm{Guts}(X)) \le -4$. If $k = 4$, the 4 annuli $A_1, \dots, A_4$ are disjoint and not homotopic to each other in the nontorus components of $\partial\, \mathrm{Guts}(X_2)$. This implies that $\chi(\partial\, \mathrm{Guts}(X_2)) \le -4$. Then it is sufficient to put $X = X_2$.

If $1 \le k \le 3$, $\mathrm{vol}\, \mathrm{Guts}(X_2) \ge 2V_8$ by Theorem 4.1. Therefore $\mathrm{vol}\, M \ge 2V_8$ by Theorem 2.5. But we need to find X in order to prove that M is obtained from 2 octahedra when the equality holds. Lemma 3.2 and Theorem 4.1 imply that there is an essential surface Y in $\mathrm{Guts}(X_2)$ such that $\chi(\partial\, \mathrm{Guts}(\mathrm{Guts}(X_2) \setminus\setminus Y)) \le -4$. Then Y intersects some of $A_1, \dots, A_k, T_{k+1}, \dots, T_4$, where $A_2, \dots, A_k, T_{k+1}, \dots, T_4$ are contained in ∂M. If A_1 is contained in ∂M or does not intersect Y, $X_3 \cup Y$ is properly embedded in M. Since $\mathrm{Guts}(X_2 \cup Y) = \mathrm{Guts}(\mathrm{Guts}(X_2) \setminus\setminus Y)$, we obtain $\chi(\partial\, \mathrm{Guts}(X_2 \cup Y)) \le -4$. Then it is sufficient to let $X = X_2 \cup Y$. If A_1 is contained in the interior of M and intersects Y, $X_3 \cup Y$ is not properly embedded in M. Suppose that $A_1 \cap Y$ is the union of l simple closed curves. Let X be the union of 2 surfaces parallel to Y, $X_2 \cap \partial\, \mathrm{Guts}(X_2)$ and $l+1$ times of $X_2 - \partial\, \mathrm{Guts}(X_2)$ (Figure 7). Since $\mathrm{Guts}(X)$ is homeomorphic to $\mathrm{Guts}(\mathrm{Guts}(X_2) \setminus\setminus Y)$, $\chi(\partial\, \mathrm{Guts}(X)) \le -4$. □

5. Realization of hyperbolic manifold

In this section we will prove that an orientable hyperbolic 3-manifold obtained from 2 ideal regular octahedra by gluing along the faces is homeomorphic to the complement of the 8^4_2 link. This completes the proof of Theorem 1.2.

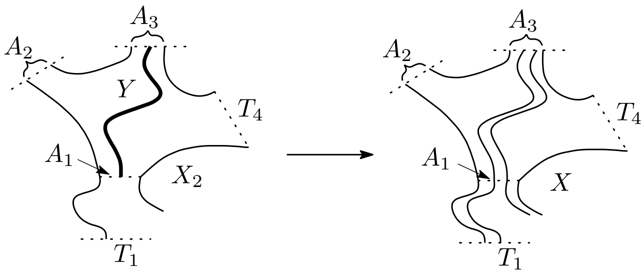

Figure 7. A construction of an essential surface the boundary of whose guts is no more than -4.

Thurston [1980, Chapter 6, Example 6.8.6] calculated the volume of the complement of the 8^4_2 link to be $2V_8$. SnapPy [Culler and Dunfield unskip] has the list of the orientable hyperbolic 3-manifolds obtained from 8 ideal regular tetrahedra by gluing along the faces. These imply the uniqueness of the minimal volume orientable hyperbolic 3-manifold with 4 cusps, but we prove it here by an elementary argument examining the possible ways of gluing along the faces of 2 octahedra.

The 12 vertices of the 2 octahedra correspond to the 4 cusps of the hyperbolic manifold. We look at the number of vertices corresponding to each cusp. Since the edge angles of an ideal regular octahedron are right angles, 4 edges of the 2 octahedra should be glued together.

Claim 5.1. *If there is a cusp consisting of one vertex x, the faces around x are glued as in Figure 8*(a)*. If there is a cusp consisting of* 2 *vertices a and b, the faces around a and b are glued as in Figure 8*(b)*.*

Proof. If there is a cusp consisting of one vertex x, the 4 edges around x are glued together, and each face around x is glued with the opposite face.

Suppose there is a cusp consisting of 2 vertices a and b. Assume that a and b are contained in one octahedron. If a and b are adjacent, no edges can be glued

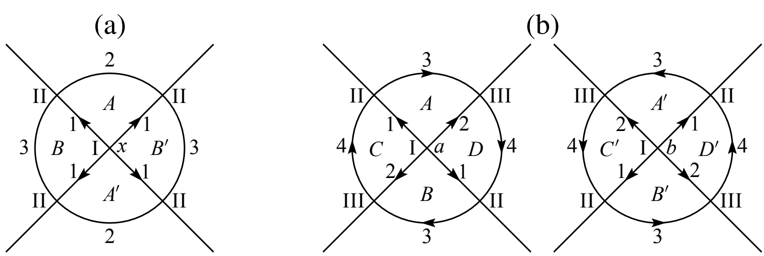

Figure 8. Face gluings for a one-vertex cusp (a) and a two-vertex cusp (b). In each case, edges with the same number are to be identified, and likewise with vertices. Unprimed faces are to be identified with their primed counterparts.

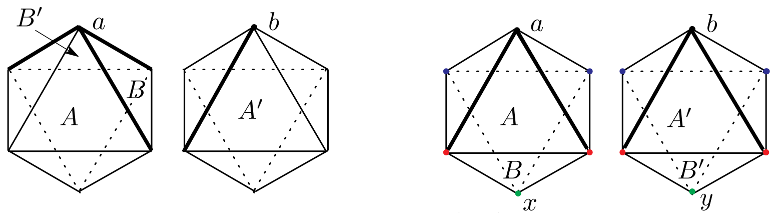

Figure 9. An impossible example in the case where two vertices form a cusp. Face A is glued to A' and B to B'.

with the edge between a and b. If b is opposite to a, we can glue no pairs of faces which are contained in different octahedra. This contradicts the connectivity. Hence a and b are contained in different octahedra.

We consider how the 8 edges around a are b are glued. Since a and b are glued, the 4 edges around a cannot be glued together. If 3 edges around a and one edge around b are glued together, 2 adjacent faces around a are glued (the left of Figure 9). Then the edge between the 2 faces can be glued with no edges. Hence 2 edges around a and 2 edges around b are glued together. Assume that adjacent edges around a are glued. Let x and y be the vertices opposite to a and b, respectively. If x and y form 2 cusps with themselves, there are 2 edges glued with no other edges. Since there are 4 cusps, there is a cusp consisting of x and y. There are 2 edges glued with no other edges even in this case (the right of Figure 9). Therefore opposite edges around a are glued and the way of gluing is determined. □

Claim 5.2. *There is no cusp consisting of* 3 *vertices.*

Proof. Assume that there is a cusp consisting of 3 vertices a, b and c. If a, b and c are vertices of one octahedron, 2 positions are possible (the left of Figure 10). If a, b and c are the vertices of one face, this face cannot be glued with another face. Otherwise, at least one of a, b and c is contained in a face of the octahedron containing a, b and c. This implies that no pair of faces of different octahedra can be glued. Hence a, b and c are not contained in one octahedron. We assume that b and c are contained in one octahedron without loss of generality. Then 2 positions are possible (the right of Figure 10). If b and c are adjacent, no edges can be glued with the edge between b and c. Hence c is opposite to b. Let x be the vertex opposite to a. Since only the 4 faces around x do not contain a, b or c, the 4 faces cannot be glued with any faces of the other octahedron.

Assume that x does not form a cusp with itself. Suppose that adjacent faces around x are glued. Then the 5 vertices except a of the octahedron containing a are glued together. There are 2 vertices y and z which form 2 cusps with themselves on the octahedron containing b and c. The 4 vertices around y are glued together

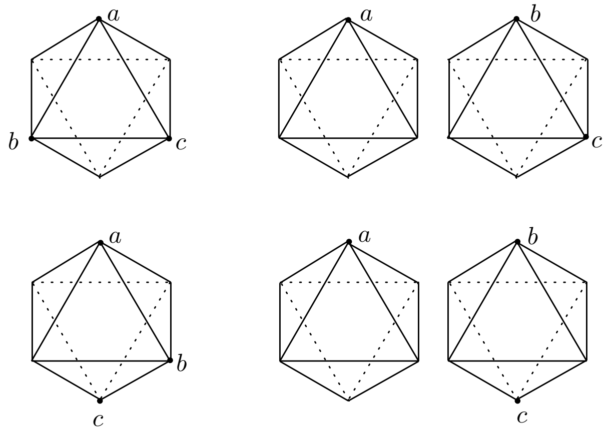

Figure 10. Positions of three vertices.

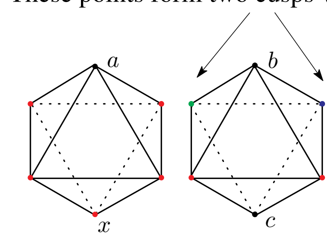

Figure 11. Five vertices of an octahedron cannot be glued together.

by Claim 5.1 (Figure 11). This contradicts that b is glued only with a and c. Hence opposite faces around x are glued.

Suppose that opposite faces A and B around x are glued with a twist, that is, the 2 vertices corresponding with x in A and B are not glued. Then 2 opposite vertices on the octahedron containing a are glued with x. Since the 5 vertices except a of the octahedron containing a cannot be glued together, the other faces C and D around x are glued with a twist. The 4 faces around a are glued with faces of the other octahedron because of the correspondence of the vertices and the fact that adjacent faces around a cannot be glued. Hence there is a vertex which forms a cusp with itself on the octahedron containing b and c (Figure 12). This contradicts that b is glued only with a and c.

Hence x forms a cusp with itself. Since 4 edges are glued together, the faces around a are glued with faces of the other octahedron. At least 3 vertices of the octahedron containing b and c are glued with the 4 vertices except a and x. Since we must obtain 4 cusps, there is a vertex that forms a cusp with itself. This is a contradiction. □

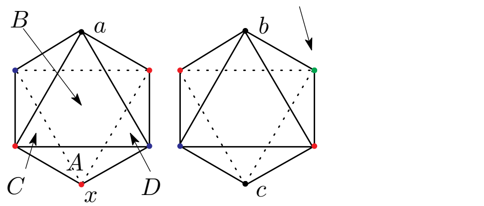

Figure 12. Opposite faces cannot be glued with a twist.

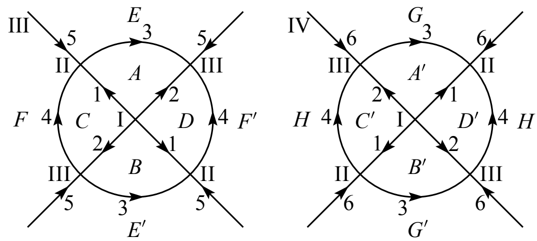

Figure 13. First gluing of the octahedra.

Claim 5.2 implies that there is a cusp consisting of one or 2 vertices. Suppose that there is a cusp consisting of one vertex x. The 4 vertices around x are glued together. The 4 faces A, B, C and D around the vertex a opposite to x are glued with faces of the other octahedron. Vertex a is glued with only one vertex b because of Claim 5.2 and the fact that 7 vertices are glued. Since the 8 vertices around a and b are glued together, the vertex y opposite to b forms a cusp with itself. The numbers of the vertices corresponding to the cusps are 1, 1, 2 and 8. By Claim 5.1 the way of gluing is determined as in Figure 13.

Suppose that there is no cusp consisting of one vertex. Then there is a cusp consisting of 2 vertices a and b. A, A', B, B', C, C', D and D' around a and b are glued as Figure 8. Since no cusp consists of one vertex, the 2 vertices x and y opposite to a and b, respectively, are glued together. The numbers of the vertices corresponding to the cusps are 2, 2, 4 and 4. The face E adjacent to A is glued with the face E' adjacent to B' because of the correspondence of the vertices and edges. The way of gluing is determined as in Figure 14.

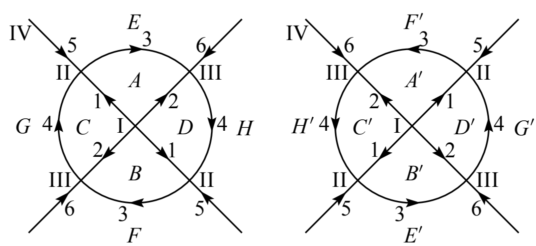

Figure 14. Second gluing of the octahedra.

Both gluings of the octahedra give homeomorphic spaces by Figure 15 and they are the 8^4_2 link complements by Figure 16.

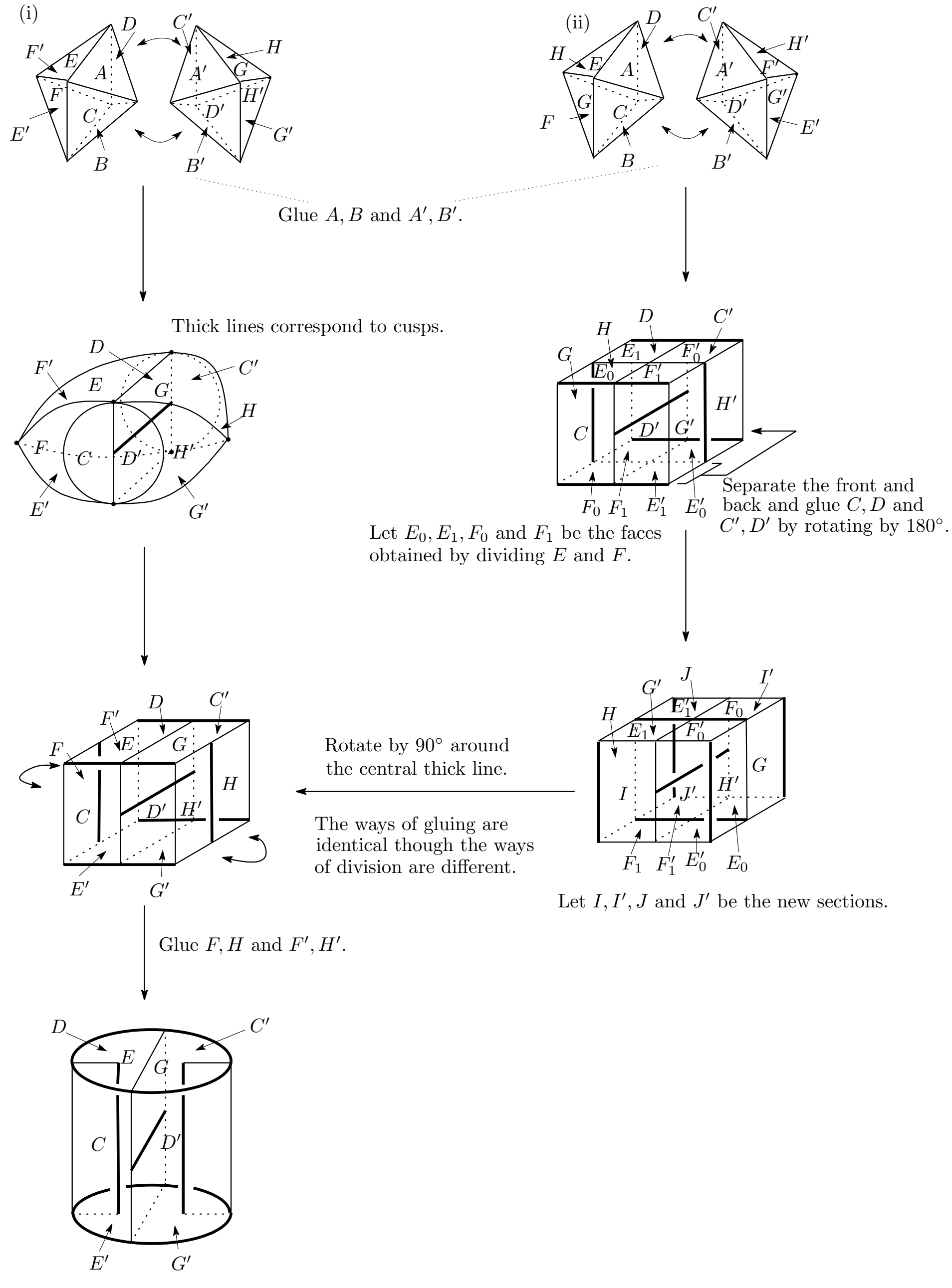

Figure 15. The gluings from Figures 13 and 14 (replicated in diagrams (i) and (ii) at the top) lead to homeomorphic spaces.

