NORMAL STATES OF TYPE III FACTORS

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Dedicated to Masamichi Takesaki on the occasion of his eightieth birthday.

Let $M$ be a factor of type III with separable predual and with normal states $\varphi_1, \ldots, \varphi_k, \omega$ with $\omega$ faithful. Let $A$ be a finite-dimensional $C^*$-subalgebra of $M$. Then it is shown that there is a unitary operator $u \in M$ such that $\varphi_i \circ \text{Ad} \ u = \omega$ on $A$ for $i = 1, \ldots, k$. This follows from an embedding result of a finite-dimensional $C^*$-algebra with a faithful state into $M$ with finitely many given states. We also give similar embedding results of $C^*$-algebras and von Neumann algebras with faithful states into $M$. Another similar result for a factor of type $\text{II}_1$ instead of type III holds.

1. Introduction

Let $M$ be a factor of type III with separable predual. Then two nonzero projections $e$ and $f$ in $M$ are equivalent, that is, there exists a partial isometry $v \in M$ such that $v^*v = e$, $vv^* = f$. If, furthermore, $e$ and $f$ are different from the identity operator $1$, then there is a unitary operator $u \in M$ such that $u^*eu = f$. This shows that there is an abundance of unitaries in $M$, so one might expect stronger results arising from these unitaries. That is what is done in the present paper. We show that if $\varphi$ and $\omega$ are faithful normal states in $M$ and $A \subset M$ is a finite-dimensional $C^*$-algebra, then there exists a unitary operator $u \in M$ such that the restrictions $\varphi \circ \text{Ad} \ u|_A$ and $\omega|_A$ are equal, where $\text{Ad} \ u$ is the inner automorphism $x \mapsto u^*xu$ of $M$. (See Corollary 2.2 for a more precise and general statement.)

This actually follows from an embedding result of a finite-dimensional $C^*$-algebra $A$ with a faithful state into $M$ with finitely given normal states. This result is then applied to obtain a similar result for the $C^*$-algebra of the compact operators on a separable Hilbert space. Furthermore, we have more general embedding results in Section 3 for $C^*$-algebras and von Neumann algebras with faithful states into a

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type III factor $M$ such that a finite number of normal states on $M$ coincide after the embedding.

If $M$ is not of type III, the corresponding result is false in general, but if $M$ is a factor of $\text{II}_1$, $\omega = \tau$ is the trace and $A \cong M_n(\mathbb{C})$, the matrix algebra of complex $n \times n$-matrices, then the corresponding result to the unitary equivalence on $A$ holds for $\omega = \tau$ and any $\varphi$. This will be shown in Section 4.

There exist results of a similar nature to the ones above in the literature. In [Connes and Størmer 1978], it has been shown that if $M$ is of type $\text{III}_1$ and $\varepsilon > 0$ then there is a unitary operator $u \in M$ with

$$\|\varphi \circ \text{Ad}u - \omega\| < \varepsilon.$$  

If one takes a pointwise weak limit point of the automorphisms of the form $\text{Ad}u$ in the above, then one finds a completely positive unital map $\pi : M \to M$ with $\varphi \circ \pi = \omega$.

In the nonseparable case, it has recently been shown by Ando and Haagerup [2013] that for some factors of type $\text{III}_1$ constructed as ultraproducts, all faithful normal states are unitarily equivalent.

In the $C^*$-algebra case it has been shown in [Kishimoto et al. 2003] that if $\varphi$ and $\omega$ are pure states of a separable $C^*$-algebra $A$ with the same kernel for their GNS-representations, then there is an asymptotically inner automorphism $\alpha$ of $A$ such that $\varphi \circ \alpha = \omega$.

Our result gives an exact equality for two states, not an approximate one, but only on a finite-dimensional $C^*$-subalgebra $A$.

2. Factors of type III

In this section we state and prove our main result.

**Theorem 2.1.** Let $M$ be a type III factor with separable predual and $\varphi_1, \ldots, \varphi_k$ normal states on $M$. Let $A$ be a finite-dimensional $C^*$-algebra and $\rho$ a faithful state on $A$. Then there exists a unital injective homomorphism $\pi : A \to M$ with

$$\varphi_i \circ \pi = \rho, \quad i = 1, \ldots, k.$$  

After proving this theorem, we will prove that it implies the following corollary.

**Corollary 2.2.** Let $M$ be a factor of type III with separable predual. Let $A$ be a finite-dimensional $C^*$-subalgebra of $M$. Let $\varphi_1, \ldots, \varphi_k$ and $\omega$ be normal states on $M$ and assume that $\omega$ is faithful. Then there exists a unitary operator $u \in M$ such that

$$\varphi_i \circ \text{Ad}u|_A = \omega|_A, \quad i = 1, \ldots, k.$$
Before starting preliminaries of our proof of Theorem 2.1, we give an outline of our method for the case \( A \cong M_d(\mathbb{C}) \).

