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We study the Knapp–Stein *R*-groups of the inner forms of SL(*N*) over a nonarchimedean local field of characteristic zero, by using a restriction from the inner forms of GL(*N*). As conjectured by Arthur, these *R*-groups are then shown to be naturally isomorphic to their dual avatars defined in terms of L-parameters. The 2-cocycles attached to *R*-groups can be described as well. The proofs are based on the results of K. Hiraga and H. Saito. We also construct examples to illustrate some new phenomena which do not occur in the case of SL(*N*) or classical groups.

1. Introduction

Let G be a connected reductive group over a local field F and $G(F)$ be the locally compact group of the F-points of G. The study of the tempered representations of $G(F)$ is a crucial ingredient of the monumental work of Harish-Chandra on his Plancherel formula. Denote by $\Pi_{\text{temp}}(G)$ the set of isomorphism classes of irreducible tempered representations, and by $\Pi_{2,\text{temp}}(G)$ its subset of representations which are square-integrable modulo the center. Roughly speaking, elements in $\Pi_{\text{temp}}(G)$ can be obtained as subrepresentations of $I_P^G(\sigma)$, where $P = MU$ is a parabolic subgroup, $\sigma \in \Pi_{2,\text{temp}}(M)$, and $I_P^G(\sigma)$ is the normalized parabolic induction. Assuming the knowledge of square-integrable representations, the study of $\Pi_{\text{temp}}(G)$ then boils down to that of the decomposition of $I_P^G(\sigma)$, for P and σ as above.

Knapp, Stein, and Silberger (for the nonarchimedean case) described the decomposition of $I_P^G(\sigma)$ in terms of the *Knapp–Stein R-group* R_σ. More precisely, we have a central extension of groups

$$1 \to \mathbb{C}^\times \to \widetilde{R}_\sigma \to R_\sigma \to 1$$

defined using the normalized intertwining operators $R_P(w, \sigma)$. It is the set $\Pi_-(\widetilde{R}_\sigma)$ of the irreducible representations of $\widetilde{R}_\sigma$ by which $\mathbb{C}^\times$ acts by $z \mapsto z \cdot \mathrm{id}$ which governs the decomposition of $I_P^G(\sigma)$. Equivalently, we are given a cohomology class $c_\sigma \in H^2(R_\sigma, \mathbb{C}^\times)$ attached to this central extension. The group R_σ itself suffices to determine whether $I_P^G(\sigma)$ is reducible or not. To extract further information, such as the description of elliptic tempered representations, some knowledge about $\widetilde{R}_\sigma$ is also needed. We refer the reader to [Arthur 1993, Section 2] for details.

On the other hand, the tempered part of the local Langlands correspondence predicts a map $\phi \mapsto \Pi_\phi$ which assigns a finite subset Π_ϕ of $\Pi_{\mathrm{temp}}(G)$ to every bounded L-parameter $\phi \in \Phi_{\mathrm{bdd}}(G)$, taken up to equivalence, such that

$$\Pi_{\mathrm{temp}}(G) = \bigsqcup_{\phi \in \Phi_{\mathrm{bdd}}(G)} \Pi_\phi.$$

The internal structure of the tempered L-packets Π_ϕ is conjectured to be controlled by the *S-group* $S_\phi := Z_{\widehat{G}}(\mathrm{Im}(\phi))$. More precisely, following [Arthur 2006], one has to introduce a central extension

$$1 \to \widetilde{Z}_\phi \to \widetilde{\mathscr{S}}_\phi \to \mathscr{S}_\phi \to 1$$

of finite groups defined in terms of S_ϕ. The L-packet Π_ϕ should be in bijection with a set $\Pi(\widetilde{\mathscr{S}}_\phi, \chi_G)$ of representations of $\widetilde{\mathscr{S}}_\phi$, where χ_G is a character of $\widetilde{Z}_\phi$ coming from Galois cohomology. The relevant definitions will be reviewed later in this article.

The tempered local Langlands correspondence is expected to behave well under normalized parabolic induction, namely, for $P = MU$ as above and $\phi_M \in \Phi_{\mathrm{bdd}}(M)$, we deduce $\phi \in \Phi_{\mathrm{bdd}}(G)$ by composing ϕ_M with the inclusion $^LM \to {}^LG$ of L-groups, which is well defined up to conjugacy. Then Π_ϕ should be the union of the irreducible constituents of $I_P^G(\sigma)$, where σ ranges over the elements of Π_{ϕ_M}. A natural question arises: is it possible to describe R_σ, or even $\widetilde{R}_\sigma$, in terms of the S-groups?

For archimedean F this has been answered by Shelstad [1982]; in that case, the extension $\widetilde{R}_\sigma \to R_\sigma$ splits and R_σ is abelian of exponent two. For general F of characteristic zero, Arthur proposed a generalization [1989b, Section 7] as follows. For every $\phi \in \Phi_{\mathrm{bdd}}(G)$ coming from $\phi_M \in \Phi_{2,\mathrm{bdd}}(M)$ (that is, a parameter for M which is square-integrable modulo the center), he introduced the *dual R-group* (also known as the *endoscopic R-group*) $R_\phi \simeq \mathscr{S}_\phi/\mathscr{S}_{\phi_M}$ and a subgroup $R_{\phi,\sigma} \subset R_\phi$ for

every $\sigma \in \Pi_{\phi_M}$. Arthur conjectures a natural isomorphism

$$R_{\phi,\sigma} \simeq R_{\sigma}.$$

This has been verified for quasisplit classical groups and unitary groups by Arthur [2013] and Mok [2012], respectively. In their construction of L-packets, the dual *R*-groups play a pivotal role through the *local intertwining relations*; see [Arthur 2013, Chapter 2]. It turns out that in these cases, we have $R_{\phi} = R_{\phi,\sigma}$ and $\widetilde{R}_{\sigma} \to R_{\sigma}$ splits; see [Arthur 2013, Section 6.5]. Similar results were obtained independently in [Ban and Zhang 2005; Goldberg 2011; Ban and Goldberg 2012] for nonarchimedean *F*. For quaternionic unitary groups, see [Hanzer 2004].

We shall assume hereafter that *F* is a nonarchimedean local field of characteristic zero.

Another good test ground for Arthur's conjectures is the group SL(*N*) and its inner forms. Indeed, the case $N = 2$ is the genesis of endoscopy [Labesse and Langlands 1979]; for general *N*, the local Langlands correspondence for the inner forms $G^{\sharp}$ of SL(*N*) is established in [Hiraga and Saito 2012], at least in the tempered case. This is based on the local Langlands correspondence for the inner forms *G* of GL(*N*), which satisfies the following nice properties:

- The L-packets Π_{ϕ} for *G* are all singletons.
- For any parabolic subgroup $P = MU$ and $\sigma \in \Pi_{\text{temp}}(M)$, the induced representation $I_P^G(\sigma)$ is irreducible.

In fact, the latter property holds for all unitary σ, known as Tadić's property (U0) [Sécherre 2009].

The (tempered) local Langlands correspondence for $G^{\sharp}$ can be obtained by restriction from $G(F)$ to $G^{\sharp}(F)$; the procedure is somehow dual to the natural projection of L-groups

$$\mathbf{pr} : {}^L G \twoheadrightarrow {}^L G^{\sharp}.$$

The same recipe can be applied to any Levi subgroup *M*, with respect to $M^{\sharp} := M \cap G^{\sharp}$.

The method of restriction provides a convenient device, but we still have to study the internal structure of L-packets for $G^{\sharp}$ and their behavior under normalized parabolic induction. For the quasisplit case $G^{\sharp} = \text{SL}(N)$, such issues can be addressed by the multiplicity-one property of Whittaker models. In that case, the Knapp–Stein *R*-groups are studied in depth in [Gelbart and Knapp 1981; 1982; Shahidi 1983; Tadić 1992; Goldberg 1994]. Roughly speaking, let $\sigma^{\sharp} \in \Pi_{2,\text{temp}}(M^{\sharp})$ which lies in the L-packet $\Pi_{\phi^{\sharp}}$. We may choose $\sigma \in \Pi_{2,\text{temp}}(M)$ so that $\sigma^{\sharp} \hookrightarrow \sigma|_{M^{\sharp}}$. Set $\pi := I_P^G(\sigma)$, which is irreducible. Then $R_{\sigma^{\sharp}}$ is described in terms of

$$X^G(\pi) = \{\eta \in (G(F)/G^{\sharp}(F))^D : \eta \otimes \pi \simeq \pi\}$$

and its analogue $X^M(\sigma)$ for the Levi subgroup M with respect to $M^\sharp$, where $(G(F)/G^\sharp(F))^D$ means the group of continuous characters of $G(F)/G^\sharp(F)$. It is then easy to relate $R_{\sigma^\sharp}$ with $R_{\phi^\sharp}$, and we deduce a canonical isomorphism $R_{\phi^\sharp} = R_{\phi^\sharp,\sigma^\sharp} \simeq R_{\sigma^\sharp}$ as well as a splitting for $\widetilde{R}_{\sigma^\sharp} \twoheadrightarrow R_{\sigma^\sharp}$. Note that we used the notations $\phi^\sharp$, $\sigma^\sharp$, etc. to denote the objects attached to $G^\sharp$ and its Levi subgroups.

Whittaker models are no longer available for the nonquasisplit inner forms $G^\sharp$ of SL(N). What saves the day is that Hiraga and Saito [2012] defined a central extension

$$1 \to \mathbb{C}^\times \to S^G(\pi) \to X^G(\pi) \to 1$$

and related it to the central extension of $\mathscr{S}$-groups alluded to above. This allows us to study the internal structure of the L-packets obtained by restriction. In our main theorem, Theorem 6.2.4, we will prove, among other things, that there is

(i) a canonical isomorphism $R_{\phi^\sharp,\sigma^\sharp} \simeq R_{\sigma^\sharp}$ as conjectured by Arthur;

(ii) a "concrete" description of the dual R-groups for $G^\sharp$, namely,

$$R_{\phi^\sharp} \simeq X^G(\pi)/X^M(\sigma),$$
$$R_{\phi^\sharp,\sigma^\sharp} \simeq Z^M(\sigma)^\perp/X^M(\sigma);$$

(iii) a description of the class $\boldsymbol{c}_{\sigma^\sharp} \in H^2(R_{\sigma^\sharp}, \mathbb{C}^\times)$ attached to $\widetilde{R}_{\sigma^\sharp} \twoheadrightarrow R_{\sigma^\sharp}$, in terms of the obstruction for extending the representation ρ of $\widetilde{\mathscr{S}}_{\phi_M^\sharp}$ to the preimage of $R_{\phi,\sigma^\sharp}$ in $\widetilde{\mathscr{S}}_{\phi^\sharp}$.

We refer the reader to Section 4 for unexplained notations. Note that the description of $\boldsymbol{c}_{\sigma^\sharp}$ is also conjectured by Arthur; see [Arthur 1996, Page 537; 2008, Section 3; 2013, Section 2.4] for further discussions.

Arthur's conjecture on R-groups for the inner forms of SL(N) is thus verified. The examples are probably more interesting, however. In Section 6.3 we will give conceptual constructions of $\phi^\sharp$ and $\sigma^\sharp$ as above such that

(i) $\widetilde{R}_{\sigma^\sharp} \twoheadrightarrow R_{\sigma^\sharp}$ is not split, or

(ii) $R_{\phi^\sharp,\sigma^\sharp} \subsetneq R_{\phi^\sharp}$.

Such phenomena do not occur to the quasisplit classical groups, the quaternionic unitary groups, or SL(N). The first example is perhaps more surprising, since $\widetilde{R}_{\sigma^\sharp} \twoheadrightarrow R_{\sigma^\sharp}$ always splits for generic inducing data. Keys [1987, Section 6] constructed a Knapp–Stein R-group with nonsplit cocycle in the nonconnected setting; our example seems to be the only known case for connected reductive groups. In both examples, the relation between $R_{\sigma^\sharp}$ and the S-groups is crucial.

In view of the possible applications to automorphic representations, one should also consider certain nontempered unitary representations, namely, those appearing in the *A-packets*; see Remark 6.2.6 for a short discussion.

Shortly after the release of the first version of our preprint, we were informed of an independent work of Choiy and Goldberg [2012] that treats the same problems except that of cocycles. Despite some overlap, their work has a completely different technical core, namely the transfer of Plancherel measures between inner forms, which should have wide-ranging applications.

Organization of this article. In Section 2, we recapitulate the formalism of normalized intertwining operators and Knapp–Stein *R*-groups. We follow the notations in [Arthur 1989a; 1993] closely. In particular, the *R*-group R_σ is regarded as a quotient of the isotropy group W_σ, instead of a subgroup.

In Section 3, we set up a general formalism of restriction of representations. These results are scattered in [Shahidi 1983; Keys 1987; Tadić 1992; Hiraga and Saito 2012], just to mention a few. In view of the possible sequels of this work, the behavior under restriction of normalized intertwining operators is treated in generality.

A special assumption is made in Section 4 (Hypothesis 4.0.2), namely, that the parabolically induced representations in question should be irreducible. We are then able to deduce finer information on *R*-groups and their cocycles in this setting. The arguments are not too difficult, but require some careful manipulations.

In Section 5, we will specialize to the inner forms of SL(*N*) and reformulate the results of Hiraga and Saito [2012] on the local Langlands correspondence and the *S*-groups. In order to study parabolic induction, we also have to generalize these results to the Levi subgroups.

In Section 6, we recapitulate Arthur's definition of dual *R*-groups via the omnipresent commutative diagram in Proposition 6.1.1. The results obtained earlier can then be easily assembled, and Arthur's conjecture on *R*-groups for the inner forms of SL(*N*) (Theorem 6.2.4) follows.

2. Preliminaries

2.1. *Conventions.* *Local fields.* Throughout the article, *F* always denotes a nonarchimedean local field of characteristic zero. We set.

- Γ_F: the absolute Galois group of *F*, defined with respect to a chosen algebraic closure $\overline{F}$;
- W_F: the Weil group of *F*;
- $\mathrm{WD}_F := W_F \times \mathrm{SU}(2)$: the Weil–Deligne group of *F*;
- $|\cdot| = |\cdot|_F$: the normalized absolute value of *F*;
- q_F: the cardinality of the residue field of *F*.

When discussing the canonical family of normalizing factors for GL(*N*) and its inner forms, we will also fix a nontrivial additive character $\psi_F : F \to \mathbb{C}^\times$.

The usual Galois cohomology over F is denoted by $H^\bullet(F, \cdot)$. The continuous cohomology of W_F is denoted by $H^\bullet_{\mathrm{cont}}(W_F, \cdots)$; the groups of continuous cocycles are denoted by $Z^\bullet_{\mathrm{cont}}(W_F, \cdots)$.

Groups and representations. For an F-group scheme G, the group of its F-points is denoted by $G(F)$; subgroups of G mean the closed subgroup schemes. The identity connected component of G is denoted by G^0. The center of G is denoted by Z_G. Centralizers (respectively normalizers) in G are denoted by $Z_G(\cdot)$ (respectively $N_G(\cdot)$). The algebraic groups over $\mathbb{C}$ are identified with their $\mathbb{C}$-points.

The derived group of G is denoted by G_{der}. Now assume G to be connected reductive. A simply connected cover of G_{der}, which is unique up to isomorphism, is denoted by $G_{\mathrm{SC}} \to G_{\mathrm{der}}$. We denote the adjoint group of G by $G_{\mathrm{AD}} := G/Z_G$. For every subgroup H of G, we denote by H_{sc} (respectively H_{ad}) the preimage of H in G_{SC} (respectively image in G_{AD}). The same formalism pertains to connected reductive $\mathbb{C}$-groups as well.

The definitions of the dual group ${}^L G = \widehat{G} \rtimes W_F$ and the L-parameters will be reviewed in Section 3.5.

The symbol $\mathrm{Ad}(\cdots)$ denotes the adjoint action of an abstract group on itself, namely $\mathrm{Ad}(x) : g \mapsto gxg^{-1}$.

For any division algebra D over F and $n \in \mathbb{Z}_{\geq 1}$, we denote by $\mathrm{GL}_D(n)$ the group of invertible elements in $\mathrm{End}_D(D^n)$, where D^n is viewed as a right D-module. It is also regarded as a connected reductive F-group.

The representations considered in this article are all over $\mathbb{C}$-vector spaces. For a connected reductive F-group G, we define the following.

- $\Pi(G)$: the set of equivalence classes of irreducible smooth representations of $G(F)$;

- $\Pi_{\mathrm{unit}}(G)$: the subset consisting of unitary (that is, unitarizable) representations;

- $\Pi_{\mathrm{temp}}(G)$: the subset consisting of tempered representations;

- $\Pi_{2,\mathrm{temp}}(G)$: the subset consisting of unitary representations which are square-integrable modulo the center.

For an abstract group S, we will also denote by $\Pi(S)$ the set of its irreducible representations up to equivalence.

The central character of $\pi \in \Pi(G)$ is denoted by ω_π. The group of morphisms (respectively the set of isomorphisms) in the category of representations of $G(F)$ is denoted by $\mathrm{Hom}_G(\cdots)$ (respectively $\mathrm{Isom}_G(\cdots)$).

For any topological group H, we set

$$H^D := \{\chi : H \to \mathbb{C}^\times, \text{ continuous character}\}.$$

For any representation π of $G(F)$ and any $\eta \in G(F)^D$, we write $\eta\pi := \eta \otimes \pi$ for abbreviation. Also note that π and $\eta\pi$ have the same underlying $\mathbb{C}$-vector spaces. If M is a subgroup of G and π is a smooth representation of $G(F)$, we shall denote the restriction of π to $M(F)$ by $\pi|_M$.

Combinatorics. Let G be a connected reductive F-group. We employ the following notations in this article. Let M be a Levi subgroup.

- $\mathcal{P}(M)$: the set of parabolic subgroups of G with Levi component M;
- $\mathcal{L}(M)$: the set of Levi subgroups of G containing M;
- $\mathcal{F}(M)$: the set of parabolic subgroups of G containing M;
- $W(M) := N_G(M)(F)/M(F)$: the Weyl group (in a generalized sense) relative to M;

The Levi decompositions are written as $P = MU$, where U denotes the unipotent radical of P. For M chosen, the opposite parabolic of $P = MU$ is denoted by $\bar{P} = M\bar{U}$. When we have to emphasize the role of G, the notations $\mathcal{P}^G(M)$, $\mathcal{L}^G(M)$, $\mathcal{F}^G(M)$, and $W^G(M)$ will be used.

Let $w \in W(M)$ with a representative $\tilde{w} \in G(F)$. For $\sigma \in \Pi(M)$, we define $\tilde{w}\sigma$ to be the representation on the same underlying vector space, with the new action

$$(\tilde{w}\sigma)(m) := \sigma(\tilde{w}^{-1}m\tilde{w}), \quad m \in M(F).$$

The equivalence class of $\tilde{w}\sigma$ depends only on $w \in W(M)$, and we will write $w\sigma$ instead, if there is no confusion.

Define $\mathcal{X}(G) := \mathrm{Hom}_{F-\mathrm{grp}}(G, \mathbb{G}_m)$ and $\mathfrak{a}_G := \mathrm{Hom}(\mathcal{X}(G), \mathbb{R})$. For every Levi subgroup M, there is a canonically split short exact sequence of finite-dimensional $\mathbb{R}$-vector spaces

$$0 \to \mathfrak{a}_G \to \mathfrak{a}_M \leftrightarrows \mathfrak{a}_M^G \to 0.$$

The linear duals of these spaces are denoted by $\mathfrak{a}_G^*$, etc. We also write $\mathfrak{a}_{G,\mathbb{C}} := \mathfrak{a}_G \otimes_{\mathbb{R}} \mathbb{C}$, etc. Sometimes we also write $\mathfrak{a}_P$ instead of $\mathfrak{a}_M$ if $P = MU$.

The Harish-Chandra map $H_G : G(F) \to \mathfrak{a}_G$ is the homomorphism characterized by

$$\langle \chi, H_G(x) \rangle = \log |\chi(x)|_F, \quad \chi \in \mathcal{X}(G).$$

For $\lambda \in \mathfrak{a}_{G,\mathbb{C}}^*$ and $\pi \in \Pi(G)$, we define $\pi_\lambda \in \Pi(G)$ by

(1) $$\pi_\lambda := e^{\langle \lambda, H_G(\cdot) \rangle} \otimes \pi.$$

Fix a minimal parabolic subgroup $P_0 = M_0 U_0$ of G. We define Δ_0, $\Delta_0^\vee$ to be the set of simple roots and coroots, which form bases of $(\mathfrak{a}_{M_0}^G)^*$ and $\mathfrak{a}_{M_0}^G$, respectively. The set of positive roots is denoted by Σ_0, and its subset of reduced roots by Σ_0^{red}. They form a bona fide root system. For every $P = MU \supset P_0$, we define

$\Delta_P \subset \Sigma_P \subset \Sigma_P^{\mathrm{red}}$ by taking the set of nonzero restrictions to $(\mathfrak{a}_M^G)^*$ of elements in $\Delta_0 \subset \Sigma_0 \subset \Sigma_0^{\mathrm{red}}$. To each $\alpha \in \Sigma_P$ we may associate the coroot $\alpha^\vee \in \mathfrak{a}_M^G$: it is defined as the restriction of the coroot in $\Delta_0^\vee$. For a given P, the objects above are independent of the choice of P_0. We can emphasize the role of G by using the notations Δ_P^G, etc. whenever needed.

Induction. We always consider a parabolic subgroup $P = MU$ of G. The modulus character of $P(F)$ is denoted by δ_P, that is,

$$(\text{left Haar measure}) = \delta_P \cdot (\text{right Haar measure}).$$

The usual smooth induction functor is denoted by $\mathrm{Ind}(\cdots)$. The normalized parabolic induction functor from P to G is denoted by $I_P^G(\cdot) := \mathrm{Ind}_P^G(\delta_P^{1/2} \otimes \cdot)$. Recall that for $\sigma \in \Pi(M)$ with underlying vector space V_σ, we use the usual model to realize $I_P^G(\sigma)$ as the space of functions $\varphi : G(F) \to V_\sigma$ such that φ is invariant under right translation by an open compact subgroup of $G(F)$, and that $\varphi(umx) = \delta_P(m)^{1/2}\sigma(m)(\varphi(x))$ for all $m \in M(F)$, $u \in U(F)$. The group $G(F)$ acts on this function space by the right regular representation.

For $\sigma, \sigma' \in \Pi(M)$ and $f \in \mathrm{Hom}_M(\sigma, \sigma')$, the induced morphism is denoted by $I_P^G(f)$; it sends φ to $f(\varphi)$.

2.2. *Normalized intertwining operators.* Our basic reference for normalized intertwining operators is [Arthur 1993]. Consider the following data.

- G: a connected reductive F-group.
- M: a Levi subgroup of G.
- $P, Q \in \mathcal{P}(M)$.
- $\sigma : M(F) \to \mathrm{Aut}_\mathbb{C}(V_\sigma)$: a smooth representation of $M(F)$ of finite length.
- $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$.

