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Bachman and Schleimer gave an upper bound for the distance of a bridge surface of a knot in a 3-manifold which admits an essential surface in the exterior. Here we give a sharper upper bound for the distance of a bridge surface of a link when the manifold admits an essential meridional sphere in the exterior.

1. Introduction

Let L be a link in a closed orientable 3-manifold M. A closed orientable surface F embedded in M is called a *Heegaard surface* of M if it cuts M into two handlebodies. We call this decomposition a *Heegaard splitting* of M. We say that L is in *bridge position* with respect to a Heegaard surface F if the intersection of L and each handlebody is *trivial*, namely, the intersection together with some arcs on F bounds mutually disjoint disks. We call F a (g, n)*-bridge surface* (or a *bridge surface* in brief) of L, where g is the genus of F and n is the half of the number $|L \cap F|$ of the components of $L \cap F$. In particular, we call a $(0, n)$-bridge surface an n*-bridge sphere* of L. Throughout this paper, we assume $n \geq 3$ for all n-bridge spheres.

Since the *distance* of a Heegaard splitting was introduced in [Hempel 2001] as a measure of complexity, it has been studied by various authors; see, for example, [Evans 2006; Hartshorn 2002; Kobayashi and Rieck 2009; Scharlemann and Tomova 2006]. This concept can be generalized to the distance of bridge surfaces of links in closed orientable 3-manifolds (see Section 2 for details). As generalizations of results from [Hartshorn 2002; Scharlemann and Tomova 2006], Bachman and Schleimer [2005] and Tomova [2007] gave upper bounds for the distance of a bridge surface of a knot in a 3-manifold when there exist essential surfaces in the knot exterior and alternate bridge surfaces, respectively, in terms of their Euler characteristics. Ido [2013] gave a refinement of the upper bound of [Tomova 2007] in the case where the genus of the bridge surface is 0.

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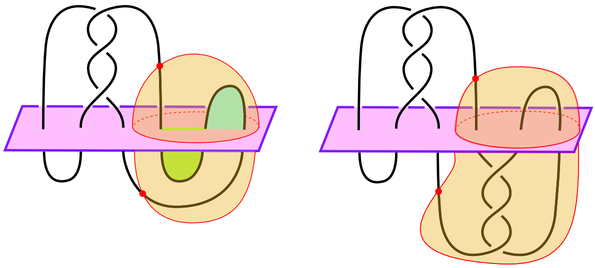

Figure 1. $d_{BS}(L, F) = 0$ and $d_T(L, F) = 1$.

In Theorem 1.1 and Corollary 1.3 below we give a refinement of Bachman and Schleimer's upper bound for the distance of bridge surfaces under some extra assumptions. (For detailed definitions, see Section 2.) For a surface S in M, we denote by S_L the surface $\mathrm{Cl}(S \setminus N(L))$, where $N(L)$ is a regular neighborhood of L in M.

Theorem 1.1. *Let L be a link in a closed orientable irreducible 3-manifold M which is in bridge position with respect to a Heegaard surface F. Suppose that there exists a c-essential sphere S in M intersecting L transversely in at least 4 points. Then $d_{BS}(L, F) \leq -\chi(S_L) = |\partial S_L| - 2$.*

Bachman and Schleimer's upper bound in this setting is $-\chi(S_L) + 2$, which equals $|\partial S_L|$.

We will denote by $d_{BS}(L, F)$ and $d_T(L, F)$ the definitions of distance given in [Bachman and Schleimer 2005] and [Tomova 2007], which disagree slightly. In general, it is easy to see that $d_{BS}(L, F) \leq d_T(L, F) \leq d_{BS}(L, F) + 2$. If we focus on bridge spheres for links in the 3-sphere S^3, we have:

Proposition 1.2. *For an n-bridge sphere F of a link L in S^3,*

- *if $d_{BS}(L, F) \geq 1$, then $d_T(L, F) = d_{BS}(L, F)$, and*
- *if $d_{BS}(L, F) = 0$, then $d_T(L, F) = 0$ or 1.*

The links and the 3-bridge spheres in Figure 1 give examples for which the two distances do not coincide, since $d_{BS}(L, F) = 0$ and $d_T(L, F) = 1$. In fact, this always holds when L is nonsplit and either L is composite or F is perturbed.

