

*Pacific
Journal of
Mathematics*

**DISTANCE OF BRIDGE SURFACES FOR LINKS WITH
ESSENTIAL MERIDIONAL SPHERES**

YEONHEE JANG

Volume 267 No. 1

January 2014

DISTANCE OF BRIDGE SURFACES FOR LINKS WITH ESSENTIAL MERIDIONAL SPHERES

YEONHEE JANG

Bachman and Schleimer gave an upper bound for the distance of a bridge surface of a knot in a 3-manifold which admits an essential surface in the exterior. Here we give a sharper upper bound for the distance of a bridge surface of a link when the manifold admits an essential meridional sphere in the exterior.

1. Introduction

Let L be a link in a closed orientable 3-manifold M . A closed orientable surface F embedded in M is called a *Heegaard surface* of M if it cuts M into two handlebodies. We call this decomposition a *Heegaard splitting* of M . We say that L is in *bridge position* with respect to a Heegaard surface F if the intersection of L and each handlebody is *trivial*, namely, the intersection together with some arcs on F bounds mutually disjoint disks. We call F a (g, n) -*bridge surface* (or a *bridge surface* in brief) of L , where g is the genus of F and n is the half of the number $|L \cap F|$ of the components of $L \cap F$. In particular, we call a $(0, n)$ -bridge surface an n -*bridge sphere* of L . Throughout this paper, we assume $n \geq 3$ for all n -bridge spheres.

Since the *distance* of a Heegaard splitting was introduced in [Hempel 2001] as a measure of complexity, it has been studied by various authors; see, for example, [Evans 2006; Hartshorn 2002; Kobayashi and Rieck 2009; Scharlemann and Tomova 2006]. This concept can be generalized to the distance of bridge surfaces of links in closed orientable 3-manifolds (see Section 2 for details). As generalizations of results from [Hartshorn 2002; Scharlemann and Tomova 2006], Bachman and Schleimer [2005] and Tomova [2007] gave upper bounds for the distance of a bridge surface of a knot in a 3-manifold when there exist essential surfaces in the knot exterior and alternate bridge surfaces, respectively, in terms of their Euler characteristics. Ido [2013] gave a refinement of the upper bound of [Tomova 2007] in the case where the genus of the bridge surface is 0.

This research is partially supported by a Grant-in-Aid for JSPS Research Fellowships for Young Scientists.

MSC2010: 57M25.

Keywords: bridge surfaces for links, distance of bridge surfaces.

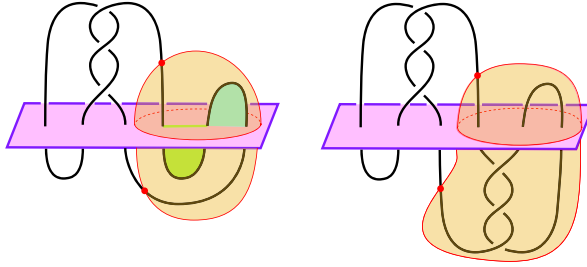


Figure 1. $d_{BS}(L, F) = 0$ and $d_T(L, F) = 1$.

In Theorem 1.1 and Corollary 1.3 below we give a refinement of Bachman and Schleimer's upper bound for the distance of bridge surfaces under some extra assumptions. (For detailed definitions, see Section 2.) For a surface S in M , we denote by S_L the surface $\text{Cl}(S \setminus N(L))$, where $N(L)$ is a regular neighborhood of L in M .

Theorem 1.1. *Let L be a link in a closed orientable irreducible 3-manifold M which is in bridge position with respect to a Heegaard surface F . Suppose that there exists a c -essential sphere S in M intersecting L transversely in at least 4 points. Then $d_{BS}(L, F) \leq -\chi(S_L) = |\partial S_L| - 2$.*

Bachman and Schleimer's upper bound in this setting is $-\chi(S_L) + 2$, which equals $|\partial S_L|$.