For every $\alpha \in \Delta_P$, we denote by r_α the smallest positive rational number such that $r_\alpha \cdot \alpha^\vee$ lies in the lattice $H_M(M(F))$. We define

$$(2) \qquad\qquad \check{\alpha} := r_\alpha \alpha^\vee.$$

By recalling (1), we form the normalized parabolic induction $I_P^G(\sigma_\lambda)$, $I_Q^G(\sigma_\lambda)$. Their underlying spaces are denoted by $I_P^G(V_{\sigma_\lambda})$, $I_Q^G(V_{\sigma_\lambda})$. The standard intertwining operator

$$J_{Q|P}(\sigma_\lambda) : I_P^G(\sigma_\lambda) \longrightarrow I_Q^G(\sigma_\lambda)$$

is defined by the absolutely convergent integral

$$(3) \qquad (J_{Q|P}(\sigma_\lambda)\varphi)(x) = \int_{U_P(F) \cap U_Q(F) \backslash U_Q(F)} \varphi(ux)\, du, \quad x \in G(F),$$

when $\langle \mathrm{Re}(\lambda), \alpha^\vee \rangle \gg 0$ for all $\alpha \in \Sigma_P^{\mathrm{red}} \cap \Sigma_{\bar{Q}}^{\mathrm{red}}$; see [Waldspurger 2003, IV.1] for the precise meaning of absolute convergence. Recall that upon choosing a special maximal compact open subgroup $K \subset G(F)$ in good position relative to M, these induced representations can be realized on a vector space that is independent of λ. It is known that $J_{Q|P}(\sigma_\lambda)$ is a rational function in the variables

$$\{q_F^{-\langle \lambda, \check{\alpha} \rangle} : \alpha \in \Delta_P\}.$$

In particular, as a function in λ, $J_{Q|P}(\sigma_\lambda)$ admits a meromorphic continuation to $\mathfrak{a}_{M,\mathbb{C}}^*$. When $\sigma \in \Pi_{\mathrm{temp}}(M)$, it is known that (3) is absolutely convergent for $\langle \mathrm{Re}(\lambda), \alpha^\vee \rangle > 0$ for all $\alpha \in \Sigma_P^{\mathrm{red}} \cap \Sigma_{\bar{Q}}^{\mathrm{red}}$. Moreover, as a meromorphic family of operators, it satisfies $\mathrm{ord}_{\lambda=0}(J_{Q|P}(\sigma_\lambda)) \geq -1$.

Henceforth we assume σ irreducible, that is, $\sigma \in \Pi(M)$. Take any $P \in \mathcal{P}(M)$. Define the j-functions as

$$(4) \qquad\qquad j(\sigma_\lambda) := J_{P|\bar{P}}(\sigma_\lambda) J_{\bar{P}|P}(\sigma_\lambda).$$

It is known that $\lambda \mapsto j(\sigma_\lambda)$ a scalar-valued meromorphic function, which is not identically zero. Moreover, $j(\sigma_\lambda)$ is independent of P and admits a product decomposition

$$j(\sigma_\lambda) = \prod_{\alpha \in \Sigma_P^{\mathrm{red}}} j_\alpha(\sigma_\lambda),$$

where j_α denotes the j-function defined relative to the Levi subgroup $M_\alpha \in \mathcal{L}(M)$ such that $\Sigma_M^{M_\alpha, \mathrm{red}} = \{\pm\alpha\}$.

Now assume $\sigma \in \Pi_{2,\mathrm{temp}}(M)$. In this paper, we define Harish-Chandra's μ-function as the meromorphic function

$$\mu(\sigma_\lambda) := j(\sigma_\lambda)^{-1}.$$

Accordingly, μ also admits a product decomposition $\mu = \prod_\alpha \mu_\alpha$. It is analytic and nonnegative for $\lambda \in i\mathfrak{a}_M^*$. Note that our definitions of j-functions and μ-functions depend on the choice of Haar measures on unipotent radicals. In particular, our μ-function differs from that in [Waldspurger 2003, V.2] by some harmless constant.

Definition 2.2.1 (cf. [Arthur 1989a, Section 2]). In this article, a family of normalizing factors is a family of meromorphic functions on the $\mathfrak{a}_{M,\mathbb{C}}^*$-orbits in $\Pi(M)$, for all Levi subgroup M of G, written as

$$r_{Q|P}(\sigma_\lambda), \quad P, Q \in \mathcal{P}(M), \sigma_\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$$

satisfying the following conditions. First of all, we define the corresponding normalized intertwining operators as

$$R_{Q|P}(\sigma_\lambda) := r_{Q|P}(\sigma_\lambda)^{-1} J_{Q|P}(\sigma_\lambda),$$

which is a meromorphic family (in λ) of intertwining operators $I_P^G(\sigma_\lambda) \to I_Q^G(\sigma_\lambda)$.

We shall also assume that a family of normalizing factors is chosen for every proper Levi subgroup.

(R$_1$) For all $P, P', P'' \in \mathscr{P}(M)$, we have $R_{P''|P}(\sigma_\lambda) = R_{P''|P'}(\sigma_\lambda) R_{P'|P}(\sigma_\lambda)$.

(R$_2$) If $\sigma \in \Pi_{\mathrm{unit}}(M)$, then

$$R_{Q|P}(\sigma_\lambda) = R_{P|Q}(\sigma_{-\bar\lambda})^*, \quad \lambda \in \mathfrak{a}_{M,\mathbb{C}}^*.$$

In particular, $R_{Q|P}(\sigma)$ is a well-defined unitary operator.

(R$_3$) This family is compatible with conjugacy, namely,

$$R_{gQg^{-1}|gPg^{-1}}(g\sigma_\lambda) = \ell(g) R_{Q|P}(\sigma_\lambda) \ell(g)^{-1}$$

for all $g \in G(F)$, where $\ell(g)$ is the map $\varphi(\,\cdot\,) \mapsto \varphi(g^{-1}\cdot)$.

(R$_4$) We have

$$r_{Q|P}(\sigma_\lambda) = \prod_{\alpha \in \Sigma_P^{\mathrm{red}} \cap \Sigma_{\bar Q}^{\mathrm{red}}} r_{\overline{P_\alpha}|P_\alpha}^{\widetilde{M}_\alpha}(\sigma_\lambda),$$

where $P_\alpha := P \cap M_\alpha$, and $r_{\overline{P_\alpha}|P_\alpha}^{M_\alpha}$ comes from the family of normalizing factors for M_α.

(R$_5$) Let $S = LU \in \mathscr{F}(M)$ containing both P and Q. Then $R_{Q|P}(\sigma_\lambda)$ is the operator deduced from $R_{P\cap L|Q\cap L}^L(\sigma_\lambda)$ by the functor $I_S^G(\,\cdot\,)$.

(R$_6$) The function $\lambda \mapsto r_{Q|P}(\sigma_\lambda)$ is rational in the variables $\{q_F^{-\langle\lambda,\check\alpha\rangle} : \alpha \in \Delta_P\}$.

(R$_7$) If $\sigma \in \Pi_{\mathrm{temp}}(M)$, the meromorphic function $\lambda \mapsto r_{Q|P}(\sigma_\lambda)$ is invertible whenever $\mathrm{Re}\langle\lambda, \alpha^\vee\rangle > 0$ for all $\alpha \in \Delta_P$.

Observe that (R$_2$) is equivalent to saying $r_{Q|P}(\sigma_\lambda) = \overline{r_{P|Q}(\sigma_{-\bar\lambda})}$ for $\sigma \in \Pi_{\mathrm{unit}}(M)$, as the unnormalized operators $J_{Q|P}(\sigma_\lambda)$ satisfy a similar condition. Similarly, (R$_3$) is equivalent to saying $r_{gQg^{-1}|gPg^{-1}}(g\sigma_\lambda) = r_{Q|P}(\sigma_\lambda)$.

The fundamental result about the normalizing factors is that they exist [Arthur 1989a, Theorem 2.1].

Remark 2.2.2. According to Langlands [1976, Appendix 2], there is a conjectural canonical family of normalizing factors $r_{Q|P}(\sigma_\lambda)$ in terms of local factors, namely,

$$r_{Q|P}(\sigma_\lambda) = \varepsilon(0, \rho_{Q|P}^\vee \circ \phi_\lambda, \psi_F)^{-1} L(0, \phi_\lambda, \rho_{Q|P}^\vee) L(1, \phi_\lambda, \rho_{Q|P}^\vee)^{-1},$$

where

- ϕ_λ is the Langlands parameter for σ_λ;
- $\hat{\mathfrak{u}}_Q$ (resp. $\hat{\mathfrak{u}}_P$) denotes the Lie algebra of the unipotent radical of the dual parabolic subgroup $\hat Q$ (respectively $\hat P$) in $\widehat{G}$;

- $\rho_{Q|P}$ is the adjoint representation of ${}^{L}M$ on $\hat{\mathfrak{u}}_{Q}/(\hat{\mathfrak{u}}_{Q} \cap \hat{\mathfrak{u}}_{P})$, and $\rho_{Q|P}^{\vee}$ denotes its contragredient;

- $\psi_{F} : F \to \mathbb{C}^{\times}$ is a chosen nontrivial additive character.

We will invoke this description only in the case $G = \mathrm{GL}_{F}(n)$. In that case, the local factors in sight are essentially those associated with pairs (ϕ_{1}, ϕ_{2}) where ϕ_{1}, ϕ_{2} are among the L-parameters parametrizing the components of σ. Such Artin local factors are known to agree with their representation-theoretic avatars, say, those defined by Rankin–Selberg convolution or by the Langlands–Shahidi method.

Remark 2.2.3. The construction of normalizing factors can be reduced to the case where M is a maximal proper Levi subgroup of G and $\sigma \in \Pi_{2,\mathrm{temp}}(M)$, as illustrated in [Arthur 1989a]. Let us give a quick sketch of this reduction.

(i) In view of (R$_4$), we are led to the case M maximal proper. Moreover, it suffices to verify (R$_3$) for the representatives in $G(F)$ of the elements in $W(M) := N_{G(F)}(M)/M(F)$, which has at most two elements.

(ii) Assume that $\sigma \in \Pi_{\mathrm{temp}}(M)$. By the classification of tempered representations, there exists a parabolic subgroup $R = M_{R}U_{R}$ of M and $\tau \in \Pi_{\mathrm{temp}}(M_{R})$ such that $\sigma \hookrightarrow I_{R}^{M}(\tau)$. The pair (M, τ) is unique up to conjugacy. There is a unique element $P(R)$ in $\mathscr{P}(M_{R})$, characterized by the properties

- $P(R) \subset P$,
- $P(R) \cap M = R$.

Consequently, parabolic induction in stages gives $I_{P}^{G} I_{R}^{M}(\tau) = I_{P(R)}^{G}(\tau)$. The same construction works when P is replaced by Q. Set

$$r_{Q|P}(\sigma_{\lambda}) := r_{Q(R)|P(R)}(\tau_{\lambda}).$$

In view of (R$_5$) together with parabolic induction in stages, we see that $R_{Q|P}(\sigma_{\lambda})$ is the restriction of $R_{Q(R)|P(R)}(\sigma_{\lambda})$ to $I_{P}^{G}(\sigma_{\lambda})$. The required conditions can be readily verified.

(iii) For general σ, we may realize it as the Langlands quotient $I_{R}^{M}(\tau_{\mu}) \twoheadrightarrow \sigma$, where $R = M_{R}U_{R}$ is a parabolic subgroup of M, $\tau \in \Pi_{\mathrm{temp}}(M_{R})$, and $\mu \in \mathfrak{a}_{M_{R}}^{*}$ satisfies $\mathrm{Re}\langle \mu, \beta^{\vee}\rangle > 0$ for all $\beta \in \Delta_{R}^{M}$. The triplet (M, τ, μ) is again unique up to conjugacy. Let $P, Q \in \mathscr{P}(M)$. Define $P(R), Q(R) \in \mathscr{P}(M_{R})$ as before and set

$$r_{Q|P}(\sigma_{\lambda}) := r_{Q(R)|P(R)}(\tau_{\lambda+\mu}).$$

Recall that

$$\mathrm{Ker}[I_{R}^{M}(\tau_{\mu}) \twoheadrightarrow \sigma] = \mathrm{Ker}(J_{\bar{R}|R}^{M}(\tau_{\mu})).$$

The condition (R$_7$) applied to τ_{μ} tells us that $\mathrm{Ker}(J_{\bar{R}|R}^{M}(\tau_{\mu})) = \mathrm{Ker}(R_{\bar{R}|R}^{M}(\tau_{\mu}))$. The same is true for μ replaced by $\mu+\lambda$. Using (R$_5$), we see that $R_{Q(R)|P(R)}(\tau_{\lambda+\mu})$

factors into $R_{Q|P}(\sigma_\lambda)$ on $I_P^G(\sigma_\lambda)$. All conditions except (R$_2$) follow from this. The proof of (R$_2$) requires more effort to deal with the unitarizability of Langlands quotients; the reader can consult [Arthur 1989a, Page 30] for details.

(iv) Reverting to our original assumption that M is maximal proper and

$$\sigma \in \Pi_{2,\text{temp}}(M),$$

it clearly remains to verify conditions (R$_1$), (R$_2$), (R$_3$), (R$_6$), (R$_7$). Furthermore, one can reduce (R$_3$) to the assertion that $r_{w\bar{P}w^{-1}|wPw^{-1}}(w(\sigma_\lambda)) = r_{\bar{P}|P}(\sigma_\lambda)$, for $w \in W(G)$ being the nontrivial element in $W(M)$ if it exists.

2.3. Knapp–Stein R-groups.

Fix a family of normalizing factors for G. Assume henceforth that M is a Levi subgroup of G and $\sigma \in \Pi_{2,\text{temp}}(M)$. Define the isotropy group

$$W_\sigma := \{w \in W(M) : w\sigma \simeq \sigma\}.$$

Fix $P \in \mathscr{P}(M)$. For $w \in W(M)$ with a representative $\tilde{w} \in G(F)$, we define the operator $r_P(\tilde{w}, \sigma) \in \text{Isom}_G(I_P^G(\sigma), I_P^G(\tilde{w}\sigma))$ by

$$(5) \qquad r_P(\tilde{w}, \sigma) : I_P^G(\sigma) \xrightarrow{R_{w^{-1}Pw|P}(\sigma)} I_{w^{-1}Pw}^G(\sigma) \xrightarrow{[\ell(\tilde{w}):\phi \mapsto \phi(\tilde{w}^{-1} \cdot)]} I_P^G(\tilde{w}\sigma).$$

We notice the following property: for any $w, w' \in W(M)$ with representatives $\tilde{w}, \tilde{w}' \in G(F)$, we have

$$(6) \qquad r_P(\tilde{w}\tilde{w}', \sigma) = r_P(\tilde{w}, \tilde{w}'\sigma) \circ r_P(\tilde{w}', \sigma).$$

Assume now $w \in W_\sigma$. Choose a representative $\tilde{w}$ of w and $\sigma(\tilde{w}) \in \text{Isom}_M(\tilde{w}\sigma, \sigma)$ to define the operator

$$R_P(\tilde{w}, \sigma) := I_P^G(\sigma(\tilde{w})) \circ r_P(\tilde{w}, \sigma).$$

The class $R_P(\tilde{w}, \sigma) \bmod \mathbb{C}^\times$ is independent of the choices of $\sigma(\tilde{w})$ and the representative $\tilde{w}$. We also have

$$R_P(\tilde{w}, \sigma) \in \text{Aut}_G(I_P^G(\sigma)),$$

$$R_P(\tilde{w}, \sigma)R_P(\tilde{w}', \sigma) = R_P(\tilde{w}\tilde{w}', \sigma) \quad \bmod \mathbb{C}^\times, \quad w, w' \in W_\sigma.$$

Now we can define the Knapp–Stein R-group as follows.

$$W_\sigma^0 := \{w \in W_\sigma : R_P(\tilde{w}, \sigma) \in \mathbb{C}^\times \text{ id}\},$$

$$R_\sigma := W_\sigma / W_\sigma^0.$$

We will also make use of the following alternative description of R_σ.

Proposition 2.3.1. *The subgroup W_σ^0 is the Weyl group of the root system on $\mathfrak{a}_M$ composed of the multiples of the roots in $\{\alpha \in \Sigma_P^{\mathrm{red}} : \mu_\alpha(\sigma) = 0\}$.*

Given any Weyl chamber $\mathfrak{a}_\sigma^+ \subset \mathfrak{a}_M$ for the aforementioned root system, there is then a unique section $R_\sigma \hookrightarrow W_\sigma$ that sends $r \in R_\sigma$ to the representative $w \in W_\sigma$ such that $w\mathfrak{a}_\sigma^+ = \mathfrak{a}_\sigma^+$. Consequently, we can write $W_\sigma = W_\sigma^0 \rtimes R_\sigma$.

In the literature, R_σ is sometimes viewed as a subgroup of W_σ in this manner; see, for example, [Goldberg 2006].

Write V_σ (respectively $I_P^G(V_\sigma)$) for the underlying vector space of the representation σ (respectively $I_P^G(\sigma)$). It follows that $w \mapsto R_P(\tilde{w}, \sigma)$ induces a projective representation of R_σ on $I_P^G(V_\sigma)$, where $\tilde{w} \in G(F)$ is any representative of $w \in W_\sigma$. We denote this projective representation provisionally by $r \mapsto R_P(r, \sigma)$, for $r \in R_\sigma$.

There is a standard way to lift $R_P(\,\cdot\,, \sigma)$ to an authentic representation of some group $\widetilde{R}_\sigma$ which sits in a central extension

$$1 \to \mathbb{C}^\times \to \widetilde{R}_\sigma \to R_\sigma \to 1,$$

such that $\mathbb{C}^\times$ acts by $z \mapsto z \cdot \mathrm{id}$. Namely, we can set $\widetilde{R}_\sigma$ to be the group of elements $(r, M[r]) \in R_\sigma \times \mathrm{Aut}_\mathbb{C}(I_P^G(V_\sigma))$ such that $M[r] \bmod \mathbb{C}^\times$ gives $R_P(r, \sigma)$. The lifted representation, denoted by $\tilde{r} \mapsto R_P(\tilde{r}, \sigma)$, is then $\tilde{r} = (r, M[r]) \mapsto M[r]$. Such a central extension by $\mathbb{C}^\times$ that lifts $R_P(\,\cdot\,, \sigma)$ is unique up to isomorphism.

Note that the central extension above can also be described by the class

$$c_\sigma \in H^2(R_\sigma, \mathbb{C}^\times)$$

coming from the $\mathbb{C}^\times$-valued 2-cocycle c_σ defined by

$$(7) \qquad R_P(\widetilde{r_1 r_2}, \sigma) = c_\sigma(r_1, r_2) R_P(\tilde{r}_1, \sigma) R_P(\tilde{r}_2, \sigma), \quad r_1, r_2 \in R_\sigma,$$

where we choose a preimage $\tilde{r} \in \widetilde{R}_\sigma$ for every $r \in R_\sigma$.

Theorem 2.3.2 (Harish-Chandra; [Silberger 1978]). *Fix a preimage $\tilde{r} \in \widetilde{R}_\sigma$ for every $r \in R_\sigma$. Then the operators $\{R_P(\tilde{r}, \sigma) : r \in R_\sigma\}$ form a basis of $\mathrm{End}_G(I_P^G(\sigma))$.*

Following Arthur, we reformulate this fundamental result as follows. Let

$$\Pi_\sigma(G) := \{\text{irreducible constituents of } I_P^G(\sigma)\}/\simeq,$$
$$\Pi_-(\widetilde{R}_\sigma) := \{\rho \in \Pi(\widetilde{R}_\sigma) : \text{ for all } z \in \mathbb{C}^\times \ \rho(z) = z \cdot \mathrm{id}\}.$$

Note that $\Pi_\sigma(G)$, $\Pi_-(\widetilde{R}_\sigma)$ are both finite sets, and each $\rho \in \Pi_-(\widetilde{R}_\sigma)$ is finite-dimensional.

Corollary 2.3.3. *Let $\mathcal{R}$ be the representation of $\widetilde{R}_\sigma \times G(F)$ on $I_P^G(V_\sigma)$ defined by*

$$\mathcal{R}(\tilde{r}, x) = R_P(\tilde{r}, \sigma) I_P^G(\sigma, x), \quad \tilde{r} \in \widetilde{R}_\sigma, x \in G(F).$$

Then there is a decomposition

$$(8) \qquad \mathcal{R} \simeq \bigoplus_{\rho \in \Pi_-(\widetilde{R}_\sigma)} \rho \boxtimes \pi_\rho,$$

where $\rho \mapsto \pi_\rho$ is a bijection from $\Pi_-(\widetilde{R}_\sigma)$ to $\Pi_\sigma(G)$, characterized by (8).

Consequently, $I_P^G(\sigma)$ is irreducible if and only if $R_\sigma = \{1\}$.

Remark 2.3.4. When G is quasisplit and σ is generic with respect to a given Whittaker datum for M, the work of Shahidi [1990] furnishes

(i) a canonical family of normalizing factors $r_{Q|P}(\sigma)$;

(ii) a canonically defined homomorphism $w \mapsto R_P(w, \sigma)$;

(iii) a canonical splitting of the central extension $1 \to \mathbb{C}^\times \to \widetilde{R}_\sigma \to R_\sigma \to 1$.

These properties are not expected in general. Indeed, we see in Example 6.3.3 that (iii) may fail.

Remark 2.3.5. The formalism above depends not only on (M, σ), but also on the choice of $P \in \mathcal{P}(M)$. One can easily pass to another choice $Q \in \mathcal{P}(M)$ by transport of structure using $R_{Q|P}(\sigma)$. For example, one has

$$r_Q(\tilde{w}, \sigma) = R_{P|Q}(\tilde{w}\sigma)^{-1} r_P(\tilde{w}, \sigma) R_{P|Q}(\sigma),$$
$$R_Q(\tilde{w}, \sigma) = R_{P|Q}(\sigma)^{-1} R_P(\tilde{w}, \sigma) R_{P|Q}(\sigma)$$

for all $w \in W(M)$ with a representative $\tilde{w} \in G(F)$ and some chosen $\sigma(\tilde{w})$.

3. Restriction

Let G, $G^\sharp$ be connected reductive F-groups such that

$$G_{\mathrm{der}} \subset G^\sharp \subset G.$$

3.1. *Restriction of representations.* In this subsection, we will review the basic results in [Tadić 1992, Section 2; Hiraga and Saito 2012, Chapter 2] concerning the restriction of a smooth representation from $G(F)$ to $G^\sharp(F)$. The objects associated to $G^\sharp$ are endowed with the superscript $\sharp$, for example, $\pi^\sharp \in \Pi(G^\sharp)$.

Proposition 3.1.1 [Silberger 1979; Tadić 1992, Lemma 2.1 and Proposition 2.2]. *Let $\pi \in \Pi(G)$. Then $\pi|_{G^\sharp}$ decomposes into a finite direct sum of smooth irreducible representations. Each irreducible constituent of $\pi|_{G^\sharp}$ has the same multiplicity.*

Conversely, every $\pi^\sharp \in \Pi(G^\sharp)$ embeds into $\pi|_{G^\sharp}$ for some $\pi \in \Pi(G)$. If the central character $\omega_{\pi^\sharp}$ is unitary, one can choose π so that ω_π is also unitary.

Proposition 3.1.2 [Tadić 1992, Corollary 2.5]. *Let $\pi_1, \pi_2 \in \Pi(G)$. The following are equivalent:*

  (i) $\mathrm{Hom}_{G^\sharp}(\pi_1, \pi_2) \neq \{0\}$.

 (ii) $\pi_1|_{G^\sharp} \simeq \pi_2|_{G^\sharp}$.

(iii) *There exists $\eta \in (G(F)/G^\sharp(F))^D$ such that $\eta\pi_1 \simeq \pi_2$.*

For $\pi \in \Pi(G)$, we define a finite "packet" of smooth irreducible representations of $G^\sharp(F)$ as

$$\Pi_\pi := \{\text{irreducible constituents of } \pi|_{G^\sharp}\}/\simeq .$$

Consequently, [Proposition 3.1.2](#) implies that $\Pi(G^\sharp) = \bigsqcup_\pi \Pi_\pi$, when π is taken over the $(G(F)/G^\sharp(F))^D$-orbits in $\Pi(G)$.