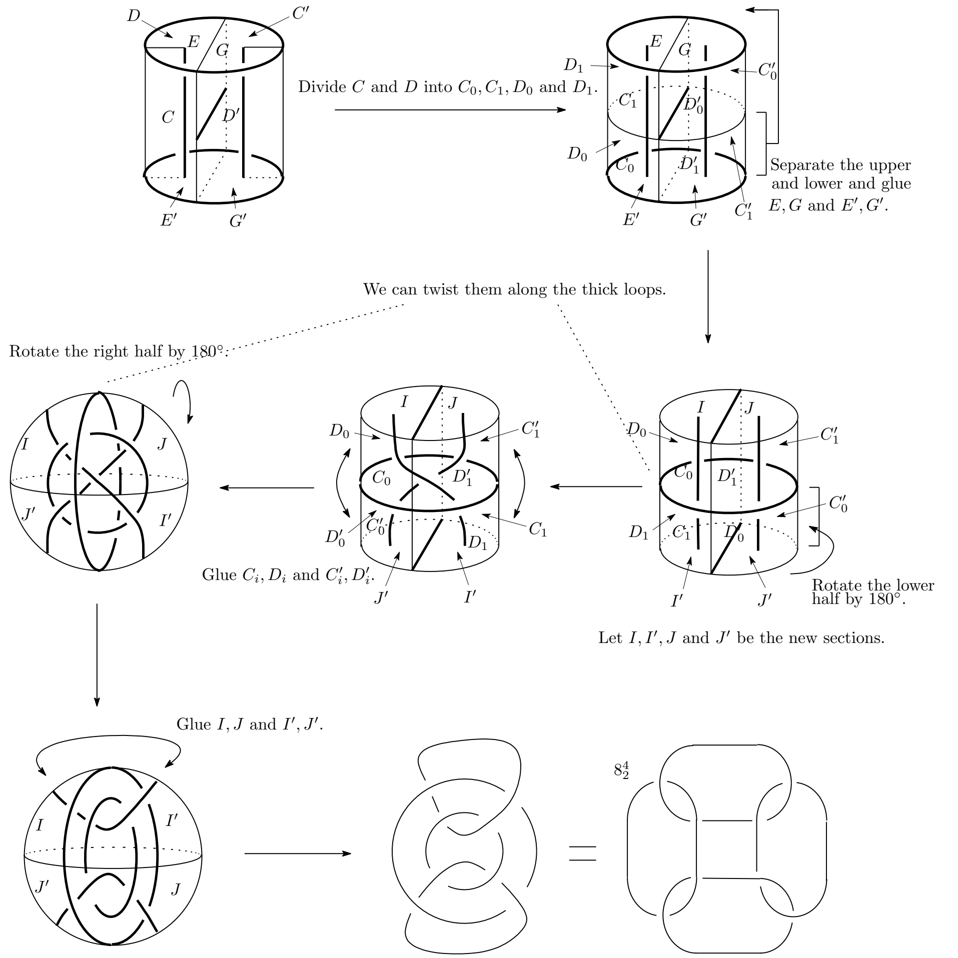

Figure 16. The space obtained in Figure 15 is the complement of the 8^4_2 link.

Acknowledgement

The author would like to express his gratitude to Takashi Tsuboi for helpful guidance and advice.

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Received September 7, 2012. Revised January 24, 2013.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 266, No. 2, 2013
dx.doi.org/10.2140/pjm.2013.266.477

ON THE WITTEN RIGIDITY THEOREM FOR STRING^c MANIFOLDS

JIANQING YU AND BO LIU

We establish family rigidity and vanishing theorems on the equivariant K-theory level for the Witten type operators on string^c manifolds introduced by Chen, Han, and Zhang.

1. Introduction

Witten [1988] derived a series of elliptic operators on the free loop space $\mathscr{L}M$ of a spin manifold M. In particular, the index of the formal signature operator on the loop space turns out to be exactly the elliptic genus constructed by Landweber and Stong [1988] and Ochanine [1987] in a topological way. Motivated by physics, Witten proposed that these elliptic operators should be rigid with respect to the circle action.

This claim of Witten was first proved by Taubes and Bott [Taubes 1989; Bott and Taubes 1989]. See also [Hirzebruch 1988; Krichever 1990] for other interesting cases. Using the modular invariance property, Kefeng Liu [1995; 1996] presented a simple and unified proof of the above result as well as various further generalizations. In particular, Liu established several new vanishing theorems.

Chen, Han, and Zhang [Chen et al. 2011] introduced a topological condition which they called the string^c condition for even-dimensional spin^c manifolds. Under this string^c condition, they constructed a Witten type genus which is the index of a Witten type operator, a linear combination of twisted spin^c Dirac operators. Furthermore, by applying Liu's method [1995; 1996], Chen, Han, and Zhang established the rigidity and vanishing theorems for this Witten type operator under the relevant anomaly cancellation condition; see [Chen et al. 2011, Theorem 3.2].

In many situations in geometry, it is rather natural and necessary to generalize the rigidity and vanishing theorems to the family case. On the equivariant Chern character level, Liu and Ma [2000; 2002] established several family rigidity and vanishing theorems. In [Liu et al. 2000; Liu et al. 2003], inspired by [Taubes 1989], Liu, Ma, and Zhang established the corresponding family rigidity and vanishing theorems on the equivariant K-theory level. As explained in [Liu et al. 2000; Liu

MSC2010: primary 58J26; secondary 58J20.
Keywords: elliptic genus, string^c manifold, K-theory.

et al. 2003], taking the Chern character might kill some torsion elements involved in the index bundle. Therefore, the rigidity and vanishing properties on the K-theory level are more subtle than those on the Chern character level.

The purpose of this paper is to establish the family rigidity and vanishing theorems on the equivariant K-theory level for the Witten type operators introduced in [Chen et al. 2011]. In fact, our main results in Theorem 2.2 may be regarded as an analogue of [Liu et al. 2000, Theorem 2.1; Liu et al. 2003, Theorems 2.1 and 2.2]. In particular, if the base manifold is a point, from our family rigidity theorem, one deduces [Chen et al. 2011, Theorem 3.2(i)]. Both the statement and the proof of Theorem 2.2 are inspired by those of [Liu et al. 2000, Theorem 2.1; Liu et al. 2003, Theorems 2.1 and 2.2], which essentially depend on the techniques developed by Taubes [1989] and Bismut and Lebeau [1991].

This paper is organized as follows. In Section 2, we state (in Theorem 2.2) and prove our main results, providing rigidity and vanishing for the family Witten type operators introduced in [Chen et al. 2011]. Section 3 is devoted to the proofs of two intermediate results, Theorems 2.8 and 2.9, which are used in the proof of Theorem 2.2.

2. Rigidity and vanishing theorems in K-theory

In this section, we establish the main results of this paper, the rigidity and vanishing theorems on the equivariant K-theory level for a family of spin^c manifolds. Such theorems hold under some anomaly cancellation assumption which is inspired by the string^c condition from [Chen et al. 2011]. For the particular case when the base manifold is a point, our results imply Theorem 3.2(i) of that reference.

This section is organized as follows. In Section 2A, we reformulate a K-theory version of the equivariant family index theorem which is proved in [Liu et al. 2003, Theorem 1.2; Liu et al. 2000, Theorem 1.1]. In Section 2B, we state our main results, the rigidity and vanishing theorems on the equivariant K-theory level for a family of spin^c manifolds. In Section 2C, we state two intermediate results on the relations between the family indices on the fixed point set, which are used to prove our main results stated in Section 2A. In Section 2D, we prove the family rigidity and vanishing theorems.

2A. ***A K-theory version of the equivariant family index theorem.*** Let M, B be two compact manifolds, and $\pi : M \to B$ a smooth fibration with compact fiber X such that $\dim X = 2l$. Let TX denote the relative tangent bundle carrying a Riemannian metric g^{TX}. We assume that TX is oriented. Let (W, h^W) be a complex Hermitian vector bundle over M.

Let (V, g^V) and $(V', g^{V'})$ be oriented real Euclidean vector bundles over M, of respective dimensions $2p$ and $2p'$. Let (L, h^L) be a complex Hermitian line bundle over M with the property that the vector bundle $U = TX \oplus V \oplus V'$ satisfies $\omega_2(U) = c_1(L) \mod 2$, where ω_2 denotes the second Stiefel–Whitney class, and c_1 denotes the first Chern class. Then the vector bundle U has a spin^c-structure. Let $S(U, L)$ be the fundamental complex spinor bundle for (U, L); see [Lawson and Michelsohn 1989, Appendix D].

Assume that there is a fiberwise S^1-action on M which lifts to V, V', L, and W, and assume the metrics g^{TX}, g^V, $g^{V'}$, h^L, and h^W are S^1-invariant. Also assume that the S^1-actions on TX, V, V', L lift to $S(U, L)$.

Let ∇^{TX} be the Levi–Civita connection on (TX, g^{TX}) along the fiber X. Let ∇^V and $\nabla^{V'}$ be S^1-invariant Euclidean connections on (V, g^V) and $(V', g^{V'})$, respectively. Let ∇^L and ∇^W be S^1-invariant Hermitian connections on (L, h^L) and (W, h^W), respectively.

The Clifford algebra bundle $C(TX)$ is the bundle of Clifford algebras over X whose fiber at $x \in X$ is the Clifford algebra $C(T_x X)$; see [Lawson and Michelsohn 1989]. Let $C(V)$ and $C(V')$ be the Clifford algebra bundles of (V, g^V) and $(V', g^{V'})$.

Let $\{e_i\}_{i=1}^{2l}$ and $\{f_j\}_{j=1}^{2p}$ be oriented orthonormal bases for (TX, g^{TX}) and (V, g^V), respectively. We denote by $c(\cdot)$ the Clifford action of $C(TX)$, $C(V)$, and $C(V')$ on $S(U, L)$. Let τ be the involution of $S(U, L)$ given by

$$\tau = (\sqrt{-1})^{l+p} c(e_1) \cdots c(e_{2l}) c(f_1) \cdots c(f_{2p}). \tag{2-1}$$

In the rest of the paper, we say that τ is the involution determined by $TX \oplus V$. We decompose $S(U, L) = S_+(U, L) \oplus S_-(U, L)$ corresponding to τ such that $\tau|_{S_\pm(U,L)} = \pm 1$. Let $\nabla^{S(U,L)}$ be the Hermitian connection on $S(U, L)$ induced by ∇^{TX}, ∇^V, $\nabla^{V'}$, and ∇^L; see [Lawson and Michelsohn 1989, Appendix D]. Then $\nabla^{S(U,L)}$ preserves the $\mathbb{Z}_2$-grading of $S(U, L)$ induced by (2-1). Let $\nabla^{S(U,L)\otimes W}$ be the Hermitian connection on $S(U, L) \otimes W$ obtained from the tensor product of $\nabla^{S(U,L)}$ and ∇^W. Let $D^X \otimes W$ be the family twisted spin^c-Dirac operator on the fiber X defined by

$$D^X \otimes W = \sum_{i=1}^{2l} c(e_i) \nabla_{e_i}^{S(U,L)\otimes W}. \tag{2-2}$$

By [Liu and Ma 2000, Proposition 1.1], the index bundle $\mathrm{Ind}_\tau(D^X \otimes W)$ over B is well-defined in the equivariant K-group $K_{S^1}(B)$. Using the same notations as in [Liu et al. 2003, (1.4)–(1.7)], we write, as an identification of virtual S^1-bundles,

$$\mathrm{Ind}_\tau(D^X \otimes W) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Ind}_\tau(D^X \otimes W, n) \otimes [n], \tag{2-3}$$

where by $[n]$ ($n \in \mathbb{Z}$) we mean the one-dimensional complex vector space on which S^1 acts as multiplication by g^n for a generator $g \in S^1$.

Let $F=\{F_\alpha\}$ be the fixed point set of the circle action on M. Then $\pi: F_\alpha\to B$ (respectively $\pi: F\to B$) is a smooth fibration with fiber Y_α (respectively Y). Let $\widetilde{\pi}: N\to F$ denote the normal bundle to F in M. Then $N=TX/TY$. We identify N as the orthogonal complement of TY in $TX|_F$. Then $TX|_F$ admits the following S^1-equivariant decomposition (see [Liu et al. 2003, (1.8)]):

$$TX|_F=\bigoplus_{v\neq 0} N_v\oplus TY, \tag{2-4}$$

where N_v is a complex vector bundle such that $g\in S^1$ acts on it by g^v with $v\in\mathbb{Z}\backslash\{0\}$. Clearly, $N=\bigoplus_{v\neq 0}N_v$. We regard N as a complex vector bundle and write $N_{\mathbb{R}}$ for the underlying real vector bundle of N. For $v\neq 0$, let $N_{v,\mathbb{R}}$ denote the underlying real vector bundle of N_v.

Similarly, let (see [Liu et al. 2003, (1.9) and (1.46)])

$$V|_F=\bigoplus_{v\neq 0} V_v\oplus V_0^{\mathbb{R}},\quad V'|_F=\bigoplus_{v\neq 0} V'_v\oplus V_0'^{\mathbb{R}},\quad W|_F=\bigoplus_{v} W_v, \tag{2-5}$$

be the S^1-equivariant decompositions of the restrictions of V, V', and W over F, respectively, where V_v, V'_v, and W_v $(v\in\mathbb{Z})$ are complex vector bundles over F on which $g\in S^1$ acts by g^v, and $V_0^{\mathbb{R}}$ and $V_0'^{\mathbb{R}}$ are the real subbundles of V and V', respectively, such that S^1 acts as identity. For $v\neq 0$, let $V_{v,\mathbb{R}}$ and $V'_{v,\mathbb{R}}$ denote the underlying real vector bundles of V_v and V'_v. Write $2p_0=\dim V_0^{\mathbb{R}}$ and $2l_0=\dim Y$.

Let us write (compare with [Liu et al. 2003, (1.47)])

$$L_F=L\otimes\Big(\bigotimes_{v\neq 0}\det N_v\otimes\bigotimes_{v\neq 0}\det V_v\otimes\bigotimes_{v\neq 0}\det V'_v\Big)^{-1}. \tag{2-6}$$

Then $TY\oplus V_0^{\mathbb{R}}\oplus V_0'^{\mathbb{R}}$ has a spin^c-structure. Let $S(TY\oplus V_0^{\mathbb{R}}\oplus V_0'^{\mathbb{R}},L_F)$ be the fundamental spinor bundle for $(TY\oplus V_0^{\mathbb{R}}\oplus V_0'^{\mathbb{R}},L_F)$. Let R be a Hermitian complex vector bundle equipped with a Hermitian connection over F. We denote by $D^Y\otimes R$ the family (twisted) spin^c Dirac operator on $S(TY\oplus V_0^{\mathbb{R}}\oplus V_0'^{\mathbb{R}},L_F)\otimes R$ defined as in (2-2) and by $D^{Y_\alpha}\otimes R$ its restriction to Y_α.

Recall that $N_{v,\mathbb{R}}$ and $V_{v,\mathbb{R}}$ are canonically oriented by their complex structures. The decompositions (2-4), (2-5) induce the orientations of TY and $V_0^{\mathbb{R}}$ respectively. Let $\{e_i\}_{i=1}^{2l_0}$, $\{f_j\}_{j=1}^{2p_0}$ be the corresponding oriented orthonormal basis of (TY,g^{TY}) and $(V_0^{\mathbb{R}},g^{V_0^{\mathbb{R}}})$. The involution of $S(TY\oplus V_0^{\mathbb{R}}\oplus V_0'^{\mathbb{R}},L_F)$ is canonically associated to that of $S(U,L)$, which we still denote by τ, which is given by

$$\tau=(\sqrt{-1})^{l_0+p_0}c(e_1)\cdots c(e_{2l_0})c(f_1)\cdots c(f_{2p_0}). \tag{2-7}$$

Let $S(TY\oplus V_0^{\mathbb{R}}\oplus V_0'^{\mathbb{R}},L_F)=S_+(TY\oplus V_0^{\mathbb{R}}\oplus V_0'^{\mathbb{R}},L_F)\oplus S_-(TY\oplus V_0^{\mathbb{R}}\oplus V_0'^{\mathbb{R}},L_F)$ be the $\mathbb{Z}_2$-grading of $S(TY\oplus V_0^{\mathbb{R}}\oplus V_0'^{\mathbb{R}},L_F)$ induced by τ.

Let $C(N_{\mathbb{R}})$ and $C(V_{v,\mathbb{R}})$ be the Clifford algebra bundle of

$$(N_{\mathbb{R}}, g^{TX}|_{N_{\mathbb{R}}}) \quad \text{and} \quad V_{v,\mathbb{R}}, g^{V}|_{V_{v,\mathbb{R}}}),$$

respectively. By [Liu et al. 2003, (1.10)], $\Lambda(\overline{N}^*)$ is a $C(N_{\mathbb{R}})$-Clifford module with the involution $\tau^N|_{\Lambda^{\text{even/odd}}(\overline{N}^*)} = \pm 1$. Similarly to [Liu et al. 2003, (1.10)], we can define the Clifford action of $C(V_{v,\mathbb{R}})$ on $\Lambda(\overline{V_v}^*)$. Then $\Lambda(\overline{V_v}^*)$ is a $C(V_{v,\mathbb{R}})$-Clifford module with the involution $\tau_v^V|_{\Lambda^{\text{even/odd}}(\overline{V_v}^*)} = \pm 1$.

By restricting to F, one has the isomorphism of $\mathbb{Z}_2$-graded $C(TX)$-Clifford modules over F as follows (compare with [Liu et al. 2003, (1.49)]):

$$\text{(2-8)} \quad (S(U, L), \tau)\big|_F \simeq \big(S(TY \oplus V_0^{\mathbb{R}} \oplus V_0'^{\mathbb{R}}, L_F), \tau\big) \mathbin{\widehat{\otimes}} (\Lambda \overline{N}^*, \tau^N) \mathbin{\widehat{\otimes}} \widehat{\bigotimes_{v\neq 0}} (\Lambda \overline{V_v}^*, \tau_v^V) \mathbin{\widehat{\otimes}} \widehat{\bigotimes_{v\neq 0}} (\Lambda \overline{V_v'}^*, \mathrm{id}),$$

where id denotes the trivial involution and $\widehat{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product (see [Lawson and Michelsohn 1989, p. 11]). Furthermore, the isomorphism (2-8) gives the identifications of the canonical connections on the bundles (compare with [Liu et al. 2003, (1.13)]).