We will denote by $d_{BS}(L, F)$ and $d_T(L, F)$ the definitions of distance given in [Bachman and Schleimer 2005] and [Tomova 2007], which disagree slightly. In general, it is easy to see that $d_{BS}(L, F) \leq d_T(L, F) \leq d_{BS}(L, F) + 2$. If we focus on bridge spheres for links in the 3-sphere S^3 , we have:

Proposition 1.2. *For an n -bridge sphere F of a link L in S^3 ,*

- *if $d_{BS}(L, F) \geq 1$, then $d_T(L, F) = d_{BS}(L, F)$, and*
- *if $d_{BS}(L, F) = 0$, then $d_T(L, F) = 0$ or 1.*

The links and the 3-bridge spheres in Figure 1 give examples for which the two distances do not coincide, since $d_{BS}(L, F) = 0$ and $d_T(L, F) = 1$. In fact, this always holds when L is nonsplit and either L is composite or F is perturbed.

The following is a direct consequence of Theorem 1.1 and Proposition 1.2.

Corollary 1.3. *Let L be a link in the 3-sphere S^3 and F an n -bridge sphere of L . Suppose that there exists a c -essential sphere S in M intersecting L transversely in at least 4 points. Then $d_T(L, F) \leq -\chi(S_L) = |\partial S_L| - 2$.*

As a consequence of Theorem 1.1 and Corollary 1.3, we obtain:

Corollary 1.4. *Let L be an arborescent link in the 3-sphere S^3 . Then $d_{BS}(L, F) \leq 2$ and $d_T(L, F) \leq 2$ for any minimal bridge sphere F of L .*

Corollary 1.5. *Let L be a link in the 3-sphere S^3 and F a minimal bridge sphere such that $d_{BS}(L, F) > 2$ or $d_T(L, F) > 2$. Then L is a hyperbolic link and the double branched covering $M_2(L)$ of S^3 branched along L is a hyperbolic manifold.*

Corollary 1.5 implies Corollary 6.2 of [Bachman and Schleimer 2005], which asserts the hyperbolicity of links admitting bridge surfaces with distance greater than 2. In fact, arborescent links are known to be hyperbolic except for some special cases (see [Bonahon and Siebenmann 2010; Futer and Guéritaud 2009; Jang 2011, Proposition 3]). On the other hand, the double branched covering $M_2(L)$ of S^3 branched along an arborescent link L is a graph manifold, and hence not hyperbolic. Thus, the latter assertion in Corollary 1.5 is meaningful. We remark that, in fact, the hyperbolicity of $M_2(L)$ implies the hyperbolicity of the link L (see [Kojima 1996; 1998]). Also, we conjecture that the assumptions on the minimality of the bridge spheres in Corollaries 1.4 and 1.5 are unnecessary. Specifically, we make the following conjectures:

- (1) $d_{BS}(L, F) \leq 2$ and $d_T(L, F) \leq 2$ for any bridge sphere F of an arborescent link L in the 3-sphere S^3 .
- (2) For a link L in S^3 which admits a bridge sphere F such that $d_{BS}(L, F) > 2$ or $d_T(L, F) > 2$, the link L is a hyperbolic link and the double branched covering $M_2(L)$ of S^3 branched along L is a hyperbolic manifold.

Statements (1) and (2) are known to be true except for 3-bridge Montesinos links (see the proof of Corollaries 1.4 and 1.5). In fact, they are true if any nonminimal bridge sphere of a 3-bridge Montesinos link has distance at most 2 (or if any nonminimal bridge sphere of a 3-bridge Montesinos link is perturbed, which implies that the distance is at most 1).

2. Definitions and notation

Our conventions mostly follow [Bachman and Schleimer 2005], though we modify some of the definitions since we treat only meridional spheres in this paper, while Bachman and Schleimer treated more general surfaces.

Throughout this paper, M is a closed orientable 3-manifold and L is a link in M . We denote the manifold $\text{Cl}(M \setminus N(L))$ by M_L . For a surface F embedded in M that intersects L transversely, we denote the surface $F \cap M_L$ by F_L and call it a *meridional surface* (with respect to L). A simple closed curve on F_L is *inessential* on F_L if it bounds a disk on F_L or it bounds an annulus on F_L together with a boundary component of F_L . We say that the curve is *essential* on F_L if it is not inessential on F_L . A *compressing disk* for F_L is a disk D embedded in M_L so that $F \cap D = \partial D$ is an essential simple closed curve on F_L . A *cut-disk* for F_L is the intersection $D^c = D \cap M_L$, where $D(\subset M)$ is a disk such that $D \cap F = \partial D$ is an