Proposition 3.1.3 [Tadić 1992, Proposition 2.7]. *Let $\pi \in \Pi(G)$ and assume that ω_π is unitary. Let P be one of the following properties of smooth irreducible representations of $G(F)$ or $G^\sharp(F)$:*

  (i) *unitary,*

 (ii) *tempered,*

(iii) *square-integrable modulo the center,*

(iv) *cuspidal.*

Then we have equivalences of the form

$$[\pi \text{ satisfies } P] \iff [\text{for some } \pi^\sharp \in \Pi_\pi, \pi^\sharp \text{ satisfies } P]$$
$$\iff [\text{for all } \pi^\sharp \in \Pi_\pi, \pi^\sharp \text{ satisfies } P].$$

Now comes the decomposition of $\pi|_{G^\sharp}$.

Definition 3.1.4. Let $\pi \in \Pi(G)$ with the underlying $\mathbb{C}$-vector space V_π. Note that $V_{\eta\pi} = V_\pi$ for all $\eta \in (G(F)/G^\sharp(F))^D$. We introduce the groups

$$X^G(\pi) := \{\eta \in (G(F)/G^\sharp(F))^D : \eta\pi \simeq \pi\},$$
$$S^G(\pi) := \langle I_\eta^G \in \mathrm{Isom}_G(\eta\pi, \pi) : \eta \in X^G(\pi)\rangle \subset \mathrm{Aut}_{G^\sharp}(\pi).$$

Observe that $\mathrm{Isom}_G(\eta\pi, \pi)$ is a $\mathbb{C}^\times$-torsor by Schur's lemma, and an element $I_\eta^G \in S^G(\pi)$ uniquely determines η. The group law is given by composition in $\mathrm{Aut}_{\mathbb{C}}(V_\pi)$, namely, by

$$\mathrm{Isom}_G(\eta\pi, \pi) \times \mathrm{Isom}_G(\eta'\pi, \pi)$$
$$= \mathrm{Isom}_G(\eta'\eta\pi, \eta'\pi) \times \mathrm{Isom}_G(\eta'\pi, \pi) \to \mathrm{Isom}_G(\eta'\eta\pi, \pi)$$

for all $\eta, \eta' \in X^G(\pi)$.

Thus we obtain a central extension of locally compact groups

(9) $$1 \to \mathbb{C}^\times \to S^G(\pi) \to X^G(\pi) \to 1,$$

where the first arrow is $z \mapsto z \cdot \mathrm{id}$ and the second one is $I_\eta^G \mapsto \eta$.

Also note that $X^G(\pi) = X^G(\xi\pi)$, $S^G(\pi) = S^G(\xi\pi)$ for any $\xi \in (G(F)/G^\sharp(F))^D$.

Note that implicit in the notations above is the reference to $G^\sharp$, which is usually clear from the context. Indications to $G^\sharp$ will be given when necessary.

It is easy to see that $X^G(\pi)$ is finite abelian. As in the setting of R-groups, we define the finite set

$$\Pi_-(S^G(\pi)) := \{\rho \in \Pi(S^G(\pi)) : \text{ for all } z \in \mathbb{C}^\times, \rho(z) = z \cdot \mathrm{id}\}.$$

Theorem 3.1.5 [Hiraga and Saito 2012, Lemma 2.5 and Corollary 2.7]. *Let $\mathfrak{S} = \mathfrak{S}(\pi)$ be the representation of $S^G(\pi) \times G^\sharp(F)$ on V_π defined by*

$$\mathfrak{S}(I, x) = I \circ \pi(x), \quad I \in S^G(\pi), x \in G^\sharp(F).$$

Then there is a decomposition

$$(10) \qquad \mathfrak{S} \simeq \bigoplus_{\rho \in \Pi_-(S^G(\pi))} \rho \boxtimes \pi_\rho^\sharp,$$

where $\rho \mapsto \pi_\rho^\sharp$ is a bijection from $\Pi_-(S^G(\pi))$ to Π_π, characterized by (10).

3.2. *Relation to parabolic induction.* Let P be a parabolic subgroup of G with a Levi decomposition $P = MU$. In this article, we denote systematically

$$P^\sharp := P \cap G^\sharp,$$
$$M^\sharp := M \cap M^\sharp.$$

Then $P^\sharp$ is a parabolic subgroup of $G^\sharp$ with Levi decomposition $P^\sharp = M^\sharp U$, since every unipotent subgroup of G is contained in G_{der}. The map $P \mapsto P^\sharp$ (respectively $M \mapsto M^\sharp$) induces a bijection between the parabolic subgroups (respectively Levi subgroups) of G and $G^\sharp$, which leaves the unipotent radicals intact. We also have a canonical identification $W(M^\sharp) = W(M)$. In what follows, we will fix Haar measures on the unipotent radicals of parabolic subgroups of G and $G^\sharp$, which are compatible with the identifications above.

Obviously, the modulus functions satisfy $\delta_P(m) = \delta_{P^\sharp}(m)$ for all $m \in M^\sharp(F)$.

Lemma 3.2.1 [Tadić 1992, Lemma 1.1]. *Let $\sigma \in \Pi(M)$. Then we have the following isomorphism between smooth representations of $G^\sharp(F)$:*

$$I_P^G(\sigma)|_{G^\sharp} \longrightarrow I_{P^\sharp}^{G^\sharp}(\sigma|_{M^\sharp}),$$
$$\varphi \mapsto \varphi|_{G^\sharp(F)},$$

which is functorial in σ.

Proof. Upon recalling the definitions of $I_P^G(\sigma)$ and $I_{P^\sharp}^{G^\sharp}(\sigma|_{M^\sharp})$ as function spaces, the assertion follows from the canonical isomorphisms

$$P^\sharp(F)\backslash G^\sharp(F) = (P^\sharp\backslash G^\sharp)(F) \xrightarrow{\sim} (P\backslash G)(F) = P(F)\backslash G(F)$$

and the fact that $\delta_P|_{M^\sharp} = \delta_{P^\sharp}$. $\qquad\square$

The next result will not be used in this article; we include it only for the sake of completeness. Recall that the normalized Jacquet functor r_P^G is the left adjoint of I_P^G. The same is true for $r_{P^\sharp}^{G^\sharp}$.

Lemma 3.2.2. *Let $\pi \in \Pi(G)$. The restriction of representations induces an isomorphism*

$$r_P^G(\pi)|_{M^\sharp} \xrightarrow{\sim} r_{P^\sharp}^{G^\sharp}(\pi|_{G^\sharp})$$

between smooth representations of $M^\sharp(F)$, which is functorial in π.

Proof. The claim is evident. $\qquad\square$

Proposition 3.2.3. *Let M be a Levi subgroup of G. Then the inclusion map $M \hookrightarrow G$ induces an isomorphism between locally compact abelian groups:*

$$M(F)/M^\sharp(F) \xrightarrow{\sim} G(F)/G^\sharp(F).$$

Proof. The inclusion map induces an isomorphism $M/M^\sharp \hookrightarrow G/G^\sharp$ as F-tori. Hence the short exact sequence $1 \to M^\sharp \to M \to M/M^\sharp \to 1$ and its avatar for G provide a commutative diagram of pointed sets with exact rows:

$$
\begin{array}{ccccccc}
1 \longrightarrow & G^\sharp(F) & \longrightarrow & G(F) & \longrightarrow & (G/G^\sharp)(F) & \longrightarrow & H^1(F, G^\sharp) \\
& \uparrow & & \uparrow & & \| & & \uparrow \\
1 \longrightarrow & M^\sharp(F) & \longrightarrow & M(F) & \longrightarrow & (M/M^\sharp)(F) & \longrightarrow & H^1(F, M^\sharp).
\end{array}
$$

Fix a parabolic subgroup P of G with a Levi decomposition $P = MU$. The rightmost vertical arrow factorizes as

$$H^1(F, M^\sharp) \to H^1(F, P^\sharp) \to H^1(F, G^\sharp).$$

The first map is an isomorphism whose inverse is induced by $P^\sharp \twoheadrightarrow P^\sharp/U = M^\sharp$. It is well known that the second map is injective, hence so is the composition. A simple diagram chasing shows $M(F)/M^\sharp(F) \xrightarrow{\sim} G(F)/G^\sharp(F)$, as asserted. $\qquad\square$

Corollary 3.2.4. *The restriction map induces an isomorphism*

$$(G(F)/G^\sharp(F))^D \xrightarrow{\sim} (M(F)/M^\sharp(F))^D.$$

Here is a trivial but important consequence: any $\eta \in (M(F)/M^\sharp(F))^D$ is invariant under the $W(M)$-action.

3.3. *Relation to intertwining operators.* Let M be a Levi subgroup of G. First of all, observe that there is a natural decomposition

$$\mathfrak{a}_M^* = \mathfrak{a}_{M^\sharp}^* \oplus \mathfrak{b}^*,$$

where

$$\mathfrak{b}^* := \mathscr{X}(M/M^\sharp) \otimes_{\mathbb{Z}} \mathbb{R} \hookrightarrow \mathfrak{a}_G^*.$$

Henceforth, we shall identify $\mathfrak{a}_{M^\sharp}^*$ as a vector subspace of $\mathfrak{a}_M^*$. We shall do the same for their complexifications. This is compatible with restrictions in the following sense:

$$(\sigma_\lambda)|_{M^\sharp} = (\sigma|_{M^\sharp})_\lambda, \quad \sigma \in \Pi(M), \ \lambda \in \mathfrak{a}_{M^\sharp,\mathbb{C}}^*.$$

Lemma 3.3.1. *Let $\sigma \in \Pi(M)$, $P, Q \in \mathscr{P}(M)$. For $\lambda \in \mathfrak{a}_{M^\sharp,\mathbb{C}}^*$ in general position, the following diagram is commutative*:

$$
\begin{array}{ccc}
I_P^G(\sigma_\lambda)|_{G^\sharp} & \xrightarrow{\ \ J_{Q|P}(\sigma_\lambda)\ \ } & I_Q^G(\sigma_\lambda)|_{G^\sharp} \\
\simeq \downarrow & & \downarrow \simeq \\
I_{P^\sharp}^{G^\sharp}(\sigma_\lambda|_{M^\sharp}) & \xrightarrow[\ J_{Q^\sharp|P^\sharp}(\sigma_\lambda|_{M^\sharp})\ ]{} & I_{Q^\sharp}^{G^\sharp}(\sigma_\lambda|_{M^\sharp}),
\end{array}
$$

where the vertical isomorphisms are those defined in Lemma 3.2.1.

Proof. It suffices to check this for $\mathrm{Re}(\lambda)$ in the cone of absolute convergence of the integrals (3) defining $J_{Q|P}$ and $J_{Q^\sharp|P^\sharp}$. The commutativity then follows from (3) and the definition of the isomorphism in Lemma 3.2.1. □

Proposition 3.3.2. *Let $\sigma \in \Pi_{2,\mathrm{temp}}(M)$, $\sigma^\sharp \in \Pi_{2,\mathrm{temp}}(M^\sharp)$ such that $\sigma^\sharp \hookrightarrow \sigma|_{M^\sharp}$. Then, for all $\lambda \in \mathfrak{a}_{M^\sharp,\mathbb{C}}^*$, we have $\mu(\sigma_\lambda) = \mu(\sigma_\lambda^\sharp)$; more precisely,*

$$\mu_\alpha(\sigma_\lambda) = \mu_\alpha(\sigma_\lambda^\sharp) \quad \text{for all } \alpha \in \Sigma_P^{\mathrm{red}} = \Sigma_{P^\sharp}^{\mathrm{red}}, \ P \in \mathscr{P}(M).$$

Proof. In view of our choice of measures on unipotent radicals, the identities of μ-functions follow from (4) and Lemma 3.3.1. □

Proposition 3.3.3. *Let $\sigma \in \Pi(M)$ and $\eta \in (G(F)/G^\sharp(F))^D$. Then we have*

$$j(\sigma) = j(\eta\sigma).$$

In particular, for $\sigma \in \Pi_{2,\mathrm{temp}}(M)$, we have $\mu(\sigma) = \mu(\eta\sigma)$.

Proof. In view of the definition of j-function (4), it suffices to observe that for all $P, Q \in \mathscr{P}(M)$ and $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ in general position, the following diagram commutes:

$$
\begin{array}{ccc}
\eta I_P^G(\sigma_\lambda) & \xrightarrow{\ J_{Q|P}(\sigma_\lambda)\ } & \eta I_Q^G(\sigma_\lambda) \\
{\scriptstyle \simeq} \downarrow & & \downarrow {\scriptstyle \simeq} \\
I_P^G(\eta\sigma_\lambda) & \xrightarrow[\ J_{Q|P}(\eta\sigma_\lambda)\ ]{} & I_Q^G(\eta\sigma_\lambda),
\end{array}
$$

where the vertical arrows are given by $\varphi(\,\cdot\,) \mapsto \eta(\,\cdot\,)\varphi(\,\cdot\,)$; note that we used the natural identification $\mathrm{Hom}_G(\pi_1, \pi_2) = \mathrm{Hom}_G(\eta\pi_1, \eta\pi_2)$ for all $\pi_1, \pi_2 \in \Pi(G)$. Indeed, the commutativity can be seen from the definition (3) of $J_{Q|P}(\,\cdot\,)$ when $\mathrm{Re}(\lambda)$ lies in the cone of absolute convergence. $\qquad\square$

Theorem 3.3.4. *One can choose a family of normalizing factors for G such that*

$$
r_{Q|P}(\sigma_\lambda) = r_{Q|P}(\eta\sigma_\lambda)
$$

for all (M, σ), $P, Q \in \mathscr{P}(M)$ and $\eta \in (G(F)/G^\sharp(F))^D$, which is unitary. Given such a family of normalizing factors, one can define normalizing factors $r_{Q^\sharp|P^\sharp}(\sigma_\lambda^\sharp)$ for those $\sigma^\sharp$ such that $\omega_{\sigma^\sharp}$ is unitary by setting

$$
(11) \qquad\qquad r_{Q^\sharp|P^\sharp}(\sigma_\lambda^\sharp) := r_{Q|P}(\sigma_\lambda), \quad \lambda \in \mathfrak{a}_{M^\sharp,\mathbb{C}}^*,
$$

where $\sigma \in \Pi(M)$ is as in Proposition 3.1.1 with ω_σ unitary.

Moreover, let σ, $\sigma^\sharp$ be as above and $\iota : \sigma^\sharp \hookrightarrow \sigma|_{M^\sharp}$ be an embedding. Let $P, Q \in \mathscr{P}(M)$, $w \in W(M)$ with a representative $\tilde{w} \in G^\sharp(F)$. The following diagrams of $G^\sharp(F)$-representations are commutative:

$$
\begin{array}{ccc}
I_P^G(\sigma_\lambda) & \xrightarrow{\ R_{Q|P}(\sigma_\lambda)\ } & I_Q^G(\sigma_\lambda) \\
\big\uparrow & & \big\uparrow \\
I_{P^\sharp}^{G^\sharp}(\sigma_\lambda^\sharp) & \xrightarrow[\ R_{Q^\sharp|P^\sharp}(\sigma_\lambda^\sharp)\ ]{} & I_{Q^\sharp}^{G^\sharp}(\sigma_\lambda^\sharp),
\end{array}
\qquad
\begin{array}{ccc}
I_P^G(\sigma_\lambda) & \xrightarrow{\ r_P(\tilde{w},\sigma_\lambda)\ } & I_P^G(\tilde{w}(\sigma_\lambda)) \\
\big\uparrow & & \big\uparrow \\
I_{P^\sharp}^{G^\sharp}(\sigma_\lambda^\sharp) & \xrightarrow[\ r_{P^\sharp}(\tilde{w},\sigma_\lambda^\sharp)\ ]{} & I_{P^\sharp}^{G^\sharp}(\tilde{w}(\sigma_\lambda^\sharp))
\end{array}
$$

for $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ in general position, where the vertical arrows are given by

$$
I_{P^\sharp}^{G^\sharp}(\sigma_\lambda^\sharp) \xrightarrow{\ I_{P^\sharp}^{G^\sharp}(\iota)\ } I_{P^\sharp}^{G^\sharp}(\sigma_\lambda|_{M^\sharp}) \xrightarrow{\ \sim\ } I_P^G(\sigma_\lambda)|_{G^\sharp}.
$$

Observe that the asserted invariance under η-twist is satisfied by Langlands' conjectural family of normalizing factors in Remark 2.2.2, since they are defined in terms of local factors through the adjoint representation of $^L M$ on the Lie algebra of $\widehat{G}$.

Proof. We will show that $r_{Q|P}(\sigma_\lambda) = r_{Q|P}(\eta\sigma_\lambda)$ by reviewing the construction in Remark 2.2.3. More precisely, we will start from the square-integrable case and show that the equality is preserved throughout the inductive construction.

To begin with, suppose that M is maximal proper and $\sigma \in \Pi_{2,\mathrm{temp}}(M)$. Suppose that $r_{Q|P}(\sigma_\lambda)$ is chosen so that (R$_1$), (R$_2$), (R$_3$), (R$_6$), (R$_7$) are satisfied. Put $r_{Q|P}(\eta\sigma_\lambda) := r_{Q|P}(\sigma_\lambda)$. Then all the conditions above except (R$_1$) are trivially satisfied for $\eta\sigma$. As for (R$_1$), all that we need to check is that when $Q = \bar{P}$,

$$r_{P|Q}(\sigma_\lambda)r_{Q|P}(\sigma_\lambda) = j(\eta\sigma_\lambda), \quad \lambda \in \mathfrak{a}^*_{M,\mathbb{C}}.$$

By Proposition 3.3.3, the right hand side is equal to $j(\sigma_\lambda)$, hence the equality holds. This completes the case of $\sigma \in \Pi_{2,\mathrm{temp}}(M)$.

Suppose now $\sigma \in \Pi_{\mathrm{temp}}(M)$. By the classification of tempered representations, we may write $\sigma \hookrightarrow I_R^M(\tau)$ for some parabolic subgroup $R = M_R U_R$ of M and $\tau \in \Pi_{2,\mathrm{temp}}(M_R)$. Twisting everything by η, we obtain $\eta\sigma \hookrightarrow I_R^M(\eta\tau)$ in the classification of tempered representations. We have $r_{Q|P}(\sigma) = r_{Q(R)|P(R)}(\tau) = r_{Q(R)|P(R)}(\eta\tau)$ by the previous case; on the other hand, the inductive construction of normalizing factors says that $r_{Q|P}(\eta\sigma) = r_{Q(R)|P(R)}(\eta\tau)$, hence $r_{Q|P}(\eta\sigma) = r_{Q|P}(\sigma)$.

The case of general σ is similar. We may write σ as the Langlands quotient $I_R^M(\tau_\mu) \twoheadrightarrow \sigma$, where $R = M_R U_R$ is as before, $\tau \in \Pi_{\mathrm{temp}}(M_R)$, and $\mathrm{Re}\langle\mu, \alpha^\vee\rangle > 0$ for all $\alpha \in \Delta_R^M$. Twisting everything by η, we have $I_R^M(\eta\tau_\mu) \twoheadrightarrow \eta\sigma$, which is still a Langlands quotient. The inductive construction of normalizing factors says that $r_{Q|P}(\sigma) = r_{Q(R)|P(R)}(\tau_{\mu+\lambda})$. Repeating the arguments for the previous case, it follows that $r_{Q|P}(\sigma) = r_{Q|P}(\eta\sigma)$.

Now we can check that

$$r_{Q^\sharp|P^\sharp}(\sigma_\lambda^\sharp) := r_{Q|P}(\sigma_\lambda)$$

is well defined. Recall that $\omega_{\sigma^\sharp}$ and ω_σ are assumed to be unitary. If σ' is another choice such that $\sigma^\sharp \hookrightarrow \sigma|_{M^\sharp}$ and $\omega_{\sigma'}$ is unitary, there exists η such that $\sigma \simeq \eta\sigma'$. This would imply that $\eta|_{Z_G(F)}$ is unitary, hence so is η itself. Therefore $r_{Q|P}(\eta\sigma_\lambda) = r_{Q|P}(\sigma_\lambda)$.

Finally, the commutativity of the diagram results from Lemma 3.3.1 and (11). $\square$

3.4. *Relation to R-groups.* In this subsection, we will fix

- a parabolic subgroup $P = MU$ of G;
- the corresponding parabolic subgroup $P^\sharp := P \cap G^\sharp = M^\sharp U$ of $G^\sharp$;
- $\sigma^\sharp \in \Pi_{2,\mathrm{temp}}(M^\sharp)$;
- $\sigma \in \Pi_{2,\mathrm{temp}}(M)$ such that $\sigma^\sharp \hookrightarrow \sigma|_{M^\sharp}$.

Given $\sigma^\sharp$, the existence of such a σ is guaranteed by Propositions 3.1.1 and 3.1.3.

Lemma 3.4.1 (cf. [Goldberg 2006, Lemma 2.3]). *Under the identification $W(M) = W(M^\sharp)$, we have $W_\sigma^0 = W_{\sigma^\sharp}^0$.*

Proof. Since both sides are generated by root reflections, it suffices to fix $\alpha \in \Sigma_P^{\mathrm{red}} = \Sigma_{P^\sharp}^{\mathrm{red}}$ and show that $s_\alpha \in W_\sigma^0$ if and only if $s_\alpha \in W_{\sigma^\sharp}^0$, where s_α denotes the root reflection with respect to α.

By [Waldspurger 2003, Proposition IV.2.2], $\mu_\alpha(\sigma) = 0$ implies $s_\alpha \sigma \simeq \sigma$. The same is true for $\sigma^\sharp$ instead of σ. According to the description of W_σ^0 (respectively $W_{\sigma^\sharp}^0$) in terms of μ-functions, we obtain

$$\mu_\alpha(\sigma) = 0 \Leftrightarrow s_\alpha \in W_\sigma^0 \quad (\text{respectively } \mu_\alpha(\sigma^\sharp) = 0 \Leftrightarrow s_\alpha \in W_{\sigma^\sharp}^0).$$

On the other hand, Proposition 3.3.2 implies $\mu_\alpha(\sigma) = \mu_\alpha(\sigma^\sharp)$. The assertion follows immediately. $\square$

Definition 3.4.2. Set

$$L(\sigma) := \{\eta \in (M(F)/M^\sharp(F))^D : \text{there exists } w \in W(M), w\sigma \simeq \eta\sigma\},$$

$$L(\sigma^\sharp) := \{\eta \in (M(F)/M^\sharp(F))^D : \text{there exists } w \in W(M), w\sigma \simeq \eta\sigma, w\sigma^\sharp \simeq \sigma^\sharp\}.$$

These are subgroups of $(M(F)/M^\sharp(F))^D$. Indeed, let

$$\eta, \eta' \in L(\sigma) \quad \text{and} \quad w, w' \in W(M)$$

such that $\eta\sigma \simeq w\sigma$, $\eta'\sigma \simeq w'\sigma$. Then one has

(12) $$\eta'\eta\sigma \simeq \eta'w\sigma = w\eta'\sigma \simeq ww'\sigma.$$

Hence $\eta\eta' \in L(\sigma)$. The case of $L(\sigma^\sharp)$ is similar. Note that $X^M(\sigma) \subset L(\sigma^\sharp) \subset L(\sigma)$.

There is an obvious counterpart for the Weyl group, namely,

$$\overline{W}_\sigma := \{w \in W(M) : \text{there exists } \eta \in (M(F)/M^\sharp(F))^D, w\sigma \simeq \eta\sigma\}.$$

It is clear that $\overline{W}_\sigma \supset W_\sigma$. On the other hand, Proposition 3.1.2 implies that $\overline{W}_\sigma \supset W_{\sigma^\sharp}$.

The following result is clear in view of the preceding definitions.