The following is a direct consequence of Theorem 1.1 and Proposition 1.2.

Corollary 1.3. *Let L be a link in the 3-sphere S^3 and F an n-bridge sphere of L. Suppose that there exists a c-essential sphere S in M intersecting L transversely in at least 4 points. Then $d_T(L, F) \leq -\chi(S_L) = |\partial S_L| - 2$.*

As a consequence of Theorem 1.1 and Corollary 1.3, we obtain:

Corollary 1.4. *Let L be an arborescent link in the 3-sphere S^3. Then $d_{BS}(L, F) \leq 2$ and $d_T(L, F) \leq 2$ for any minimal bridge sphere F of L.*

Corollary 1.5. *Let L be a link in the 3-sphere S^3 and F a minimal bridge sphere such that $d_{BS}(L, F) > 2$ or $d_T(L, F) > 2$. Then L is a hyperbolic link and the double branched covering $M_2(L)$ of S^3 branched along L is a hyperbolic manifold.*

Corollary 1.5 implies Corollary 6.2 of [Bachman and Schleimer 2005], which asserts the hyperbolicity of links admitting bridge surfaces with distance greater than 2. In fact, arborescent links are known to be hyperbolic except for some special cases (see [Bonahon and Siebenmann 2010; Futer and Guéritaud 2009; Jang 2011, Proposition 3]). On the other hand, the double branched covering $M_2(L)$ of S^3 branched along an arborescent link L is a graph manifold, and hence not hyperbolic. Thus, the latter assertion in Corollary 1.5 is meaningful. We remark that, in fact, the hyperbolicity of $M_2(L)$ implies the hyperbolicity of the link L (see [Kojima 1996; 1998]). Also, we conjecture that the assumptions on the minimality of the bridge spheres in Corollaries 1.4 and 1.5 are unnecessary. Specifically, we make the following conjectures:

(1) $d_{BS}(L, F) \leq 2$ and $d_T(L, F) \leq 2$ for any bridge sphere F of an arborescent link L in the 3-sphere S^3.

(2) For a link L in S^3 which admits a bridge sphere F such that $d_{BS}(L, F) > 2$ or $d_T(L, F) > 2$, the link L is a hyperbolic link and the double branched covering $M_2(L)$ of S^3 branched along L is a hyperbolic manifold.

Statements (1) and (2) are known to be true except for 3-bridge Montesinos links (see the proof of Corollaries 1.4 and 1.5). In fact, they are true if any nonminimal bridge sphere of a 3-bridge Montesinos link has distance at most 2 (or if any nonminimal bridge sphere of a 3-bridge Montesinos link is perturbed, which implies that the distance is at most 1).

2. Definitions and notation

Our conventions mostly follow [Bachman and Schleimer 2005], though we modify some of the definitions since we treat only meridional spheres in this paper, while Bachman and Schleimer treated more general surfaces.

Throughout this paper, M is a closed orientable 3-manifold and L is a link in M. We denote the manifold $\mathrm{Cl}(M \setminus N(L))$ by M_L. For a surface F embedded in M that intersects L transversely, we denote the surface $F \cap M_L$ by F_L and call it a *meridional surface* (with respect to L). A simple closed curve on F_L is *inessential* on F_L if it bounds a disk on F_L or it bounds an annulus on F_L together with a boundary component of F_L. We say that the curve is *essential* on F_L if it is not inessential on F_L. A *compressing disk* for F_L is a disk D embedded in M_L so that $F \cap D = \partial D$ is an essential simple closed curve on F_L. A *cut-disk* for F_L is the intersection $D^c = D \cap M_L$, where $D (\subset M)$ is a disk such that $D \cap F = \partial D$ is an