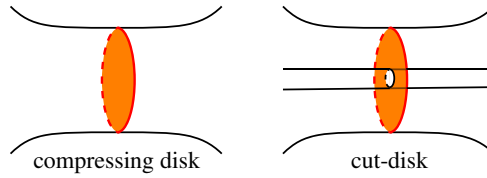


Figure 2. c-disks.

essential simple closed curve on F_L and $|D \cap N(L)|$ is a meridian disk (i.e., D intersects L transversely in one point). A c -disk for F_L is either a compressing disk or a cut-disk for F_L (see Figure 2). We say that a surface $F \subset M$ is c -essential if there are no c -disks for F_L , F_L is not boundary parallel in M_L and F is not a 2-sphere that bounds a 3-ball in M_L .

Let L be a link in M which is in a bridge position with respect to a Heegaard surface F of M . We denote by $\mathcal{C}(F_L)$ the *curve complex* of F_L , that is, each vertex of $\mathcal{C}(F_L)$ corresponds to the isotopy class of an essential simple loop in F_L and $k + 1$ distinct vertices form a k -simplex if and only if there are mutually disjoint representatives of the corresponding isotopy classes. For two vertices v and v' of $\mathcal{C}(F_L)$, we denote by $d(v, v')$ the number of 1-simplexes in the shortest path (of 1-simplexes) connecting v and v' . For two sets A and B of vertices of $\mathcal{C}(F_L)$, we define $d(A, B)$ by the minimum of $\{d(v, v') \mid v \in A, v' \in B\}$. Let V_0 and V_1 be the closures of the two components of $M \setminus F$, and let $H_i = V_i \cap M_L$ ($i = 0, 1$). For each $i = 0, 1$, we denote by $\mathcal{D}_{BS}(H_i)$ (resp. $\mathcal{D}_T(H_i)$) the set of the vertices of $\mathcal{C}(F_L)$ with representatives bounding c -disks (resp. compressing disks) in H_i . We define the distances $d_{BS}(L, F)$ and $d_T(L, F)$ of L with respect to F as $d(\mathcal{D}_{BS}(H_0), \mathcal{D}_{BS}(H_1))$ and $d(\mathcal{D}_T(H_0), \mathcal{D}_T(H_1))$, respectively.

Let V be a handlebody and T the union of trivial arcs properly embedded in V . We say that a finite graph Σ properly embedded in V is a *spine* of (V, T) if $V \setminus \Sigma$ is homeomorphic to $\partial V \times [0, 1)$ and the projection $V \setminus \Sigma \cong \partial V \times [0, 1) \rightarrow [0, 1)$ has no maxima on T . Let L be a link in M which is in a bridge position with respect to a Heegaard surface F of M , and let V_0 and V_1 be the closures of the two components of $M \setminus F$. For each $i = 0, 1$, let Σ_i be the spine of $(V_i, L \cap V_i)$ and let $p_i : V_i \setminus \Sigma_i (\cong \partial V_i \times [0, 1)) \rightarrow [0, 1)$ be the projection as above. Define maps $\varphi_0 : [0, 1) \rightarrow (0, \frac{1}{2}]$ and $\varphi_1 : [0, 1) \rightarrow [\frac{1}{2}, 1)$ by $\varphi_0(t) = \frac{1}{2}(1-t)$ and $\varphi_1(t) = \frac{1}{2}(1+t)$. A *sweep-out* of F with respect to L is a map $h : M \rightarrow [0, 1]$ defined by $h(\Sigma_i) = i$ and $h|_{V_i \setminus \Sigma_i} = \varphi_i \circ p_i$ ($i = 0, 1$).

3. Proof of the main theorem

In this section, we prove Theorem 1.1, and also Proposition 1.2 and Corollary 1.3.