Lemma 3.4.3. *There is a homomorphism given by*

$$\overline{\Gamma} : \overline{W}_\sigma \longrightarrow L(\sigma)/X^M(\sigma) w \mapsto \text{the } [\eta \bmod X^M(\sigma)] \text{ such that } w\sigma \simeq \eta\sigma,$$

which satisfies the following.

(i) *$\overline{\Gamma}$ is surjective.*

(ii) *$\mathrm{Ker}(\overline{\Gamma}) = W_\sigma$.*

(iii) *The preimage of $L(\sigma^\sharp)/X^M(\sigma)$ is $W_{\sigma^\sharp}W_\sigma$.*

Definition 3.4.4. Let

$$\Gamma : \overline{W}_\sigma / W_\sigma \xrightarrow{\sim} L(\sigma) / X^M(\sigma)$$

be the isomorphism obtained from $\overline{\Gamma}$ in the previous Lemma.

Proposition 3.4.5 [Goldberg 2006, Proposition 3.2]. *Set*

$$R_\sigma[\sigma^\sharp] := (W_\sigma \cap W_{\sigma^\sharp}) / W_{\sigma^\sharp}^0,$$

which is legitimate by Lemma 3.4.1. This is a subgroup of $R_{\sigma^\sharp}$.

(i) *The homomorphism $\overline{\Gamma}$ induces an isomorphism*

$$\Gamma : R_{\sigma^\sharp} / R_\sigma[\sigma^\sharp] \xrightarrow{\sim} L(\sigma^\sharp) / X^M(\sigma).$$

(ii) *If $R_\sigma = \{1\}$, or equivalently, if $I_P^G(\sigma)$ is irreducible, then Γ induces an isomor-phism $R_{\sigma^\sharp} \xrightarrow{\sim} L(\sigma^\sharp) / X^M(\sigma)$. Consequently, $R_{\sigma^\sharp}$ is abelian in this case.*

Proof. Lemma 3.4.3 gives an isomorphism

$$W_{\sigma^\sharp} / (W_\sigma \cap W_{\sigma^\sharp}) \xrightarrow{\sim} L(\sigma^\sharp) / X^M(\sigma)$$

that can be viewed as a restriction of Γ. By Lemma 3.4.1, we can take the quotients by $W_\sigma^0 = W_{\sigma^\sharp}^0$ on the left hand side. The first assertion follows immediately.

For the second assertion, it suffices to note that $R_\sigma[\sigma^\sharp]$ embeds into R_σ as well, since $W_\sigma^0 = W_{\sigma^\sharp}^0$. $\qquad\square$

3.5. *L-parameters.* Let G be a connected reductive F-group equipped with a quasisplit inner twist

$$\psi : G \times_F \overline{F} \to G^* \times_F \overline{F}.$$

We identify $\widehat{G}$ with $\widehat{G^*}$, thus $^L G = {}^L G^*$. The reader should recall that the definition of the complex reductive group $\widehat{G^*}$ and the Γ_F-action thereof depend on the choice of a Γ_F-stable splitting $(B^*, T^*, (E_\alpha)_{\alpha \in \Delta(B^*, T^*)})$ (also known as an F-splitting) of $G^*(\overline{F})$; see [Kottwitz 1984, Section 1]. These choices permit us to define a correspondence $M^* \leftrightarrow {}^L M^*$ between the conjugacy classes of Levi subgroups of G^* and their dual avatars inside $^L G^*$. Using the inner twist ψ, it also makes sense to say if a Levi subgroup M^* of G^* comes from G; this notion only depends on the conjugacy classes of Levi subgroups.

For any Levi subgroup M of G, there is a canonical bijection between $W^G(M)$ and $W^{\widehat{G}}(\widehat{M})$ coming from the bijection between roots and coroots.

An L-parameter for G^* is a homomorphism

$$\phi : \mathrm{WD}_F \to {}^L G^* = {}^L G$$

such that

- ϕ is an L-homomorphism, that is, the composition of ϕ with the projection $^{L}G \rightarrow W_F$ equals $\mathrm{WD}_F \rightarrow W_F$;

- ϕ is continuous;

- the projection of $\mathrm{Im}(\phi)$ to $\widehat{G}$ is formed of semisimple elements.

Two L-parameters ϕ_1, ϕ_2 are called equivalent, denoted by $\phi_1 \sim \phi_2$, if they are conjugate by $\widehat{G}$. We say that ϕ is bounded if the projection of $\mathrm{Im}(\phi)$ to $\widehat{G}$ is bounded (that is, relatively compact); this property depends only on the equivalence class of ϕ.

Given an L-parameter ϕ, we define

$$S_\phi := Z_{\widehat{G}}(\mathrm{Im}(\phi)).$$

The connected component S_ϕ^0 is a connected reductive subgroup of $\widehat{G}$. We record the following basic properties.

(i) The Levi subgroups $^{L}M^* \subset {}^{L}M$ which contain $\mathrm{Im}(\phi)$ minimally are conjugate by S_ϕ^0.

(ii) Letting $^{L}M^*$ be a Levi subgroup containing $\mathrm{Im}(\phi)$ minimally, $Z_{\widehat{M^*}}^{\Gamma_F, 0}$ is a maximal torus of S_ϕ^0.

Indeed, these assertions follow from [Borel 1979, Proposition 3.6] and its proof applied to the subgroup $\mathrm{Im}(\phi)$ of ^{L}G.

So far, everything depends only on the quasisplit inner form G^*. We say that ϕ is G-relevant if M_ϕ^* corresponds to a Levi subgroup of G; in this case, we write $M_\phi^* = M_\phi$. Put

$$\Phi(G) := \{\phi : \mathrm{WD}_F \rightarrow {}^{L}G, \ \phi \text{ is a } G\text{-relevant L-parameter}\}/\sim,$$

$$\Phi_{\mathrm{bdd}}(G) := \{\phi \in \Phi(G) : \phi \text{ is bounded}\},$$

$$\Phi_{2,\mathrm{bdd}}(G) := \{\phi \in \Phi_{\mathrm{bdd}}(G) : M_\phi = G\}.$$

Since the relevance condition is vacuous if $G = G^*$, we have $\Phi(G) \subset \Phi(G^*)$, etc.

Now let $G^\sharp$ be a subgroup of G such that $G_{\mathrm{der}} \subset G^\sharp \subset G$. We will study the lifting of L-parameters from $G^\sharp$ to G, which is in some sense dual to the restriction of representations. In what follows, the L-groups of G and $G^\sharp$ will be defined using compatible choices of quasisplit inner twists and F-splittings.

There is a natural, Γ_F-equivariant central extension

$$1 \rightarrow \widehat{Z}^\sharp \rightarrow \widehat{G} \xrightarrow{\mathbf{pr}} \widehat{G^\sharp} \rightarrow 1$$

which is dual to $G^\sharp \rightarrow G$; here $\widehat{Z}^\sharp$ is the $\mathbb{C}$-torus dual to $G/G^\sharp$.

For $\phi \in \Phi(G)$, we shall set $\phi^\sharp := \mathbf{pr} \circ \phi \in \Phi(G^\sharp)$. When this equality holds, ϕ is called a lifting of $\phi^\sharp$.

Theorem 3.5.1. *For any $\phi^\sharp \in \Phi(G^\sharp)$, there exists a lifting $\phi \in \Phi(G)$ of $\phi^\sharp$ which is unique up to twists by $H^1_{\mathrm{cont}}(W_F, \widehat{Z}^\sharp)$. If $\phi^\sharp \in \Phi_{\mathrm{bdd}}(G^\sharp)$ (respectively $\phi^\sharp \in \Phi_{2,\mathrm{bdd}}(G^\sharp)$), then ϕ can be chosen so that $\phi \in \Phi_{\mathrm{bdd}}(G)$ (respectively $\phi \in \Phi_{2,\mathrm{bdd}}(G)$).*

Note that by local class field theory, $H^1_{\mathrm{cont}}(W_F, \widehat{Z}^\sharp)$ parametrizes the continuous characters of $(G(F)/G^\sharp(F))^D$.

Proof. The existential part is just [Labesse 1985, Théorème 8.1] and the uniqueness follows easily. Assume that $\phi^\sharp \in \Phi_{\mathrm{bdd}}(G^\sharp)$ (respectively $\phi^\sharp \in \Phi_{2,\mathrm{bdd}}(G^\sharp)$) and let ϕ be any lifting of $\phi^\sharp$; we have to show that there exists a continuous 1-cocycle $a : W_F \to \widehat{Z}^\sharp$ such that the twisted L-parameter $a\phi$ is bounded (respectively bounded and satisfying $M_{a\phi} = M_\phi = G$).

Note that there exists a central isogeny of connected reductive groups

$$G^\sharp \times C \to G$$

given by multiplication, where C is some subtorus of Z_G^0. Hence $C \to G/G^\sharp$ is also an isogeny. By duality, we obtain a Γ_F-equivariant central isogeny of connected reductive complex groups

$$\widehat{G} \to \widehat{G^\sharp} \times \widehat{C}.$$

Let ϕ' be the composition of ϕ (projected to the $\widehat{G}$ component) with the afore-mentioned central isogeny. Let us show that $\mathrm{Im}(\phi')$ is bounded upon twisting ϕ. The first component of ϕ' is automatically bounded since $\phi^\sharp$ is. On the other hand, $\widehat{C}$ is isogenous to $\widehat{Z}^\sharp$, therefore, upon replacing ϕ by $a\phi$ for some suitable 1-cocycle $a : W_F \to \widehat{Z}^\sharp$, the second component can be made bounded. Hence $a\phi$ is a bounded L-parameter.

To finish the proof, it remains to observe that, assuming $\phi^\sharp = \mathbf{pr} \circ \phi$, the preimage of $M^\sharp_{\phi^\sharp}$ in $\widehat{G}$ is equal to M_ϕ. $\square$

Here we record a construction related to inner forms which will be required later. Recall that the inner forms of G^* are parametrized by $H^1(F, G^*_{\mathrm{AD}})$. Kottwitz [1984, Section 6] defined the "abelianization" map $\mathrm{ab}^1 : H^1(F, G^*_{\mathrm{AD}}) \to (Z^{\Gamma_F}_{\widehat{G}_{\mathrm{sc}}})^D$ between pointed sets. Hence we can associate to G a character χ_G of $Z^{\Gamma_F}_{\widehat{G}_{\mathrm{sc}}}$, in the following way:

(13) {inner forms of G^*} $= H^1(F, G^*_{\mathrm{AD}}) \xrightarrow{\mathrm{ab}^1} (Z^{\Gamma_F}_{\widehat{G}_{\mathrm{sc}}})^D$, $G \mapsto [\chi_G : Z^{\Gamma_F}_{\widehat{G}_{\mathrm{sc}}} \to \mathbb{C}^\times]$.

4. Restriction, continued

This section is devoted to the study of restriction under parabolic induction. As before, we fix connected reductive F-groups G, $G^\sharp$ such that $G_{\mathrm{der}} \subset G^\sharp \subset G$. We

also fix a Levi subgroup M of G and $P \in \mathscr{P}(M)$. The bijection between Levi subgroups (respectively parabolic subgroups) $M \mapsto M^\sharp$ (respectively $P \mapsto P^\sharp$) is defined in Section 3.2.

The normalizing factors for G, $G^\sharp$ are chosen as in Theorem 3.3.4 for the representations with unitary central character.

Let $\sigma \in \Pi(M)$. We shall make the following (rather restrictive) hypothesis on σ throughout this section.

Hypothesis 4.0.2. We assume that $\pi := I_P^G(\sigma)$ is irreducible.

4.1. *Embedding of central extensions.*

Proposition 4.1.1. *Let $\sigma \in \Pi(M)$ and $\pi := I_P^G(\sigma) \in \Pi(G)$.*

(i) *Under the identification of Corollary 3.2.4, we have $X^M(\sigma) \hookrightarrow X^G(\pi)$.*

(ii) *Let $\omega \in X^M(\sigma)$, $I_\omega^M \in \mathrm{Isom}_M(\omega\sigma, \sigma)$. Define the operator I_ω^G as the composition of*

$$A_\omega : \omega I_P^G(\sigma) \to I_P^G(\omega\sigma), \quad \varphi \mapsto \omega(\,\cdot\,)\varphi(\,\cdot\,),$$

with $I_P^G(I_\omega^M) : I_P^G(\omega\sigma) \xrightarrow{\sim} I_P^G(\sigma)$. Then $I_\omega^G \in \mathrm{Isom}_G(\omega\pi, \pi)$.

(iii) *We have the following commutative diagram of groups with exact rows:*

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & S^G(\pi) & \longrightarrow & X^G(\pi) & \longrightarrow & 1 \\
& & \Big\| & & \Big\uparrow & & \Big\uparrow & & \\
1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & S^M(\sigma) & \longrightarrow & X^M(\sigma) & \longrightarrow & 1,
\end{array}
$$

where the arrow $S^M(\sigma) \to S^G(\pi)$ is the map $I_\omega^M \mapsto I_\omega^G$ defined above.

Proof. It follows from the definition that $I_\omega^G \in S^G(\pi)$ for all $\omega \in X^M(\sigma)$, hence $X^M(\sigma) \subset X^G(\pi)$. On the other hand, I_ω^G is simply the map $\varphi \mapsto \omega(\,\cdot\,)I_\omega^M(\varphi(\,\cdot\,))$. It is clear that $I_\omega^M \mapsto I_\omega^G$ is a group homomorphism. The commutativity of the diagram is then clear. $\qquad\square$

We denote by $\boldsymbol{K}_0(\Pi_-(S^M(\sigma)))$ the space of virtual characters of $S^M(\sigma)$ generated by the elements of $\Pi_-(S^M(\sigma))$. Similarly, $\boldsymbol{K}_0(\Pi_\sigma)$ denotes the space of virtual characters of $M^\sharp(F)$ generated by the elements of Π_σ. The bijection $\rho \mapsto \pi_\rho^\sharp$ in Theorem 3.1.5 extends to an isomorphism $\boldsymbol{K}_0(\Pi_-(S^M(\sigma))) \xrightarrow{\sim} \boldsymbol{K}_0(\Pi_\sigma)$. Assuming $\pi = I_P^G(\sigma)$ irreducible, we have the analogous isomorphism

$$\boldsymbol{K}_0(\Pi_-(S^G(\pi))) \xrightarrow{\sim} \boldsymbol{K}_0(\Pi_\pi),$$

as well as the linear maps

$$\mathrm{Ind}_{S^M(\sigma)}^{S^G(\pi)} : \boldsymbol{K}_0(\Pi_-(S^M(\sigma))) \to \boldsymbol{K}_0(\Pi_-(S^G(\pi))),$$

$$I_{P^\sharp}^{G^\sharp} : \boldsymbol{K}_0(\Pi_\sigma) \to \boldsymbol{K}_0(\Pi_\pi),$$

given by the usual induction and normalized parabolic induction, respectively. Note that Lemma 3.2.1 is invoked here.

Proposition 4.1.2. *The following diagram commutes*:

$$
\begin{array}{ccc}
\boldsymbol{K}_0(\Pi_-(S^G(\pi))) & \xrightarrow{\ \simeq\ } & \boldsymbol{K}_0(\Pi_\pi) \\
{\scriptstyle\mathrm{Ind}_{SM(\sigma)}^{S^G(\pi)}}\big\uparrow & & \big\uparrow{\scriptstyle I_{P^\sharp}^{G^\sharp}} \\
\boldsymbol{K}_0(\Pi_-(S^M(\sigma))) & \xrightarrow[\ \simeq\ ]{} & \boldsymbol{K}_0(\Pi_\sigma).
\end{array}
$$

To prove this, some harmonic analysis on the groups $S^M(\sigma)$, $S^G(\pi)$ is needed. These groups are infinite; nonetheless, the usual theory carries over, as we are only concerned about the representations in $\Pi_-(S^G(\pi))$, $\Pi_-(S^M(\sigma))$ or their contragredients. The worried reader may reduce $S^G(\pi) \to X^G(\pi)$ (respectively $S^M(\sigma) \to X^M(\sigma)$) to a central extension by $\mu_m := \{z \in \mathbb{C}^\times : z^m = 1\}$ for some $m \in \mathbb{Z}$, which is always possible.

Proof. Let $\sigma^\sharp \in \Pi_\sigma$ and $\rho \in \Pi_-(S^M(\sigma))$ be the corresponding element. Define

$$
\tau := \mathrm{Ind}_{SM(\sigma)}^{S^G(\pi)}(\rho) \in \boldsymbol{K}_0(\Pi_-(S^G(\pi))),
$$
$$
\pi^\sharp := I_{P^\sharp}^{G^\sharp}(\sigma^\sharp) \in \boldsymbol{K}_0(\Pi_\pi).
$$

We have to show that τ corresponds to $\pi^\sharp$. To begin with, set

$$
\sigma[I_\omega^M] := \sigma(\cdot) \circ I_\omega^M : M(F) \to \mathrm{Aut}_\mathbb{C}(V_\sigma), \quad I_\omega^M \in S^M(\sigma),
$$

where V_σ is the underlying vector space of σ. Then $(\sigma[I_\omega^M], \sigma)$ is a smooth ω-representation of $M(F)$, that is,

$$
\sigma[I_\omega^M](xy) = \omega(y)\sigma[I_\omega^M](x)\sigma(y), \quad x, y \in M(F).
$$

This notion appears in the study of automorphic induction, and more generally it fits into the formalism of twisted endoscopy; cf. [Lemaire 2010, Section 0.4]. It is easy to see that $\Theta_\sigma[I_\sigma^M] := \mathrm{Tr}\,\sigma[I_\sigma^M]$ is well defined as a distribution on $M(F)$. We may restrict $\sigma[I_\omega^M]$ to $M^\sharp(F)$; by abuse of notations, the corresponding distribution, which is also well defined by Proposition 3.1.1, is again denoted by $\Theta_\sigma[I_\sigma^M]$. The same definition applies to π.

Theorem 3.1.5 implies the following identity of distributions on $M^\sharp(F)$:

$$
(14) \qquad \Theta_{\sigma^\sharp} = \frac{1}{|X^M(\sigma)|} \sum_{\omega \in X^M(\sigma)} \mathrm{Tr}(\rho^\vee)(I_\omega^M) \cdot \Theta_\sigma[I_\omega^M],
$$

where $\rho^\vee$ is the contragredient of ρ and $I_\omega^M \in S^M(\sigma)$ is any preimage ω; the summand does not depend on the choice of I_ω^M.

Define $Z^M(\sigma)$ to be the subgroup of elements $\omega \in X^M(\sigma)$ such that every preimage of ω in $S^M(\sigma)$ is central. Define $Z^G(\pi)$ similarly. The sum in (14) can be taken over $Z^M(\sigma)$, since $\rho|_{\mathbb{C}^\times} = \mathrm{id}$ implies that $\mathrm{Tr}(\rho^\vee)$ is zero outside the center.

Let $\pi_1^\sharp \in K_0(\Pi_\pi)$ be the character corresponding to τ. By the same reasoning, there is an identity of distributions on $G^\sharp(F)$

$$(15) \qquad \Theta_{\pi_1^\sharp} = \frac{1}{|X^G(\pi)|} \sum_{\eta \in Z^G(\pi)} \mathrm{Tr}(\tau^\vee)\left(I_\eta^G\right) \cdot \Theta_\pi[I_\eta^G],$$

where $I_\eta^G \in S^G(\pi)$ is any preimage of η, as before. It remains to show that $\Theta_{\pi_1^\sharp}(f^\sharp) = \Theta_{\pi^\sharp}(f^\sharp)$ for every $f^\sharp \in C_c^\infty(G^\sharp(F))$.

Choose a special maximal compact open subgroup $K \subset G(F)$ in good position relative to M, and set $K^\sharp := K \cap G^\sharp(F)$. Equip K and $K^\sharp$ with appropriate Haar measures that are compatible with the Iwasawa decomposition; see [Waldspurger 2003, I.1]. The parabolic descent of characters implies

$$(16) \qquad \Theta_{\pi^\sharp}(f^\sharp) = \Theta_{\sigma^\sharp}(f_{P^\sharp}^\sharp) = \frac{1}{|X^M(\sigma)|} \sum_{\omega \in Z^M(\sigma)} \mathrm{Tr}(\rho^\vee)\left(I_\omega^M\right) \cdot \Theta_\sigma[I_\sigma^M](f_{P^\sharp}^\sharp),$$

where

$$f_{P^\sharp}^\sharp(m) = \delta_P^{1/2}(m) \iint_{U(F) \times K^\sharp} f^\sharp(k^{-1}muk)\, du\, dk, \quad m \in M^\sharp(F).$$

Since $\tau = \mathrm{Ind}_{S^M(\sigma)}^{S^G(\pi)}(\rho)$, we have

$$\mathrm{Tr}(\tau^\vee)(I_\eta^G) = \begin{cases} \dfrac{1}{|X^M(\sigma)|} \displaystyle\sum_{\xi \in X^G(\pi)} \mathrm{Tr}(\rho^\vee)((I_\xi^G)^{-1} I_\eta^M I_\xi^G), & \text{if } \eta \in X^M(\sigma),\ I_\eta^M \mapsto I_\eta^G, \\ 0, & \text{otherwise}, \end{cases}$$

where $I_\xi^G \in S^G(\pi)$ is any preimage of ξ; cf. [Serre 1967, Proposition 20]. This may be rewritten as

$$\mathrm{Tr}(\tau^\vee)(I_\eta^G) = \begin{cases} \dfrac{|X^G(\pi)|}{|X^M(\sigma)|} \mathrm{Tr}(\rho^\vee)(I_\eta^M), & \text{if } \eta \in X^M(\sigma) \cap Z^G(\pi),\ I_\eta^M \mapsto I_\eta^G, \\ 0, & \text{otherwise}. \end{cases}$$

We claim that $\Theta_\pi[I_\omega^G](f^\sharp) = \Theta_\sigma[I_\omega^M](f_{P^\sharp}^\sharp)$ if $I_\sigma^M \mapsto I_\sigma^G \in S^G(\pi)$. First of all, note that $\Theta_\pi[I_\omega^G]$ is the normalized parabolic induction of $\Theta_\sigma[I_\omega^M]$ in the setting of ω-representations [Lemaire 2010, Sections 1.7 and 3.8]. Hence we have the parabolic descent of ω-characters [Lemaire 2010, Théorème 3.8.2], namely,

$$\Theta_\pi[I_\omega^G](f) = \Theta_\sigma[I_\omega^M](f_{P,\omega}), \quad f \in C_c^\infty(G(F)),$$

where

$$f_{P,\omega}(m) = \delta_P^{1/2}(m) \iint_{U(F) \times K} \omega(k) f(k^{-1} m u k) \, du \, dk, \quad m \in M(F).$$

To prove the claim, let us sketch how to "restrict" the ω-character relation above to $G^{\sharp}(F)$. There exists a compact open subgroup $C \subset Z_G(F)$ such that

(i) $C \cap G^{\sharp}(F) = \{1\}$,

(ii) $C \subset K$,

(iii) ω and ω_{σ} are trivial on C, and

(iv) the multiplication maps $C \times G^{\sharp}(F) \hookrightarrow G(F)$ and $C \times M^{\sharp}(F) \hookrightarrow M(F)$ are submersive.

Define $\mathbb{1}_C$ to be the constant function 1 on C. Choose the unique Haar measure on C such that the submersions above preserve measures locally. Given that $f^{\sharp} \in C_c^{\infty}(G^{\sharp}(F))$, we set $f = \mathrm{vol}(C)^{-1} \mathbb{1}_C \otimes f^{\sharp}$ on $C \times G^{\sharp}(F)$, and zero elsewhere. For such f, by inspecting the proof of [Lemaire 2010, Proposition 1.8.1], we may redefine $f_{P,\omega}$ by taking the double integral of $f(k^{-1} m u k)$ over $U(F) \times K^{\sharp}$, so that $f_{P,\omega} = \mathrm{vol}(C)^{-1} \mathbb{1}_C \otimes f_{P^{\sharp}}^{\sharp}$ on $C \times M^{\sharp}(F)$ and zero elsewhere. Therefore

$$\Theta_{\pi}[I_{\omega}^G](f) = \Theta_{\pi}[I_{\omega}^G](f^{\sharp}),$$

$$\Theta_{\sigma}[I_{\sigma}^M](f_{P,\omega}) = \Theta_{\sigma}[I_{\sigma}^M](f_{P^{\sharp}}^{\sharp}).$$

Hence our claim follows.