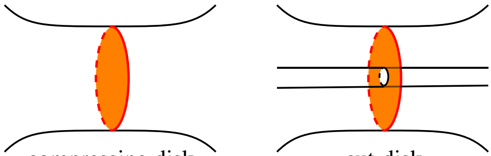

Figure 2. c-disks.

essential simple closed curve on F_L and $|D \cap N(L)|$ is a meridian disk (i.e., D intersects L transversely in one point). A *c-disk* for F_L is either a compressing disk or a cut-disk for F_L (see Figure 2). We say that a surface $F \subset M$ is *c-essential* if there are no c-disks for F_L, F_L is not boundary parallel in M_L and F is not a 2-sphere that bounds a 3-ball in M_L.

Let L be a link in M which is in a bridge position with respect to a Heegaard surface F of M. We denote by $\mathscr{C}(F_L)$ the *curve complex* of F_L, that is, each vertex of $\mathscr{C}(F_L)$ corresponds to the isotopy class of an essential simple loop in F_L and $k+1$ distinct vertices form a k-simplex if and only if there are mutually disjoint representatives of the corresponding isotopy classes. For two vertices v and v' of $\mathscr{C}(F_L)$, we denote by $d(v, v')$ the number of 1-simplexes in the shortest path (of 1-simplexes) connecting v and v'. For two sets A and B of vertices of $\mathscr{C}(F_L)$, we define $d(A, B)$ by the minimum of $\{d(v, v') \mid v \in A, v' \in B\}$. Let V_0 and V_1 be the closures of the two components of $M \setminus F$, and let $H_i = V_i \cap M_L$ $(i = 0, 1)$. For each $i = 0, 1$, we denote by $\mathscr{D}_{BS}(H_i)$ (resp. $\mathscr{D}_T(H_i)$) the set of the vertices of $\mathscr{C}(F_L)$ with representatives bounding c-disks (resp. compressing disks) in H_i. We define the distances $d_{BS}(L, F)$ and $d_T(L, F)$ of L with respect to F as $d(\mathscr{D}_{BS}(H_0), \mathscr{D}_{BS}(H_1))$ and $d(\mathscr{D}_T(H_0), \mathscr{D}_T(H_1))$, respectively.

Let V be a handlebody and T the union of trivial arcs properly embedded in V. We say that a finite graph Σ properly embedded in V is a *spine* of (V, T) if $V \setminus \Sigma$ is homeomorphic to $\partial V \times [0, 1)$ and the projection $V \setminus \Sigma \cong \partial V \times [0, 1) \to [0, 1)$ has no maxima on T. Let L be a link in M which is in a bridge position with respect to a Heegaard surface F of M, and let V_0 and V_1 be the closures of the two components of $M \setminus F$. For each $i = 0, 1$, let Σ_i be the spine of $(V_i, L \cap V_i)$ and let $p_i : V_i \setminus \Sigma_i (\cong \partial V_i \times [0, 1)) \to [0, 1)$ be the projection as above. Define maps $\varphi_0 : [0, 1) \to \left(0, \frac{1}{2}\right]$ and $\varphi_1 : [0, 1) \to \left[\frac{1}{2}, 1\right)$ by $\varphi_0(t) = \frac{1}{2}(1-t)$ and $\varphi_1(t) = \frac{1}{2}(1+t)$. A *sweep-out* of F with respect to L is a map $h : M \to [0, 1]$ defined by $h(\Sigma_i) = i$ and $h|_{V_i \setminus \Sigma_i} = \varphi_i \circ p_i$ $(i = 0, 1)$.

3. Proof of the main theorem

In this section, we prove Theorem 1.1, and also Proposition 1.2 and Corollary 1.3.