Proof of Theorem 1.1. If $d_{BS}(L, F) \leq 1$, then $d_{BS}(L, F) < -\chi(S_L) = |\partial S_L| - 2$ always holds since $|\partial S_L| \geq 4$ by the hypothesis. Hence, we may assume that $d_{BS}(L, F) \geq 2$. Let H_0, H_1, Σ_0 and Σ_1 be as in the previous section, and let $h : M \rightarrow [0, 1]$ be a sweep-out of F with respect to L . Set $F_L(t) = h^{-1}(t) \cap M_L$. Let $H_0(t)$ be the closure of the component of $M_L \setminus F_L(t)$ that contains Σ_0 , and $H_1(t)$ the closure of $M_L \setminus H_0(t)$. Let ϵ_0 be chosen just larger than the radius of $N(L)$ but small enough so that S meets $H_0(\epsilon_0)$ and $H_1(1 - \epsilon_0)$ in c-disks for $F_L(\epsilon_0)$ and $F_L(1 - \epsilon_0)$. Then the surface $F_L(t)$ is homeomorphic to F_L for every value $t \in [\epsilon_0, 1 - \epsilon_0]$, and we can take a homeomorphism

$$\Phi : \bigcup_{t=\epsilon_0}^{1-\epsilon_0} F_L(t) \rightarrow F_L \times [\epsilon_0, 1 - \epsilon_0]$$

such that $\Phi(F_L(t)) = F_L \times \{t\}$. Let $\pi = \text{pr}_1 \circ \Phi$, where $\text{pr}_1 : F_L \times [\epsilon_0, 1 - \epsilon_0] \rightarrow F_L$ is the projection onto the first factor. Hence, for a loop γ on $F_L(t)$, the image $\pi(\gamma)$ is a loop on F_L .

Note: The results referred to throughout this proof are from [Bachman and Schleimer 2005].

We assume that the essential meridional sphere S is in *standard position* as in the proof of the main theorem of that reference. Namely,

- Each boundary component of S_L lies on $\partial F_L(t)$ for some $t \in (\epsilon_0, 1 - \epsilon_0)$. If some boundary component of S is contained in $\partial F_L(t)$, we consider t a critical value for S .
- All critical points of $h|_{S_L}$ are nondegenerate (i.e., maxima, minima, or saddles). We will refer to any such critical point whose height is between ϵ_0 and $1 - \epsilon_0$ and to any meridional boundary component as a *critical submanifold* (of S).
- The heights of any two critical submanifolds of S are distinct.

Let t_0 be the supremum of $t \in [\epsilon_0, 1 - \epsilon_0]$ such that there is a loop in $S \cap F_L(t)$ which bounds a c-disk for $F_L(t)$ in $H_0(t)$. Likewise, let t_1 be the infimum of $t \in [\epsilon_0, 1 - \epsilon_0]$ such that some loop in $S \cap F_L(t)$ bounds a c-disk for $F_L(t)$ in $H_1(t)$. Since $d_{BS}(L, F) \geq 2$, we may assume that $\epsilon_0 < t_0 < t_1 < 1 - \epsilon_0$ by Claims 5.4–5.6.

Choose $\epsilon > 0$ sufficiently small so that there is no critical values in $[t_0 - \epsilon, t_0 + \epsilon]$ and in $[t_1 - \epsilon, t_1 + \epsilon]$ other than t_0 and t_1 . By the definition of t_0 , there is a loop $\gamma_0 \subset S \cap F_L(t_0 - \epsilon)$ which bounds a c-disk for $F_L(t_0 - \epsilon)$ in $H_0(t_0 - \epsilon)$. Similarly, there is a loop $\gamma_1 \subset S \cap F_L(t_1 + \epsilon)$ which bounds a c-disk for $F_L(t_1 + \epsilon)$ in $H_0(t_1 + \epsilon)$.

We see that $S \cap F_L(t_0 + \epsilon)$ contains a loop essential on S_L . To this end, assume on the contrary that every component of $S \cap F_L(t_0 + \epsilon)$ is inessential on S_L . By the definition of t_0 , a component of $S \cap F_L(t_0 + \epsilon)$ is inessential also on $F_L(t_0 + \epsilon)$ since, otherwise, $S \cap H_0(t_0 + \epsilon)$ is a c-disk. Note that there is no essential spheres

or decomposing spheres for L by the assumption that $d_{BS}(L, F) \geq 2$ together with Theorem 1. Hence, we can isotope S so that $S_L \subset H_1(t_0 + \epsilon)$, which is impossible by Claim 5.2. Similarly, we can see that $S \cap F_L(t_1 - \epsilon)$ contains a loop essential on S_L . Cut S_L along loops on $S \cap F_L(t_0 + \epsilon)$ and $S \cap F_L(t_1 - \epsilon)$ which are essential on S_L . Let S' be the closure of one of the components which meets both $F_L(t_0 + \epsilon)$ and $F_L(t_1 - \epsilon)$. Note that every loop on S_L is separating since S is a sphere, and that every component of $S_L \setminus S'$ contains at least two boundary components of S_L . Thus, the Euler characteristic $\chi(S')$ is bigger than or equal to $\chi(S_L) + 2$.