Let S^1 act on $L|_F$ by sending $g \in S^1$ to g^{l_c} ($l_c \in \mathbb{Z}$) on F. Then l_c is locally constant on F. Following [Liu et al. 2003, (1.50)], we define the following elements in $K(F)[\![q^{1/2}]\!]$:

$$\text{(2-9)} \quad \begin{aligned} R(q) = {} & q^{\frac{1}{2}(\sum_v |v| \dim N_v - \sum_v v \dim V_v - \sum_v v \dim V_v' + l_c)} \bigotimes_{v>0} (\mathrm{Sym}_{q^v}(N_v) \otimes \det N_v) \\ & \otimes \bigotimes_{v<0} \mathrm{Sym}_{q^{-v}}(\overline{N}_v) \otimes \bigotimes_{v\neq 0} \Lambda_{-q^v}(V_v) \otimes \bigotimes_{v\neq 0} \Lambda_{q^v}(V_v') \otimes \Big(\sum_v q^v W_v\Big) \\ = {} & \sum_n R_n q^n \end{aligned}$$

and

$$\text{(2-10)} \quad \begin{aligned} R'(q) = {} & q^{1/2(-\sum_v |v| \dim N_v - \sum_v v \dim V_v - \sum_v v \dim V_v' + l_c)} \bigotimes_{v>0} \mathrm{Sym}_{q^{-v}}(\overline{N}_v) \\ & \otimes \bigotimes_{v<0} (\mathrm{Sym}_{q^v}(N_v) \otimes \det N_v) \otimes \bigotimes_{v\neq 0} \Lambda_{-q^v}(V_v) \\ & \otimes \bigotimes_{v\neq 0} \Lambda_{q^v}(V_v') \otimes \Big(\sum_v q^v W_v\Big) = \sum_n R_n' q^n. \end{aligned}$$

As explained in [Liu et al. 2003, p. 139], since $TX \oplus V \oplus V' \oplus L$ is spin, one gets

$$\text{(2-11)} \qquad \sum_v v \dim N_v + \sum_v v \dim V_v + \sum_v v \dim V_v' + l_c \equiv 0 \mod 2.$$

Therefore, $R(q)$, $R'(q) \in K(F)[\![q]\!]$.

The following theorem was essentially proved in [Liu et al. 2003, Theorem 1.2].

Theorem 2.1. *For $n \in \mathbb{Z}$, the following identity holds in $K(B)$:*

$$\operatorname{Ind}_\tau(D^X \otimes W, n) = \sum_\alpha (-1)^{\sum_{0<v} \dim N_v} \operatorname{Ind}_\tau(D^{Y_\alpha} \otimes R_n) = \sum_\alpha (-1)^{\sum_{v<0} \dim N_v} \operatorname{Ind}_\tau(D^{Y_\alpha} \otimes R'_n). \tag{2-12}$$

2B. ***Family rigidity and vanishing theorems.*** Let $\pi : M \to B$ be a fibration of compact manifolds with compact fiber X and $\dim X = 2l$. We assume that S^1 acts fiberwise on M and TX has an S^1-invariant spin^c structure. Let K_X be the S^1-equivariant complex line bundle over M which is induced by the S^1-invariant spin^c structure of TX. Let $S(TX, K_X)$ be the complex spinor bundle of (TX, K_X); see [Lawson and Michelsohn 1989, Appendix D].

Let V be an even-dimensional real vector bundle over M. We assume that V has an S^1-invariant spin structure. Let $S(V) = S^+(V) \oplus S^-(V)$ be the spinor bundle of V. Let W be an S^1-equivariant complex vector bundle over M. Let $K_W = \det(W)$ be the determinant line bundle of W.

We define the following elements in $K(M)[\![q^{1/2}]\!]$:

$$R_1(V) = \big(S^+(V) + S^-(V)\big) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(V),$$

$$R_2(V) = \big(S^+(V) - S^-(V)\big) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^n}(V), \tag{2-13}$$

$$R_3(V) = \bigotimes_{n=1}^{\infty} \Lambda_{-q^{n-1/2}}(V), \qquad R_4(V) = \bigotimes_{n=1}^{\infty} \Lambda_{q^{n-1/2}}(V),$$

$$Q_1(W) = \bigotimes_{n=0}^{\infty} \Lambda_{q^n}(\overline{W}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(W) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^{n-1/2}}(\overline{W}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^{n-1/2}}(W) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^{n-1/2}}(\overline{W}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^{n-1/2}}(W).$$

For $N \in \mathbb{Z}$, $N \geq 1$, let $y = e^{2\pi i/N} \in \mathbb{C}$. Let G_y be the multiplicative group generated by y. Following [Witten 1988], as in [Liu et al. 2000, Section 2.1], we consider the fiberwise action G_y on W and $\overline{W}$ by sending $y \in G_y$ to y on W and y^{-1} on $\overline{W}$. Then G_y acts naturally on $Q_1(W)$.

Let $H^*_{S^1}(M, \mathbb{Z}) = H^*(M \times_{S^1} ES^1, \mathbb{Z})$ denote the S^1-equivariant cohomology group of M, where ES^1 is the universal S^1-principal bundle over the classifying space BS^1 of S^1. So $H^*_{S^1}(M, \mathbb{Z})$ is a module over $H^*(BS^1, \mathbb{Z})$ induced by the projection $\bar{\pi} : M \times_{S^1} ES^1 \to BS^1$. Let $p_1(\cdot)_{S^1}$ denote the first S^1-equivariant

Pontryagin class and $\omega_2(\cdot)_{S^1}$ the second S^1-equivariant Stiefel–Whitney class. As $V \times_{S^1} ES^1$ is spin over $M \times_{S^1} ES^1$, one knows that $\frac{1}{2}p_1(V)_{S^1}$ is well-defined in $H^*_{S^1}(M, \mathbb{Z})$; see [Taubes 1989, pp. 456–457]. Recall that

$$H^*(BS^1, \mathbb{Z}) = \mathbb{Z}[\![u]\!] \tag{2-14}$$

with u a generator of degree 2.

In the following, we denote by $D^X \otimes R$ the family twisted spin^c Dirac operator acting fiberwise on $S(TX, K_X) \otimes R$. Recall that if $\mathrm{Ind}(D^X \otimes R, n)$ vanishes for $n \neq 0$, we say that $D^X \otimes R$ is rigid on the equivariant K-theory level for the S^1-action.

Now we can state the main results of this paper, which can be thought of as analogous to [Liu et al. 2000, Theorem 2.1].

Theorem 2.2. *Assume $w_2(W)_{S^1} = w_2(TX)_{S^1}$, $\frac{1}{2}p_1(V + 3W - TX)_{S^1} = e \cdot \bar{\pi}^* u^2$ ($e \in \mathbb{Z}$) in $H^*_{S^1}(M, \mathbb{Z})$, and $c_1(W) = 0 \mod N$. For $i = 1, 2, 3, 4$, consider the family of $G_y \times S^1$-equivariant twisted spin^c Dirac operators*

$$D^X \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \bigotimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX) \otimes R_i(V) \otimes Q_1(W). \tag{2-15}$$

(i) *If $e = 0$, these operators are rigid on the equivariant K-theory level for the S^1-action.*

(ii) *If $e < 0$, the index bundles of these operators are zero in $K_{G_y \times S^1}(B)$. In particular, these index bundles are zero in $K_{G_y}(B)$.*

Remark 2.3. As explained in [Liu et al. 2000, Remark 2.1], $w_2(W)_{S^1} = w_2(TX)_{S^1}$ means that $\frac{1}{2}p_1(3W - TX)_{S^1}$ is well defined and that $c_1(K_W \otimes K_X^{-1})_{S^1} = 0 \mod 2$. By [Hattori and Yoshida 1976, Corollary 1.2], the S^1-action on M can be lifted to $(K_W \otimes K_X^{-1})^{1/2}$ and is compatible with the S^1-action on $K_W \otimes K_X^{-1}$.

Take $N = 1$, that is, we forget the G_y-action on W and remove the corresponding assumption $c_1(W) = 0 \mod N$. Furthermore, take $W = K_X$ and $V = 0$. Then an interesting consequence of Theorem 2.2 is the following family rigidity and vanishing property, which may be thought of as an extension of [Liu et al. 2003, Theorem 2.3] to the spin^c case. When the base manifold is a point, it turns out to be exactly [Chen et al. 2011, Theorem 3.2(i)].

Corollary 2.4. *Assume $\frac{1}{2}p_1(3K_X - TX)_{S^1} = e \cdot \bar{\pi}^* u^2$ ($e \in \mathbb{Z}$) in $H^*_{S^1}(M, \mathbb{Z})$. Consider the family of S^1-equivariant twisted spin^c Dirac operators*

$$D^X \otimes \bigotimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX) \otimes Q_1(K_X). \tag{2-16}$$

(i) *If* $e = 0$, *these operators are rigid on the equivariant* K*-theory level for the* S^1*-action.*

(ii) *If* $e < 0$, *the index bundles of these operators are zero in* $K_{S^1}(B)$. *In particular, these index bundles are zero in* $K(B)$.

Remark 2.5. The operators in (2-16) are the Witten type operators introduced in [Chen et al. 2011]. By taking $N = 1$, $W = K_X$, $V = 0$, and letting the base manifold B be a point in [Liu et al. 2000, Theorem 2.1], we get [Chen et al. 2011, Theorem 3.2(ii)]. It is rather natural to establish an analogue of [Liu et al. 2000, Theorem 2.1], which corresponds to [Chen et al. 2011, Theorem 3.2(i)]. That is one of the motivations of Theorem 2.2.

Actually, as in [Liu et al. 2000; Liu et al. 2003], our proof of Theorem 2.2 works under the following slightly weaker hypothesis. Let us first explain some notations.

For each $n > 1$, consider $\mathbb{Z}_n \subset S^1$, the cyclic subgroup of order n. We have the $\mathbb{Z}_n$-equivariant cohomology of M defined by

$$H^*_{\mathbb{Z}_n}(M, \mathbb{Z}) = H^*(M \times_{\mathbb{Z}_n} ES^1, \mathbb{Z}),$$

and there is a natural "forgetful" map

$$\alpha(S^1, \mathbb{Z}_n) : M \times_{\mathbb{Z}_n} ES^1 \to M \times_{S^1} ES^1$$

which induces a pullback

$$\alpha(S^1, \mathbb{Z}_n)^* : H^*_{S^1}(M, \mathbb{Z}) \to H^*_{\mathbb{Z}_n}(M, \mathbb{Z}).$$

We denote by $\alpha(S^1, 1)$ the arrow which forgets the S^1-action. Thus

$$\alpha(S^1, 1)^* : H^*_{S^1}(M, \mathbb{Z}) \to H^*(M, \mathbb{Z})$$

is induced by the inclusion of M into $M \times_{S^1} ES^1$ as a fiber over BS^1.

Finally, note that if $\mathbb{Z}_n$ acts trivially on a space Y, then there is a new arrow $t^* : H^*(Y, \mathbb{Z}) \to H^*_{\mathbb{Z}_n}(Y, \mathbb{Z})$ induced by the projection $t : Y \times_{\mathbb{Z}_n} ES^1 = Y \times B\mathbb{Z}_n \to Y$.

Let $\mathbb{Z}_\infty = S^1$. For each $1 < n \le +\infty$, let $i : M(n) \to M$ be the inclusion of the fixed point set of $\mathbb{Z}_n \subset S^1$ in M, and so i induces $i_{S^1} : M(n) \times_{S^1} ES^1 \to M \times_{S^1} ES^1$.

In the rest of this paper, we suppose that there exists some integer $e \in \mathbb{Z}$ such that, for $1 < n \le +\infty$,

$$\begin{aligned}(2\text{-}17) \quad \alpha(S^1, \mathbb{Z}_n)^* \circ i^*_{S^1}\big(\tfrac{1}{2} p_1(V + 3W - TX)_{S^1} - e \cdot \bar{\pi}^* u^2\big) \\ = t^* \circ \alpha(S^1, 1)^* \circ i^*_{S^1}\big(\tfrac{1}{2} p_1(V + 3W - TX)_{S^1}\big).\end{aligned}$$

As indicated in [Liu et al. 2000, Remark 2.4], the relation (2-17) clearly follows from the hypothesis of Theorem 2.2 by pulling back and forgetting. Thus it is a weaker hypothesis.

We can now state a slightly more general version of Theorem 2.2.

Theorem 2.6. *Let the hypothesis be as in (2-17).*

(i) *If $e = 0$, the index bundles of the twisted spin^c Dirac operators in Theorem 2.2 are rigid on the equivariant K-theory level for the S^1-action.*

(ii) *If $e < 0$, the index bundles of the twisted spin^c Dirac operators in Theorem 2.2 are zero as elements in $K_{G_y \times S^1}(B)$, and, in particular, these index bundles are zero in $K_{G_y}(B)$.*

The rest of this section is devoted to a proof of Theorem 2.6.

2C. ***Two recursive formulas.*** Let $F = \{F_\alpha\}$ be the fixed point set of the circle action. Then $\pi : F \to B$ is a fibration with compact fiber denoted by $Y = \{Y_\alpha\}$.

As in [Liu et al. 2000, (2.5)], we may and we will assume that

$$TX|_F = TY \oplus \bigoplus_{v>0} N_v,$$
$$TX|_F \otimes_{\mathbb{R}} \mathbb{C} = TY \otimes_{\mathbb{R}} \mathbb{C} \oplus \bigoplus_{v>0} (N_v \oplus \bar{N}_v), \tag{2-18}$$

where N_v are complex vector bundles on which S^1 acts by sending $g \in S^1$ to g^v. We also assume that (see [Liu et al. 2000, (2.6)])

$$V|_F = V_0^{\mathbb{R}} \oplus \bigoplus_{v>0} V_v, \quad W|_F = \bigoplus_v W_v, \tag{2-19}$$

where V_v, W_v are complex vector bundles on which S^1 acts by sending g to g^v, and $V_0^{\mathbb{R}}$ is a real vector bundle on which S^1 acts as identity.

By (2-18), as in [Liu et al. 2000, (2.7)] , there is a natural isomorphism between the $\mathbb{Z}_2$-graded $C(TX)$-Clifford modules over F,

$$S(TX, K_X)|_F \simeq S\Big(TY, K_X \otimes \bigotimes_{v>0} (\det N_v)^{-1}\Big) \widehat{\otimes} \widehat{\bigotimes_{v>0}} \Lambda N_v. \tag{2-20}$$

For a complex vector bundle R over F, let $D^Y \otimes R$ and $D^{Y_\alpha} \otimes R$ be the twisted spin^c Dirac operators on $S\big(TY, K_X \bigotimes_{v>0} (\det N_v)^{-1})\big) \otimes R$ over F and F_α, respectively.

We introduce the following locally constant functions on F (see [Liu et al. 2000, (2.8)]):

$$e(N) = \sum_{v>0} v^2 \dim N_v, \quad d'(N) = \sum_{v>0} v \dim N_v,$$
$$e(V) = \sum_{v>0} v^2 \dim V_v, \quad d'(V) = \sum_{v>0} v \dim V_v, \tag{2-21}$$
$$e(W) = \sum_{v} v^2 \dim W_v, \quad d'(W) = \sum_{v} v \dim W_v.$$

As in [Liu et al. 2000, (2.9)], we write

$$L(N)=\bigotimes_{v>0}(\det N_v)^v,\quad L(V)=\bigotimes_{v>0}(\det V_v)^v,$$
$$L(W)=\bigotimes_{v\neq 0}(\det W_v)^v,\quad L=L(N)^{-1}\otimes L(V)\otimes L(W)^3. \tag{2-22}$$

By using (2-17) and computing as in [Liu et al. 2000, (2.10)–(2.11)], we know that

$$c_1(L)=0,\quad e(V)+3\cdot e(W)-e(N)=2e, \tag{2-23}$$

which means L is a trivial complex line bundle over each component F_α of F, and S^1 acts on L by sending g to g^{2e}, and G_y acts on L by sending y to $y^{3d'(W)}$. From [Liu et al. 2000, Lemma 2.1], we know that $d'(W) \bmod N$ is constant on each connected component of M. Thus we can extend L to a trivial complex line bundle over M, and we extend the S^1-action on it by sending $g\in S^1$ on the canonical section 1 of L to $g^{2e}\cdot 1$, and G_y acts on L by sending y to $y^{3d'(W)}$.

In what follows, if $R(q)=\sum_{m\in\frac12\mathbb{Z}}q^mR_m\in K_{S^1}(M)[\![q^{1/2}]\!]$, we also denote $\operatorname{Ind}(D^X\otimes R_m,h)$ by $\operatorname{Ind}(D^X\otimes R(q),m,h)$. For $i=1,2,3,4$, set

$$R_{i1}=(K_W\otimes K_X^{-1})^{1/2}\otimes R_i(V)\otimes Q_1(W). \tag{2-24}$$

As in [Liu et al. 2000, Proposition 2.1], by using Theorem 2.1, we first express the global equivariant family index via the family indices on the fixed point set.