All in all, (15) becomes

$$\Theta_{\pi_1^{\sharp}}(f^{\sharp}) = \frac{1}{|X^M(\sigma)|} \sum_{\omega \in X^M(\sigma) \cap Z^G(\pi)} \mathrm{Tr}(\rho^{\vee})(I_{\omega}^M) \cdot \Theta_{\sigma}[I_{\omega}^M](f_{P^{\sharp}}^{\sharp}),$$

where $I_{\sigma}^M \in S^M(\sigma)$ is any preimage of ω and $I_{\sigma}^M \mapsto I_{\sigma}^G \in S^G(\pi)$. In comparison with (16), it suffices to show that $\Theta_{\sigma}[I_{\omega}^M](f_{P^{\sharp}}^{\sharp}) = 0$ if $\omega \in Z^M(\sigma)$ but $\omega \notin Z^G(\pi)$. Indeed, for $I \in S^G(\pi)$, we have

$$\Theta_{\sigma}[I_{\omega}^M](f_{P^{\sharp}}^{\sharp}) = \Theta_{\pi}[I_{\omega}^G](f^{\sharp}) = \mathrm{Tr}\,\mathfrak{S}(I_{\omega}^G, f^{\sharp}) = \mathrm{Tr}\,\mathfrak{S}(I^{-1} I_{\omega}^G I, f^{\sharp}),$$

since $\mathfrak{S}$ is a representation of $S^G(\pi) \times G^{\sharp}(F)$. Since I is arbitrary and $\mathfrak{S}|_{\mathbb{C}^{\times} \times \{1\}} = \mathrm{id}$, we conclude that $\Theta_{\sigma}[I_{\omega}^M](f_{P^{\sharp}}^{\sharp}) \neq 0$ only if $\omega \in Z^G(\pi)$. $\qquad\square$

4.2. Description of R-groups. Let $w \in W(M)$ with a chosen representative

$$\tilde{w} \in G^{\sharp}(F).$$

Recall the operator $r_P(\tilde{w}, \sigma) : I_P^G(\sigma) \to I_P^G(\tilde{w}\sigma)$ defined in (5), which is the composition of $R_{w^{-1}Pw|P}(\sigma) : I_P^G(\sigma) \to I_{w^{-1}Pw|P}^G(\sigma)$ with the isomorphism

$$\ell(\tilde{w}) : I_{w^{-1}Pw}^G(\sigma) \to I_P^G(\tilde{w}\sigma), \quad \varphi(\cdot) \mapsto \varphi(\tilde{w}^{-1}\cdot).$$

For $\eta \in (G(F)/G^\sharp(F))^D$, recall the isomorphism A_η defined as

$$\eta I_P^G(\sigma) \to I_P^G(\eta\sigma), \quad \varphi(\cdot) \mapsto \eta(\cdot)\varphi(\cdot).$$

Note that the representations σ, $\eta\sigma$, $\tilde{w}\sigma$, and $\eta\tilde{w}\sigma$ share the same underlying vector space V_σ. As usual, we will compose the operators above after appropriate twists by η or $\tilde{w}$. For example, given η, η', we may define $A_\eta A_{\eta'}$, which is equal to $A_{\eta\eta'} : \eta\eta' I_P^G(\sigma) \to I_P^G(\eta\eta'\sigma)$.

Proposition 4.2.1. *Let*

$$L(\sigma) \subset (M(F)/M^\sharp(F))^D$$

be the subgroup defined in Definition 3.4.2. Upon identifying $M(F)/M^\sharp(F)$ and $G(F)/G^\sharp(F)$, we have

$$L(\sigma) \subset X^G(\pi).$$

If $\sigma \in \Pi_{2,\mathrm{temp}}(M)$, equality holds.

Proof. Let $\eta \in L(\sigma)$. By definition, there exists $w \in W(M)$ with a representative $\tilde{w} \in G^\sharp(F)$ such that $\eta\sigma \simeq \tilde{w}\sigma$. Hence $\eta\pi \simeq I_P^G(\eta\sigma) \simeq I_P^G(\tilde{w}\sigma)$. There is also an isomorphism $r_P(\tilde{w}^{-1}, \tilde{w}\sigma) : I_P^G(\tilde{w}\sigma) \xrightarrow{\sim} I_P^G(\sigma)$. Hence $\eta\pi \simeq \pi$.

Assume $\sigma \in \Pi_{2,\mathrm{temp}}(M)$ and let $\eta \in X^G(\pi)$. Then $I_P^G(\eta\sigma) \simeq I_P^G(\sigma)$. By [Waldspurger 2003, Proposition III.4.1], there exists $w \in W(M)$ with $w\sigma \simeq \eta\sigma$. $\square$

Henceforth we take $w \in \overline{W}_\sigma$. Take any $\eta \in L(\sigma)$ whose class modulo $X^M(\sigma)$ equals $\overline{\Gamma}(w)$ (see Lemma 3.4.3). Then η is of finite order by Proposition 4.2.1; in particular, η is unitary. For any isomorphism

$$i : \eta\sigma \xrightarrow{\sim} \tilde{w}\sigma,$$

we deduce an isomorphism

$$\mathrm{Ad}(i) : S^M(\eta\sigma) \to S^M(\tilde{w}\sigma), \quad I \mapsto iIi^{-1} =: \mathrm{Ad}(i)I.$$

By the obvious identifications $S^M(\sigma) = S^M(\eta\sigma) = S^M(\tilde{w}\sigma)$ (without using i), one can view $\mathrm{Ad}(i)$ as an automorphism of $S^M(\sigma)$. It leaves $\mathbb{C}^\times$ intact and covers the identity map $X^M(\eta\sigma) \xrightarrow{\sim} X^M(\tilde{w}\sigma)$, and hence induces a bijection

$$\Pi_-(S^M(\sigma)) \to \Pi_-(S^M(\sigma)), \quad \rho \mapsto \rho \circ \mathrm{Ad}(i).$$

We shall write

$$\tilde{w}\rho := \rho \circ \mathrm{Ad}(i).$$

The puzzling notation will be justified by Lemma 4.2.3.

Henceforth, we adopt the following convention: for $I_\eta^G \in S^G(\pi)$, we regard $\mathrm{Ad}(I_\eta^G)$ as an automorphism of $S^M(\sigma)$ via the embedding $S^M(\sigma) \hookrightarrow S^G(\pi)$ provided by Proposition 4.1.1.

Lemma 4.2.2. *Let η, $\tilde{w}$, and $i : \eta\sigma \xrightarrow{\sim} \tilde{w}\sigma$ be as above. As automorphisms of $S^M(\sigma)$, we have*

$$\mathrm{Ad}(i) = \mathrm{Ad}(I_\eta^G)$$

for every $I_\eta^G \in S^G(\pi)$ in the preimage of η.

Proof. Given η, the assertion is independent of the choice of I_η^G. Let us consider the specific choice as follows:

$$I_\eta^G := r_P(\tilde{w}^{-1}, \tilde{w}\sigma) \circ I_P^G(i) \circ A_\eta : \eta I_P^G(\sigma) \to I_P^G(\sigma).$$

By the definition of the embedding $S^M(\sigma) \hookrightarrow S^G(\pi)$ and that of A_η, one sees that $\mathrm{Ad}(A_\eta)$ induces the identity map $S^M(\sigma) \xrightarrow{\sim} S^M(\eta\sigma)$. Similarly, the functorial properties of $r_P(\tilde{w}^{-1}, \cdot)$ imply that $\mathrm{Ad}(r_P(\tilde{w}^{-1}, \tilde{w}\sigma))$ induces the identity map $S^M(\tilde{w}\sigma) \xrightarrow{\sim} S^M(\sigma)$. One can readily check that $\mathrm{Ad}(I_P^G(i))$ induces $\mathrm{Ad}(i) : S^M(\eta\sigma) \xrightarrow{\sim} S^M(\tilde{w}\sigma)$. Hence the assertion follows. $\qquad\square$

Before stating the next result, recall that σ, $\tilde{w}\sigma$, and $\eta\sigma$ share the same underlying space V_σ.

Lemma 4.2.3. *Let $\rho \in \Pi_-(S^M(\sigma))$ and $\sigma^\sharp \in \Pi_\sigma$. By identifying the groups $S^M(\sigma)$, $S^M(\tilde{w}\sigma)$, and $S^M(\eta\sigma)$, the following are equivalent:*

(i) $\rho \in \Pi_-(S^M(\sigma))$ *corresponds to* $\sigma^\sharp \hookrightarrow \sigma|_{M^\sharp}$.

(ii) $\rho \in \Pi_-(S^M(\tilde{w}\sigma))$ *corresponds to* $\tilde{w}\sigma^\sharp \hookrightarrow \tilde{w}\sigma|_{M^\sharp}$.

(iii) $\tilde{w}\rho \in \Pi_-(S^M(\eta\sigma))$ *corresponds to* $\tilde{w}\sigma^\sharp \hookrightarrow \eta\sigma|_{M^\sharp}$.

(iv) $\tilde{w}\rho \in \Pi_-(S^M(\sigma))$ *corresponds to* $\tilde{w}\sigma^\sharp \hookrightarrow \sigma|_{M^\sharp}$.

Proof. Recall that ρ corresponds to $\sigma^\sharp \hookrightarrow \sigma|_{M^\sharp}$ means that $\rho \boxtimes \sigma^\sharp \hookrightarrow \mathfrak{S}(\sigma)$, where $\mathfrak{S}(\sigma)$ is the $S^M(\sigma) \times M^\sharp(F)$-representation on V_σ defined in Theorem 3.1.5.

The first two properties are equivalent by a transport of structure via $\mathrm{Ad}(\tilde{w}) : M^\sharp \to M^\sharp$. The last two properties are evidently equivalent. Finally, the equivalence between the second and the third properties follows by pulling ρ back via $\mathrm{Ad}(i) : S^M(\eta\sigma) \xrightarrow{\sim} S^M(\tilde{w}\sigma)$. $\qquad\square$

Now, recall that $Z^M(\sigma)$ is the projection to $X^M(\sigma)$ of the center of $S^M(\sigma)$. Temporarily fix a preimage I_η^G for every $\eta \in X^G(\pi)$ and define

(17) $\quad Z^M(\sigma)^\perp := \{\eta \in X^G(\pi) : \text{ for all } \omega \in Z^M(\sigma),\ I_\eta^G I_\omega^G = I_\omega^G I_\eta^G\} \supset X^M(\sigma).$

Proposition 4.2.4. *Let σ, w, and η be as before. For any $\sigma^{\sharp} \in \Pi_{\sigma}$ (respectively $\rho \in \Pi_{-}(S^{M}(\sigma))$), we have $\tilde{w}\sigma^{\sharp} \simeq \sigma^{\sharp}$ (respectively $\tilde{w}\rho \simeq \rho$) if and only if $\eta \in Z^{M}(\sigma)^{\perp}$.*

Proof. Let $\rho \in \Pi_{-}(S^{M}(\sigma))$ be the representation corresponding to $\sigma^{\sharp} \in \Pi_{\sigma}$ by Theorem 3.1.5. By Lemma 4.2.3, it suffices to show that $\tilde{w}\rho \simeq \rho$ if and only if $\eta \in Z_{M}(\sigma)^{\perp}$.

The elements in $\Pi_{-}(S^{M}(\sigma))$ are described by a variant of the Stone–von Neumann theorem for the central extension $1 \to \mathbb{C}^{\times} \to S^{M}(\sigma) \to X^{M}(\sigma) \to 1$. Namely, consider the data (L, ρ_0) where

- L is a maximal abelian subgroup of $S^{M}(\sigma)$;
- ρ_0 is an irreducible representation of L such that $\rho_0(z) = z$ for all $z \in \mathbb{C}^{\times} \subset L$.

Then $\rho := \mathrm{Ind}_{L}^{S^{M}(\sigma)}(\rho_0)$ is an element of $\Pi_{-}(S^{M}(\sigma))$. Every $\rho \in \Pi_{-}(S^{M}(\sigma))$ arises in this way. Moreover, the isomorphism class of ρ is determined by its central character. These facts are standard consequences of Mackey's theory. See [Kazhdan and Patterson 1984, 0.3] and the remark after Proposition 4.1.2.

We have $\tilde{w}\rho = \rho \circ \mathrm{Ad}(I_{\eta}^{G})$ by Lemma 4.2.2. To conclude the proof, it suffices to show that $\mathrm{Ad}(I_{\eta}^{G})$ fixes the central character of ρ if and only if $\eta \in Z^{M}(\sigma)^{\perp}$. This is immediate. $\square$

Corollary 4.2.5. *Assume $\sigma \in \Pi_{2,\mathrm{temp}}(M)$ and $\sigma^{\sharp} \in \Pi_{\sigma}$. Then we have $L(\sigma^{\sharp}) = Z^{M}(\sigma)^{\perp}$, and the map Γ in Proposition 3.4.5 is an isomorphism*

$$\Gamma : R_{\sigma^{\sharp}} \to Z^{M}(\sigma)^{\perp} / X^{M}(\sigma), \quad w W_{\sigma^{\sharp}}^{0} \mapsto \eta X^{M}(\sigma)$$

where $w \in W_{\sigma^{\sharp}}$ and $\eta \in Z^{M}(\sigma)^{\perp}$ satisfy the relation

$$w\sigma \simeq \eta\sigma.$$

Proof. This results immediately from the definition of $L(\sigma^{\sharp})$. $\square$

4.3. *Cocycles.*

Definition 4.3.1. Suppose for a moment that H is a finite group and N is a normal subgroup of H. Let ρ be an irreducible representation of N and assume that $h\rho := \rho \circ \mathrm{Ad}(h)^{-1} \simeq \rho$ for all $h \in H$. This is a necessary condition for extending ρ to an irreducible representation of H, but not sufficient in general. Recall the following construction of an obstruction $\boldsymbol{c}_{\rho} \in H^{2}(H/N, \mathbb{C}^{\times})$ for extending ρ, where $\mathbb{C}^{\times}$ is equipped with the trivial H/N-action. We can choose intertwining operators $\rho(h) \in \mathrm{Isom}_{N}(h\rho, \rho)$ for each $h \in H$, such that

$$\rho(nh) = \rho(n)\rho(h), \quad \rho(hn) = \rho(h)\rho(n)$$

for every $h \in H$, $n \in N$. Note that either of the equations above implies the other. There is a $\mathbb{C}^\times$-valued 2-cocycle c_ρ characterized by

$$(18) \qquad \rho(h_1 h_2) = c_\rho(h_1, h_2) \rho(h_1) \rho(h_2), \quad h_1, h_2 \in H.$$

One readily checks that c_ρ factors through $H/N \times H/N$, and thus defines a class $\boldsymbol{c}_\rho \in H^2(H/N, \mathbb{C}^\times)$. This cohomology class only depends on ρ itself.

The formalism can also be generalized to the case where H, N are central extensions of finite groups by $\mathbb{C}^\times$, and $\rho(z) = z \cdot \mathrm{id}$ for all $z \in \mathbb{C}^\times$.

Let us return to the formalism of the previous subsection. In particular, we assume $P = MU \subset G$ and $\sigma \in \Pi_{2,\mathrm{temp}}(M)$ with the underlying vector space V_σ. Set $\pi := I_P^G(\sigma)$ as usual. For every $\eta \in Z^M(\sigma)^\perp$, we fix $w \in W(M)$, a representative $\tilde{w} \in G^\sharp(F)$, and an isomorphism

$$i : \eta\sigma \xrightarrow{\sim} \tilde{w}\sigma.$$

Let $\rho \in \Pi_-(S^M(\sigma))$ be corresponding to $\sigma^\sharp \in \Pi_\sigma$. Proposition 4.2.4 implies that $\tilde{w}\rho \simeq \rho$ for every η as above. Equivalently, $\rho \circ \mathrm{Ad}((I_\eta^G)^{-1}) \simeq \rho$ for every $I_\eta^G \in S^G(\pi)$ in the preimage of η by Lemma 4.2.2. We will use the shorthand

$$^\eta\rho := \rho \circ \mathrm{Ad}((I_\eta^G)^{-1}).$$

As in Definition 4.3.1, one considers the problem of extending ρ to the preimage of $Z^M(\sigma)^\perp$ in $S^G(\pi)$. Recall that $Z^M(\sigma)^\perp / X^M(\sigma) = R_{\sigma^\sharp}$ by Corollary 4.2.5. The goal of this subsection is to describe the obstruction class $\boldsymbol{c}_\rho \in H^2(R_{\sigma^\sharp}, \mathbb{C}^\times)$ so obtained.

Recall that in Theorem 3.1.5, we have defined an $S^M(\sigma) \times M^\sharp(F)$-representation $\mathfrak{S} = \mathfrak{S}(\sigma)$ on V_σ. Analogously, we define $\mathfrak{S}(\eta\sigma)$ and $\mathfrak{S}(\tilde{w}\sigma)$; all of them are realized on V_σ. We fix an embedding $\iota : \rho \boxtimes \sigma^\sharp \hookrightarrow \mathfrak{S}(\sigma)$ of $S^M(\sigma) \times M^\sharp(F)$-representations. By Lemma 4.2.3, the same map gives $\iota : \rho \boxtimes \tilde{w}\sigma^\sharp \hookrightarrow \mathfrak{S}(\tilde{w}\sigma)$ and $\iota : \rho \boxtimes \sigma^\sharp \hookrightarrow \mathfrak{S}(\eta\sigma)$ with appropriate equivariances.

Lemma 4.3.2. *For η, $\tilde{w}$, and $\sigma^\sharp$ fixed as before, we define $\mathfrak{S}'(\eta\sigma)$ to be the $S^M(\eta\sigma) \times M^\sharp(F)$-representation on V_σ defined by*

$$\mathfrak{S}'(\eta\sigma)(I, x) = \mathfrak{S}(\eta\sigma)(\mathrm{Ad}(I_\eta^G)^{-1}I, x), \quad I \in S^M(\eta\sigma), x \in M^\sharp(F).$$

Then the map ι induces an embedding of $S^M(\eta\sigma) \times M^\sharp(F)$-representations

$$\iota : {}^\eta\rho \boxtimes \sigma^\sharp \hookrightarrow \mathfrak{S}'(\eta\sigma),$$

and there exists a unique equivariant isomorphism

$$\alpha \boxtimes \sigma^\sharp(\tilde{w})^{-1} : {}^\eta\rho \boxtimes \sigma^\sharp \xrightarrow{\sim} \rho \boxtimes \tilde{w}\sigma^\sharp,$$

for some $\alpha \in \mathrm{Isom}_{S^M(\sigma)}(^\eta\rho, \rho)$ and $\sigma^\sharp(\tilde{w}) \in \mathrm{Isom}_{M^\sharp}(\tilde{w}\sigma^\sharp, \sigma^\sharp)$, which makes the following diagram commutative:

$$
\begin{array}{ccc}
\mathfrak{S}'(\eta\sigma) & \xrightarrow[\simeq]{i} & \mathfrak{S}(\tilde{w}\sigma) \\
\big\uparrow{\scriptstyle\iota} & & \big\uparrow{\scriptstyle\iota} \\
^\eta\rho \boxtimes \sigma^\sharp & \xrightarrow[\alpha\boxtimes\sigma^\sharp(\tilde{w})^{-1}]{} & \rho \boxtimes \tilde{w}\sigma^\sharp.
\end{array}
$$

Observe that the pair $(\alpha, \sigma^\sharp(\tilde{w})^{-1})$ is unique up to $\{(z, z^{-1}) : z \in \mathbb{C}^\times\}$.

Proof. The $S^M(\eta\sigma)$-action on $\mathfrak{S}'(\eta\sigma)$ makes i equivariant. The leftmost vertical arrow comes from the original embedding $\iota : \rho\boxtimes\sigma^\sharp \hookrightarrow \mathfrak{S}(\eta\sigma)$ by an $\mathrm{Ad}(I^G_\eta)^{-1}$-twist. The images of the vertical arrows are characterized as the $\sigma^\sharp$ (respectively $\tilde{w}\sigma^\sharp$)-isotypic parts under the $M^\sharp(F)$-action. Proposition 4.2.4 implies that $\sigma^\sharp \simeq \tilde{w}\sigma^\sharp$. Therefore there must exist an equivariant isomorphism $^\eta\rho \boxtimes \sigma^\sharp \xrightarrow{\sim} \rho \boxtimes \tilde{w}\sigma^\sharp$ that makes the diagram commute. Such an isomorphism must be of the form $\alpha \boxtimes \sigma^\sharp(\tilde{w})^{-1}$. $\qquad\square$

Lemma 4.3.3. *Write $r_P := r_P(\tilde{w}, \sigma)$ and $r_{P^\sharp} := r_{P^\sharp}(\tilde{w}, \sigma^\sharp)$. There is a commutative diagram*

$$
\begin{array}{ccc}
I^G_P(\sigma) & \xrightarrow{r_P} & I^G_P(\tilde{w}\sigma) \\
\big\uparrow{\scriptstyle I^{G^\sharp}_{P^\sharp}(\iota)} & & \big\uparrow{\scriptstyle I^{G^\sharp}_{P^\sharp}(\iota)} \\
\rho \boxtimes I^{G^\sharp}_{P^\sharp}(\sigma^\sharp) & \xrightarrow[\mathrm{id}\boxtimes r_{P^\sharp}]{} & \rho \boxtimes I^{G^\sharp}_{P^\sharp}(\tilde{w}\sigma^\sharp)
\end{array}
$$

whose arrows are equivariant for the $S^M(\tilde{w}\sigma) \times G^\sharp(F)$ and $S^M(\sigma) \times G^\sharp(F)$-actions.

Proof. Without loss of generality, we may assume $\rho = \mathrm{Hom}_{M^\sharp}(\sigma^\sharp, \sigma)$, that is, the multiplicity space. The embedding $\iota : \rho\boxtimes\sigma^\sharp \hookrightarrow \sigma$ can be taken to be $\epsilon \otimes v \mapsto \epsilon(v)$. Then the commutativity of the diagram follows by applying Theorem 3.3.4 to each $\epsilon \in \mathrm{Hom}_{M^\sharp}(\sigma^\sharp, \sigma)$. The equivariance of the horizontal arrows results from Theorem 3.3.4 and the functorial properties of $r_P(\tilde{w}, \cdot)$, $r_{P^\sharp}(\tilde{w}, \cdot)$. $\qquad\square$

Lemma 4.3.4. *With the notations of Lemma 4.3.2, there is a commutative diagram*

$$
\begin{array}{ccc}
I^G_P(\eta\sigma) & \xrightarrow{r_P(\tilde{w},\sigma)^{-1}\circ I^G_P(i)} & I^G_P(\sigma) \\
\big\uparrow{\scriptstyle I^{G^\sharp}_{P^\sharp}(\iota)} & & \big\uparrow{\scriptstyle I^{G^\sharp}_{P^\sharp}(\iota)} \\
^\eta\rho \boxtimes I^{G^\sharp}_{P^\sharp}(\sigma^\sharp) & \xrightarrow[\alpha\boxtimes R_{P^\sharp}(\tilde{w},\sigma^\sharp)^{-1}]{} & \rho \boxtimes I^{G^\sharp}_{P^\sharp}(\sigma^\sharp),
\end{array}
$$

where we set $R_{P^\sharp}(\tilde{w}, \sigma^\sharp) := \sigma^\sharp(\tilde{w}) \circ r_{P^\sharp}(\tilde{w}, \sigma^\sharp)$, by using the pair $(\alpha, \sigma^\sharp(\tilde{w})^{-1})$ of isomorphisms in Lemma 4.3.2.