Let α_0 (resp. α_1) be a component of $\partial S' \cap F_L(t_0 + \epsilon)$ (resp. $\partial S' \cap F_L(t_1 - \epsilon)$). By Claim 5.9, every loop of $S \cap F_L(t)$ for every regular value $t \in [t_0, t_1]$ of $h|_S$ is either essential on both $F_L(t)$ and S_L or inessential on both $F_L(t)$ and S_L . In particular, the loops α_0 and α_1 are essential also on $F_L(t_0 + \epsilon)$ and $F_L(t_1 - \epsilon)$, respectively. Since we chose a sufficiently small ϵ , we may assume that the images $\pi(\gamma_0)$ and $\pi(\alpha_0)$ on F_L are disjoint. Similarly, we assume that $\pi(\gamma_1)$ and $\pi(\alpha_1)$ on F_L are disjoint. By Claim 5.7 and Lemma 5.12, we see that the distance $d_{BS}(\pi(\alpha_0), \pi(\alpha_1))$ is bounded above by the number of *essential* critical submanifolds on S' . (Here, an essential critical submanifold is a critical submanifold P of S' such that neither of the boundary components of a small horizontal neighborhood of P in S' does not bound a disk on S' . See [Bachman and Schleimer 2005] for detail.) Note that the number of essential critical submanifolds on S' equals $-\chi(S')$.

Hence, we have

$$\begin{aligned} d_{BS}(\pi(\gamma_0), \pi(\gamma_1)) &\leq d_{BS}(\pi(\gamma_0), \pi(\alpha_0)) + d_{BS}(\pi(\alpha_0), \pi(\alpha_1)) + d_{BS}(\pi(\alpha_1), \pi(\gamma_1)) \\ &\leq d_{BS}(\pi(\alpha_0), \pi(\alpha_1)) + 2 \\ &\leq -\chi(S') + 2 \\ &\leq -\chi(S_L). \end{aligned}$$

This completes the proof of Theorem 1.1. □

Proof of Proposition 1.2. Let V_0 and V_1 be the closures of the two components of $S^3 \setminus F$, and let $H_i = \text{Cl}(V_i \setminus N(L))$ ($i = 0, 1$).

We first assume that $d_{BS}(L, F) = n \geq 1$, and let c_0, \dots, c_n are essential loops on F_L realizing the distance $d_{BS}(L, F)$. Namely, c_0 and c_n bounds c-disk in H_0 and H_1 , respectively, and $c_{i-1} \cap c_i = \emptyset$ for $i = 1, \dots, n$. Assume that c_0 bounds a cut-disk D^c in H_0 . Since V_0 is a 3-ball by the hypothesis and c_0 is essential in F_L , $H_0 \setminus D^c$ has two components H_0^1 and H_0^2 neither of which is homeomorphic to a solid torus, and c_1 lies on ∂H_0^1 and ∂H_0^2 , say ∂H_0^1 . Then, we can find a compressing disk D for F_L in ∂H_0^2 , disjoint from $D^c \cup c_1$, and we replace c_0 with ∂D . Similarly, in the case where c_n bounds a cut-disk in H_1 , we can replace c_n with a loop c'_n which bounds a compressing disk in H_1 and is disjoint from c_{n-1} . Hence, we have $d_T(L, F) = n = d_{BS}(L, F)$.

Assume that $d_{BS}(L, F) = 0$. Then there is a loop c which bounds c -disks in both H_0 and H_1 . By using an argument similar to that for the previous case, we can find loops c' and c'' that bound compressing disks in H_0 and H_1 , respectively, and are mutually disjoint. Hence, we have $d_T(L, F) \leq 1$. \square

Proof of Corollary 1.3. By Proposition 1.2, we have $d_T(L, F) = \max\{1, d_{BS}(L, F)\}$. Since $d_{BS}(L, F) \leq -\chi(S_L)$ by Theorem 1.1 and $-\chi(S_L) \geq 2$ by the hypothesis, we have $d_T(L, F) \leq -\chi(S_L)$. \square

4. Applications

In this section, we prove Corollaries 1.4 and 1.5.