Proposition 2.7. *For $m\in\frac12\mathbb{Z}$, $h\in\mathbb{Z}$, $1\le i\le 4$, we have the following identity in $K_{G_y}(B)$:*

$$\begin{aligned}\operatorname{Ind}\Big(D^X\otimes\bigotimes_{n=1}^{\infty}\operatorname{Sym}_{q^n}(TX)\otimes R_{i1},m,h\Big)\\ =\sum_\alpha(-1)^{\sum_{v>0}\dim N_v}\operatorname{Ind}\Big(D^{Y_\alpha}\otimes\bigotimes_{n=1}^{\infty}\operatorname{Sym}_{q^n}(TX|_F)\otimes R_{i1}\\ \otimes\operatorname{Sym}\Big(\bigoplus_{v>0}N_v\Big)\bigotimes_{v>0}\det N_v,m,h\Big).\end{aligned} \tag{2-25}$$

To simplify the notation, we use the same convention as in [Liu et al. 2000, p. 945]. For $n_0\in\mathbb{N}^*$, we define a number operator P on $K_{S^1}(M)[\![q^{1/n_0}]\!]$ in the following way: if $R(q)=\bigoplus_{n\in(1/n_0)\mathbb{Z}}R_nq^n\in K_{S^1}(M)[\![q^{1/n_0}]\!]$, P acts on $R(q)$ by multiplication by n on R_n. From now on, we simply denote $\operatorname{Sym}_{q^n}(TX)$, $\Lambda_{q^n}(V)$, and $\Lambda_{q^n}(W)$ by $\operatorname{Sym}(TX_n)$, $\Lambda(V_n)$, and $\Lambda(W_n)$. In this way, P acts on TX_n, V_n, and W_n by multiplication by n, and the action of P on $\operatorname{Sym}(TX_n)$, $\Lambda(V_n)$, and $\Lambda(W_n)$ is naturally induced by the corresponding action of P on TX_n, V_n, and W_n. So the eigenspace of $P=n$ is just given by the coefficient of q^n of the corresponding element $R(q)$. For $R(q)=\bigoplus_{n\in(1/n_0)\mathbb{Z}}R_nq^n\in K_{S^1}(M)[\![q^{1/n_0}]\!]$, we also denote $\operatorname{Ind}(D^X\otimes R_m,h)$ by $\operatorname{Ind}(D^X\otimes R(q),m,h)$.

For $p \in \mathbb{N}$, we introduce the following elements in $K_{S^1}(F)[\![q]\!]$ (see [Liu et al. 2000, (3.6)]):

$$\begin{aligned} \mathcal{F}_p(X) &= \bigotimes_{n=1}^{\infty} \mathrm{Sym}(TY_n) \otimes \bigotimes_{v>0} \Big(\bigotimes_{n=1}^{\infty} \mathrm{Sym}(N_{v,n}) \bigotimes_{n>pv} \mathrm{Sym}(\bar{N}_{v,n}) \Big), \\ \mathcal{F}'_p(X) &= \bigotimes_{v>0} \bigotimes_{0 \le n \le pv} (\mathrm{Sym}(N_{v,-n}) \otimes \det N_v), \\ \mathcal{F}^{-p}(X) &= \mathcal{F}_p(X) \otimes \mathcal{F}'_p(X). \end{aligned} \tag{2-26}$$

Then, from (2-18), over F, we have

$$\mathcal{F}^0(X) = \bigotimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX|_F) \otimes \mathrm{Sym}\Big(\bigoplus_{v>0} N_v \Big) \otimes \bigotimes_{v>0} \det N_v. \tag{2-27}$$

We now state two intermediate results on the relations between the family indices on the fixed point set. These two recursive formulas are used in the next subsection to prove Theorem 2.6.

Theorem 2.8 (compare with [Liu et al. 2000, Theorem 2.3]). *For $1 \le i \le 4$, h, $p \in \mathbb{Z}$, $p > 0$, $m \in \frac{1}{2}\mathbb{Z}$, the following identity holds in $K_{G_y}(B)$:*

$$\begin{aligned} &\sum_\alpha (-1)^{\sum_{v>0} \dim N_v} \mathrm{Ind}\big(D^{Y_\alpha} \otimes \mathcal{F}^0(X) \otimes R_{i1}, m, h\big) \\ &\quad = \sum_\alpha (-1)^{pd'(N) + \sum_{v>0} \dim N_v} \\ &\qquad\qquad \times \mathrm{Ind}\big(D^{Y_\alpha} \otimes \mathcal{F}^{-p}(X) \otimes R_{i1}, m + \tfrac{1}{2} p^2 e(N) + \tfrac{1}{2} p d'(N), h\big). \end{aligned} \tag{2-28}$$

The proof of Theorem 2.8 will be given in Sections 3B–3D.

Theorem 2.9 (compare with [Liu et al. 2000, Theorem 2.4]). *For each α, $1 \le i \le 4$, h, $p \in \mathbb{Z}$, $p > 0$, $m \in \frac{1}{2}\mathbb{Z}$, the following identity holds in $K_{G_y}(B)$:*

$$\begin{aligned} &\mathrm{Ind}\big(D^{Y_\alpha} \otimes \mathcal{F}^{-p}(X) \otimes R_{i1}, m + \tfrac{1}{2} p^2 e(N) + \tfrac{1}{2} p d'(N), h\big) \\ &\qquad = (-1)^{pd'(W)} \mathrm{Ind}(D^{Y_\alpha} \otimes \mathcal{F}^0(X) \otimes R_{i1} \otimes L^{-p}, m + ph + p^2 e, h). \end{aligned} \tag{2-29}$$

The proof of Theorem 2.9 will be given in Section 3A.

2D. *A proof of Theorem 2.6.*

Proof. As $\frac{1}{2} p_1(3W - TX)_{S^1} \in H^*_{S^1}(X, \mathbb{Z})$ is well defined, one has the same identity as in [Liu et al. 2000, (2.27)]:

$$d'(N) + d'(W) = 0 \mod 2. \tag{2-30}$$

From Proposition 2.7, Theorems 2.8 and 2.9, and (2-30), for $1 \le i \le 4$, h, $p \in \mathbb{Z}$, $p > 0$, $m \in \frac{1}{2}\mathbb{Z}$, we get the following identity (compare with [Liu et al. 2000, (2.28)]):

$$\operatorname{Ind}\Big(D^X \otimes \bigotimes_{n=1}^{\infty} \operatorname{Sym}_{q^n}(TX) \otimes R_{i1}, m, h\Big) = \operatorname{Ind}\Big(D^X \otimes \bigotimes_{n=1}^{\infty} \operatorname{Sym}_{q^n}(TX) \otimes R_{i1} \otimes L^{-p}, m', h\Big), \tag{2-31}$$

with

$$m' = m + ph + p^2 e. \tag{2-32}$$

By (2-13) and (2-24), if $m < 0$ or $m' < 0$, either side of (2-31) is identically zero, which completes the proof of Theorem 2.6. In fact:

- (i) Assume that $e = 0$. Let $h \in \mathbb{Z}$, $m_0 \in \frac{1}{2}\mathbb{Z}$, $h \neq 0$ be fixed. If $h > 0$, we take $m' = m_0$. Then, for p large enough, we get $m < 0$ in (2-31). If $h < 0$, we take $m = m_0$. Then, for p large enough, we get $m' < 0$ in (2-31).
- (ii) Assume that $e < 0$. For $h \in \mathbb{Z}$, $m_0 \in \frac{1}{2}\mathbb{Z}$, we take $m = m_0$. Then, for p large enough, we get $m' < 0$ in (2-31). □

Remark 2.10. We point out here that there is a $\mathbb{Z}/k$ version of Theorem 2.6, which is an analogue of [Liu and Yu ≥ 2013, Theorem 4.4]. In fact, by using the mod k localization formula for $\mathbb{Z}/k$ circle actions on $\mathbb{Z}/k$ spin^c manifolds established in [Liu and Yu ≥ 2013, Theorem 2.7] (see also [Zhang 2003, Theorem 2.1] for the spin case), our proof of Theorem 2.6 can be applied to the case of $\mathbb{Z}/k$ manifolds with little modification.

Remark 2.11 (compare with [Liu et al. 2000, Remark 2.5]). If M is connected, by (2-31), for $1 \leq i \leq 4$, in $K_{G_y}(B)$, we get

$$\operatorname{Ind}\Big(D^X \otimes \bigotimes_{n=1}^{\infty} \operatorname{Sym}_{q^n}(TX) \otimes R_{i1}\Big) = \operatorname{Ind}\Big(D^X \otimes \bigotimes_{n=1}^{\infty} \operatorname{Sym}_{q^n}(TX) \otimes R_{i1}\Big) \otimes [3d'(W)], \tag{2-33}$$

where by $[3d'(W)]$ we mean the one-dimensional complex vector space on which $y \in G_y$ acts by multiplication by $y^{3d'(W)}$. In particular, if B is a point and $3d'(W) \neq 0 \mod N$, we get the vanishing theorem for string^c manifolds analogue to the result of [Hirzebruch 1988, Section 10].

3. Proofs of Theorems 2.8 and 2.9

In this section, we prove the two intermediate results stated in Section 2C and used in Section 2D to prove our main results.

In Section 3A, following [Liu et al. 2000, Section 3.2], we prove Theorem 2.9. In Section 3B, we introduce the same refined shift operators as in [Liu et al. 2000, Section 4.2]. In Section 3C, we construct the twisted spin^c Dirac operator on

$M(n_j)$, the fixed point set of the naturally induced $\mathbb{Z}_{n_j}$-action on M. In Section 3D, by applying the S^1-equivariant index theorem in Section 2A, we finally prove Theorem 2.8.

3A. ***A proof of Theorem 2.9.*** We start with some notation and conventions.

Let H be the canonical basis of $\mathrm{Lie}(S^1) = \mathbb{R}$, that is,

$$\exp(tH) = \exp(2\sqrt{-1}\pi t),$$

for $t \in \mathbb{R}$. On the fixed point F, let $\boldsymbol{J}_H$ denote the operator which computes the weight of the S^1-action on $\Gamma(F, E|_F)$ for any S^1-equivariant vector bundle E over M. Then $\boldsymbol{J}_H$ can be explicitly given by (see [Liu et al. 2003, (3.2)])

$$\boldsymbol{J}_H = \frac{1}{2\pi\sqrt{-1}} \mathscr{L}_H|_{\Gamma(F,E|_F)}, \tag{3-1}$$

where $\mathscr{L}_H$ denotes the infinitesimal action of H on $\Gamma(M, E)$.

Recall that the $\mathbb{Z}_2$-grading on

$$S(TX, K_X) \otimes \bigotimes_{n=1}^{\infty} \mathrm{Sym}(TX_n)$$

is induced by the $\mathbb{Z}_2$-grading on $S(TX, K_X)$, and the $\mathbb{Z}_2$-grading on

$$S\Big(TY, K_X \otimes \bigotimes_{v>0} (\det N_v)^{-1}\Big) \otimes \mathscr{F}^{-p}(X)$$

is induced by the one on $S\big(TY, K_X \otimes \bigotimes_{v>0} (\det N_v)^{-1}\big)$. Write

$$\begin{aligned} &Q_W^1 = \bigotimes_{n=0}^{\infty} \Lambda(\overline{W}_n) \otimes \bigotimes_{n=1}^{\infty} \Lambda(W_n), \quad && Q_W^2 = \bigotimes_{n\in\mathbb{N}+\frac12} \Lambda(\overline{W}_n) \otimes \bigotimes_{n\in\mathbb{N}+\frac12} \Lambda(W_n), \\ &F_V^1 = S(V) \otimes \bigotimes_{n=1}^{\infty} \Lambda(V_n), && F_V^2 = \bigotimes_{n\in\mathbb{N}+\frac12} \Lambda(V_n). \end{aligned} \tag{3-2}$$

There are two natural $\mathbb{Z}_2$-gradings on F_V^1, F_V^2 (respectively Q_W^1, Q_W^2). The first grading is induced by the $\mathbb{Z}_2$-grading of $S(V)$ and the forms of homogeneous degrees in $\bigotimes_{n=1}^{\infty} \Lambda(V_n)$, $\bigotimes_{n\in\mathbb{N}+\frac12} \Lambda(V_n)$ (respectively Q_W^2). We define $\tau_e|_{F_V^{i\pm}} = \pm 1$ $(i = 1, 2)$ (respectively $\tau_e|_{Q_W^{2\pm}} = \pm 1$) to be the involution defined by this $\mathbb{Z}_2$-grading. The second grading is the one for which F_V^i and Q_W^i $(i = 1, 2)$ are purely even, that is, $F_V^{i+} = F_V^i$, $Q_W^{i+} = Q_W^i$. We denote by $\tau_s = \mathrm{id}$ the involution defined by this $\mathbb{Z}_2$-grading. Set $Q(W) = Q_W^1 \otimes Q_W^2 \otimes Q_W^2$. We denote by τ_1 the $\mathbb{Z}_2$-grading on $Q(W)$ defined by

$$(Q(W), \tau_1) = (Q_W^1, \tau_s) \,\widehat{\otimes}\, (Q_W^2, \tau_e) \,\widehat{\otimes}\, (Q_W^2, \tau_s). \tag{3-3}$$

Then the coefficients of q^n ($n \in \frac{1}{2}\mathbb{Z}$) in (2-13) of $R_1(V)$, $R_2(V)$, $R_3(V)$, $R_4(V)$, $Q_1(W)$ are exactly the $\mathbb{Z}_2$-graded vector subbundles of (F_V^1, τ_s), (F_V^1, τ_e), (F_V^2, τ_e), (F_V^2, τ_s), $(Q(W), \tau_1)$, respectively, on which P acts by multiplication by n.

Furthermore, we denote by τ_e (respectively τ_s) the $\mathbb{Z}_2$-grading on

$$S(TX, K_X) \otimes \bigotimes_{n=1}^{\infty} \mathrm{Sym}(TX_n) \otimes F_V^i$$

($i = 1, 2$) induced by the above $\mathbb{Z}_2$-gradings. We denote by τ_{e1} (respectively τ_{s1}) the $\mathbb{Z}_2$-grading on $S(TX, K_X) \otimes \bigotimes_{n=1}^{\infty} \mathrm{Sym}(TX_n) \otimes F_V^i \otimes Q(W)$ ($i = 1, 2$) defined by

$$\tau_{e1} = \tau_e \mathbin{\widehat{\otimes}} \tau_1, \qquad \tau_{s1} = \tau_s \mathbin{\widehat{\otimes}} \tau_1. \tag{3-4}$$

We still denote by τ_{e1} (respectively τ_{s1}) the $\mathbb{Z}_2$-grading on

$$S\Big(TY, K_X \bigotimes_{v>0} (\det N_v)^{-1}\Big) \otimes \mathscr{F}^{-p}(X) \otimes F_V^i \otimes Q(W)$$

($i = 1, 2$) which is induced as in (3-4).

By (2-19), as in (2-20), there is a natural isomorphism between the $\mathbb{Z}_2$-graded $C(V)$-Clifford modules over F,

$$S(V)|_F \simeq S\Big(V_0^{\mathbb{R}}, \bigotimes_{v>0} (\det V_v)^{-1}\Big) \otimes \widehat{\bigotimes_{v>0}} \Lambda V_v. \tag{3-5}$$

Let $V_0 = V_0^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Using (2-19) and (3-5), we rewrite (3-2) on the fixed point set F as follows:

$$\begin{aligned}
Q_W^1 &= \bigotimes_{n=0}^{\infty} \Lambda\Big(\bigoplus_v \overline{W}_{v,n}\Big) \otimes \bigotimes_{n=1}^{\infty} \Lambda\Big(\bigoplus_v W_{v,n}\Big),\\
Q_W^2 &= \bigotimes_{n\in\mathbb{N}+\frac{1}{2}} \Lambda\Big(\bigoplus_v \overline{W}_{v,n}\Big) \otimes \bigotimes_{n\in\mathbb{N}+\frac{1}{2}} \Lambda\Big(\bigoplus_v W_{v,n}\Big),\\
F_V^1 &= \bigotimes_{n=1}^{\infty} \Lambda\Big(V_{0,n} \oplus \bigoplus_{v>0} (V_{v,n} \oplus \overline{V}_{v,n})\Big) \otimes S\Big(V_0^{\mathbb{R}}, \bigotimes_{v>0} (\det V_v)^{-1}\Big) \otimes \bigotimes_{v>0} \Lambda V_{v,0},\\
F_V^2 &= \bigotimes_{n\in\mathbb{N}+\frac{1}{2}} \Lambda\Big(V_{0,n} \oplus \bigoplus_{v>0} (V_{v,n} \oplus \overline{V}_{v,n})\Big).
\end{aligned} \tag{3-6}$$

We can reformulate Theorem 2.9 as follows.