Proof. This is the concatenation of the diagram in Lemma 4.3.3 and the one in Lemma 4.3.2, after applying $I_{P^\sharp}^{G^\sharp}(\,\cdot\,)$. □

Proposition 4.3.5. *Let c_ρ be the obstruction of extending ρ to the preimage of $Z^M(\sigma)^\perp$ in $S^G(\pi)$, and $c_{\sigma^\sharp}$ the class attached to $R_{\sigma^\sharp}$ in (7). Then we have*

$$c_\rho = c_{\sigma^\sharp}^{-1}$$

in $H^2(R_{\sigma^\sharp}, \mathbb{C}^\times)$.

Proof. Fix $\iota : \rho \boxtimes \sigma^\sharp \hookrightarrow \sigma$. Also fix a set of representatives $\tilde{w} \in G^\sharp(F)$ for each $w \in R_{\sigma^\sharp}$. For each η, together with the auxiliary choice $i : \eta\sigma \xrightarrow{\sim} \tilde{w}\sigma$, the top row of the diagram in Lemma 4.3.4 gives an operator $I_\eta^G \circ A_\eta^{-1} : I_P^G(\eta\sigma) \xrightarrow{\sim} I_P^G(\sigma)$ for some $I_\eta^G \in S^G(\pi)$. The isomorphism $A_\eta : \eta I_P^G(\sigma) \xrightarrow{\sim} I_P^G(\sigma)$ has no effect after restriction. Therefore Lemma 4.3.4 asserts that I_η^G is pulled-back to $\alpha \boxtimes R_{P^\sharp}(\tilde{w}, \sigma^\sharp)^{-1}$ under $I_{P^\sharp}^{G^\sharp}(\iota)$.

Now we can forget i and vary I_η^G in the preimage of η in $S^G(\pi)$, which is a $\mathbb{C}^\times$-torsor. Regard $\alpha = \alpha(I_\eta^G)$ as a function of I_η^G; it is well defined once we have pinned down the operator $\sigma^\sharp(\tilde{w})$ coupled with α.

Suppose that η is replaced by $\eta\omega$, where $\omega \in X^M(\sigma)$; accordingly, I_η^G is replaced by $I_\eta^G I_\omega^G$, where $I_\omega^M \in S^M(\sigma)$ lies in the preimage of ω and $I_\omega^M \mapsto I_\omega^G$. This does not affect the chosen data $\tilde{w}$ and ι. On the other hand, the diagram in Lemma 4.3.4 says that $\alpha \boxtimes R_{P^\sharp}(\tilde{w}, \sigma^\sharp)^{-1}$ is replaced by

$$\alpha \circ \rho(I_\omega^M) \boxtimes R_{P^\sharp}(\tilde{w}, \sigma^\sharp)^{-1}.$$

It follows that we can pin down the operators $\sigma^\sharp(\tilde{w})$, and introduce a well-defined function

$$I_\eta^G \mapsto \alpha(I_\eta^G) \in \mathrm{Isom}_{S^M(\sigma)}(^\eta\rho, \rho),$$

for every I_η^G in the preimage of $\eta \in Z^M(\sigma)^\perp$ in $S^G(\pi)$, such that

- I_η^G is pulled-back to $\alpha(I_\eta^G) \boxtimes R_{P^\sharp}(\tilde{w}, \sigma^\sharp)^{-1}$ under $I_{P^\sharp}^{G^\sharp}(\iota)$;
- $\alpha(I_\eta^G I_\omega^G) = \alpha(I_\eta^G)\rho(I_\omega^M)$ for every $\omega \in X^M(\sigma)$ and I_ω^M in its preimage.

Such a family of intertwining operators meets the requirements of Definition 4.3.1. Thus the obstruction can be accounted by the $\mathbb{C}^\times$-valued 2-cocycle c_ρ given by

$$\alpha(I_\xi^G I_\eta^G) = c_\rho(w_\xi, w_\eta)\alpha(I_\xi^G)\alpha(I_\eta^G), \quad \xi, \eta \in Z^M(\sigma)^\perp,$$

where $w_\eta \in R_{\sigma^\sharp}$ denotes the element determined by η as in Corollary 4.2.5. The same is true for w_ξ.

On the other hand, write $w_\eta \mapsto \tilde{w}_\eta$ for the map that picks the chosen representative for $w_\eta \in R_{\sigma^\sharp}$. Equation (7) defines a 2-cocycle $c_{\sigma^\sharp}$. For every $\xi, \eta \in Z^M(\sigma)^\perp$, we obtain

$$R_{P^\sharp}(\tilde{w}_{\xi\eta}, \sigma^\sharp) = R_{P^\sharp}(\tilde{w}_{\eta\xi}, \sigma^\sharp) = c_{\sigma^\sharp}(w_\eta, w_\xi) R_{P^\sharp}(\tilde{w}_\eta, \sigma^\sharp) R_{P^\sharp}(\tilde{w}_\xi, \sigma^\sharp).$$

All in all, the pull-back of $I_\xi^G I_\eta^G$ by $I_{P^\sharp}^{G^\sharp}(\iota)$ equals

$$c_\rho(w_\xi, w_\eta) c_{\sigma^\sharp}(w_\eta, w_\xi)^{-1} \cdot \text{(the pull-back of } I_\xi^G) \circ \text{(the pull-back of } I_\eta^G).$$

Therefore $c_\rho(w_\xi, w_\eta) = c'_{\sigma^\sharp}(w_\xi, w_\eta) := c_{\sigma^\sharp}(w_\eta, w_\xi)$. It is routine to check that $c'_{\sigma^\sharp} : (R_{\sigma^\sharp})^2 \to \mathbb{C}^\times$ is also a 2-cocycle. Denote by $\boldsymbol{c}'_{\sigma^\sharp}$ the cohomology class of $c'_{\sigma^\sharp}$. It remains to show that $\boldsymbol{c}'_{\sigma^\sharp} = \boldsymbol{c}_{\sigma^\sharp}^{-1}$. We use the following observation: let A be a finite abelian group acting trivially on $\mathbb{C}^\times$; we claim that there is an injective group homomorphism

$$\text{comm} : H^2(A, \mathbb{C}^\times) \to \text{Hom}\left(\bigwedge^2 A, \mathbb{C}^\times\right), \quad \boldsymbol{c} \mapsto [x \wedge y \mapsto c(y, x) c(x, y)^{-1}],$$

where c is any 2-cocycle representing the class $\boldsymbol{c}$. Indeed, let

$$1 \to \mathbb{C}^\times \to \tilde{A} \to A \to 1$$

be the central extension corresponding to $\boldsymbol{c}$. Then $(x, y) \mapsto c(y, x) c(x, y)^{-1}$ is just the commutator pairing of this central extension. The injectivity results from the elementary fact that such an extension splits if and only if $\tilde{A}$ is commutative.

Apply this to $A = R_{\sigma^\sharp}$. Since $\text{comm}(\boldsymbol{c}'_{\sigma^\sharp}) = \text{comm}(\boldsymbol{c}_{\sigma^\sharp}^{-1})$, we deduce $\boldsymbol{c}'_{\sigma^\sharp} = \boldsymbol{c}_{\sigma^\sharp}^{-1}$, as asserted. $\qquad\square$

5. The inner forms of SL(N)

5.1. *The groups.* Fix $N \in \mathbb{Z}_{\geq 1}$ and let $G^* := \text{GL}_F(N)$. Let A be a central simple algebra over F of dimension N^2. There exist $n \in \mathbb{Z}_{\geq 1}$ and a central division algebra D over F satisfying

$$n^2 \cdot \dim_F D = N^2,$$

such that A is isomorphic to $\text{End}_D(D^n)$. The division F-algebra D is uniquely determined by A. We put

$$\text{Nrd} := \text{the reduced norm of } A,$$

$$\text{GL}_D(n) := A^\times,$$

$$\text{SL}_D(n) := \text{Ker}(\text{Nrd} : A^\times \to \mathbb{G}_m).$$

We can regard $A^\times$ as a reductive F-group. It is well known that $A \mapsto A^\times$ induces a bijection between the central simple F-algebras of dimension N^2 and the inner

forms of G^*. Given A, or equivalently, given (n, D) as above, we shall always write

$$G := GL_D(n).$$

Under an inner twist $\psi : G \times_F \bar{F} \xrightarrow{\sim} G^* \times_G \bar{F}$, the determinant map $\det : G^* \to \mathbb{G}_m$ corresponds to $\mathrm{Nrd} : G \to \mathbb{G}_m$. Since the parametrization of the inner forms of an F-group G^* only depend on G^*_{AD}, the map $A \mapsto SL_D(n)$ establishes a bijection between the central simple F-algebras of dimension N^2 and the inner forms of $SL_N(F)$. We write

$$G^\sharp := SL_D(n) = G_{der}.$$

Note that $G(F)/G^\sharp(F) = (G/G^\sharp)(F) = F^\times$, since $H^1(F, G^\sharp)$ is trivial by the Hasse principle.

As mentioned in Section 3.5, the inner twist ψ gives a correspondence between Levi subgroups: the Levi subgroups of G is of the form

$$M = \prod_{i=1}^r GL_D(n_i), \quad n_1 + \cdots + n_r = n,$$

and the corresponding Levi subgroup of G^*, well defined up to conjugacy, is simply

$$M^* = \prod_{i=1}^r GL_F(n_i \cdot \dim_F D).$$

The L-groups of G and $G^\sharp$ are easily described. We have

$$\widehat{G} = \widehat{G^*} = GL(N, \mathbb{C}),$$
$$\widehat{G^\sharp} = PGL(N, \mathbb{C}),$$
$$\widehat{G}_{SC} = (\widehat{G^\sharp})_{SC} = SL(N, \mathbb{C}),$$
$$Z_{\widehat{G}_{SC}} = Z_{(\widehat{G^\sharp})_{SC}} = \mu_N(\mathbb{C}) := \{z \in \mathbb{C}^\times : z^N = 1\}.$$

These complex groups are endowed with the trivial Galois action, thus $^L G = \widehat{G} \times W_F$ and $^L G^\sharp = \widehat{G^\sharp} \times W_F$. The inclusion $G^\sharp \hookrightarrow G$ is dual to the quotient homomorphism $GL(N, \mathbb{C}) \to PGL(N, \mathbb{C})$.

It is also possible to describe the characters $\chi_G = \chi_{G^\sharp}$ in (13) explicitly. Observe that Γ_F acts trivially on $Z_{\widehat{G}_{SC}}$, and one can identify the Pontryagin dual of $Z_{\widehat{G}_{SC}} = \mu_N(\mathbb{C})$, denoted by $Z^D_{\widehat{G}_{SC}}$, with $\mathbb{Z}/N\mathbb{Z}$: a class $e \in \mathbb{Z}/N\mathbb{Z}$ corresponds to the character $z \mapsto z^e$. For the inner form $G = GL_D(n)$ of $G^* = GL_F(N)$, we have

(19) $\chi_G \in Z^D_{\widehat{G}_{SC}}$ corresponds to $(n \bmod N) \in \mathbb{Z}/N\mathbb{Z}$.

Later on, the results of Section 4 will be applied to the tempered representations of $G(F)$. This is justified by the following general result.

Theorem 5.1.1 [Sécherre 2009]. *Let $P = MU$ be a parabolic subgroup of G and $\sigma \in \Pi_{\text{unit}}(M)$. Then $I_P^G(\sigma)$ is irreducible. In particular, Hypothesis 4.0.2 is satisfied by σ.*

Note that the tempered case is already established in [Deligne et al. 1984].

5.2. *Local Langlands correspondences.* This subsection is a summary of [Hiraga and Saito 2012, Chapter 11].

Local Langlands correspondence for $\text{GL}_D(n)$. Using the local Langlands correspondence for G^*, we can define the notion of G^*-generic elements in $\Phi(G)$: a parameter $\phi \in \Phi(G) \subset \Phi(G^*)$ is called G^*-generic if it parametrizes a generic representation of $G^*(F)$. This defines a subset $\Phi_{G^*\text{-gen}}(G)$ of $\Phi(G)$.

Theorem 5.2.1 [Hiraga and Saito 2012, Lemmas 11.1 and 11.2]. *Let $G = \text{GL}_D(n)$ and $G^* = \text{GL}_F(N)$ as in Section 5.1. There exists a subset $\Pi_{G^*\text{-gen}}(G)$ of $\Pi(G)$ satisfying*

- $\Pi_{G^*\text{-gen}}(G) \supset \Pi_{\text{temp}}(G)$,
- $\Pi_{G^*\text{-gen}}(G)$ *is stable under twists by* $(G(F)/G^\sharp(F))^D$,

and a canonically defined bijection between $\Pi_{G^*\text{-gen}}(G)$ and $\Phi_{G^*\text{-gen}}(G)$, denoted by $\pi \leftrightarrow \phi$, such that

$$
\begin{array}{ccc}
\Pi_{G^*\text{-gen}}(G) & \xleftarrow{\ \sim\ } & \Phi_{G^*\text{-gen}}(G) \\
\big\uparrow & & \big\uparrow \\
\Pi_{\text{temp}}(G) & \xleftarrow{\ \sim\ } & \Phi_{\text{bdd}}(G) \\
\big\uparrow & & \big\uparrow \\
\Pi_{2,\text{temp}}(G) & \xleftarrow{\ \sim\ } & \Phi_{2,\text{bdd}}(G).
\end{array}
$$

The correspondence satisfies the following compatibility properties.

(i) *When $G = G^*$, the usual Langlands correspondence for $\text{GL}_F(N)$ is recovered.*

(ii) *Given $\pi \leftrightarrow \phi$ and $\boldsymbol{a} \in H^1_{\text{cont}}(W_F, Z_{\widehat{G}})$, let η be the character of $G(F)$ deduced from $\boldsymbol{a}$ by local class field theory. Then we have $\omega\pi \leftrightarrow a\phi$.*

(iii) *Given a Levi subgroup $M = \prod_{i \in I} \text{GL}_D(n_i)$ of G, let $\sigma := \boxtimes_{i \in I} \sigma_i \in \Pi_{\text{temp}}(M)$. Let $\phi_M \in \Phi_{\text{bdd}}(M)$ such that $\sigma \leftrightarrow \phi_M$ and let ϕ be the composition of ϕ_M with some L-embedding ${}^L M \hookrightarrow {}^L G$. Then, for any $P \in \mathscr{P}(M)$, we have*

$$I_P^G(\sigma) \leftrightarrow \phi.$$

Note that, in the last assertion, $I_P^G(\sigma)$ is irreducible according to Theorem 5.1.1.

The definitions of $\Pi_{G^*\text{-gen}}(G)$ and $\pi \leftrightarrow \phi$ are based upon the local Langlands correspondence for G^* and the Jacquet–Langlands correspondence for essentially

square-integrable representations. We refer the reader to [Hiraga and Saito 2012, Section 11] for details; the compatibility properties are also implicit. Only the tempered/bounded case of the theorem will be used in this article.

Local Langlands correspondence for $SL_D(n)$. Let $G = GL_D(n)$ and $G^\sharp = G_{der} = SL_D(n)$, so that the formalism in Section 3 is applicable. The idea is to define the packets $\Pi_{\phi^\sharp}$ via restriction, by combining the results in Sections 3.1 and 3.5. Let $\Phi_{G^*\text{-gen}}(G^\sharp)$ be the set of $\phi^\sharp \in \Phi(G^\sharp)$ such that $\phi \in \Phi_{G^*\text{-gen}}(G)$ for some lifting ϕ of $\phi^\sharp$ (hence for all liftings, since twisting by characters does not affect G^*-genericity). For any $\phi^\sharp \in \Phi_{G^*\text{-gen}}(G^\sharp)$, define the corresponding packet by

$$\Pi_{\phi^\sharp} := \Pi_\pi, \quad \pi \leftrightarrow \phi \text{ for some lifting } \phi \in \Phi_{G^*\text{-gen}}(G).$$

By Proposition 3.1.2 and Theorem 3.5.1, the definition of $\Pi_{\phi^\sharp}$ does not depend on the choice of lifting.

On the other hand, set

$$\Pi_{G^*\text{-gen}}(G^\sharp) = \bigsqcup_\pi \Pi_\pi,$$

where π ranges over the $(G(F)/G^\sharp(F))^D$-orbits in $\Pi_{G^*\text{-gen}}(G)$. Our version of the local Langlands correspondence for $G^\sharp$ is stated as follows.

Theorem 5.2.2 [Hiraga and Saito 2012, Chapter 12]. *We have*

$$\Pi_{temp}(G^\sharp) \subset \Pi_{G^*\text{-gen}}(G^\sharp),$$

and there is a decomposition

$$(20) \qquad \Pi_{G^*\text{-gen}}(G^\sharp) = \bigsqcup_{\phi^\sharp \in \Phi_{G^*\text{-gen}}(G^\sharp)} \Pi_{\phi^\sharp},$$

which restricts to

$$\Pi_{temp}(G^\sharp) = \bigsqcup_{\phi^\sharp \in \Phi_{temp}(G^\sharp)} \Pi_{\phi^\sharp},$$

$$\Pi_{2,temp}(G^\sharp) = \bigsqcup_{\phi^\sharp \in \Phi_{2,temp}(G^\sharp)} \Pi_{\phi^\sharp}.$$

Proof. The assertion follows from Proposition 3.1.3, Theorems 5.2.1 and 3.5.1, and Proposition 3.1.2. $\square$

Note that each packet $\Pi_{\phi^\sharp}$ is finite. From the endoscopic point of view, in order to justify the correspondence (20), one has to explicate

(i) the internal structure of the packets $\Pi_{\phi^\sharp}$,

(ii) their relation to S-groups,

(iii) the endoscopic character identities for $G^\sharp$.

We will recall the definition of S-groups (or, more precisely, their component groups, called $\mathscr{S}$-groups) in the next subsection, then summarize its relation to the internal structure of packets; this is one of the main results in [Hiraga and Saito 2012]. The character identities will not be used in this article; we refer the interested reader to [Hiraga and Saito 2012, Theorem 12.7].

Normalizing factors. Choose a nontrivial additive character $\psi_F : F \to \mathbb{C}^\times$. Now we can exhibit a canonical family of normalizing factors for G and $G^\sharp$ with respect to ψ_F.

Let us begin with G. According to the construction in Remark 2.2.3, it suffices to consider the case of inducing representations $\sigma \in \Pi_{2,\text{temp}}(M)$, where M is a Levi subgroup of G. When $D = F$, or equivalently, $G = G^* = \text{GL}_F(N)$, the formula in Remark 2.2.2 furnishes a family of normalizing factors in the tempered case, by the Langlands–Shahidi method. To pass to the nonquasisplit case, we use the preservation of μ-functions by Jacquet–Langlands correspondence [Aubert and Plymen 2005, Theorem 7.2] (up to a harmless constant depending only on D and n).

From Theorem 3.3.4, we deduce a canonical family of normalizing factors for $G^\sharp$, at least for the inducing representations $\sigma^\sharp$ whose central character is unitary. In what follows, the normalized intertwining operators for G and $G^\sharp$ are assumed to be defined with respect to these factors.

5.3. *Identification of S-groups.* *Generalities.* To begin with, we summarize the definition of the S-groups in the nonquasisplit case by following [Arthur 2006].

Definition 5.3.1. Let G be a connected reductive F-group. Choose a quasisplit inner twist $\psi : G \times_F \bar{F} \to G^* \times_F \bar{F}$ as well as an F-splitting for $G^*(\bar{F})$ to define the L-groups. Let $\phi \in \Phi(G^*)$. We set

$$S_{\phi,\text{ad}} := Z_{\widehat{G}}(\text{Im}(\phi))/Z_{\widehat{G}}^{\Gamma_F} \xrightarrow{\sim} (Z_{\widehat{G}}(\text{Im}(\phi))Z_{\widehat{G}})/Z_{\widehat{G}} \subset \widehat{G}_{\text{AD}},$$

$$S_{\phi,\text{sc}} := \text{the preimage of } S_{\phi,\text{ad}} \text{ in } \widehat{G}_{\text{SC}},$$

$$\mathscr{S}_\phi := \pi_0(S_{\phi,\text{ad}}, 1),$$

$$\widetilde{\mathscr{S}}_\phi := \pi_0(S_{\phi,\text{sc}}, 1).$$

From the central extension $1 \to Z_{\widehat{G}_{\text{SC}}} \to S_{\phi,\text{sc}} \to S_{\phi,\text{ad}} \to 1$, we obtain another central extension

$$1 \to \widetilde{Z}_\phi \to \widetilde{\mathscr{S}}_\phi \to \mathscr{S}_\phi \to 1,$$

where

$$\widetilde{Z}_\phi := Z_{\widehat{G}_{\text{SC}}}/(Z_{\widehat{G}_{\text{SC}}} \cap S_{\phi,\text{sc}}^0) = \text{Im}[Z_{\widehat{G}_{\text{SC}}} \to \widetilde{\mathscr{S}}_\phi].$$

Remark 5.3.2. When G is an inner form of SL(*N*), we recover the definition of the modified S-groups in [Hiraga and Saito 2012].

The relevance condition of L-parameters intervenes in the following result. Recall that we have defined a character χ_G of $Z_{\widehat{G}_{SC}}^{\Gamma_F}$ in (13).

Lemma 5.3.3 [Hiraga and Saito 2012, Lemma 9.1]. *If* $\phi \in \Phi(G)$, *then*

$$\chi_G : Z_{\widehat{G}_{SC}}^{\Gamma_F} \to \mathbb{C}^\times$$

is trivial on $Z_{\widehat{G}_{SC}} \cap S_{\phi,sc}^0$.

By abuse of notations, the so-obtained character of $Z_{\widehat{G}_{SC}}^{\Gamma_F}/(Z_{\widehat{G}_{SC}} \cap S_{\phi,sc}^0) \subset \widetilde{Z}_\phi$ is still denoted by χ_G. Also note that χ_G depends only on G_{AD}.

Proof. Let us reproduce the proof in [Hiraga and Saito 2012] here. Let $M = M_\phi$ be a minimal Levi subgroup of G through which ϕ factorizes (we used the relevance condition here). By the recollections in Section 3.5, $Z_{\widehat{M}_{sc}}^{\Gamma_F,0}$ is a maximal torus in $S_{\phi,sc}^0$. Therefore

$$S_{\phi,sc}^0 \cap Z_{\widehat{G}_{SC}} = Z_{\widehat{M}_{sc}}^{\Gamma_F,0} \cap Z_{\widehat{G}_{SC}}^{\Gamma_F},$$

and the last group is contained in $\mathrm{Ker}(\chi_G)$ by [Arthur 1999, Corollary 2.2]. $\square$

Consider the familiar situation $G_{der} \subset G^\sharp \subset G$ (cf. Section 3.5), so that we have the Γ_F-equivariant central extension

$$1 \to \widehat{Z}^\sharp \to \widehat{G} \xrightarrow{\mathbf{pr}} \hat{G}^\sharp \to 1.$$

Let $\phi \in \Phi(G)$ and $\phi^\sharp := \mathbf{pr} \circ \phi \in \Phi(G^\sharp)$. The definitions above pertain to $(G^\sharp, \phi^\sharp)$ as well. Set

$$X^G(\phi) := \{\boldsymbol{a} \in H_{cont}^1(W_F, \widehat{Z}^\sharp) : \boldsymbol{a}\phi \sim \phi\}.$$

This is a finite abelian group (cf. the proof of Theorem 3.5.1).