A (2-string) *trivial tangle* is a pair of a 3-ball and the union of two arcs *trivially embedded* in the 3-ball, that is, the arcs together with some arcs on the boundary of the 3-ball bound disjoint disks. A *rational tangle* is an ambient isotopy class of a trivial tangle with its boundary fixed. It is well known that rational tangles can be parametrized by rational numbers, called the *slopes* of rational tangles. A *Montesinos pair* is a pair of a 3-manifold and a 1-submanifold which is built from the pair, called a *hollow Montesinos pair*, (illustrated in either half of Figure 3) by plugging some of the holes with rational tangles of finite slopes.

An *arborescent link* is a link in the 3-sphere S^3 obtained by gluing some Montesinos pairs in their boundaries. In particular, we call a link obtained from a hollow Montesinos pair of the form shown on the left in Figure 3 by plugging the holes with rational tangles of finite slopes r_1, r_2, \dots, r_m a *Montesinos link*, and denote it by $M_1(r_1, r_2, \dots, r_m)$. We call m the *length* of the Montesinos link $M_1(r_1, r_2, \dots, r_m)$ when neither of r_1, r_2, \dots, r_m is an integer. Similarly, we denote by $M_2(r_1, r_2, \dots, r_m)$ the arborescent link obtained from a hollow Montesinos pair of the form shown on the right in Figure 3 by plugging the holes with rational tangles of finite slopes r_1, r_2, \dots, r_m .

Lemma 4.1. *Let L be an arborescent link in S^3 which has bridge index at least 3, and suppose that L does not admit an essential Conway sphere (i.e., a c -essential sphere in S^3 intersecting L transversely in exactly 4 points). Then L is equivalent to a Montesinos link of length 3 as illustrated in Figure 4. In that figure, each circle with a rational number r_i ($i = 1, 2, 3$) inside represents a rational tangle of slope r_i .*

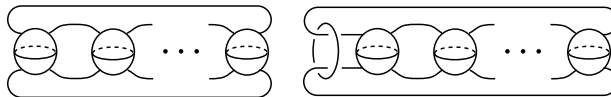


Figure 3. Hollow Montesinos pairs.

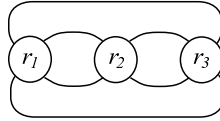


Figure 4. A Montesinos link $M_1(r_1, r_2, r_3)$.

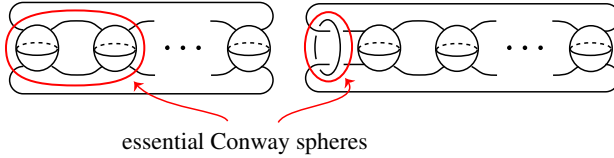


Figure 5. Essential Conway spheres in Montesinos pairs.

Proof. Let L be an arborescent link in S^3 and suppose that L does not admit an essential Conway sphere. Then L is obtained from a Montesinos pair of one of the forms shown in Figure 3 by plugging the holes with rational tangles of finite slopes (see [Bonahon and Siebenmann 2010, Theorem 3.4] or [Jang 2011, Theorem 4]). That is, L is equivalent to a Montesinos link $M_1(r_1, r_2, \dots, r_{m_1})$ or an arborescent link $M_2(r_1, r_2, \dots, r_{m_2})$ for some rational numbers r_i 's. Moreover, the m_1 and m_2 cannot be bigger than 3 and 1, respectively, since otherwise L admits an essential Conway sphere as illustrated in Figure 5, which contradicts the hypothesis.