Proof of Theorem 1.1. If $d_{BS}(L, F) \leq 1$, then $d_{BS}(L, F) < -\chi(S_L) = |\partial S_L| - 2$ always holds since $|\partial S_L| \geq 4$ by the hypothesis. Hence, we may assume that $d_{BS}(L, F) \geq 2$. Let H_0, H_1, Σ_0 and Σ_1 be as in the previous section, and let $h : M \to [0, 1]$ be a sweep-out of F with respect to L. Set $F_L(t) = h^{-1}(t) \cap M_L$. Let $H_0(t)$ be the closure of the component of $M_L \setminus F_L(t)$ that contains Σ_0, and $H_1(t)$ the closure of $M_L \setminus H_0(t)$. Let ϵ_0 be chosen just larger than the radius of $N(L)$ but small enough so that S meets $H_0(\epsilon_0)$ and $H_1(1-\epsilon_0)$ in c-disks for $F_L(\epsilon_0)$ and $F_L(1-\epsilon_0)$. Then the surface $F_L(t)$ is homeomorphic to F_L for every value $t \in [\epsilon_0, 1-\epsilon_0]$, and we can take a homeomorphism

$$\Phi : \bigcup_{t=\epsilon_0}^{1-\epsilon_0} F_L(t) \to F_L \times [\epsilon_0, 1-\epsilon_0]$$

such that $\Phi(F_L(t)) = F_L \times \{t\}$. Let $\pi = \mathrm{pr}_1 \circ \Phi$, where $\mathrm{pr}_1 : F_L \times [\epsilon_0, 1-\epsilon_0] \to F_L$ is the projection onto the first factor. Hence, for a loop γ on $F_L(t)$, the image $\pi(\gamma)$ is a loop on F_L.

Note: The results referred to throughout this proof are from [Bachman and Schleimer 2005].

We assume that the essential meridional sphere S is in *standard position* as in the proof of the main theorem of that reference. Namely,

- Each boundary component of S_L lies on $\partial F_L(t)$ for some $t \in (\epsilon_0, 1-\epsilon_0)$. If some boundary component of S is contained in $\partial F_L(t)$, we consider t a critical value for S.
- All critical points of $h|_{S_L}$ are nondegenerate (i.e., maxima, minima, or saddles). We will refer to any such critical point whose height is between ϵ_0 and $1-\epsilon_0$ and to any meridional boundary component as a *critical submanifold* (of S).
- The heights of any two critical submanifolds of S are distinct.

Let t_0 be the supremum of $t \in [\epsilon_0, 1-\epsilon_0]$ such that there is a loop in $S \cap F_L(t)$ which bounds a c-disk for $F_L(t)$ in $H_0(t)$. Likewise, let t_1 be the infimum of $t \in [\epsilon_0, 1-\epsilon_0]$ such that some loop in $S \cap F_L(t)$ bounds a c-disk for $F_L(t)$ in $H_1(t)$. Since $d_{BS}(L, F) \geq 2$, we may assume that $\epsilon_0 < t_0 < t_1 < 1-\epsilon_0$ by Claims 5.4–5.6.

Choose $\epsilon > 0$ sufficiently small so that there is no critical values in $[t_0-\epsilon, t_0+\epsilon]$ and in $[t_1-\epsilon, t_1+\epsilon]$ other than t_0 and t_1. By the definition of t_0, there is a loop $\gamma_0 \subset S \cap F_L(t_0-\epsilon)$ which bounds a c-disk for $F_L(t_0-\epsilon)$ in $H_0(t_0-\epsilon)$. Similarly, there is a loop $\gamma_1 \subset S \cap F_L(t_1+\epsilon)$ which bounds a c-disk for $F_L(t_1+\epsilon)$ in $H_0(t_1+\epsilon)$.