Lemma 5.3.4. *Let* $s \in S_{\phi^\sharp,ad}$, *regarded as an element of* $\widehat{G^\sharp}/Z_{\widehat{G}^\sharp}^{\Gamma_F}$. *Then* s *determines a class* $\boldsymbol{a} \in H_{cont}^1(W_F, \widehat{Z}^\sharp)$ *characterized by*

$$(21) \qquad \tilde{s}\phi(w)\tilde{s}^{-1} = a(w)\phi(w), \quad w \in \mathrm{WD}_F,$$

where

- $\tilde{s} \in \widehat{G}$ *is a lifting of* s,
- $a : W_F \to \widehat{Z}^\sharp$ *is some 1-cocycle representing* $\boldsymbol{a}$, *inflated to* WD_F.

This induces an exact sequence

$$\mathscr{S}_\phi \to \mathscr{S}_{\phi^\sharp} \to X^G(\phi) \to 1.$$

Proof. Choose a lifting $\tilde{s} \in \widehat{G}$. Since s centralizes $\phi^\sharp$, there exists a continuous function $a : \mathrm{WD}_F \to \widehat{Z}^\sharp$ satisfying (21). It is straightforward to check that a is inflated from a 1-cocycle $W_F \to \widehat{Z}^\sharp$. The 1-cocycle a does depend on the choice of $\tilde{s}$, but its class $\boldsymbol{a} \in H^1_{\mathrm{cont}}(W_F, \widehat{Z}^\sharp)$ is uniquely determined by s; it is also obvious that $s \mapsto \boldsymbol{a}$ is a homomorphism. Conversely, every s that satisfies (21) for some $\tilde{s}, a$ clearly belongs to $S_{\phi^\sharp, \mathrm{ad}}$. Hence the image of $s \mapsto \boldsymbol{a}$ equals $X^G(\phi)$, by the very definition of $X^G(\phi)$.

If s is mapped to the trivial class in $X^G(\phi)$, we may choose $\tilde{s}$ so that $\tilde{s}\phi\tilde{s}^{-1} = \phi$; therefore s comes from $S_{\phi, \mathrm{ad}}$, and vice versa. Hence we have an exact sequence of locally compact groups

$$S_{\phi, \mathrm{ad}} \to S_{\phi^\sharp, \mathrm{ad}} \to X^G(\phi) \to 1.$$

By a connectedness argument, we may pass from the S-groups to the $\mathcal{S}$-groups that give the asserted exact sequence. $\qquad\square$

The case of the inner forms of SL(N). Let us revert to the situation where

$$G = \mathrm{GL}_D(n),$$
$$G^* = \mathrm{GL}_F(N),$$
$$G^\sharp = \mathrm{SL}_D(n),$$
$$\chi_G : Z_{\widehat{G}_{\mathrm{SC}}} = \mu_N(\mathbb{C}) \to \mathbb{C}^\times.$$

It is well known that $\mathcal{S}_\phi = \{1\}$ for every $\phi \in \Phi(G^*)$. Indeed, $Z_{\widehat{G}}(\mathrm{Im}(\phi))$ is a principal Zariski open subset in some linear subspace of $\mathrm{Mat}_{N \times N}(\mathbb{C})$, and thus is connected; so is its quotient by $Z_{\widehat{G}}^{\Gamma_F} = Z_{\widehat{G}} = \mathbb{C}^\times$.

Let $\phi^\sharp \in \Phi(G^\sharp)$ with a lifting $\phi \in \Phi(G)$. Hence Lemma 5.3.4 yields a canonical isomorphism

$$\mathcal{S}_{\phi^\sharp} \xrightarrow{\sim} X^G(\phi).$$

Assume henceforth that $\phi^\sharp \in \Phi_{G^*\text{-gen}}(G^\sharp)$, so $\phi \in \Phi_{G^*\text{-gen}}(G)$ as well. The local Langlands correspondence for G (Theorem 5.2.1) is thus applicable. Since the local Langlands correspondence is compatible with twisting by characters, we have $X^G(\phi) = X^G(\pi)$, where $\pi \leftrightarrow \phi$. Therefore we deduce the natural isomorphism

(22) $$\mathcal{S}_{\phi^\sharp} \xrightarrow{\sim} X^G(\pi), \quad \text{where } \pi \leftrightarrow \phi, \ \phi^\sharp = \mathbf{pr} \circ \phi.$$

Also observe that χ_G induces a character of $\widetilde{Z}_{\phi^\sharp}$ by Lemma 5.3.3, since Γ_F acts trivially on $Z_{\widehat{G}_{\mathrm{SC}}}$.

Theorem 5.3.5 [Hiraga and Saito 2012, Lemma 12.5]. *Let $\phi^\sharp \in \Phi_{G^*\text{-gen}}(G^\sharp)$ with a chosen lifting $\phi \in \Phi_{G^*\text{-gen}}(G)$. Let $\pi \in \Pi_{G^*\text{-gen}}(G)$ such that $\pi \leftrightarrow \phi$*

by Theorem 5.2.1. *Then there exists a homomorphism*

$$\Lambda : \widetilde{\mathscr{S}}_{\phi^\sharp} \to S^G(\pi)$$

such that the following diagram is commutative with exact rows:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \widetilde{Z}_{\phi^\sharp} & \longrightarrow & \widetilde{\mathscr{S}}_{\phi^\sharp} & \longrightarrow & \mathscr{S}_{\phi^\sharp} & \longrightarrow & 1 \\
& & \downarrow{\scriptstyle \chi_G} & & \downarrow{\scriptstyle \Lambda} & & \downarrow{\scriptstyle \simeq} & & \\
1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & S^G(\pi) & \longrightarrow & X^G(\pi), & \longrightarrow & 1
\end{array}
$$

where the rightmost vertical arrow is that of (22).

Moreover, Λ *is unique up to* $\mathrm{Hom}(X^G(\pi), \mathbb{C}^\times)$, *that is, up to the automorphisms of the lower central extension, and upon identifying* $\mathscr{S}_{\phi^\sharp}$ *and* $X^G(\pi)$, *this diagram is a push-forward of central extensions by* χ_G.

Note that the assertions about uniqueness and the push-forward are evident; the upshot is the existence of Λ.

Let $\phi^\sharp$, ϕ, π be as above. Put

$$\Pi(\widetilde{\mathscr{S}}_{\phi^\sharp}, \chi_G) := \{\rho \in \Pi(\widetilde{\mathscr{S}}_{\phi^\sharp}) : \text{for all } z \in \widetilde{Z}_{\phi^\sharp}, \ \rho(z) = \chi_G(z)\,\mathrm{id}\}.$$

The homomorphism Λ in Theorem 5.3.5 induces a bijection

$$\Pi(\widetilde{\mathscr{S}}_{\phi^\sharp}, \chi_G) \xrightarrow{\sim} \Pi_-(S^G(\pi)).$$

Recall that in the local Langlands correspondence for $G^\sharp$ (Theorem 5.2.2), the packet $\Pi_{\phi^\sharp}$ attached to $\phi^\sharp$ is defined as Π_π, the set of irreducible constituents of $\pi|_{G^\sharp}$. Combining Theorem 3.1.5 with Theorem 5.3.5, we arrive at the following description of the packet $\Pi_{\phi^\sharp}$.

Corollary 5.3.6. *Let* $\phi^\sharp$, ϕ, π *be as above. Let* $\mathrm{Hom}(X^G(\pi), \mathbb{C}^\times)$ *act on* $\Pi_{\phi^\sharp}$ *via the canonical isomorphisms* $\Pi_{\phi^\sharp} = \Pi_\pi = \Pi_-(S^G(\pi))$. *Then there is a bijection*

$$\Pi(\widetilde{\mathscr{S}}_{\phi^\sharp}, \chi_G) \xrightarrow{\sim} \Pi_{\phi^\sharp},$$

which is canonical up to the $\mathrm{Hom}(X^G(\pi), \mathbb{C}^\times)$-*action on* $\Pi_{\phi^\sharp}$.

When G is quasisplit, χ_G will be trivial and $\Pi(\widetilde{\mathscr{S}}_{\phi^\sharp}, \chi_G) = \Pi(\mathscr{S}_{\phi^\sharp})$; the bijection in Corollary 5.3.6 can then be normalized by choosing a Whittaker datum for $G^\sharp$; cf. [Hiraga and Saito 2012, Chapter 3]. In general, however, there is no reason to expect a canonical choice of the bijection $\Pi(\widetilde{\mathscr{S}}_{\phi^\sharp}, \chi_G) \xrightarrow{\sim} \Pi_{\phi^\sharp}$.

5.4. *Generalization.* Consider the following abstract setting.

- Let M, $M^\sharp$, $M_0^\sharp$ be connected reductive F-groups such that M has a split inner form M^*, and

$$M_{\mathrm{der}} \subset M_0^\sharp \subset M^\sharp \subset M.$$

- For $\pi \in \Pi(M)$, let $S^M(\pi)$ and $X^M(\pi)$ be the groups defined in Section 3.1 relative to $M^\sharp$, and denote by $S_0^M(\pi)$, $X_0^M(\pi)$ the groups defined relative to $M_0^\sharp$.

- Assume that there are subsets $\Pi_{\mathrm{gen}}(M)$ and $\Phi_{\mathrm{gen}}(M)$ of $\Pi(M)$ and $\Phi(M)$, respectively, together with a "local Langlands correspondence" $\pi \leftrightarrow \phi$ between $\Pi_{\mathrm{gen}}(M)$ and $\Phi_{\mathrm{gen}}(M)$ that is compatible with twist by characters, as in Theorem 5.2.1. We define $\Phi_{\mathrm{gen}}(M^\sharp)$ (respectively $\Phi_{\mathrm{gen}}(M_0^\sharp)$) to be the set of L-parameters that lift to $\Pi_{\mathrm{gen}}(M^\sharp)$ (respectively $\Pi_{\mathrm{gen}}(M_0^\sharp)$) via Theorem 3.5.1.

- Assume that $\mathcal{S}_\phi = \{1\}$ for every $\phi \in \Phi_{\mathrm{gen}}(M)$.

Let $\phi \in \Phi_{\mathrm{gen}}(M)$ and $\pi \in \Pi_{\mathrm{gen}}(M)$ such that $\pi \leftrightarrow \phi$. As before, we deduce L-parameters $\phi^\sharp \in \Phi_{\mathrm{gen}}(M^\sharp)$ and $\phi_0^\sharp \in \Phi_{\mathrm{gen}}(M_0^\sharp)$. First of all, let $\bullet$ be one of the subscripts "ad" or "sc". We have

$$S_{\phi^\sharp,\bullet} \subset S_{\phi_0^\sharp,\bullet} \quad \text{and} \quad S_{\phi^\sharp,\bullet}^0 \subset S_{\phi_0^\sharp,\bullet}^0.$$

In view of the definitions in Section 5.3, we deduce natural isomorphisms μ, $\tilde{\mu}$ that fit into the following commutative diagram:

(23)

$$
\begin{array}{ccc}
\mathcal{S}_{\phi^\sharp} & \xrightarrow{\ \mu\ } & \mathcal{S}_{\phi_0^\sharp} \\
\uparrow & & \uparrow \\
\widetilde{\mathcal{S}}_{\phi^\sharp} & \xrightarrow{\ \tilde{\mu}\ } & \widetilde{\mathcal{S}}_{\phi_0^\sharp} \\
\uparrow & & \uparrow \\
\widetilde{Z}_{\phi^\sharp} & \xrightarrow{\ \tilde{\mu}\ } & \widetilde{Z}_{\phi_0^\sharp}.
\end{array}
$$

$$\chi_M \searrow \qquad \swarrow \chi_M$$
$$\mathbb{C}^\times$$

Secondly, we have $X^M(\pi) \subset X_0^M(\pi)$ as subgroups of $M(F)^D$. Consequently, $S^M(\pi) \subset S_0^M(\pi)$. Iterating the arguments for (22), we obtain the commutative diagram

(24)

$$
\begin{array}{ccc}
\mathcal{S}_{\phi^\sharp} & \xrightarrow{\ \simeq\ } & X^M(\pi) \\
\mu \downarrow & & \downarrow \\
\mathcal{S}_{\phi_0^\sharp} & \xrightarrow{\ \simeq\ } & X_0^M(\pi).
\end{array}
$$

Theorem 5.4.1. *Let $\phi, \phi^\sharp, \phi_0^\sharp$, and π be as above. Assume that there exists a homomorphism $\Lambda_0 : \widetilde{\mathscr{S}}_{\phi_0^\sharp} \to S_0^M(\pi)$ such that the following diagram is commutative with exact rows*:

(25)
$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \widetilde{Z}_{\phi_0^\sharp} & \longrightarrow & \widetilde{\mathscr{S}}_{\phi_0^\sharp} & \longrightarrow & \mathscr{S}_{\phi_0^\sharp} & \longrightarrow & 1 \\
 & & \chi_M \downarrow & & \downarrow \Lambda_0 & & \downarrow \simeq & & \\
1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & S_0^M(\pi) & \longrightarrow & X_0^M(\pi) & \longrightarrow & 1.
\end{array}
$$

Then, by setting $\Lambda := \Lambda_0 \circ \tilde{\mu} : \widetilde{\mathscr{S}}_{\phi^\sharp} \to S_0^M(\pi)$ (cf. (23)), the image of Λ lies in $S^M(\pi)$ and the analogous diagram below is commutative:

(26)
$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \widetilde{Z}_{\phi^\sharp} & \longrightarrow & \widetilde{\mathscr{S}}_{\phi^\sharp} & \longrightarrow & \mathscr{S}_{\phi^\sharp} & \longrightarrow & 1 \\
 & & \chi_M \downarrow & & \downarrow \Lambda & & \downarrow \simeq & & \\
1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & S^M(\pi) & \longrightarrow & X^M(\pi) & \longrightarrow & 1.
\end{array}
$$

Consequently, there is a bijection

$$
\Pi(\widetilde{\mathscr{S}}_{\phi^\sharp}, \chi_G) \xrightarrow{\sim} \Pi_{\phi^\sharp},
$$

which is canonical up to the $\mathrm{Hom}(X^G(\pi), \mathbb{C}^\times)$-action on $\Pi_{\phi^\sharp}$.

Proof. Let $\tilde{s} \in \widetilde{\mathscr{S}}_{\phi^\sharp}$, denote by s its image in $\mathscr{S}_{\phi^\sharp}$, and set $s_0 := \mu(s) \in \mathscr{S}_{\phi_0^\sharp}$. Let $\eta \in X_0^M(\pi)$ be the character coming from $\Lambda(\tilde{s}) := \Lambda_0(\tilde{\mu}(\tilde{s})) \in S_0^M(\pi)$. Then, by (25) and (23), η is the image of s_0 under $\mathscr{S}_{\phi_0^\sharp} \xrightarrow{\sim} X_0^M(\pi)$; using (24), it is also the image of s under $\mathscr{S}_{\phi^\sharp} \xrightarrow{\sim} X^M(\pi)$. If we can show $\Lambda(\tilde{s}) \in S^M(\pi)$ for all $\tilde{s}$, the rightmost square in (26) will commute. Since the square

$$
\begin{array}{ccc}
S^M(\pi) & \longrightarrow & X^M(\pi) \\
\big\uparrow & & \big\uparrow \\
S_0^M(\pi) & \longrightarrow & X_0^M(\pi)
\end{array}
$$

is commutative and cartesian for trivial reasons, it follows that $\Lambda(\tilde{s}) \in S^M(\pi)$. Hence the image of Λ lies in $S^M(\pi)$. This also finishes the commutativity of the rightmost square in (26).

Consider the leftmost square in (26). It follows from (23) that, for all $z \in \widetilde{Z}_{\phi^\sharp}$, we have

$$
\Lambda(z) = \Lambda_0(\tilde{\mu}(z)) = \chi_M(\tilde{\mu}(z)) = \chi_M(z).
$$

Hence the leftmost square is commutative as well.

The bijection $\Pi(\widetilde{\mathcal{F}}_{\phi^\sharp}, \chi_G) \xrightarrow{\sim} \Pi_{\phi^\sharp}$ follows easily from the previous assertions, as in the proof of Corollary 5.3.6. □

Remark 5.4.2. The conditions of Theorem 5.4.1 are satisfied if M is a Levi subgroup of $\mathrm{GL}_D(n)$, say of the form

$$M = \prod_{i \in I} \mathrm{GL}_D(n_i), \quad \sum_{i \in I} n_i = n$$

and

$$M_0^\sharp := M_{\mathrm{der}} = \prod_{i \in I} \mathrm{SL}_D(n_i).$$

We simply set $\Pi_{\mathrm{gen}}(M) := \Pi_{M^*\text{-gen}}(M)$ and $\Phi_{\mathrm{gen}}(M) := \Phi_{M^*\text{-gen}}(M)$ by a straightforward generalization of the definitions in Section 5.2. The correspondence $\pi \leftrightarrow \phi$ follows from that in Theorem 5.2.1, applied to each index $i \in I$. The group $M^\sharp$ can be any intermediate group between $M_0^\sharp$ and M, including the important case where

$$M^\sharp := M \cap \mathrm{SL}_D(n) = \left\{ (x_i)_{i \in I} \in M : \prod_{i \in I} \mathrm{Nrd}(x_i) = 1 \right\}.$$

Therefore, Theorem 5.3.5 and Corollary 5.3.6 can be generalized to the Levi subgroups of $\mathrm{SL}_D(n)$.

Indeed, it suffices to verify the commutativity of the diagram (25). Writing $\pi = \boxtimes_{i \in I} \pi_i$ and $\phi = (\phi_i)_{i \in I}$, the results in Section 5.3 applied to each $i \in I$ gives a commutative diagram similar to (25), except that its bottom row is the central extension

(27) $$1 \to (\mathbb{C}^\times)^I \to \prod_{i \in I} S^{\mathrm{GL}_D(n_i)}(\pi_i) \to X_0^G(\pi) \to 1$$

and χ_M is replaced by

$$\prod_{i \in I} \chi_{\mathrm{GL}_D(n_i)} : \widetilde{Z}_{\phi_0^\sharp} \to (\mathbb{C}^\times)^I.$$

To obtain the desired short exact sequence, it remains to take the push-forward of (27) by the multiplication map $(\mathbb{C}^\times)^I \to \mathbb{C}^\times$.

6. The dual R-groups

6.1. *A commutative diagram.* As in Section 5.1, we take

$$G = \mathrm{GL}_D(n), \quad G^* = \mathrm{GL}_F(N), \quad G^\sharp = \mathrm{SL}_D(n).$$

We also fix a Levi subgroup M of G and set $M^\sharp := M \cap G^\sharp$.

To define the dual groups ${}^L G$, ${}^L M$, etc., we fix a quasisplit inner twist $\psi : G \times_F \overline{F} \xrightarrow{\sim} G^* \times_F \overline{F}$ which restricts to a quasisplit inner twist $M \times_F \overline{F} \xrightarrow{\sim} M^* \times_F \overline{F}$,

as well as an F-splitting for G^* that is compatible with M^*. Therefore there is a canonical L-embedding ${}^L M \hookrightarrow {}^L G$. The same is true for ${}^L G^\sharp$ and ${}^L M^\sharp$.

As usual, the natural projections ${}^L G \to {}^L G^\sharp$ and ${}^L M \to {}^L M^\sharp$ are denoted by **pr**. Put

$$A_{\widehat{M^\sharp}} := Z_{\widehat{M^\sharp}} = Z_{\widehat{M^\sharp}}^{\Gamma_F, 0} \hookrightarrow \widehat{G^\sharp}.$$

Consider $\phi_M \in \Phi_{2,\mathrm{bdd}}(M)$. Let ϕ be its composition with ${}^L M \hookrightarrow {}^L G$. Set

$$\phi_M^\sharp := \mathbf{pr} \circ \phi_M \in \Phi_{2,\mathrm{bdd}}(M^\sharp) \quad \text{and} \quad \phi^\sharp := \mathbf{pr} \circ \phi \in \Phi_{\mathrm{bdd}}(G^\sharp).$$

Every $\phi^\sharp \in \Phi_{\mathrm{bdd}}(G^\sharp)$ is obtained in this way (recall Theorem 3.5.1).

The construction of the dual R-group associated to $\phi^\sharp$, denoted by $R_{\phi^\sharp}$, is given as follows. Define

$$\begin{aligned}
N_{\phi^\sharp,\mathrm{ad}} &:= N_{S_{\phi^\sharp,\mathrm{ad}}}(A_{\widehat{M^\sharp}}), \\
\mathfrak{N}_{\phi^\sharp} &:= \pi_0(N_{\phi^\sharp,\mathrm{ad}}, 1), \\
W_{\phi^\sharp} &:= W(S_{\phi^\sharp,\mathrm{ad}}, A_{\widehat{M^\sharp}}) \hookrightarrow W^{\widehat{G}}(\widehat{M}), \\
W_{\phi^\sharp}^0 &:= W(S_{\phi^\sharp,\mathrm{ad}}^0, A_{\widehat{M^\sharp}}) \lhd W_{\phi^\sharp}, \\
R_{\phi^\sharp} &:= W_{\phi^\sharp} / W_{\phi^\sharp}^0.
\end{aligned}$$

The meaning of $W(\cdots, \cdots)$ is as follows: for any pair of complex groups $a \subset A$, the symbol $W(A, a)$ denotes the group $N_A(a)/Z_A(a)$. Note that $W_{\phi^\sharp}^0$ is the Weyl group associated to some root system, as $S_{\phi^\sharp,\mathrm{ad}}^0$ is connected and reductive.

Since the centralizer of $A_{\widehat{M^\sharp}}$ in the connected reductive group $S_{\phi^\sharp,\mathrm{ad}}^0$ is connected, there exists a canonical injection $W_{\phi^\sharp}^0 \hookrightarrow \mathfrak{N}_{\phi^\sharp}$. From the results recalled in Section 3.5, the torus $A_{\widehat{M^\sharp}}$ is a maximal torus in $S_{\phi^\sharp,\mathrm{ad}}^0$. Using the conjugacy of maximal tori, one sees that the inclusion map $N_{\phi^\sharp,\mathrm{ad}} \hookrightarrow S_{\phi^\sharp,\mathrm{ad}}$ induces a canonical isomorphism $\mathfrak{N}_{\phi^\sharp} / W_{\phi^\sharp}^0 \xrightarrow{\sim} \mathcal{S}_{\phi^\sharp}$.

On the other hand, we also have canonical injections

$$\begin{aligned}
\mathcal{S}_{\phi_M^\sharp} &\hookrightarrow \mathcal{S}_{\phi^\sharp}, \\
\mathcal{S}_{\phi_M^\sharp} &\hookrightarrow \mathfrak{N}_{\phi^\sharp}.
\end{aligned}$$

The first one follows from the fact that

$$Z_{\widehat{M^\sharp}}^{\Gamma_F} = Z_{\widehat{G^\sharp}}^{\Gamma_F} Z_{\widehat{M^\sharp}}^{\Gamma_F, 0};$$

see [Arthur 1999, Lemma 1.1]. The injectivity of the second map follows; moreover, its image is characterized as the elements fixing $A_{\widehat{M^\sharp}}$ pointwise.

The relations among these groups are recapitulated in the following result.

Proposition 6.1.1 [Arthur 1989b, Section 7]. *The groups above fit into a commutative diagram*

whose rows and columns are exact.

The arrow $\mathscr{S}_{\phi^\sharp} \to R_{\phi^\sharp}$ is uniquely determined by the other terms in this diagram; cf. the proof of Lemma 6.2.1 below.

The same constructions can be applied to ϕ and ϕ_M. The corresponding objects are denoted by W_ϕ, W_ϕ^0, etc.