Theorem 3.1. *For each $\alpha, h, p \in \mathbb{Z}$, $p > 0$, $m \in \frac{1}{2}\mathbb{Z}$, for $i = 1, 2$, $\tau = \tau_{e1}$ or τ_{s1}, the following identity holds in $K_{G_y}(B)$:*

$$\begin{aligned}
&\mathrm{Ind}_\tau\Big(D^{Y_\alpha} \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \mathscr{F}^{-p}(X) \otimes F_V^i \otimes Q(W),\\
&\qquad\qquad m + \tfrac{1}{2}p^2 e(N) + \tfrac{1}{2}p d'(N), h\Big)\\
&= (-1)^{pd'(W)} \mathrm{Ind}_\tau\Big(D^{Y_\alpha} \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \mathscr{F}^0(X) \otimes F_V^i \otimes Q(W) \otimes L^{-p},\\
&\qquad\qquad m + ph + p^2 e, h\Big).
\end{aligned} \tag{3-7}$$

Following [Taubes 1989] in spirit, we introduce the same shift operators as in [Liu et al. 2000, (3.9)]. For $p \in \mathbb{N}$, we set

$$
\begin{aligned}
r_* : N_{v,n} &\to N_{v,n+pv}, \quad & r_* : \overline{N}_{v,n} &\to \overline{N}_{v,n-pv},\\
r_* : V_{v,n} &\to V_{v,n+pv}, \quad & r_* : \overline{V}_{v,n} &\to \overline{V}_{v,n-pv},\\
r_* : W_{v,n} &\to W_{v,n+pv}, \quad & r_* : \overline{W}_{v,n} &\to \overline{W}_{v,n-pv}.
\end{aligned}
\tag{3-8}
$$

Proposition 3.2. *For $p \in \mathbb{Z}$, $p > 0$, $i = 1, 2$, there are natural isomorphisms of vector bundles over F:*

$$
r_*(\mathscr{F}^{-p}(X)) \simeq \mathscr{F}^0(X) \otimes L(N)^p, \quad r_*(F_V^i) \simeq F_V^i \otimes L(V)^{-p}.
\tag{3-9}
$$

For any $p \in \mathbb{Z}$, $p > 0$, $i = 1, 2$, there are natural $G_y \times S^1$-equivariant isomorphisms of vector bundles over F,

$$
r_*(Q_W^i) \simeq Q_W^i \otimes L(W)^{-p}.
\tag{3-10}
$$

In particular, one gets the $G_y \times S^1$-equivariant bundle isomorphism

$$
r_*(Q(W)) \simeq Q(W) \otimes L(W)^{-3p}.
\tag{3-11}
$$

Proof. By Proposition 3.1 of [Liu et al. 2000], only the $i = 2$ case in (3-10) needs to be proved.

Using Equations (3.14)–(3.16) of the same reference, we have a natural $G_y \times S^1$-equivariant isomorphisms of vector bundles over F:

$$
\begin{aligned}
\bigotimes_{\substack{n\in\mathbb{N}+\frac{1}{2},v>0\\ 0<n<pv}} \Lambda^{i_n}(\overline{W}_{v,n-pv}) &\simeq \bigotimes_{\substack{n\in\mathbb{N}+\frac{1}{2},v>0\\ 0<n<pv}} \Lambda^{\dim W_v - i_n}(W_{v,-n+pv}) \otimes \bigotimes_{v>0} (\det \overline{W}_v)^{pv},\\
\bigotimes_{\substack{n\in\mathbb{N}+\frac{1}{2},v<0\\ 0<n<-pv}} \Lambda^{i'_n}(W_{v,n+pv}) &\simeq \bigotimes_{\substack{n\in\mathbb{N}+\frac{1}{2},v<0\\ 0<n<-pv}} \Lambda^{\dim W_v - i'_n}(\overline{W}_{v,-n-pv}) \otimes \bigotimes_{v>0} (\det W_v)^{-pv}.
\end{aligned}
\tag{3-12}
$$

From (2-22) and (3-12), we get (3-10) for the case $i = 2$. □

The following proposition, which is an analogue of [Liu et al. 2000, Proposition 3.2], is deduced from Proposition 3.2.

Proposition 3.3. *For $p \in \mathbb{Z}$, $p > 0$, $i = 1, 2$, the G_y-equivariant isomorphism of vector bundles over F induced by (3-9), (3-11), denoted by*

$$
\begin{aligned}
r_* : S\Big(TY, K_X \otimes \bigotimes_{v>0} (\det N_v)^{-1}\Big) \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \mathscr{F}^{-p}(X) \otimes F_V^i \otimes Q(W)\\
\longrightarrow S\Big(TY, K_X \otimes \bigotimes_{v>0} (\det N_v)^{-1}\Big) \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \mathscr{F}^0(X) \otimes F_V^i \otimes Q(W) \otimes L^{-p},
\end{aligned}
\tag{3-13}
$$

satisfies the identities

$$\begin{aligned} r_*^{-1}\cdot \boldsymbol{J}_H\cdot r_* &= \boldsymbol{J}_H, \\ r_*^{-1}\cdot P\cdot r_* &= P+p\boldsymbol{J}_H+p^2e-\tfrac{1}{2}p^2e(N)-\tfrac{1}{2}pd'(N). \end{aligned} \tag{3-14}$$

For the $\mathbb{Z}_2$-gradings, we have

$$r_*^{-1}\tau_e r_*=\tau_e,\quad r_*^{-1}\tau_s r_*=\tau_s,\quad r_*^{-1}\tau_1 r_*=(-1)^{pd'(W)}\tau_1. \tag{3-15}$$

Proof. By the proof of [Liu et al. 2000, Proposition 3.2], we need to compute the action of $r_*^{-1}\cdot P\cdot r_*$ on

$$\bigotimes_{\substack{n\in\mathbb{N}+\frac{1}{2},v>0\\ 0<n<pv}}\Lambda^{i_n}(\overline{W}_{v,n})\otimes\bigotimes_{\substack{n\in\mathbb{N}+\frac{1}{2},v<0\\ 0<n<-pv}}\Lambda^{i'_n}(W_{v,n}).$$

In fact, by (3-12),

$$\begin{aligned} r_*^{-1}\cdot P\cdot r_* &= \sum_{\substack{n\in\mathbb{N}+\frac{1}{2},\, v>0\\ 0<n<pv}}(\dim W_v-i_n)(-n+pv)+\sum_{\substack{n\in\mathbb{N}+\frac{1}{2},\, v<0\\ 0<n<-pv}}(\dim W_v-i'_n)(-n-pv) \\ &= P+p\boldsymbol{J}_H+\tfrac{1}{2}p^2e(W). \end{aligned} \tag{3-16}$$

By [Liu et al. 2000, (3.21)–(3.23)], (2-21)–(2-23), and (3-16), we deduce the second line of (3-14). The first line of (3-14) is obvious.

Consider the $\mathbb{Z}_2$-gradings. The first two identities of (3-15) were proved in [Liu et al. 2003, (3.18)]. τ_1 changes only on

$$\bigotimes_{\substack{n\in\mathbb{N}+\frac{1}{2},v>0\\ 0<n<pv}}\Lambda^{i_n}(\overline{W}_{v,n})\otimes\bigotimes_{\substack{n\in\mathbb{N}+\frac{1}{2},v<0\\ 0<n<-pv}}\Lambda^{i'_n}(W_{v,n}).$$

From (2-21) and (3-12), we get the third identity of (3-15). This completes the proof of Proposition 3.3. □

Theorem 3.1 is a direct consequence of Proposition 3.3. This also completes the proof of Theorem 2.9. □

The rest of this section is devoted to a proof of Theorem 2.8.

3B. ***The refined shift operators.*** We first introduce a partition of [0, 1] as in [Liu et al. 2000, pp. 942–943]. Set

$$J=\{v\in\mathbb{N}\mid \text{there exists } \alpha \text{ such that } N_v\neq 0 \text{ on } F_\alpha\}$$

and

$$\Phi=\bigl\{\beta\in(0,1]\mid \text{there exists } v\in J \text{ such that } \beta v\in\mathbb{Z}\bigr\}. \tag{3-17}$$

We order the elements in Φ so that

$$\Phi = \{\beta_i \mid 1 \le i \le J_0,\ J_0 \in \mathbb{N} \text{ and } \beta_i < \beta_{i+1}\}.$$

Then, for any integer $1 \le i \le J_0$, there exist $p_i, n_i \in \mathbb{N}$, $0 < p_i \le n_i$, with $(p_i, n_i) = 1$ such that

$$\beta_i = p_i/n_i. \tag{3-18}$$

Clearly, $\beta_{J_0} = 1$. We also set $p_0 = 0$ and $\beta_0 = 0$.

For $0 \le j \le J_0$, $p \in \mathbb{N}^*$, we write

$$\begin{aligned} I_j^p &= \Big\{(v,n) \in \mathbb{N}\times\mathbb{N} \,\Big|\, v \in J,\ (p-1)v < n \le pv,\ \frac{n}{v} = p-1+\frac{p_j}{n_j}\Big\},\\ \bar{I}_j^p &= \Big\{(v,n) \in \mathbb{N}\times\mathbb{N} \,\Big|\, v \in J,\ (p-1)v < n \le pv,\ \frac{n}{v} > p-1+\frac{p_j}{n_j}\Big\}. \end{aligned} \tag{3-19}$$

Clearly, I_0^p is the empty set. We define $\mathcal{F}_{p,j}(X)$ as in [Liu et al. 2000, (2.21)], analogously to (2-26). More specifically, we set

$$\begin{aligned} \mathcal{F}_{p,j}(X) &= \bigotimes_{n=1}^{\infty}\mathrm{Sym}(TY_n)\otimes\bigotimes_{v>0}\Big(\bigotimes_{n=1}^{\infty}\mathrm{Sym}(N_{v,n})\otimes\bigotimes_{n>(p-1)v+\frac{p_j}{n_j}v}\mathrm{Sym}(\bar{N}_{v,n})\Big)\\ &\qquad\otimes\bigotimes_{\substack{v>0\\ 0\le n\le(p-1)v+\lfloor\frac{p_j}{n_j}v\rfloor}}(\mathrm{Sym}(N_{v,-n})\otimes\det N_v)\\ &= \mathcal{F}_p(X)\otimes\mathcal{F}'_{p-1}(X)\otimes\bigotimes_{(v,n)\in\bar{I}_j^p}\mathrm{Sym}(\bar{N}_{v,n})\otimes\bigotimes_{(v,n)\in\bigcup_{i=0}^{j}I_i^p}(\mathrm{Sym}(N_{v,-n})\otimes\det N_v), \end{aligned} \tag{3-20}$$

where, for $s \in \mathbb{R}$, the notation $\lfloor s\rfloor$ denotes the greatest integer not exceeding s. Then

$$\mathcal{F}_{p,0}(X) = \mathcal{F}^{-p+1}(X), \quad \mathcal{F}_{p,J_0}(X) = \mathcal{F}^{-p}(X). \tag{3-21}$$

From the construction of β_i, we know that, for $v \in J$, there is no integer in $((p_{j-1}/n_{j-1})v, (p_j/n_j)v)$. Furthermore (see [Liu et al. 2000, (4.24)]),

$$\Big\lfloor\frac{p_{j-1}}{n_{j-1}}v\Big\rfloor = \begin{cases}\lfloor(p_j/n_j)v\rfloor - 1 & \text{if } v \equiv 0 \mod (n_j),\\ \lfloor(p_j/n_j)v\rfloor & \text{if } v \not\equiv 0 \mod (n_j).\end{cases} \tag{3-22}$$

We use the same shift operators r_{j*}, $1 \le j \le J_0$ as in [Liu et al. 2000, (4.21)], which refine the shift operator r_* defined in (3-8). For $p \in \mathbb{N}\backslash\{0\}$, set

$$\begin{aligned} r_{j*}&: N_{v,n} \to N_{v,n+(p-1)v+p_jv/n_j}, & r_{j*}&: \bar{N}_{v,n} \to \bar{N}_{v,n-(p-1)v-p_jv/n_j},\\ r_{j*}&: V_{v,n} \to V_{v,n+(p-1)v+p_jv/n_j}, & r_{j*}&: \bar{V}_{v,n} \to \bar{V}_{v,n-(p-1)v-p_jv/n_j},\\ r_{j*}&: W_{v,n} \to W_{v,n+(p-1)v+p_jv/n_j}, & r_{j*}&: \overline{W}_{v,n} \to \overline{W}_{v,n-(p-1)v-p_jv/n_j}. \end{aligned} \tag{3-23}$$

For $1 \le j \le J_0$, we define $\mathscr{F}(\beta_j)$, $F_V^1(\beta_j)$, $F_V^2(\beta_j)$, $Q_W^1(\beta_j)$, and $Q_W^2(\beta_j)$ over F as follows (compare with [Liu et al. 2000, (4.13)]):

(3-24)

$$\begin{aligned}
\mathscr{F}(\beta_j) = \bigotimes_{0<n\in\mathbb{Z}} \operatorname{Sym}(TY_n) \otimes \bigotimes_{\substack{v>0\\ v\equiv 0, n_j/2 \bmod n_j}} \bigotimes_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v} \operatorname{Sym}(N_{v,n}\oplus \overline{N}_{v,n})&\\
\otimes \bigotimes_{0<v'<n_j/2} \operatorname{Sym}\Big(\bigoplus_{v\equiv v',-v' \bmod n_j} \Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v} N_{v,n} \oplus \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v} \overline{N}_{v,n}\Big)\Big)&,
\end{aligned}$$

$$\begin{aligned}
F_V^1(\beta_j) = \Lambda\Big(\bigoplus_{0<n\in\mathbb{Z}} V_{0,n} \bigoplus_{\substack{v>0\\ v\equiv 0, n_j/2 \bmod n_j}} \Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v} V_{v,n} \oplus \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v} \overline{V}_{v,n}\Big)&\\
\bigoplus_{0<v'<n_j/2} \Big(\bigoplus_{v\equiv v',-v' \bmod n_j} \Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v} V_{v,n} \oplus \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v} \overline{V}_{v,n}\Big)\Big)\Big)&,
\end{aligned}$$

$$\begin{aligned}
F_V^2(\beta_j) = \Lambda\Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{1}{2}} V_{0,n} \bigoplus_{\substack{v>0\\ v\equiv 0, n_j/2 \bmod n_j}} \Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v+\frac{1}{2}} V_{v,n} \oplus \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v+\frac{1}{2}} \overline{V}_{v,n}\Big)&\\
\bigoplus_{0<v'<n_j/2} \Big(\bigoplus_{v\equiv v',-v' \bmod n_j} \Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v+\frac{1}{2}} V_{v,n} \oplus \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v+\frac{1}{2}} \overline{V}_{v,n}\Big)\Big)\Big)&,
\end{aligned}$$

$$Q_W^1(\beta_j) = \Lambda\Big(\bigoplus_{v} \Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v} W_{v,n} \oplus \bigoplus_{0\le n\in\mathbb{Z}-\frac{p_j}{n_j}v} \overline{W}_{v,n}\Big)\Big),$$

$$Q_W^2(\beta_j) = \Lambda\Big(\bigoplus_{v} \Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v+\frac{1}{2}} W_{v,n} \oplus \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v+\frac{1}{2}} \overline{W}_{v,n}\Big)\Big).$$

Using (3-22), Equations (3-24), and computing directly, we get an analogue of [Liu et al. 2000, Proposition 4.1] which refines Proposition 3.2:

Proposition 3.4. *For $p \in \mathbb{Z}$, $p > 0$, $1 \le j \le J_0$, there are natural isomorphisms of vector bundles over F:*

$$\begin{aligned}
r_{j*}(\mathscr{F}_{p,j-1}(X)) \simeq&\\
\mathscr{F}(\beta_j) \otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} \operatorname{Sym}(\overline{N}_{v,0}) \otimes \bigotimes_{v>0} (\det N_v)^{\lfloor \frac{p_j}{n_j}v\rfloor + (p-1)v+1} \otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} (\det N_v)^{-1}&,
\end{aligned}$$

$$r_{j*}(\mathscr{F}_{p,j}(X)) \simeq \mathscr{F}(\beta_j) \otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} \operatorname{Sym}(N_{v,0}) \otimes \bigotimes_{v>0} (\det N_v)^{\lfloor \frac{p_j}{n_j}v\rfloor + (p-1)v+1},$$

$$\begin{aligned}
r_{j*}(F_V^1) \simeq S\Big(V_0^{\mathbb{R}}, \bigotimes_{v>0} (\det V_v)^{-1}\Big) \otimes F_V^1(\beta_j)&\\
\otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} \Lambda(V_{v,0}) \otimes \bigotimes_{v>0} (\det \overline{V}_v)^{\lfloor \frac{p_j}{n_j}v\rfloor + (p-1)v}&,
\end{aligned}$$

$$r_{j*}(F_V^2) \simeq F_V^2(\beta_j) \otimes \bigotimes_{\substack{v>0 \\ v\equiv n_j/2 \bmod n_j}} \Lambda(V_{v,0}) \otimes \bigotimes_{v>0} (\det \overline{V}_v)^{\lfloor \frac{p_j}{n_j} v + \frac{1}{2} \rfloor + (p-1)v}.$$

For $p \in \mathbb{Z}$, $p > 0$, $1 \le j \le J_0$, there are natural $G_y \times S^1$-equivariant isomorphisms of vector bundles over F,

$$\text{(3-25)}\quad r_{j*}(Q_W^1) \simeq Q_W^1(\beta_j) \otimes \bigotimes_{\substack{v>0 \\ v\equiv 0 \bmod n_j}} \det W_v \otimes \bigotimes_{v>0} (\det \overline{W}_v)^{\lfloor \frac{p_j}{n_j} v \rfloor + (p-1)v+1} \otimes \bigotimes_{v<0} (\det W_v)^{\lfloor -\frac{p_j}{n_j} v \rfloor - (p-1)v},$$

$$r_{j*}(Q_W^2) \simeq Q_W^2(\beta_j) \otimes \bigotimes_{\substack{v>0 \\ v\equiv n_j/2 \bmod n_j}} \Lambda(W_{v,0}) \otimes \bigotimes_{\substack{v<0 \\ v\equiv n_j/2 \bmod n_j}} \Lambda(\overline{W}_{v,0}) \otimes \bigotimes_{v>0} (\det \overline{W}_v)^{\lfloor \frac{p_j}{n_j} v + \frac{1}{2} \rfloor + (p-1)v} \otimes \bigotimes_{v<0} (\det W_v)^{\lfloor -\frac{p_j}{n_j} v + \frac{1}{2} \rfloor - (p-1)v}.$$