We see that $S \cap F_L(t_0+\epsilon)$ contains a loop essential on S_L. To this end, assume on the contrary that every component of $S \cap F_L(t_0+\epsilon)$ is inessential on S_L. By the definition of t_0, a component of $S \cap F_L(t_0+\epsilon)$ is inessential also on $F_L(t_0+\epsilon)$ since, otherwise, $S \cap H_0(t_0+\epsilon)$ is a c-disk. Note that there is no essential spheres or decomposing spheres for L by the assumption that $d_{BS}(L, F) \geq 2$ together with Theorem 1. Hence, we can isotope S so that $S_L \subset H_1(t_0+\epsilon)$, which is impossible by Claim 5.2. Similarly, we can see that $S \cap F_L(t_1-\epsilon)$ contains a loop essential on S_L. Cut S_L along loops on $S \cap F_L(t_0+\epsilon)$ and $S \cap F_L(t_1-\epsilon)$ which are essential on S_L. Let S' be the closure of one of the components which meets both $F_L(t_0+\epsilon)$ and $F_L(t_1-\epsilon)$. Note that every loop on S_L is separating since S is a sphere, and that every component of $S_L \setminus S'$ contains at least two boundary components of S_L. Thus, the Euler characteristic $\chi(S')$ is bigger than or equal to $\chi(S_L)+2$.

Let α_0 (resp. α_1) be a component of $\partial S' \cap F_L(t_0+\epsilon)$ (resp. $\partial S' \cap F_L(t_1-\epsilon)$). By Claim 5.9, every loop of $S \cap F_L(t)$ for every regular value $t \in [t_0, t_1]$ of $h|_S$ is either essential on both $F_L(t)$ and S_L or inessential on both $F_L(t)$ and S_L. In particular, the loops α_0 and α_1 are essential also on $F_L(t_0+\epsilon)$ and $F_L(t_1-\epsilon)$, respectively. Since we chose a sufficiently small ϵ, we may assume that the images $\pi(\gamma_0)$ and $\pi(\alpha_0)$ on F_L are disjoint. Similarly, we assume that $\pi(\gamma_1)$ and $\pi(\alpha_1)$ on F_L are disjoint. By Claim 5.7 and Lemma 5.12, we see that the distance $d_{BS}(\pi(\alpha_0), \pi(\alpha_1))$ is bounded above by the number of *essential* critical submanifolds on S'. (Here, an essential critical submanifold is a critical submanifold P of S' such that neither of the boundary components of a small horizontal neighborhood of P in S' does not bound a disk on S'. See [Bachman and Schleimer 2005] for detail.) Note that the number of essential critical submanifolds on S' equals $-\chi(S')$.

Hence, we have

$$\begin{aligned} d_{BS}(\pi(\gamma_0), \pi(\gamma_1)) &\leq d_{BS}(\pi(\gamma_0), \pi(\alpha_0)) + d_{BS}(\pi(\alpha_0), \pi(\alpha_1)) + d_{BS}(\pi(\alpha_1), \pi(\gamma_1)) \\ &\leq d_{BS}(\pi(\alpha_0), \pi(\alpha_1)) + 2 \\ &\leq -\chi(S') + 2 \\ &\leq -\chi(S_L). \end{aligned}$$

This completes the proof of Theorem 1.1. □

Proof of Proposition 1.2. Let V_0 and V_1 be the closures of the two components of $S^3 \setminus F$, and let $H_i = \mathrm{Cl}(V_i \setminus N(L))$ $(i = 0, 1)$.

We first assume that $d_{BS}(L, F) = n \geq 1$, and let $c_0, \ldots, c_n$ are essential loops on F_L realizing the distance $d_{BS}(L, F)$. Namely, c_0 and c_n bounds c-disk in H_0 and H_1, respectively, and $c_{i-1} \cap c_i = \varnothing$ for $i = 1, \ldots, n$. Assume that c_0 bounds a cut-disk D^c in H_0. Since V_0 is a 3-ball by the hypothesis and c_0 is essential in F_L, $H_0 \setminus D^c$ has two components H_0^1 and H_0^2 neither of which is homeomorphic to a solid torus, and c_1 lies on ∂H_0^1 and ∂H_0^2, say ∂H_0^1. Then, we can find a compressing disk D for F_L in ∂H_0^2, disjoint from $D^c \cup c_1$, and we replace c_0 with ∂D. Similarly, in the case where c_n bounds a cut-disk in H_1, we can replace c_n with a loop c_n' which bounds a compressing disk in H_1 and is disjoint from c_{n-1}. Hence, we have $d_T(L, F) = n = d_{BS}(L, F)$.