Upon identifying $W^{\widehat{G}}(\widehat{M})$ and $W^G(M)$, we can make $W_{\phi^\sharp}$ act on the tempered L-packet $\Pi_{\phi^\sharp}$. For any $\sigma^\sharp \in \Pi_{\phi^\sharp}$, define

$$W_{\phi^\sharp, \sigma^\sharp} := \mathrm{Stab}_{W_{\phi^\sharp}}(\sigma^\sharp),$$
$$W_{\phi^\sharp, \sigma^\sharp}^0 := \mathrm{Stab}_{W_{\phi^\sharp}^0}(\sigma^\sharp),$$
$$R_{\phi^\sharp, \sigma^\sharp} := W_{\phi^\sharp, \sigma^\sharp} / W_{\phi^\sharp, \sigma^\sharp}^0.$$

The last object $R_{\phi^\sharp, \sigma^\sharp}$, viewed as a subgroup of $R_{\phi^\sharp}$, is what we want to compare with the Knapp–Stein R-group $R_{\sigma^\sharp}$.

6.2. *Identification of R-groups.* Retain the notations of the previous subsection and fix a parabolic subgroup $P \in \mathscr{P}(M)$. We shall always make the identification $W^G(M) = W^{G^\sharp}(M^\sharp)$. The results in Section 4 are applicable to tempered representations of $M(F)$ by Theorem 5.1.1.

Henceforth, let $\sigma \in \Pi_{2,\mathrm{temp}}(M)$ (respectively $\pi \in \Pi_{\mathrm{temp}}(G)$) be the representations corresponding to ϕ_M (respectively ϕ) by Theorem 5.2.1. Then we have

$$\pi \simeq I_P^G(\sigma).$$

Recall that we have defined canonical isomorphisms

$$\mathscr{S}_{\phi^{\sharp}} \xrightarrow{\sim} X^{G}(\pi) \quad \text{and} \quad \mathscr{S}_{\phi_{M}^{\sharp}} \xrightarrow{\sim} X^{M}(\sigma)$$

in Section 5.3-5.4. By inspecting the construction in Lemma 5.3.4, we see that these two isomorphisms are compatible with the embeddings $\mathscr{S}_{\phi_{M}^{\sharp}} \hookrightarrow \mathscr{S}_{\phi}$ and $X^{M}(\sigma) \hookrightarrow X^{G}(\sigma)$. Therefore we obtain $\gamma : X^{G}(\pi)/X^{M}(\sigma) \xrightarrow{\sim} \mathscr{S}_{\phi^{\sharp}}/\mathscr{S}_{\phi_{M}^{\sharp}}$.

Lemma 6.2.1. *Define an isomorphism*

$$\widehat{\Gamma}^{-1} : X^{G}(\pi)/X^{M}(\sigma) \xrightarrow{\gamma} \mathscr{S}_{\phi^{\sharp}}/\mathscr{S}_{\phi_{M}^{\sharp}} \xrightarrow{\sim} W_{\phi^{\sharp}}/W_{\phi^{\sharp}}^{0} =: R_{\phi^{\sharp}},$$

where the second arrow is given by Proposition 6.1.1. Then it is characterized by the equation

$$\eta\sigma \simeq w\sigma$$

whenever

$$\widehat{\Gamma}^{-1}(\eta \bmod X^{M}(\sigma)) = w \bmod W_{\phi^{\sharp}}^{0},$$

for all $\eta \in X^{G}(\pi)$ and $w \in W_{\phi^{\sharp}}$.

As the notation suggests, we set $\widehat{\Gamma}$ to be the inverse of $\widehat{\Gamma}^{-1}$.

Proof. The equation $\eta\sigma \simeq w\sigma$ clearly characterizes $\widehat{\Gamma}^{-1}$. Let $\eta \in X^{G}(\pi)$ and $\boldsymbol{a} \in H_{\text{cont}}^{1}(W_{F}, \widehat{Z}^{\sharp})$ which corresponds to η, together with a chosen 1-cocycle a in the cohomology class of $\boldsymbol{a}$. Since $L(\sigma) = X^{G}(\pi)$ by Proposition 4.2.1, there exists $w \in W^{G}(M)$ such that $\eta\sigma \simeq w\sigma$. On the dual side, it implies that there exists $t \in N_{\widehat{G}}(\widehat{M})$ representing w, such that

$$t\phi t^{-1} = a\phi.$$

This implies that $t \bmod Z_{\widehat{G}}$ belongs to $S_{\phi^{\sharp}, \text{ad}}$, and its class $[t]$ in $\mathscr{S}_{\phi^{\sharp}}$ corresponds to η (recall the construction in Lemma 5.3.4).

On the other hand, we have $t \in N_{\phi^{\sharp}, \text{ad}}$ and its class [t] in $\mathfrak{N}_{\phi^{\sharp}}$ is mapped to $[t]$ under the arrow $\mathfrak{N}_{\phi^{\sharp}} \twoheadrightarrow \mathscr{S}_{\phi^{\sharp}}$ in Proposition 6.1.1. One can also apply the arrow $\mathfrak{N}_{\phi^{\sharp}} \twoheadrightarrow W_{\phi^{\sharp}}$ to [t]; since $W_{\phi^{\sharp}}$ is identified as a subgroup of $W^{G}(M)$, the image is simply w.

Upon some contemplation of the diagram in Proposition 6.1.1, one can see that the image of $[t]$ under $\mathscr{S}_{\phi^{\sharp}} \to R_{\phi^{\sharp}}$ is just w modulo $W_{\phi^{\sharp}}^{0}$, completing the proof. $\square$

Recall that we have defined the group $\overline{W}_{\sigma} \subset W^{G}(M)$. In view of Lemma 3.4.3 and Proposition 4.2.1, we have a canonical isomorphism

$$\Gamma : \overline{W}_{\sigma}/W_{\sigma} \xrightarrow{\sim} X^{G}(\pi)/X^{M}(\sigma).$$

This is to be compared with $\widehat{\Gamma} : W_{\phi^{\sharp}}/W_{\phi^{\sharp}}^{0} \xrightarrow{\sim} X^{G}(\pi)/X^{M}(\sigma)$.

Proposition 6.2.2. *We have*

  (i) $\overline{W}_\sigma = W_{\phi^\sharp}$,

 (ii) $W_\sigma = W^0_{\phi^\sharp}$,

(iii) $\Gamma = \widehat{\Gamma}$.

In particular, $R_{\phi^\sharp} \simeq X^G(\pi)/X^M(\sigma)$.

Proof. The first assertion follows from the definition of $\overline{W}_\sigma$ and Theorem 3.5.1. Hence Γ and $\widehat{\Gamma}$ can be regarded as two surjective homomorphisms from $W_{\phi^\sharp}$ onto $X^G(\pi)/X^M(\sigma)$. However, they admit the same characterization (of the form $\eta\sigma \simeq w\sigma$) by Lemmas 6.2.1 and 3.4.3, and hence are equal. This proves the remaining two assertions. $\qquad\square$

Proposition 6.2.3. *For all $\sigma^\sharp \in \Pi_{\phi^\sharp}$, we have that*

  (i) $W_{\sigma^\sharp} = W_{\phi^\sharp,\sigma^\sharp}$,

 (ii) $W^0_{\sigma^\sharp} = W^0_{\phi^\sharp,\sigma^\sharp}$,

(iii) *the restriction of $\widehat{\Gamma}$ to $W_{\phi^\sharp,\sigma^\sharp}/W^0_{\phi^\sharp,\sigma^\sharp}$ induces an isomorphism*

$$W_{\phi^\sharp,\sigma^\sharp}/W^0_{\phi^\sharp,\sigma^\sharp} \xrightarrow{\sim} Z^M(\sigma)^\perp/X^M(\sigma).$$

In particular, $R_{\phi^\sharp,\sigma^\sharp} = R_{\sigma^\sharp}$, *and the isomorphisms Γ, $\widehat{\Gamma}$ from these R-groups onto $Z^M(\sigma)^\perp/X^M(\sigma)$ coincide (recall Proposition 3.4.5 and Corollary 4.2.5).*

Remainder: the group $Z^M(\sigma)^\perp$ above is defined in (17).

Proof. Our proof is based on the previous result. The first assertion follows immediately from the disjointness of tempered L-packets. By Lemma 3.4.1 and the fact that $W_\sigma = W^0_\sigma$, we have

$$\begin{aligned} W^0_{\phi^\sharp,\sigma^\sharp} &= W^0_{\phi^\sharp} \cap W_{\sigma^\sharp} = W_\sigma \cap W_{\phi^\sharp} \\ &= W^0_{\sigma^\sharp} \cap W_{\sigma^\sharp} = W^0_{\sigma^\sharp}. \end{aligned}$$

The second assertion follows and the third assertion is then immediate from Proposition 4.2.4. $\qquad\square$

Note that the proof for the isomorphism $\widehat{\Gamma} : R_{\phi^\sharp,\sigma^\sharp} \xrightarrow{\sim} Z^M(\sigma)^\perp/X^M(\sigma)$ is independent of the Knapp–Stein theory.

The behavior of the local Langlands correspondence (Theorem 5.2.2) for $G^\sharp$ and its Levi subgroups can now be summarized as follows.

Theorem 6.2.4. *Let G, $G^\sharp$ and $P = MU$, $P^\sharp = M^\sharp U$ be as before. For $\phi^\sharp_M$ in $\Phi_{\mathrm{bdd}}(M^\sharp)$, let $\phi^\sharp \in \Phi_{\mathrm{bdd}}(G^\sharp)$ be the composition of $\phi^\sharp_M$ with ${}^L M^\sharp \hookrightarrow {}^L G^\sharp$.*

(i) *For every* $\rho \in \Pi(\widetilde{\mathcal{S}}_{\phi_M^\sharp}, \chi_M)$, *parametrizing an irreducible representation* $\sigma^\sharp \in \Pi_{\phi_M^\sharp}$, *the map*

$$I_{P^\sharp}^{G^\sharp}(\sigma^\sharp),$$

regarded as a virtual character of $G^\sharp(F)$, *corresponds to that of* $\mathrm{Ind}_{\widetilde{\mathcal{S}}_{\phi_M^\sharp}}^{\widetilde{\mathcal{S}}_{\phi_M^\sharp}}(\rho)$.

(ii) *For any* $\sigma^\sharp$ *as above,* $I_{P^\sharp}^{G^\sharp}(\sigma^\sharp)$ *is irreducible if and only if* $Z^M(\sigma)^\perp = X^M(\sigma)$ *for some (equivalently, for any)* $\sigma \in \Pi_{\mathrm{temp}}(M)$ *such that* $\sigma^\sharp \hookrightarrow \sigma|_{M^\sharp}$.

(iii) *If* $\phi_M^\sharp \in \Phi_{2,\mathrm{bdd}}(M^\sharp)$, *we have natural isomorphisms*

$$R_{\phi^\sharp} \simeq X^G(\pi)/X^M(\sigma),$$
$$R_{\phi^\sharp, \sigma^\sharp} \simeq R_{\sigma^\sharp} \simeq Z^M(\sigma)^\perp/X^M(\sigma),$$

where we set $\pi := I_P^G(\sigma) \in \Pi_{\mathrm{temp}}(G)$, *for any choice of* $\sigma \in \Pi_{2,\mathrm{temp}}(M)$ *such that* $\sigma^\sharp \hookrightarrow \sigma|_{M^\sharp}$.

(iv) *For* $\phi_M^\sharp$, $\sigma^\sharp$, *and* ρ *as above, the class* $\boldsymbol{c}_{\sigma^\sharp} \in H^2(R_{\sigma^\sharp}, \mathbb{C}^\times)$ *of* (7) *corresponds to* $\boldsymbol{c}_\rho^{-1}$, *where* $\boldsymbol{c}_\rho \in H^2(R_{\phi^\sharp, \sigma^\sharp}, \mathbb{C}^\times)$ *is the obstruction for extending* ρ *to a representation of the preimage in* $\widetilde{\mathcal{S}}_{\phi^\sharp}$ *of* $R_{\phi^\sharp, \sigma^\sharp}$ *(see* Definition 4.3.1*).*

If $G^\sharp$ *is quasisplit,* $Z^M(\sigma)^\perp = X^G(\pi)$ *and* $\widetilde{R}_{\sigma^\sharp} \to R_{\sigma^\sharp}$ *splits.*

As mentioned in the Introduction, this settles Arthur's conjectures on R-groups for $G^\sharp$.

Proof. The first part is nothing but a special case of Proposition 4.1.2. The second part then results from the proof of Proposition 4.2.4; the independence of the choice of σ is clear. The third part results from Propositions 6.2.2 and 6.2.3. The fourth part is the combination of Proposition 4.3.5 and Theorem 5.4.1.

Finally, $S^G(\pi)$ is commutative when $G^\sharp$ is quasisplit, as $\chi_G = 1$. Hence we have $Z^M(\sigma)^\perp = X^G(\pi)$ and ρ can always be extended in that case. $\qquad\square$

Remark 6.2.5. The decomposition of $I_{P^\sharp}^{G^\sharp}(\sigma^\sharp)$ depends on $\phi_M^\sharp$, but not on the element $\sigma^\sharp$. This is not expected to hold for other groups.

Remark 6.2.6. We have limited ourselves to the tempered representations. However, if the local Langlands correspondence (Theorem 5.2.2) and Theorem 5.3.5 can be extended to Arthur parameters $\psi^\sharp : \mathrm{WD}_F \times \mathrm{SU}(2) \to {}^L G^\sharp$ (see [Arthur 1989b], Section 6]), our results should be applicable to Arthur packets $\Pi_{\psi^\sharp}$ as well, except the part concerning the Knapp–Stein R-groups $R_{\sigma^\sharp}$. Note that the crucial lifting Theorem 3.5.1 also holds for Arthur parameters; see [Labesse 1985, Remarque 8.2].

6.3. *Examples.* The next example on R-groups will be constructed using Steinberg representations, whose definitions are reviewed below.

Definition 6.3.1. For the moment, we assume G to be any connected reductive F-group. Fix a minimal parabolic subgroup P_0 of G. The Steinberg representation St_G of G is the virtual character of $G(F)$ given by

$$\mathrm{St}_G := \sum_{\substack{P \supset P_0 \\ P = MU}} (-1)^{\dim \mathfrak{a}_M^G} I_P^G(\delta_P^{-1/2} \mathbb{1}_M),$$

where the sum ranges over the parabolic subgroups P containing P_0 and $\mathbb{1}_M$ denotes the trivial representation of $M(F)$.

The basic fact [Casselman 1973] is that St_G comes from a smooth irreducible representation in $\Pi_{2,\mathrm{temp}}(G)$, which we denote by the same symbol St_G. It is clearly independent of the choice of P_0.

Lemma 6.3.2. *For G as in Definition 6.3.1 and a subgroup $G^\sharp$ satisfying*

$$G_{\mathrm{der}} \subset G^\sharp \subset G,$$

we have

$$\mathrm{St}_G|_{G^\sharp} \simeq \mathrm{St}_{G^\sharp}.$$

In particular, the group $X^G(\mathrm{St}_G)$ defined in Section 3.1 is trivial.

Proof. Recall the bijection $P \mapsto P^\sharp := P \cap G^\sharp$ between the parabolic subgroups of G and $G^\sharp$. Since $(\mathbb{1}_L)|_{L^\sharp} = \mathbb{1}_{L^\sharp}$ for any Levi subgroup L of G, the first isomorphism follows by comparing the formulas defining St_G and $\mathrm{St}_{G^\sharp}$, together with Lemma 3.2.1. Hence the restriction of St_G to $G^\sharp$ is irreducible. It follows from Theorem 3.1.5 that $X^G(\mathrm{St}_G) = \{1\}$. □

Let us revert to the setting $G = \mathrm{GL}_D(n)$ and $G^\sharp = \mathrm{SL}_D(n)$.

Example 6.3.3. We now set out to construct an example in which $\widetilde{R}_{\sigma^\sharp} \to R_{\sigma^\sharp}$ does not split for every $\sigma^\sharp \hookrightarrow \sigma|_M$.

First of all, there exists $\mathrm{GL}_D(m)$, for some choice of D, m, and a representation $\tau \in \Pi_{2,\mathrm{temp}}(\mathrm{GL}_D(m))$ such that $S^{\mathrm{GL}_D(m)}(\sigma)$ is noncommutative. Indeed, for $m = 1$ and D equal to the quaternion algebra over F, Arthur exhibits [2006, Page 215] an L-parameter $\phi_\tau \in \Phi_{2,\mathrm{temp}}(D^\times)$ such that

- $\widetilde{\mathcal{S}}_{\phi_\tau^\sharp}$ is isomorphic to the quaternion group of order 8;
- $\widetilde{Z}_{\phi_\tau}$ corresponds to $\{\pm 1\}$.

In fact, ϕ_τ factors through a homomorphism $\mathrm{Gal}(K/F) \to \mathrm{PGL}(2, \mathbb{C})$, where K is a biquadratic extension of F, whose image is generated by the elements $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in $\mathrm{PGL}(2, \mathbb{C})$.

Since $\chi_{D^\times}$ is injective on $\widetilde{Z}_{\phi_\tau}$ by (19), Theorem 5.3.5 entails that $S^{D^\times}(\tau)$ is noncommutative.

Take $\eta, \omega \in X^{\mathrm{GL}_D(m)}(\tau)$ so that their preimages in $S^{\mathrm{GL}_D(m)}(\tau)$ do not commute. Let c (respectively d) be the order of η (respectively ω). Put $\mathrm{St} := \mathrm{St}_{\mathrm{GL}_D(m)}$ and take

$$M := \mathrm{GL}_D(m) \times \prod_{\substack{1 \leq i \leq c \\ 1 \leq j \leq d}} \mathrm{GL}_D(m),$$

$$G := \mathrm{GL}_D(m(cd+1)),$$

$$\sigma := \tau \boxtimes \boxtimes_{\substack{1 \leq i \leq c \\ 1 \leq j \leq d}} \eta^{i-1} \omega^{j-1} \mathrm{St},$$

$$\pi := I_P^G(\sigma) \quad \text{for some } P \in \mathcal{P}(M).$$

Here $X^M(\sigma)$ is defined relatively to $M^\sharp := G^\sharp \cap M$ where $G^\sharp = \mathrm{SL}_D(m(cd+1))$. The presence of St forces $X^M(\sigma)$ to be trivial, by Lemma 6.3.2. Hence $\sigma^\sharp := \sigma|_{M^\sharp}$ is irreducible, and it is parametrized by the 1-dimensional character $\chi_M : \widetilde{Z}_{\phi_M^\sharp} \to \mathbb{C}^\times$. According to Theorem 6.2.4, the central extension $\widetilde{R}_{\sigma^\sharp} \twoheadrightarrow R_{\sigma^\sharp}$ splits if and only if ρ can be extended to $\widetilde{\mathcal{S}}_{\phi^\sharp}$. This is the case if and only if $S^G(\pi) \twoheadrightarrow X^G(\pi)$ splits, by Theorem 5.3.5. Hence it suffices to show the noncommutativity of $S^G(\pi)$.

Put $L := \mathrm{GL}_D(m) \times \mathrm{GL}_D(mcd) \in \mathcal{L}^G(M)$ and set $\nu := I_{P \cap L}^L(\sigma)$. We claim $\eta, \omega \in S^L(\nu)$. Indeed, the $\mathrm{GL}_D(m)$-component of L does not cause any problem. As for the $\mathrm{GL}_D(mcd)$-component, take representatives $\tilde{w}_\eta, \tilde{w}_\eta$ in $\mathrm{SL}_D(mcd)$ of the cyclic permutations

$$w_\eta : 1 \to \cdots \to c \to 1,$$

$$w_\omega : 1 \to \cdots \to d \to 1$$

of the indexes i and j, respectively. Then the intertwining operators J_η, J_ω are given by the operators in (5) using $\tilde{w}_\eta, \tilde{w}_\omega$. Furthermore, J_η and J_ω commute with each other; this follows from (6) and the obvious fact that $\tilde{w}_\eta$ and $\tilde{w}_\omega$ can be chosen to commute.

From our choice of η, ω, it follows that the preimages of η, ω in $S^L(\nu)$ do not commute. Since $S^L(\nu) \hookrightarrow S^G(\pi)$ by Proposition 4.1.1, $S^G(\pi)$ is noncommutative, as required.

Example 6.3.4. Now we set out to show that the inclusion $R_{\phi^\sharp, \sigma^\sharp} \subset R_{\phi^\sharp}$ is proper in general. By Theorem 6.2.4 and the notations therein, it amounts to showing that $Z^M(\sigma)^\perp \subsetneq X^G(\pi)$ in general.

As in the previous example, we take some $m \geq 1$, a central division F-algebra D, and $\tau \in \Pi_{2,\mathrm{temp}}(\mathrm{GL}_D(m))$ such that $X^{\mathrm{GL}_D(m)}(\tau)$ contains η, ω with noncommuting preimages in $S^{\mathrm{GL}_D(m)}(\sigma)$. Take another $\tau' \in \Pi_{2,\mathrm{temp}}(\mathrm{GL}_D(m))$ such that

$X^{\mathrm{GL}_D(m)}(\tau') = \langle \eta \rangle$. Denote by d the order of ω. We take

$$M := \mathrm{GL}_D(m) \times \prod_{1 \le j \le d} \mathrm{GL}_D(m),$$

$$G := \mathrm{GL}_D(m(d+1)),$$

$$\sigma := \tau \boxtimes \boxtimes_{1 \le j \le d} \omega^{j-1} \tau',$$

$$\pi := I_P^G(\sigma) \quad \text{for some } P \in \mathscr{P}(M).$$

Therefore $X^M(\sigma) = \langle \eta \rangle$ (defined relative to $M^\sharp = M \cap G^\sharp$ with $G^\sharp := \mathrm{SL}_D(m(d+1))$ as before), and $S^M(\sigma)$ is commutative. In particular $Z^M(\sigma) = X^M(\sigma) = \langle \eta \rangle$.

On the other hand, a variant of the arguments in the previous example show that $\omega, \eta \in X^G(\pi)$ with noncommuting preimages in $S^G(\pi)$. Hence $\omega \in X^G(\pi)$ and $\omega \notin Z^M(\sigma)^\perp$, as required.

Note that such τ, τ' do exist when D is the quaternion algebra over F and $m = 1$; in that case η, ω are identified with quadratic characters of $F^\times$. Indeed, a candidate of τ is given in the previous example. On the other hand, to construct τ' for a given η, we are reduced to constructing $\tau'' \in \Pi_{2,\mathrm{temp}}(\mathrm{GL}_F(2))$ with $X^{\mathrm{GL}_F(2)}(\tau'') = \{1, \eta\}$ and then taking τ to be the Jacquet–Langlands transfer of τ''.

To finish the construction, let E be the quadratic extension of F determined by η and let $\theta : E^\times \to \mathbb{C}^\times$ be a continuous character. Set $\tau'' := \mathrm{Ind}_{E/F}(\theta)$ (the local automorphic induction; cf. [Jacquet and Langlands 1970, Theorem 4.6]). Then $\eta \tau'' = \tau''$. From [Labesse and Langlands 1979, Pages 738–739], one sees that τ'' is cuspidal and $|X^{\mathrm{GL}_F(2)}(\tau'')| = 2$ for general θ, which suffices to conclude.

Acknowledgements

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KUOK FAI CHAO
SHANGHAI CENTER FOR MATHEMATICAL SCIENCES
FUDAN UNIVERSITY
NO. 220 HANDAN ROAD
SHANGHAI, 200433
CHINA
kchao@fudan.edu.cn

WEN-WEI LI
MORNINGSIDE CENTER OF MATHEMATICS
ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE
CHINESE ACADEMY OF SCIENCES
55, ZHONGGUANCUN EAST ROAD
BEIJING, 100190
CHINA

wwli@math.ac.cn

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