Proof. By [Liu et al. 2000, Proposition 4.1], we need only prove the second isomorphism in (3-25). In fact, using [Liu et al. 2000, (3.14)], we have the natural $G_y \times S^1$-equivariant isomorphisms of vector bundles over F:

$$\text{(3-26)}\quad \bigotimes_{\substack{0<n\in\mathbb{Z}+\frac{1}{2},\, v>0 \\ n-(p-1)v-(p_j/n_j)v\le 0}} \Lambda^{i_n}(\overline{W}_{v,n-(p-1)v-(p_j/n_j)v}) \simeq \bigotimes_{v>0} (\det \overline{W}_v)^{\lfloor \frac{p_j}{n_j} v + \frac{1}{2} \rfloor + (p-1)v} \otimes \bigotimes_{\substack{0<n\in\mathbb{Z}+\frac{1}{2},\, v>0 \\ n-(p-1)v-(p_j/n_j)v\le 0}} \Lambda^{\dim W_v - i_n}(W_{v,-n+(p-1)v+(p_j/n_j)v}),$$

$$\text{(3-27)}\quad \bigotimes_{\substack{0<n\in\mathbb{Z}+\frac{1}{2},\, v<0 \\ n+(p-1)v+(p_j/n_j)v\le 0}} \Lambda^{i'_n}(W_{v,n+(p-1)v+(p_j/n_j)v}) \simeq \bigotimes_{v<0} (\det W_v)^{\lfloor -\frac{p_j}{n_j} v + \frac{1}{2} \rfloor - (p-1)v} \otimes \bigotimes_{\substack{0<n\in\mathbb{Z}+\frac{1}{2},\, v<0 \\ n+(p-1)v+(p_j/n_j)v\le 0}} \Lambda^{\dim W_v - i'_n}(\overline{W}_{v,-n-(p-1)v-(p_j/n_j)v}).$$

From the last equation in (3-24), together with (3-26) and (3-27), we get the second isomorphism in (3-25). The proof of Proposition 3.4 is complete. □

3C. ***The spin^c Dirac operators on $M(n_j)$.*** Recall that there is a nontrivial circle action on M which can be lifted to the circle actions on V and W.

For $n \in \mathbb{N}\backslash\{0\}$, let $\mathbb{Z}_n \subset S^1$ denote the cyclic subgroup of order n. Let $M(n_j)$ be the fixed point set of the induced $\mathbb{Z}_{n_j}$ action on M. Then

$$\pi : M(n_j) \to B$$

is a fibration with compact fiber $X(n_j)$. Let $N(n_j) \to M(n_j)$ be the normal bundle to $M(n_j)$ in M. As in [Bott and Taubes 1989, p. 151] (see also [Liu et al. 2000, Section 4.1; Liu et al. 2003, Section 4.1; Taubes 1989]), we see that $N(n_j)$ and V can be decomposed, as real vector bundles over $M(n_j)$, into

$$\begin{aligned} N(n_j) &= \bigoplus_{0<v<n_j/2} N(n_j)_v \oplus N(n_j)^{\mathbb{R}}_{n_j/2}, \\ V|_{M(n_j)} &= V(n_j)^{\mathbb{R}}_0 \oplus \bigoplus_{0<v<n_j/2} V(n_j)_v \oplus V(n_j)^{\mathbb{R}}_{n_j/2}, \end{aligned} \tag{3-28}$$

where $V(n_j)^{\mathbb{R}}_0$ is the real vector bundle on which $\mathbb{Z}_{n_j}$ acts by identity, and $N(n_j)^{\mathbb{R}}_{n_j/2}$ and $V(n_j)^{\mathbb{R}}_{n_j/2}$ are defined to be zero if n_j is odd. Moreover, for $0 < v < n_j/2$, $N(n_j)_v$ and $V(n_j)_v$ each admit a unique complex structure making them into complex vector bundles on which $g \in \mathbb{Z}_{n_j}$ acts by g^v. We also denote by $V(n_j)_0$, $V(n_j)_{n_j/2}$, and $N(n_j)_{n_j/2}$ the corresponding complexification of $V(n_j)^{\mathbb{R}}_0$, $V(n_j)^{\mathbb{R}}_{n_j/2}$, and $N(n_j)^{\mathbb{R}}_{n_j/2}$.

Similarly, we also have the following $\mathbb{Z}_{n_j}$-equivariant decomposition of W over $M(n_j)$ into complex vector bundles:

$$W|_{M(n_j)} = \bigoplus_{0\le v<n_j} W(n_j)_v, \tag{3-29}$$

where for $0 \le v < n_j$, $g \in \mathbb{Z}_{n_j}$ acts on $W(n_j)_v$ by sending g to g^v.

By [Liu et al. 2000, Lemma 4.1] (which generalizes [Bott and Taubes 1989, Lemmas 9.4 and 10.1] and [Taubes 1989, Lemma 5.1]), we know that the vector bundles $TX(n_j)$ and $V(n_j)^{\mathbb{R}}_0$ are orientable and even-dimensional. Thus $N(n_j)$ is orientable over $M(n_j)$. By (3-28), $V(n_j)^{\mathbb{R}}_{n_j/2}$ and $N(n_j)^{\mathbb{R}}_{n_j/2}$ are also orientable and even-dimensional. In what follows, we fix the orientations of $N(n_j)^{\mathbb{R}}_{n_j/2}$ and $V(n_j)^{\mathbb{R}}_{n_j/2}$. Then $TX(n_j)$ and $V(n_j)^{\mathbb{R}}_0$ are naturally oriented by (3-28) and the orientations of TX, V, $N(n_j)^{\mathbb{R}}_{n_j/2}$ and, $V(n_j)^{\mathbb{R}}_{n_j/2}$. Let $W(n_j)^{\mathbb{R}}_{n_j/2}$ be the underlying real vector bundle of $W(n_j)_{n_j/2}$, which are canonically oriented by its complex structure.

By (2-18), (2-19), (3-28), and (3-29), we get identifications of complex vector bundles over F (see [Liu et al. 2000, (4.9) and (4.12)]): for $0 < v \le n_j/2$,

$$\begin{aligned} N(n_j)_v|_F &= \bigoplus_{\substack{v'>0\\ v'\equiv v \bmod n_j}} N_{v'} \oplus \bigoplus_{\substack{v'>0\\ v'\equiv -v \bmod n_j}} \bar{N}_{v'}, \\ V(n_j)_v|_F &= \bigoplus_{\substack{v'>0\\ v'\equiv v \bmod n_j}} V_{v'} \oplus \bigoplus_{\substack{v'>0\\ v'\equiv -v \bmod n_j}} \bar{V}_{v'}, \end{aligned} \tag{3-30}$$

and for $0 \le v < n_j$,

$$W(n_j)_v|_F = \bigoplus_{v'\equiv v \bmod n_j} W_{v'}. \tag{3-31}$$

We also get identifications of real vector bundles over F (see [Liu et al. 2000, (4.11)]):

$$\begin{aligned} TX(n_j)|_F &= TY \oplus \bigoplus_{\substack{v>0\\ v\equiv 0 \bmod n_j}} N_v, & N(n_j)^{\mathbb{R}}_{n_j/2}|_F &= \bigoplus_{\substack{v>0\\ v\equiv n_j/2 \bmod n_j}} N_v,\\ V(n_j)^{\mathbb{R}}_0|_F &= V_0^{\mathbb{R}} \oplus \bigoplus_{\substack{v>0\\ v\equiv 0 \bmod n_j}} V_v, & V(n_j)^{\mathbb{R}}_{n_j/2}|_F &= \bigoplus_{\substack{v>0\\ v\equiv n_j/2 \bmod n_j}} V_v. \end{aligned} \tag{3-32}$$

Moreover, we have an identifications of complex vector bundles over F:

$$\begin{aligned} TX(n_j)|_F \otimes_{\mathbb{R}} \mathbb{C} &= TY \otimes_{\mathbb{R}} \mathbb{C} \oplus \bigoplus_{\substack{v>0\\ v\equiv 0 \bmod n_j}} (N_v \oplus \overline{N}_v),\\ V(n_j)_0|_F &= V_0^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \oplus \bigoplus_{\substack{v>0\\ v\equiv 0 \bmod n_j}} (V_v \oplus \overline{V}_v). \end{aligned} \tag{3-33}$$

As $(p_j, n_j) = 1$, we know that, for $v \in \mathbb{Z}$, $(p_j/n_j)v \in \mathbb{Z}$ if and only if $(v/n_j) \in \mathbb{Z}$. Also, $(p_j/n_j)v \in \mathbb{Z} + \frac{1}{2}$ if and only if $(v/n_j) \in \mathbb{Z} + \frac{1}{2}$. Also, if $v \equiv -v' \bmod n_j$, then

$$\{n \mid 0 < n \in \mathbb{Z} + (p_j/n_j)v\} = \{n \mid 0 < n \in \mathbb{Z} - (p_j/n_j)v'\}.$$

Using the identifications (3-30), (3-31), and (3-33), we can rewrite $\mathscr{F}(\beta_j)$, $F_V^1(\beta_j)$, $F_V^2(\beta_j)$, $Q_W^1(\beta_j)$, and $Q_W^2(\beta_j)$ over F defined in (3-24) as follows (compare with [Liu et al. 2000, (4.7)]):

$$\begin{aligned} \mathscr{F}(\beta_j) = \bigotimes_{0<n\in\mathbb{Z}} &\operatorname{Sym}(TX(n_j)_n)\\ &\otimes \bigotimes_{0<v<n_j/2} \operatorname{Sym}\Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v} N(n_j)_{v,n} \oplus \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v} \overline{N(n_j)_{v,n}}\Big)\\ &\otimes \bigoplus_{0<n\in\mathbb{Z}+\frac{1}{2}} \operatorname{Sym}(N(n_j)_{n_j/2,n}), \end{aligned} \tag{3-34}$$

$$\begin{aligned} F_V^1(\beta_j) = \Lambda\Big(&\bigoplus_{0<n\in\mathbb{Z}} V(n_j)_{0,n}\\ &\oplus \bigoplus_{0<v<n_j/2} \Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v} V(n_j)_{v,n} \oplus \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v} \overline{V(n_j)_{v,n}}\Big)\\ &\oplus \bigoplus_{0<n\in\mathbb{Z}+\frac{1}{2}} V(n_j)_{n_j/2,n}\Big), \end{aligned} \tag{3-35}$$

$$\begin{aligned} F_V^2(\beta_j) = \Lambda\Big(&\bigoplus_{0<n\in\mathbb{Z}} V(n_j)_{n_j/2,n}\\ &\oplus \bigoplus_{0<v<n_j/2} \Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v+\frac{1}{2}} V(n_j)_{v,n} \oplus \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v+\frac{1}{2}} \overline{V(n_j)_{v,n}}\Big)\\ &\oplus \bigoplus_{0<n\in\mathbb{Z}+\frac{1}{2}} V(n_j)_{0,n}\Big), \end{aligned} \tag{3-36}$$

$$Q_W^1(\beta_j)=\Lambda\Big(\bigoplus_{0\le v<n_j}\Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v}W(n_j)_{v,n}\oplus\bigoplus_{0\le n\in\mathbb{Z}-\frac{p_j}{n_j}v}\overline{W(n_j)}_{v,n}\Big)\Big),\tag{3-37}$$

$$Q_W^2(\beta_j)=\Lambda\Big(\bigoplus_{0\le v<n_j}\Big(\bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v+\frac{1}{2}}W(n_j)_{v,n}\oplus\bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v+\frac{1}{2}}\overline{W(n_j)}_{v,n}\Big)\Big).\tag{3-38}$$

We indicate here that $\mathscr{F}(\beta_j)$, $F_V^1(\beta_j)$, $F_V^2(\beta_j)$, $Q_W^1(\beta_j)$, and $Q_W^2(\beta_j)$ in (3-24) are the restrictions of the corresponding vector bundles in the right side of (3-34)–(3-38) over $M(n_j)$, which will still be denoted as $\mathscr{F}(\beta_j)$, $F_V^1(\beta_j)$, $F_V^2(\beta_j)$, $Q_W^1(\beta_j)$, and $Q_W^2(\beta_j)$. Write

$$Q_W(\beta_j)=Q_W^1(\beta_j)\otimes Q_W^2(\beta_j)\otimes Q_W^2(\beta_j),\tag{3-39}$$

which we now think of as a vector bundle over $M(n_j)$.

We now define the spin^c Dirac operators on $M(n_j)$. The following lemma follows from the proof of [Bott and Taubes 1989, Lemmas 11.3 and 11.4].

Lemma 3.5 (compare with [Liu et al. 2000, Lemma 4.2]). *Assume that (2-17) holds. Let*

$$L(n_j)=\bigotimes_{0<v<n_j/2}\Big(\det(N(n_j)_v)\otimes\det(\overline{V(n_j)}_v)\otimes\big(\det(\overline{W(n_j)}_v)\otimes\det(W(n_j)_{n_j-v})\big)^3\Big)^{(r(n_j)+1)v}\tag{3-40}$$

be the complex line bundle over $M(n_j)$. Then $L(n_j)$ has an n_j-th root over $M(n_j)$.

Moreover, $U_1:=TX(n_j)\oplus V(n_j)_0^{\mathbb{R}}\oplus W(n_j)_{n_j/2}^{\mathbb{R}}\oplus W(n_j)_{n_j/2}^{\mathbb{R}}$ has a spin^c structure defined by

$$L_1:=K_X\otimes\bigotimes_{0<v<n_j/2}\big(\det(N(n_j)_v)\otimes\det(\overline{V(n_j)}_v)\big)\otimes(\det(W(n_j)_{n_j/2}))^3\otimes L(n_j)^{r(n_j)/n_j},$$

and $U_2:=TX(n_j)\oplus V(n_j)_{n_j/2}^{\mathbb{R}}\oplus W(n_j)_{n_j/2}^{\mathbb{R}}\oplus W(n_j)_{n_j/2}^{\mathbb{R}}$ has a spin^c structure defined by

$$L_2:=K_X\otimes\bigotimes_{0<v<n_j/2}\det(N(n_j)_v)\otimes(\det(W(n_j)_{n_j/2}))^3\otimes L(n_j)^{r(n_j)/n_j}.$$

We remark that in order to define an S^1- or G_y- action on $L(n_j)^{r(n_j)/n_j}$, we must replace the S^1- or G_y-action by its n_j-fold action. Here, by abusing notation, we still speak of an S^1- or G_y-action without causing any confusion.

Let $S(U_1,L_1)$ and $S(U_2,L_2)$ be the fundamental complex spinor bundles for (U_1,L_1) and (U_2,L_2); see [Lawson and Michelsohn 1989, Appendix D]. There are two $\mathbb{Z}_2$-gradings on these bundles. The first grading, denoted by τ_s, is induced by the involutions on $S(U_1,L_1)$ and $S(U_2,L_2)$ determined by $TX(n_j)\oplus W(n_j)_{n_j/2}^{\mathbb{R}}$ as in (2-1). The second grading, which we denote by τ_e, is induced by the involution on $S(U_1,L_1)$ determined by $TX(n_j)\oplus V(n_j)_0^{\mathbb{R}}\oplus W(n_j)_{n_j/2}^{\mathbb{R}}$, and by the involution on $S(U_2,L_2)$ determined by $U_2=TX(n_j)\oplus V(n_j)_{n_j/2}^{\mathbb{R}}\oplus W(n_j)_{n_j/2}^{\mathbb{R}}$, as in (2-1).

In what follows, by $D^{X(n_j)}$ we mean the S^1-equivariant spin^c Dirac operator on $S(U_1, L_1)$ or $S(U_2, L_2)$ over $M(n_j)$.