We note that $M_2(r_1)$ is equivalent to the Montesinos link $M_1(-1/2, 1/2, -1/r_1)$. Moreover, we can easily see that $M_1(r_1, r_2, \dots, r_{m_1})$ admits a 2-bridge presentation if the length of $M_1(r_1, r_2, \dots, r_{m_1})$ is 1 or 2, which contradicts the assumption that the bridge index of L is at least 3. Thus, L is equivalent to a Montesinos link of length 3, which is the desired result. \square

Proof of Corollary 1.4. Let L be an arborescent link in S^3 and F a bridge sphere of L . If there is an essential tori or an essential Conway sphere in the complement of L , then the distances $d_{BS}(L, F)$ and $d_T(L, F)$ are at most 2 by [Bachman and Schleimer 2005, Theorem 5.1] together with Theorem 1.1 and Corollary 1.3. Thus, in the rest of the proof, we assume that there is no essential tori or essential Conway spheres. By Lemma 4.1, the link L is equivalent to a Montesinos link of length 3 (see Figure 4).

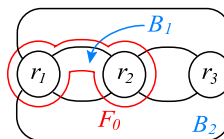


Figure 6. A 3-bridge sphere for a Montesinos link.

Assume that F is a minimal bridge sphere (that is, a 3-bridge sphere) of L . By [Jang 2013], we may assume that F is (equivalent to) the 3-bridge sphere F_0 in Figure 6 without loss of generality. Let B_1 be the 3-ball bounded by F containing two of the three rational tangles and B_2 the other 3-ball bounded by F (see Figure 6), and let H_i be the closure of $B_i \setminus N(L)$ ($i = 1, 2$). Let c_0 , c_1 and c_2 be the loops on F_L as illustrated in Figure 7. Then c_0 bounds a cut-disk in H_1 , c_2 bounds a compressing disk in H_2 , and c_1 is disjoint from $c_0 \cup c_2$. These imply $d_{BS}(L, F) \leq 2$. Moreover, by Proposition 1.2, we have $d_T(L, F) \leq 2$. \square

Proof of Corollary 1.5. If the distances are greater than two, then, by [Bachman and Schleimer 2005, Theorem 5.1] together with Theorem 1.1 and Corollary 1.3, there is no essential tori in the exterior of L , no essential Conway spheres for L , no essential spheres nor essential annuli. By [Bachman and Schleimer 2005, Corollary 6.2], L is a hyperbolic link. Moreover, the double branched cover $M_2(L)$ of S^3 branched along L has a trivial JSJ decomposition. Thus, $M_2(L)$ is either a Seifert fibered space or a hyperbolic manifold. In the former case, we obtain that either L is a Montesinos link or the complement of L admits a Seifert fibration, which contradicts Corollary 1.4 or the fact that L is hyperbolic, accordingly. Hence, $M_2(L)$ must be hyperbolic. \square

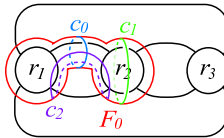


Figure 7. Curves realizing distance 2.

Acknowledgements

The author would like to thank Professor Tsuyoshi Kobayashi and Professor Makoto Sakuma for their helpful comments. She also appreciates the referee for careful reading and valuable suggestions.

References

- [Bachman and Schleimer 2005] D. Bachman and S. Schleimer, “Distance and bridge position”, *Pacific J. Math.* **219**:2 (2005), 221–235. MR 2007a:57028 Zbl 1086.57011
- [Bonahon and Siebenmann 2010] F. Bonahon and L. C. Siebenmann, “New geometric splittings of classical knots, and the classification and symmetries of arborescent knots”, preprint, 2010, <http://www-bcf.usc.edu/~fbonahon/Research/Preprints/BonSieb.pdf>.
- [Evans 2006] T. Evans, “High distance Heegaard splittings of 3-manifolds”, *Topology Appl.* **153**:14 (2006), 2631–2647. MR 2007j:57020 Zbl 1107.57011