Assume that $d_{BS}(L, F) = 0$. Then there is a loop c which bounds c-disks in both H_0 and H_1. By using an argument similar to that for the previous case, we can find loops c' and c'' that bound compressing disks in H_0 and H_1, respectively, and are mutually disjoint. Hence, we have $d_T(L, F) \leq 1$. □

Proof of Corollary 1.3. By Proposition 1.2, we have $d_T(L, F) = \max\{1, d_{BS}(L, F)\}$. Since $d_{BS}(L, F) \leq -\chi(S_L)$ by Theorem 1.1 and $-\chi(S_L) \geq 2$ by the hypothesis, we have $d_T(L, F) \leq -\chi(S_L)$. □

4. Applications

In this section, we prove Corollaries 1.4 and 1.5.

A (2-string) *trivial tangle* is a pair of a 3-ball and the union of two arcs *trivially embedded* in the 3-ball, that is, the arcs together with some arcs on the boundary of the 3-ball bound disjoint disks. A *rational tangle* is an ambient isotopy class of a trivial tangle with its boundary fixed. It is well known that rational tangles can be parametrized by rational numbers, called the *slopes* of rational tangles. A *Montesinos pair* is a pair of a 3-manifold and a 1-submanifold which is built from the pair, called a *hollow Montesinos pair*, (illustrated in either half of Figure 3) by plugging some of the holes with rational tangles of finite slopes.

An *arborescent link* is a link in the 3-sphere S^3 obtained by gluing some Montesinos pairs in their boundaries. In particular, we call a link obtained from a hollow Montesinos pair of the form shown on the left in Figure 3 by plugging the holes with rational tangles of finite slopes $r_1, r_2, \ldots, r_m$ a *Montesinos link*, and denote it by $M_1(r_1, r_2, \ldots, r_m)$. We call m the *length* of the Montesinos link $M_1(r_1, r_2, \ldots, r_m)$ when neither of $r_1, r_2, \ldots, r_m$ is an integer. Similarly, we denote by $M_2(r_1, r_2, \ldots, r_m)$ the arborescent link obtained from a hollow Montesinos pair of the form shown on the right in Figure 3 by plugging the holes with rational tangles of finite slopes $r_1, r_2, \ldots, r_m$.

Lemma 4.1. *Let L be an arborescent link in S^3 which has bridge index at least* 3, *and suppose that L does not admit an essential Conway sphere* (*i.e.*, *a c-essential sphere in S^3 intersecting L transversely in exactly* 4 *points*). *Then L is equivalent to a Montesinos link of length* 3 *as illustrated in Figure 4. In that figure*, *each circle with a rational number r_i* ($i = 1, 2, 3$) *inside represents a rational tangle of slope r_i.*

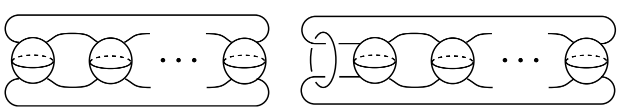

Figure 3. Hollow Montesinos pairs.

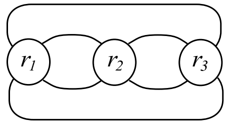

Figure 4. A Montesinos link $M_1(r_1, r_2, r_3)$.

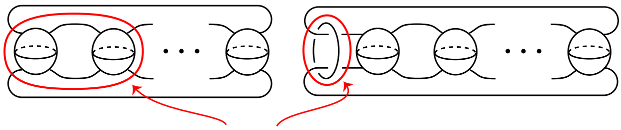

Figure 5. Essential Conway spheres in Montesinos pairs.