Corresponding to (2-8), by (3-30) and (3-31), we define $S(U_1, L_1)'$ and $S(U_2, L_2)'$ equipped with involutions τ_s' and τ_e' as follows (compare with [Liu et al. 2000, (4.16)]):

$$(3\text{-}41)\quad (S(U_1, L_1)', \tau_s'/\tau_e') = \Big(S\Big(TY \oplus V_0^{\mathbb{R}}, L_1 \otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} (\det N_v \otimes \det V_v)^{-1} \otimes \bigotimes_{v\equiv n_j/2 \bmod n_j} (\det W_v)^{-2}\Big), \tau_s'/\tau_e'\Big) \otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} \Lambda_{\pm 1}(V_v) \otimes \bigotimes_{v\equiv n_j/2 \bmod n_j} \Lambda_{-1}(W_v) \otimes \bigotimes_{v\equiv n_j/2 \bmod n_j} \Lambda(W_v)$$

and

$$(3\text{-}42)\quad (S(U_2, L_2)', \tau_s'/\tau_e') = S\Big(TY, L_2 \otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} (\det N_v)^{-1} \otimes \bigotimes_{\substack{v>0\\ v\equiv n_j/2 \bmod n_j}} (\det V_v)^{-1} \otimes \bigotimes_{v\equiv n_j/2 \bmod n_j} (\det W_v)^{-2}\Big) \otimes \bigotimes_{\substack{v>0\\ v\equiv n_j/2 \bmod n_j}} \Lambda_{\pm 1}(V_v) \otimes \bigotimes_{v\equiv n_j/2 \bmod n_j} \Lambda_{-1}(W_v) \otimes \bigotimes_{v\equiv n_j/2 \bmod n_j} \Lambda(W_v).$$

Then, by (2-8), for $i = 1, 2$, we have the following isomorphisms of Clifford modules over F preserving the $\mathbb{Z}_2$-gradings (compare with [Liu et al. 2000, (4.17)]):

$$(3\text{-}43)\qquad (S(U_i, L_i), \tau_s/\tau_e)|_F \simeq (S(U_i, L_i)', \tau_s'/\tau_e') \otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} \Lambda_{-1}(N_v).$$

As in [Liu et al. 2000, pp. 952], we introduce formally the following complex line bundles over F:

$$(3\text{-}44)\quad L_1' = \Big(L_1^{-1} \otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} (\det N_v \otimes \det V_v) \otimes \bigotimes_{v\equiv n_j/2 \bmod n_j} (\det W_v)^2 \otimes \bigotimes_{v>0} (\det N_v \otimes \det V_v)^{-1} \otimes K_X\Big)^{1/2}$$

and

$$(3\text{-}45)\quad L_2' = \Big(L_2^{-1} \otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} \det N_v \otimes \bigotimes_{\substack{v>0\\ v\equiv n_j/2 \bmod n_j}} \det V_v \otimes \bigotimes_{v\equiv n_j/2 \bmod n_j} (\det W_v)^2 \otimes \bigotimes_{v>0} (\det N_v)^{-1} \otimes K_X\Big)^{1/2}.$$

In fact, from (2-8), Lemma 3.5, and the assumption that V is spin, one verifies easily that $c_1(L_i'^2) = 0 \mod 2$ for $i = 1, 2$, which implies that L_1' and L_2' are well-defined complex line bundles over F.

Then, by [Liu et al. 2000, (3.14)], (3-41)–(3-45), and the definitions of L_1, L_2, we get the following identifications of Clifford modules over F (compare with [Liu et al. 2000, (4.19)]):

$$
\begin{aligned}
(3\text{-}46)\quad & (S(U_1, L_1)' \otimes L_1', (\tau_s'/\tau_e') \otimes \mathrm{id}) \\
&= S\Big(TY, K_X \otimes \bigotimes_{v>0} (\det N_v)^{-1}\Big) \otimes (S(V_0^{\mathbb{R}}, \bigotimes_{v>0} (\det V_v)^{-1}), \mathrm{id}/\tau) \\
&\otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} \Lambda_{\pm 1}(V_v) \otimes \bigotimes_{\substack{v>0\\ v\equiv n_j/2 \bmod n_j}} \Lambda_{-1}(W_v) \otimes \bigotimes_{\substack{v<0\\ v\equiv n_j/2 \bmod n_j}} \Lambda_{-1}(\overline{W}_v) \\
&\otimes \bigotimes_{\substack{v>0\\ v\equiv n_j/2 \bmod n_j}} \Lambda(W_v) \otimes \bigotimes_{\substack{v<0\\ v\equiv n_j/2 \bmod n_j}} \Lambda(\overline{W}_v) \otimes \bigotimes_{\substack{v<0\\ v\equiv n_j/2 \bmod n_j}} (\det W_v)^2
\end{aligned}
$$

and

$$
\begin{aligned}
(3\text{-}47)\quad & (S(U_2, L_2)' \otimes L_2', (\tau_s'/\tau_e') \otimes \mathrm{id}) \\
&= S\Big(TY, K_X \otimes \bigotimes_{v>0} (\det N_v)^{-1}\Big) \otimes \bigotimes_{\substack{v>0\\ v\equiv n_j/2 \bmod n_j}} \Lambda_{\pm 1}(V_v) \otimes \bigotimes_{\substack{v>0\\ v\equiv n_j/2 \bmod n_j}} \Lambda_{-1}(W_v) \\
&\otimes \bigotimes_{\substack{v<0\\ v\equiv n_j/2 \bmod n_j}} \Lambda_{-1}(\overline{W}_v) \otimes \bigotimes_{\substack{v>0\\ v\equiv n_j/2 \bmod n_j}} \Lambda(W_v) \otimes \bigotimes_{\substack{v<0\\ v\equiv n_j/2 \bmod n_j}} \Lambda(\overline{W}_v) \otimes \bigotimes_{\substack{v<0\\ v\equiv n_j/2 \bmod n_j}} (\det W_v)^2.
\end{aligned}
$$

Now we compare the $\mathbb{Z}_2$-gradings in (3-46) and (3-47). Set (compare with [Liu et al. 2000, (4.20)])

$$
(3\text{-}48)\quad
\begin{aligned}
\Delta(n_j, N) &= \sum_{n_j/2 < v' < n_j} \sum_{0<v,\, v\equiv v' \bmod n_j} \dim N_v + o(N(n_j)^{\mathbb{R}}_{n_j/2}), \\
\Delta(n_j, V) &= \sum_{n_j/2 < v' < n_j} \sum_{0<v,\, v\equiv v' \bmod n_j} \dim V_v + o(V(n_j)^{\mathbb{R}}_{n_j/2}), \\
\Delta(n_j, W) &= \sum_{v<0,\, v\equiv n_j/2 \bmod n_j} \dim W_v,
\end{aligned}
$$

where $o(N(n_j)^{\mathbb{R}}_{n_j/2})$ and $o(V(n_j)^{\mathbb{R}}_{n_j/2})$ equal 0 or 1 depending on whether the given orientation on $N(n_j)^{\mathbb{R}}_{n_j/2}$ and $V(n_j)^{\mathbb{R}}_{n_j/2}$ agrees or disagrees with the complex orientation of

$$
\bigoplus_{\substack{v>0\\ v\equiv n_j/2 \mod n_j}} N_v \quad \text{and} \quad \bigoplus_{\substack{v>0\\ v\equiv n_j/2 \mod n_j}} V_v,
$$

respectively.

As explained in [Liu et al. 2003, p. 166], for the $\mathbb{Z}_2$-gradings induced by τ_s, the differences of the $\mathbb{Z}_2$-gradings of (3-46) and (3-47) are both

$$(-1)^{\Delta(n_j,N)+\Delta(n_j,W)};$$

for the $\mathbb{Z}_2$-gradings induced by τ_e, the difference of the $\mathbb{Z}_2$-gradings of (3-46) (respectively (3-47)) is

$$(-1)^{\Delta(n_j,N)+\Delta(n_j,V)+\Delta(n_j,W)}$$

(respectively $(-1)^{\Delta(n_j,N)+o(V(n_j)^{\mathbb{R}}_{n_j/2})+\Delta(n_j,W)}$).

Lemma 3.6 (compare with [Liu et al. 2000, Lemma 4.3]). *Let us write*

$$\begin{aligned} L(\beta_j)_1 = L_1' \otimes \bigotimes_{v>0} (\det N_v)^{\lfloor \frac{p_j}{n_j} v\rfloor + (p-1)v+1} \otimes \bigotimes_{v>0} (\det \overline{V}_v)^{\lfloor \frac{p_j}{n_j} v\rfloor + (p-1)v} \\ \otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} (\det N_v)^{-1} \otimes \bigotimes_{v<0} (\det W_v)^{\lfloor -\frac{p_j}{n_j} v\rfloor + 2\lfloor -\frac{p_j}{n_j} v+\frac{1}{2}\rfloor - 3(p-1)v} \\ \otimes \bigotimes_{v>0} (\det \overline{W}_v)^{\lfloor \frac{p_j}{n_j} v\rfloor + 2\lfloor \frac{p_j}{n_j} v+\frac{1}{2}\rfloor + 3(p-1)v+1} \\ \otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} \det W_v \otimes \bigotimes_{\substack{v<0\\ v\equiv n_j/2 \bmod n_j}} (\det \overline{W}_v)^2 \end{aligned}$$

and

$$\begin{aligned} L(\beta_j)_2 = L_2' \otimes \bigotimes_{v>0} (\det N_v)^{\lfloor \frac{p_j}{n_j} v\rfloor + (p-1)v+1} \otimes \bigotimes_{v>0} (\det \overline{V}_v)^{\lfloor \frac{p_j}{n_j} v+\frac{1}{2}\rfloor + (p-1)v} \\ \otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} (\det N_v)^{-1} \otimes \bigotimes_{v<0} (\det W_v)^{\lfloor -\frac{p_j}{n_j} v\rfloor + 2\lfloor -\frac{p_j}{n_j} v+\frac{1}{2}\rfloor - 3(p-1)v} \\ \otimes \bigotimes_{v>0} (\det \overline{W}_v)^{\lfloor \frac{p_j}{n_j} v\rfloor + 2\lfloor \frac{p_j}{n_j} v+\frac{1}{2}\rfloor + 3(p-1)v+1} \\ \otimes \bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}} \det W_v \otimes \bigotimes_{\substack{v<0\\ v\equiv n_j/2 \bmod n_j}} (\det \overline{W}_v)^2. \end{aligned}$$

Then $L(\beta_j)_1$ and $L(\beta_j)_2$ can be extended naturally to $G_y \times S^1$-equivariant complex line bundles over $M(n_j)$ which we will still denote by $L(\beta_j)_1$ and $L(\beta_j)_2$.

Proof. We introduce the following line bundle over $M(n_j)$:

$$\begin{aligned} L^{\omega}(\beta_j) = \bigotimes_{0<v<n_j/2} \Big(\det(N(n_j)_v) \otimes \det(\overline{V(n_j)}_v) \\ \otimes (\det(\overline{W(n_j)}_v) \otimes \det(W(n_j)_{n_j-v}))^3\Big)^{-\omega(v)-r(n_j)v}. \end{aligned} \tag{3-49}$$

where, as in [Liu et al. 2003, (4.35)], we define ω by

$$\left\lfloor \frac{p_j}{n_j} v \right\rfloor = \frac{p_j}{n_j} v - \frac{\omega(v)}{n_j}.$$

As in [Liu et al. 2003, (4.38); Liu et al. 2000, (4.28)], Lemma 3.5 implies that $L^{\omega}(\beta_j)^{1/n_j}$ is well-defined over $M(n_j)$. Direct calculation shows that

$$L(\beta_j)_1 = L^{-(p-1)-p_j/n_j} \otimes L^{\omega}(\beta_j)^{1/n_j} \otimes \bigotimes_{0<v<n_j/2} \det(\overline{W(n_j)}_v) \otimes (\det(\overline{W(n_j)}_{n_j/2}))^2 \otimes \bigotimes_{1\le m\le p_j/2} \bigotimes_{m-\frac{1}{2}<(p_j/n_j)v<m} (\det(\overline{W(n_j)}_v) \otimes \det(W(n_j)_{n_j-v}))^2$$

and

$$L(\beta_j)_2 = L^{-(p-1)-p_j/n_j} \otimes L^{\omega}(\beta_j)^{1/n_j} \otimes \bigotimes_{0<v<n_j/2} \det(\overline{W(n_j)}_v) \otimes (\det(\overline{W(n_j)}_{n_j/2}))^2 \otimes \bigotimes_{1\le m\le p_j/2} \bigotimes_{m-\frac{1}{2}<(p_j/n_j)v<m} \Big((\det(\overline{W(n_j)}_v) \otimes \det(W(n_j)_{n_j-v}))^2 \otimes \det(\overline{V(n_j)}_v)\Big).$$

The proof of Lemma 3.6 is complete. ∎

To simplify the notation, we introduce the following locally constant functions on F (compare with [Liu et al. 2003, (4.45); Liu et al. 2000, (4.30)]):

$$\begin{aligned}(3\text{-}50)\quad \varepsilon_W^1 = -\frac{1}{2}\sum_{v>0}(\dim W_v)\cdot\bigg(&\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v\Big)\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v+1\Big)\\ &-\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v\Big)\Big(2\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v\Big)+1\Big)\bigg)\\ -\frac{1}{2}\sum_{v<0}(\dim W_v)\cdot\bigg(&\Big(-\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor-(p-1)v\Big)\Big(-\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor-(p-1)v+1\Big)\\ &+\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v\Big)\Big(2\Big(-\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor-(p-1)v\Big)+1\Big)\bigg),\end{aligned}$$

$$\begin{aligned}(3\text{-}51)\quad \varepsilon_W^2 = -\frac{1}{2}\sum_{v>0}(\dim W_v)\cdot\bigg(&\Big(\Big\lfloor\frac{p_j}{n_j}v+\frac{1}{2}\Big\rfloor+(p-1)v\Big)^2\\ &-2\Big(\frac{p_j}{n_j}v+(p-1)v\Big)\Big(\Big\lfloor\frac{p_j}{n_j}v+\frac{1}{2}\Big\rfloor+(p-1)v\Big)\bigg)\\ -\frac{1}{2}\sum_{v<0}(\dim W_v)\cdot\bigg(&\Big(\Big\lfloor-\frac{p_j}{n_j}v+\frac{1}{2}\Big\rfloor-(p-1)v\Big)^2\\ &+2\Big(\frac{p_j}{n_j}v+(p-1)v\Big)\Big(\Big\lfloor-\frac{p_j}{n_j}v+\frac{1}{2}\Big\rfloor-(p-1)v\Big)\bigg),\end{aligned}$$

$$\begin{aligned}(3\text{-}52)\quad \varepsilon_1 = \frac{1}{2}\sum_{v>0}(\dim N_v-\dim V_v)\bigg(&\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v\Big)\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v+1\Big)\\ &-\Big(\frac{p_j}{n_j}v+(p-1)v\Big)\Big(2\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v\Big)+1\Big)\bigg),\end{aligned}$$

$$\begin{aligned}
\varepsilon_2 = \frac{1}{2}\sum_{v>0}(\dim N_v)\cdot\Big(&\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v\Big)\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v+1\Big)\\
&-\Big(\frac{p_j}{n_j}v+(p-1)v\Big)\Big(2\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v\Big)+1\Big)\Big)\\
&-\frac{1}{2}\sum_{v>0}(\dim V_v)\cdot\Big(\Big(\Big\lfloor\frac{p_j}{n_j}+\frac{1}{2}\Big\rfloor+(p-1)v\Big)^2\\
&-2\Big(\frac{p_j}{n_j}v+(p-1)v\Big)\Big(\Big\lfloor\frac{p_j}{n_j}+\frac{1}{2}\Big\rfloor+(p-1)v\Big)\Big).
\end{aligned}\tag{3-53}$$

As in [Liu et al. 2000, (2.23)], for $0\le j\le J_0$, we set

$$\begin{aligned}
e(p,\beta_j,N) &= \frac{1}{2}\sum_{v>0}(\dim N_v)\cdot\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v\Big)\\
&\qquad\times\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v+1\Big),\\
d'(p,\beta_j,N) &= \sum_{v>0}(\dim N_v)\cdot\Big(\Big\lfloor\frac{p_j}{n_j}v\Big\rfloor+(p-1)v\Big).
\end{aligned}\tag{3-54}$$

Then $e(p,\beta_j,N)$ and $d'(p,\beta_j,N)$ are locally constant functions on F. In particular, we have

$$\begin{aligned}
e(p,\beta_0,N) &= \tfrac{1}{2}(p-1)^2e(N)+\tfrac{1}{2}(p-1)d'(N),\\
e(p,\beta_{J_0},N) &= \tfrac{1}{2}p^2e(N)+\tfrac{1}{2}pd'(N),\\
d'(p,\beta_{J_0},N) &= d'(p+1,\beta_0,N)=pd'(N).
\end{aligned}\tag{3-55}$$

Proposition 3.7 (compare with [Liu et al. 2000, Proposition 4.2]). *For $i=1,2$, the G_y-equivariant isomorphisms of complex vector bundles over F induced by Proposition 3.4 and (3-46)–(3-47),*

$$\begin{aligned}
r_{i1}: S\Big(TY, K_X\otimes\bigotimes_{v>0}(\det N_v)^{-1}\Big)&\otimes(K_W\otimes K_X^{-1})^{1/2}\otimes\mathscr{F}_{p,j-1}(X)\otimes F_V^i\otimes Q(W)\\
\longrightarrow S(U_i,L_i)'&\otimes(K_W\otimes K_X^{-1})^{1/2}\otimes\mathscr{F}(\beta_j)\otimes F_V^i(\beta_j)\\
&\otimes Q_W(\beta_j)\otimes L(\beta_j)_i\otimes\bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}}\mathrm{Sym}(\overline{N}_{v,0})
\end{aligned}$$

and

$$\begin{aligned}
r_{i2}: S\Big(TY, K_X\otimes\bigotimes_{v>0}(\det N_v)^{-1}\Big)&\otimes(K_W\otimes K_X^{-1})^{1/2}\otimes\mathscr{F}_{p,j}(X)\otimes F_V^i\otimes Q(W)\\
\longrightarrow S(U_i,L_i)'&\otimes(K_W\otimes K_X^{-1})^{1/2}\otimes\mathscr{F}(\beta_j)\otimes F_V^i(\beta_j)\\
&\otimes Q_W(\beta_j)\otimes L(\beta_j)_i\otimes\bigotimes_{\substack{v>0\\ v\equiv 0 \bmod n_j}}(\mathrm{Sym}(N_{v,0})\otimes\det N_v)
\end{aligned}$$

have the following properties:

(i) *For $i = 1, 2$ and $\gamma = 1, 2$, we have*

$$r_{i\gamma}^{-1} \cdot \boldsymbol{J}_H \cdot r_{i\gamma} = \boldsymbol{J}_H, \quad r_{i\gamma}^{-1} \cdot P \cdot r_{i\gamma} = P + \left(\frac{p_j}{n_j} + (p-1)\right)\boldsymbol{J}_H + \varepsilon_{i\gamma}, \tag{3-56}$$

where the $\varepsilon_{i\gamma}$ are given by

$$\begin{aligned} \varepsilon_{i1} &= \varepsilon_i + \varepsilon_W^1 + 2\varepsilon_W^2 - e(p, \beta_{j-1}, N), \\ \varepsilon_{i2} &= \varepsilon_i + \varepsilon_W^1 + 2\varepsilon_W^2 - e(p, \beta_j, N). \end{aligned} \tag{3-57}$$

(ii) *For $i = 1, 2$ and $\gamma = 1, 2$, we have*

$$r_{i\gamma}^{-1}\tau_e r_{i\gamma} = (-1)^{\mu_i}\tau_e, \quad r_{i\gamma}^{-1}\tau_s r_{i\gamma} = (-1)^{\mu_3}\tau_s, \quad r_{i\gamma}^{-1}\tau_1 r_{i\gamma} = (-1)^{\mu_4}\tau_1, \tag{3-58}$$

where the μ_i are given by

$$\begin{aligned} \mu_1 &= -\sum_{v>0}(\dim V_v)\left\lfloor \frac{p_j}{n_j}v \right\rfloor + \Delta(n_j, N) + \Delta(n_j, V) + \Delta(n_j, W) \bmod 2, \\ \mu_2 &= -\sum_{v>0}(\dim V_v)\cdot\left\lfloor \frac{p_j}{n_j}v + \frac{1}{2} \right\rfloor + \Delta(n_j, N) + o(V(n_j)^{\mathbb{R}}_{n_j/2}) + \Delta(n_j, W) \bmod 2, \\ \mu_3 &= \Delta(n_j, N) + \Delta(n_j, W) \bmod 2, \\ \mu_4 &= \sum_{v>0}(\dim W_v)\cdot\left(\left\lfloor \frac{p_j}{n_j}v + \frac{1}{2} \right\rfloor + (p-1)v\right) \\ &\qquad + \sum_{v<0}(\dim W_v)\cdot\left(\left\lfloor -\frac{p_j}{n_j}v + \frac{1}{2} \right\rfloor - (p-1)v\right) \bmod 2. \end{aligned}$$

Proof. By the proof of [Liu et al. 2000, Proposition 4.2], we need to compute the action of $r_*^{-1} \cdot P \cdot r_*$ on

$$\bigotimes_{\substack{0<n\in\mathbb{Z}+\frac{1}{2},\, v>0 \\ n-(p-1)v-(p_j/n_j)v\le 0}} \Lambda^{i_n}(\overline{W}_{v,n}) \otimes \bigotimes_{\substack{0<n\in\mathbb{Z}+\frac{1}{2},\, v<0 \\ n+(p-1)v+(p_j/n_j)v\le 0}} \Lambda^{i'_n}(W_{v,n}).$$

In fact, by (3-26) and (3-27), as in (3-16), we get

$$\begin{aligned} r_*^{-1} \cdot P \cdot r_* &= \sum_{\substack{0<n\in\mathbb{Z}+\frac{1}{2},\, v>0 \\ n-(p-1)v-(p_j/n_j)v\le 0}} (\dim W_v - i_n)\left(-n + (p-1)v + \frac{p_j}{n_j}v\right) \\ &\qquad + \sum_{\substack{0<n\in\mathbb{Z}+\frac{1}{2},\, v<0 \\ n+(p-1)v+(p_j/n_j)v\le 0}} (\dim W_v - i'_n)\left(-n - (p-1)v - \frac{p_j}{n_j}v\right) \\ &= P + \left(p - 1 + \frac{p_j}{n_j}\right)\boldsymbol{J}_H + \varepsilon_W^2. \end{aligned} \tag{3-59}$$

By [Liu et al. 2000, (4.36)–(4.38)] and (3-59), we deduce the second identity in (3-56). The first identity in (3-56) is obvious.

Consider the $\mathbb{Z}_2$-gradings. By [Liu et al. 2003, (4.49)–(4.50)] and the discussion following (3-48), we get the first two identities in (3-58). Observe that τ_1 changes only on

$$\bigotimes_{\substack{0<n\in\mathbb{Z}+\frac{1}{2},\, v>0\\ n-(p-1)v-(p_j/n_j)v\le 0}} \Lambda^{i_n}(\overline{W}_{v,n}) \otimes \bigotimes_{\substack{0<n\in\mathbb{Z}+\frac{1}{2},\, v<0\\ n+(p-1)v+(p_j/n_j)v\le 0}} \Lambda^{i'_n}(W_{v,n}).$$

From (3-26) and (3-27), we get the third identity in (3-58). □

3D. *A proof of Theorem 2.8.*

Lemma 3.8 (compare with [Liu et al. 2000, Lemmas 4.4 and 4.6]). *For each connected component M' of $M(n_j)$, the following functions are independent on the connected components of F in M':*

$$\tag{3-60}\begin{aligned} &\varepsilon_i+\varepsilon_W^1+2\varepsilon_W^2, && i=1,2,\\ &d'(p,\beta_j,N)+\mu_i+\mu_4 \bmod 2, && i=1,2,3,\\ &d'(p,\beta_{j-1},N)+\sum_{0<v}\dim N_v+\mu_i+\mu_4 \bmod 2, && i=1,2,3. \end{aligned}$$

Proof. Recall that $\lfloor \frac{p_j}{n_j}v\rfloor=\frac{p_j}{n_j}v-\frac{\omega(v)}{n_j}$. By using (3-31), we explicitly express ε_W^1 and ε_W^2 defined in (3-50)–(3-51) as follows:

$$\tag{3-61}\begin{aligned}\varepsilon_W^1=\tfrac{1}{2}(p-1+p_j/n_j)^2e(W)+\tfrac{1}{8}\dim W(n_j)_{n_j/2}\\ +\tfrac{1}{2}\sum_{0<v<n_j/2}\frac{\omega(v)\omega(-v)}{n_j^2}(\dim W(n_j)_v+\dim W(n_j)_{n_j-v}),\end{aligned}$$

and

$$\tag{3-62}\begin{aligned}\varepsilon_W^2=\tfrac{1}{2}(p-1+p_j/n_j)^2e(W)-\tfrac{1}{8}\dim W(n_j)_{n_j/2}\\ -\tfrac{1}{2}\sum_{0\le m\le (p_j-1)/2}\ \sum_{m<\frac{p_j}{n_j}v<m+\frac{1}{2}}\left(\frac{\omega(v)}{n_j}\right)^2(\dim W(n_j)_v+\dim W(n_j)_{n_j-v})\\ -\tfrac{1}{2}\sum_{0\le m\le p_j/2}\ \sum_{m-\frac{1}{2}<\frac{p_j}{n_j}v<m}\left(\frac{\omega(-v)}{n_j}\right)^2(\dim W(n_j)_v+\dim W(n_j)_{n_j-v}).\end{aligned}$$

By using (2-23), (3-61), (3-62), and the explicit expressions of ε_i given in [Liu et al. 2003, (4.56)–(4.57)], we know the functions in the first line of (3-60) are independent on the connected components of F in M'.

Now consider the functions in the rest of the lines of (3-60). By (2-30), (3-30), (3-32), (3-48) and [Liu et al. 2000, Lemma 4.5], we get

$$d'(p,\beta_j,N)+\mu_i+\mu_4 \equiv \sum_{0<m\le p_j/2}\ \sum_{\substack{0<v<n_j/2\\ m-\frac12<\frac{p_i}{n_j}v<m}} \dim N(n_j)_v+\tfrac12\dim_{\mathbb R} N(n_j)^{\mathbb R}_{n_j/2} + \sum_{v>0}(\dim N_v)\Big\lfloor\frac{p_j}{n_j}v+\frac12\Big\rfloor+\sum_{v>0}(\dim W_v)\Big\lfloor\frac{p_j}{n_j}v+\frac12\Big\rfloor + \sum_{v<0}(\dim W_v)\Big\lfloor-\frac{p_j}{n_j}v+\frac12\Big\rfloor+o\big(N(n_j)^{\mathbb R}_{n_j/2}\big)+\Delta(n_j,W) \bmod 2. \tag{3-63}$$

But, by [Liu et al. 2000, Lemma 4.5], as $w_2(W\oplus TX)_{S^1}=0$, we know that, modulo 2,

$$\sum_{v>0}(\dim N_v)\Big\lfloor\frac{p_j}{n_j}v+\frac12\Big\rfloor+\sum_{v>0}(\dim W_v)\Big\lfloor\frac{p_j}{n_j}v+\frac12\Big\rfloor + \sum_{v<0}(\dim W_v)\Big\lfloor-\frac{p_j}{n_j}v+\frac12\Big\rfloor+o(N(n_j)^{\mathbb R}_{n_j/2})+\Delta(n_j,W) \tag{3-64}$$

is independent on the connected components of F in M'. Thus, the independence on the connected components of F in M' of the functions in the second line of (3-60) is proved, which, combined with [Liu et al. 2000, (4.42)], implies the same independent property of the functions in the third line of (3-60). □

By (3-34)–(3-39) and Lemma 3.6, we know that the Dirac operator

$$D^{X(n_j)}\otimes\mathcal{F}(\beta_j)\otimes F_V^i(\beta_j)\otimes Q_W(\beta_j)\otimes L(\beta_j)_i$$

($i=1,2$) is well-defined on $M(n_j)$. Observe that (2-12) in Theorem 2.1 is compatible with the G_y-action. Thus, by using Proposition 3.7, Lemma 3.8 and applying Theorem 2.1 to each connected component of $M(n_j)$ separately, we deduce that, for $i=1,2$, $1\le j\le J_0$, $m\in(1/2)\mathbb{Z}$, $h\in\mathbb{Z}$, $\tau=\tau_{e1}$ or τ_{s1},

$$\begin{aligned}&\sum_\alpha(-1)^{d'(p,\beta_{j-1},N)+\sum_{v>0}\dim N_v}\operatorname{Ind}_\tau\big(D^{Y_\alpha}\otimes(K_W\otimes K_X^{-1})^{1/2}\\&\qquad\otimes\mathcal{F}_{p,j-1}(X)\otimes F_V^i\otimes Q(W),m+e(p,\beta_{j-1},N),h\big)\\&=\sum_\beta(-1)^{d'(p,\beta_{j-1},N)+\sum_{v>0}\dim N_v+\mu}\operatorname{Ind}_\tau\bigg(D^{X(n_j)}\otimes(K_W\otimes K_X^{-1})^{1/2}\otimes\mathcal{F}(\beta_j)\\&\qquad\otimes F_V^i(\beta_j)\otimes Q_W(\beta_j)\otimes L(\beta_j)_i,m+\varepsilon_i+\varepsilon_W^1+2\varepsilon_W^2+\Big(\frac{p_j}{n_j}+(p-1)\Big)h,h\bigg)\\&=\sum_\alpha(-1)^{d'(p,\beta_j,N)+\sum_{v>0}\dim N_v}\operatorname{Ind}_\tau\big(D^{Y_\alpha}\otimes(K_W\otimes K_X^{-1})^{1/2}\otimes\mathcal{F}_{p,j}(X)\\&\qquad\otimes F_V^i\otimes Q(W),m+e(p,\beta_j,N),h\big),\end{aligned}\tag{3-65}$$

where $\sum_\beta$ means the sum over all the connected components of $M(n_j)$. In (3-65), if $\tau = \tau_{s1}$, $\mu = \mu_3 + \mu_4$; if $\tau = \tau_{e1}$, $\mu = \mu_i + \mu_4$. Combining (3-55) with (3-65), we get (2-28). The proof of Theorem 2.8 is complete.

Acknowledgements

The authors wish to thank Professor Weiping Zhang for helpful discussions.

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Received July 11, 2012.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 266, No. 2, 2013
dx.doi.org/10.2140/pjm.2013.266.509

ACKNOWLEDGEMENT

The editors gratefully acknowledge the valuable advice of the referees who helped them select and better the papers appearing in 2013 in the *Pacific Journal of Mathematics* (reports dated January 1 through November 11, 2013).

Colin Adams, Roi Decampo Alvarez, Henning Haahr Andersen, Cung The Anh, Martin Arkowitz, Christophe Bavard, Daniel Benoit, Don Blasius, David Peter Blecher, Brian H. Bowditch, Khristo Boyadzhiev, Elena Braverman, Simon Brendle, Robert F. Brown, Nathaniel Brown, Winfried Bruns, Ulrich Bunke, Henrique Bursztyn, Zbigniew B ocki, Danny Calegari, G. Carlier, Jeffrey Case, Fabrizio Catanese, Abhijit Champanerkar, Xiao-Li Chao, Vyjayanthi Chari, Moira Chas, Bing-Long Chen, Jih-Hsin Cheng, Mathieu Colin, Stephen Curran, Galia Dafni, Matthew B. Day, Jason DeBlois, Martin Deraux, Trond Digernes, Ivan Dimitrov, Qing Ding, Stephen R. Doty, Tedi Draghici, Ken Dykema, Ed Effros, Jose Espinar, John Etnyre, Alberto Facchini, Elisha Falbel, Yi Fang, Alex Feingold, Sérgio R. Fenley, Leonor Ferrer, Dorel Fetcu, Tobias Finis, Rita Fioresi, Giovanni P. Galdi, José A. Gálvez, Pedro A. García-Sánchez, Yuxin Ge, Claus Gerhardt, Eknath Ghate, Amalendu Ghosh, Anthony Giaquinto, David Goldberg, María del Mar González, Dmitry Gourevitch, C. Robin Graham, Robert Greene, Li Guo, M. J. D. Hamilton, Chenxu He, Haruzo Hida, Kaoru Hiraga, Eriko Hironaka, Craig D. Hodgson, Derek Holt, Kazuhiro Ichihara, Alexander Iosevich, Tadeusz Januszkiewicz, Peter Jossen, Uwe Kahler, Nicholas M. Katz, William H. Kazez, Ely Kerman, Ju-Lee Kim, Bruce Kleiner, Dessislava Hristova Kochloukova, Mustafa Korkmaz, Brett Kotschwar, Dirk Kusin, Robert Kusner, Enrico Laeng, Claude LeBrun, David B. Leep, Melvin Leok, Shengjia Li, Aihua Li, Haizhong Li, Junfeng Li, Song-Ying Li, Tian-Jun Li, Xiangdong Li, Yi Li, Nan Li, David Li-Bland, Xian-Gao Liu, Maciej Malicki, Romanos Malikiosis, Francisco Marcellan, Paolo Maremonti, Dario Martelli, David Martínez Torres, Howard Masur, Volodymyr Mazorchuk, Dermot McCarthy, Mark Meilstrup, Massimiliano Mella, Chikako Mese, Changxing Miao, Erich Miersemann, Pablo Mira, Yoshihiko Mitsumatsu, Ali Mohammadian, Anne Moreau, Frank Morgan, Adriano Moura, Vicente Muñoz, Katsuyuki Naoi, Frédéric Naud, Gabriele Nebe, Bruce Olberding, Juan de Dios Perez, N. Christopher Phillips, Sorin Popa, Edmund Puczylowski, Heydar Radjavi, Robert C. Rhoades, Yo’av Rieck, Brooks Roberts, Marc Rosso, Min Ru, Zeev Rudnick, Peter Russell, Patrick Ryan, Paolo Salani, Gregory Sankaran, Jacques Sauloy, Alistair Rowland

John Savage, Gordan Savin, Murray Schacher, Florent Schaffhauser, Ralf Schiffler, Natasa Sesum, Zhongmin Shen, Yuguang Shi, Dimitri Shlyakhtenko, Thomas Sinclair, Yannick Sire, Jian Song, Rabah Souam, Steven Spallone, Viktor Starkov, András I. Stipsicz, Xiaotao Sun, Sergei Tabachnikov, Jennifer Taback, Giorgio Talenti, Jinggang Tan, Nicolas Templier, Susan Tolman, Maggy Tomova, Peter Miles Topping, Burt Totaro, Martin Traizet, Hsian-Hua Tseng, Vladimir Turaev, Thomas Unger, Ramón Vázquez-Lorenzo, T. N. Venkataramana, Stefano Vidussi, Thomas Vogel, Nathalie Wahl, Genevieve S Walsh, Changyou Wang, Meng Wang, Erxiao Wang, Yi Wang, Micah Warren, Cameron Wickham, George Willis, Hao Xu, Yi-Song Yang, Xiaokui Yang, Paul Yang, Wei Yuan, Weiping Zhang, Entao Zhao, Meng Zhu.

CONTENTS

Volume 266, no. 1 and no. 2

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PACIFIC JOURNAL OF MATHEMATICS

Volume 266 No. 2 December 2013