- [Futer and Guéritaud 2009] D. Futer and F. Guéritaud, “Angled decompositions of arborescent link complements”, *Proc. Lond. Math. Soc.* (3) **98**:2 (2009), 325–364. MR 2009m:57008 Zbl 1163.57003
- [Hartshorn 2002] K. Hartshorn, “Heegaard splittings of Haken manifolds have bounded distance”, *Pacific J. Math.* **204**:1 (2002), 61–75. MR 2003a:57037 Zbl 1065.57021
- [Hempel 2001] J. Hempel, “3-manifolds as viewed from the curve complex”, *Topology* **40**:3 (2001), 631–657. MR 2002f:57044 Zbl 0985.57014
- [Ido 2013] A. Ido, “An estimation of Hempl distance for bridge spheres”, preprint, 2013.
- [Jang 2011] Y. Jang, “Classification of 3-bridge arborescent links”, *Hiroshima Math. J.* **41**:1 (2011), 89–136. MR 2012i:57011 Zbl 1231.57006
- [Jang 2013] Y. Jang, “Classification of 3-bridge spheres of 3-bridge arborescent links”, *J. Math. Soc. Japan* **65**:1 (2013), 97–136. MR 3034400 Zbl 06152429
- [Kobayashi and Rieck 2009] T. Kobayashi and Y. Rieck, “Manifolds admitting both strongly irreducible and weakly reducible minimal genus Heegaard splittings”, *Comm. Anal. Geom.* **17**:4 (2009), 637–649. MR 2011d:57054 Zbl 1235.57014
- [Kojima 1996] S. Kojima, “Nonsingular parts of hyperbolic 3-cone-manifolds”, pp. 115–122 in *Topology and Teichmüller spaces* (Katinkulta, 1995), edited by S. Kojima et al., World Scientific, Singapore, 1996. MR 99i:57028 Zbl 0928.57010
- [Kojima 1998] S. Kojima, “Deformations of hyperbolic 3-cone-manifolds”, *J. Differential Geom.* **49**:3 (1998), 469–516. MR 2000d:57023 Zbl 0990.57004
- [Scharlemann and Tomova 2006] M. Scharlemann and M. Tomova, “Alternate Heegaard genus bounds distance”, *Geom. Topol.* **10** (2006), 593–617. MR 2007b:57040 Zbl 1128.57022
- [Tomova 2007] M. Tomova, “Multiple bridge surfaces restrict knot distance”, *Algebr. Geom. Topol.* **7** (2007), 957–1006. MR 2008h:57014 Zbl 1142.57005

Received July 4, 2012. Revised May 30, 2013.

YEONHEE JANG
DEPARTMENT OF MATHEMATICS
NARA WOMEN’S UNIVERSITY
KITAUYOYA NISHIMACHI
NARA 630-8526
JAPAN
yeonheejang@cc.nara-wu.ac.jp

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

EDITORS

V. S. Varadarajan (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
pacific@math.ucla.edu

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Don Blasius
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI
CALIFORNIA INST. OF TECHNOLOGY
INST. DE MATEMÁTICA PURA E APLICADA
KEIO UNIVERSITY
MATH. SCIENCES RESEARCH INSTITUTE
NEW MEXICO STATE UNIV.
OREGON STATE UNIV.

STANFORD UNIVERSITY
UNIV. OF BRITISH COLUMBIA
UNIV. OF CALIFORNIA, BERKELEY
UNIV. OF CALIFORNIA, DAVIS
UNIV. OF CALIFORNIA, LOS ANGELES
UNIV. OF CALIFORNIA, RIVERSIDE
UNIV. OF CALIFORNIA, SAN DIEGO
UNIV. OF CALIF., SANTA BARBARA

UNIV. OF CALIF., SANTA CRUZ
UNIV. OF MONTANA
UNIV. OF OREGON
UNIV. OF SOUTHERN CALIFORNIA
UNIV. OF UTAH
UNIV. OF WASHINGTON
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.


See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2014 is US \$410/year for the electronic version, and \$535/year for print and electronic. Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2014 Mathematical Sciences Publishers

PACIFIC JOURNAL OF MATHEMATICS

Volume 267 No. 1 January 2014

Numerical study of unbounded capillary surfaces	1
YASUNORI AOKI and HANS DE STERCK	
Dual R -groups of the inner forms of $SL(N)$	35
KUOK FAI CHAO and WEN-WEI LI	
Automorphisms and quotients of quaternionic fake quadrics	91
AMIR DŽAMBIĆ and XAVIER ROULLEAU	
Distance of bridge surfaces for links with essential meridional spheres	121
YEONHEE JANG	
Normal states of type III factors	131
YASUYUKI KAWAHIGASHI, YOSHIKO OGATA and ERLING STØRMER	
Eigenvalues and entropies under the harmonic-Ricci flow	141
YI LI	
Quantum extremal loop weight modules and monomial crystals	185
MATHIEU MANSUY	
Lefschetz fibrations with small slope	243
NAOYUKI MONDEN	



0030-8730(2014)267:1;1-7