After diagonalizing the density matrix of \( \rho \), what we have to find is a system of matrix units \( \{e_{ij}\} \) in \( M \) for which we have \( \varphi_n(e_{ij}) = \delta_{ij}\lambda_i \) for all \( n = 1, \ldots, k \) and \( i, j = 1, \ldots, d \), where the \( \lambda_i \) are eigenvalues of the density matrix of \( \rho \). We first choose \( e_{ii} \) satisfying this condition. Then we choose \( e_{12}, e_{13}, \ldots, e_{1d} \) inductively so that we have various identities saying that the values of certain linear functionals applied to a certain partial isometry are all zero at each induction step. This is done by a version of a noncommutative Lyapunov theorem, and what we need is a special case of [Akemann and Anderson 1991, Theorem 2.5(1)]. Since the statement and its proof are short, we include them here in the form we need, for the sake of convenience of the reader.

**Lemma 2.3.** Let \( M \) be a nonatomic von Neumann algebra and \( \Phi : M \to \mathbb{C}^n \) a \( \sigma \)-weakly continuous linear map. Then for any \( a \in M_{+, 1} \), there exists a projection \( p \in M \) such that \( \Phi(p) = \Phi(a) \).

**Proof.** Let \( D := \{ x \in M_{+, 1} \mid \Phi(x) = \Phi(a) \} \),
where \( M_{+, 1} \) denotes the positive operators in the unit ball of \( M \). Then \( D \) is a nonempty \( \sigma \)-weakly compact convex set. Therefore, by the Krein–Milman theorem, there exists an extremal point \( b \) of \( D \). We show \( b \) is a projection. If \( b \) were not a projection, then there exists \( \delta \in (0, \frac{1}{2}) \) such that the spectral projection \( p \) of \( b \) corresponding to \( (\delta, 1 - \delta) \) is nonzero. By the assumption on \( M \), \( pM_{sa}p \) is an infinite-dimensional real linear space while its range with respect to \( \Phi \) is finite-dimensional. This implies the existence of a nonzero \( y \in pM_{sa}p \) such that \( \Phi(y) = 0 \). Setting \( t := \delta/\|y\| \), we have \( b \pm ty \in D \). As we have \( b = (b + ty)/2 + (b - ty)/2 \), this contradicts the fact that \( b \) is extremal in \( D \).

We now construct appropriate matrix units by induction on the size of matrix units.

**Lemma 2.4.** Let \( M \) be a type III factor with separable predual and \( \varphi_1, \ldots, \varphi_n \) normal states on \( M \). Let \( \lambda_i > 0, i = 1, \ldots, m \) with \( \sum_i \lambda_i = 1 \). Then there exists a system of matrix units \( \{e_{ij}\}_{i, j = 1, \ldots, m} \) such that \( \varphi_l(e_{ij}) = \delta_{ij}\lambda_i \) for all \( l = 1, \ldots, m \).

**Proof.** For a projection \( p \in M \) satisfying \( 0 \leq \varphi_l(p) = \lambda < 1 \) for \( l = 1, \ldots, n \) and \( 0 \leq t \leq 1 - \lambda \), there exists a projection \( q \) orthogonal to \( p \) such that \( \varphi_l(q) = t \). To see this, we consider a \( \sigma \)-weakly continuous linear map \( \Phi : M_{\tilde{p}} \to \mathbb{C}^n \), where we write \( \tilde{p} = 1 - p \), given by \( \Phi(x) = (\varphi_l(x))_{l=1}^n \), and apply Lemma 2.3 for \( a = t \tilde{p}/(1 - \lambda) \).

Using this fact inductively, we have \( \{e_{ii}\} \).
We next define partial isometries $u_{i1}, i = 1, \ldots, m$, inductively such that $e_{ij} = u_{i1}u_{j1}^*$ satisfy the conditions of the lemma. Let $u_{11} = e_{11}$ and assume that we have found $u_{i1}, i = 1, \ldots, k$ with $k < m$. Let $v$ be a partial isometry in $M$ with $v^*v = e_{11}$, $vv^* = e_{k+1,k+1}$. Then define a map

$$\Phi : e_{11}Me_{11} \to \mathbb{C}^{nk}$$

$$\Phi(x) := (\varphi_l(vxu_{j1}^*))_{l=1,\ldots,n, j=1,\ldots,k}.$$  

This map $\Phi$ is $\sigma$-weakly continuous and linear, so by using Lemma 2.3 with $a = e_{11}/2$, we obtain a projection $p \in e_{11}Me_{11}$ such that $\Phi(p) = \Phi(e_{11})/2$. Define $u_{k+1,1} := vp - v(1-p)$.