Proof. Let L be an arborescent link in S^3 and suppose that L does not admit an essential Conway sphere. Then L is obtained from a Montesinos pair of one of the forms shown in Figure 3 by plugging the holes with rational tangles of finite slopes (see [Bonahon and Siebenmann 2010, Theorem 3.4] or [Jang 2011, Theorem 4]). That is, L is equivalent to a Montesinos link $M_1(r_1, r_2, \ldots, r_{m_1})$ or an arborescent link $M_2(r_1, r_2, \ldots, r_{m_2})$ for some rational numbers r_i's. Moreover, the m_1 and m_2 cannot be bigger than 3 and 1, respectively, since otherwise L admits an essential Conway sphere as illustrated in Figure 5, which contradicts the hypothesis.

We note that $M_2(r_1)$ is equivalent to the Montesinos link $M_1(-1/2, 1/2, -1/r_1)$. Moreover, we can easily see that $M_1(r_1, r_2, \ldots, r_{m_1})$ admits a 2-bridge presentation if the length of $M_1(r_1, r_2, \ldots, r_{m_1})$ is 1 or 2, which contradicts the assumption that the bridge index of L is at least 3. Thus, L is equivalent to a Montesinos link of length 3, which is the desired result. □

Proof of Corollary 1.4. Let L be an arborescent link in S^3 and F a bridge sphere of L. If there is an essential tori or an essential Conway sphere in the complement of L, then the distances $d_{BS}(L, F)$ and $d_T(L, F)$ are at most 2 by [Bachman and Schleimer 2005, Theorem 5.1] together with Theorem 1.1 and Corollary 1.3. Thus, in the rest of the proof, we assume that there is no essential tori or essential Conway spheres. By Lemma 4.1, the link L is equivalent to a Montesinos link of length 3 (see Figure 4).

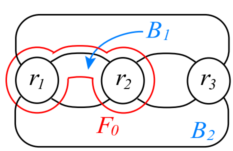

Figure 6. A 3-bridge sphere for a Montesinos link.

Assume that F is a minimal bridge sphere (that is, a 3-bridge sphere) of L. By [Jang 2013], we may assume that F is (equivalent to) the 3-bridge sphere F_0 in Figure 6 without loss of generality. Let B_1 be the 3-ball bounded by F containing two of the three rational tangles and B_2 the other 3-ball bounded by F (see Figure 6), and let H_i be the closure of $B_i \setminus N(L)$ $(i = 1, 2)$. Let c_0, c_1 and c_2 be the loops on F_L as illustrated in Figure 7. Then c_0 bounds a cut-disk in H_1, c_2 bounds a compressing disk in H_2, and c_1 is disjoint from $c_0 \cup c_2$. These imply $d_{BS}(L, F) \leq 2$. Moreover, by Proposition 1.2, we have $d_T(L, F) \leq 2$. □

Proof of Corollary 1.5. If the distances are greater than two, then, by [Bachman and Schleimer 2005, Theorem 5.1] together with Theorem 1.1 and Corollary 1.3, there is no essential tori in the exterior of L, no essential Conway spheres for L, no essential spheres nor essential annuli. By [Bachman and Schleimer 2005, Corollary 6.2], L is a hyperbolic link. Moreover, the double branched cover $M_2(L)$ of S^3 branched along L has a trivial JSJ decomposition. Thus, $M_2(L)$ is either a Seifert fibered space or a hyperbolic manifold. In the former case, we obtain that either L is a Montesinos link or the complement of L admits a Seifert fibration, which contradicts Corollary 1.4 or the fact that L is hyperbolic, accordingly. Hence, $M_2(L)$ must be hyperbolic. □

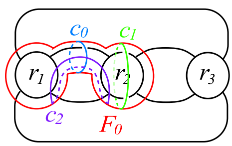

Figure 7. Curves realizing distance 2.

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