Since $p \leq e_{11}$, an easy computation shows that $u_{k+1,1}^*u_{k+1,1} = e_{11}$, $u_{k+1,1}u_{k+1,1}^* = e_{k+1,k+1}$. Let $e_{k+1,j} = u_{k+1,1}u_{j1}^*$ and $e_{j,k+1} = u_{j1}u_{k+1,1}^*$. Then the $e_{ij}, i, j \leq k+1$, form a set of matrix units, and using the definition of $\Phi$ and that $\Phi(p) = \Phi(e_{11})/2$, we get for all $l$

$$\varphi_l(u_{k+1,1}u_{j1}^*) = \varphi_l((2vp - v)u_{j1}^*)$$

$$= 2\varphi_l(vpu_{j1}^*) - \varphi_l(vu_{j1}^*)$$

$$= 0.$$

Thus

$$\varphi_l(e_{j,k+1}) = \varphi_l(u_{j1}u_{k+1,1}^*) = \varphi_l(u_{k+1,1}u_{j1}^*) = 0,$$

completing the proof of the lemma. \hfill \square

**Proof of Theorem 2.1.** First we consider the case $A = M_m(\mathbb{C})$. We choose a system of matrix units $\{v_{ij}\}_{i,j=1,\ldots,m}$ of $A = M_m(\mathbb{C})$ which diagonalizes the density matrix $D_\rho$ of $\rho$, that is, $D_\rho = \sum_{i=1}^m \lambda_i v_{ii}v_{ii}^*$. As $\rho$ is faithful, we have $\lambda_i > 0$ for all $i$.

By Lemma 2.4, we obtain a system of matrix units $\{e_{ij}\}_{i,j=1,\ldots,m}$ in $M$ satisfying

(1)  
$$\varphi_n(e_{ij}) = \delta_{ij}\lambda_i, \quad n = 1, \ldots, k, \quad i, j = 1, \ldots, m.$$

Define

$$\pi : M_m(\mathbb{C}) \to M, \quad \pi(v_{ij}) = e_{ij}.$$  

Then $\pi$ gives a unital homomorphism satisfying the desired condition.

For the general case $A \simeq \bigoplus_{k=1}^b M_{n_k}(\mathbb{C})$, let $m = \sum_{k=1}^b n_k$. Let $\hat{\rho}$ be a faithful extension of $\rho$ to $M_m(\mathbb{C})$. Applying the above result to $M_{n_i}(\mathbb{C})$ and $\hat{\rho}$, there exists a unital homomorphism $\hat{\pi} : M_m(\mathbb{C}) \to M$ such that

$$\varphi_n \circ \hat{\pi} = \hat{\rho}, \quad n = 1, \ldots, k.$$  

The restriction $\pi := \hat{\pi}|_A$ gives a unital homomorphism from $A$ to $M$ satisfying $\varphi_n \circ \pi = \rho$, for $n = 1, \ldots, k.$ \hfill \square
Proof of Corollary 2.2. Let $p$ be the unit of $A$. Considering $A \oplus \mathbb{C}(1-p)$ instead of $A$, we may assume that $A$ contains the unit of $M$ from the beginning.

First we consider the case $A \cong M_m(\mathbb{C})$, $m \in \mathbb{N}$. Let $\{f_{ij}\}_{i,j=1,\ldots,m}$, $\{v_{ij}\}_{i,j=1,\ldots,m}$ be systems of matrix units of $A$ and $M_m(\mathbb{C})$, respectively. Let $\gamma : M_m(\mathbb{C}) \to A$ be an isomorphism given by $\gamma(v_{ij}) = f_{ij}$.

Then $\rho := \omega \circ \gamma$ is a faithful state on $M_m(\mathbb{C})$. From Theorem 2.1, there exists a unital homomorphism $\pi : M_m(\mathbb{C}) \to M$ such that $\varphi_n \circ \pi = \rho$, $n = 1, \ldots, k$.

The algebras $\mathbb{A}$ and $\pi(M_m(\mathbb{C}))$ are subalgebras of $M$ isomorphic to $M_m(\mathbb{C})$ with complete sets of matrix units $\{f_{ij}\}$ and $\{\pi(v_{ij})\}$. As in [Haagerup and Musat 2011, Lemma 2.1], if $v \in M$ is a partial isometry with $v^*v = \pi(v_{11})$ and $vv^* = f_{11}$, then $u := \sum_{i=1}^m \pi(v_{i1})v^*f_{ii}$ is a unitary in $M$ satisfying $uf_{ij}u^* = \pi(v_{ij})$. Hence we have

$$\varphi_n \circ \text{Ad} u(f_{ij}) = \varphi_n(\pi(v_{ij})) = \rho(v_{ij}) = \omega \circ \gamma(v_{ij}) = \omega(f_{ij}),$$

that is, $\varphi_n \circ \text{Ad} u|_A = \omega|_A$ for $n = 1, \ldots, k$.

For the general case $A \cong \bigoplus_{l=1}^b M_n(\mathbb{C})$, let $\{f_{ij}^{(l)}\}_{i,j=1,\ldots,n_l}$ be a system of matrix units of $M_n(\mathbb{C})$ for each $l = 1, \ldots, b$. As $M$ is of type III, for all $l = 1, \ldots, b$, the nonzero projections $f_{11}^{(l)}$ and $s_{11}^{(l)}$ are mutually equivalent. Hence, there exist partial isometries $v^{(l)} \in M$ such that $v^{(l)*}v^{(l)} = f_{11}^{(l)}$ and $v^{(l)}v^{(l)*} = f_{11}^{(l)}$. Set $w_{(k,i)(l,j)} := f_{11}^{(k)}v^{(k)*}v^{(l)}f_{11}^{(l)}$, for $k, l = 1, \ldots, b$, $i = 1, \ldots, n_k$, and $j = 1, \ldots, n_l$. Then we have

$$w_{(k,i)(l,j)}^* = f_{11}^{(l)}v^{(l)*}f_{11}^{(k)}v_{(i,j)}^{(k)} = w_{(l,j)(k,i)}$$

and

$$w_{(k,i)(l,j)}w_{(l',j')(k',i')} = f_{11}^{(k)}v^{(k)*}f_{11}^{(l')}v_{(j',i')}^{(l')}f_{11}^{(k')}v^{(k)*}f_{11}^{(k')}f_{11}^{(l')}$$

for $l' = 1, \ldots, b$, $i' = 1, \ldots, n_l$, and $j' = 1, \ldots, n_l$. Set

$$\sum_{(k,i)} w_{(k,i)(l,j)} = \sum_{i,k} f_{11}^{(k)}v^{(k)*}f_{11}^{(l)}v_{(i,j)}^{(k)} = \sum_{(k,i)} f_{11}^{(k)} = 1.$$

Hence $\{w_{(k,i)(l,j)}\}_{(k,i),(l,j)}$ gives a system of matrix units of a $C^*$-subalgebra $B$ of $M$ isomorphic to $M_m$, for $m := \sum_{k=1}^bn_k$. As $w_{(k,i)(k,j)} = f_{11}^{(k)}f_{11}^{(k)} = f_{11}^{(k)}$, $\{w_{(k,i)(l,j)}\}$ is an extension of $\{f_{11}^{(k)}\}$ and $A$ is a subalgebra of $B$. We apply the above argument to $B \cong M_m(\mathbb{C})$ and obtain a unitary $u$ in $M$ such that $\varphi_i \circ \text{Ad} u|_B = \omega|_B$.

In particular, we obtain $\varphi_i \circ \text{Ad} u|_A = \omega|_A$ for $i = 1, \ldots, k$. \qed

3. Embedding of operator algebras with faithful states

The above theorem can be extended to the algebra of the compact operators as follows.
**Theorem 3.1.** Let \( K(\mathcal{H}) \) denote the set of all the compact operators on a separable Hilbert space \( \mathcal{H} \). Let \( \rho \) be a faithful state on \( K(\mathcal{H}) \). Let \( M \) be a factor of type III with separable predual, \( \varphi_1, \varphi_2, \ldots, \varphi_k \) normal states on \( M \). Then there exists a homomorphism \( \pi \) of \( K(\mathcal{H}) \) into \( M \) such that
\[
\varphi_n \circ \pi = \rho, \quad n = 1, \ldots, k.
\]

**Proof.** We may assume that \( \mathcal{H} \) is infinite-dimensional and \( \varphi_1 \) is faithful — for example, by adding a faithful state to the set of all the \( \varphi_i \).

Let \( \{v_{ij}\} \) be a system of matrix units of \( K(\mathcal{H}) \) diagonalizing the density matrix \( D_\rho \) of \( \rho \), that is, \( D_\rho = \sum_{i=1}^{\infty} \lambda_i v_{ii} \). As \( \rho \) is faithful, we have \( \lambda_i > 0 \) for all \( i \).

We claim that there exists a system of matrix units \( \{e_{ij}\}_{i,j \in \mathbb{N}} \) in \( M \) satisfying
\[
\varphi_n(e_{ij}) = \delta_{ij} \lambda_i, \quad n = 1, \ldots, k, \quad i, j = 1, 2, \ldots.
\]
This is proved in the same way as in the proof of Theorem 2.1. \( \square \)

A slight rewriting of the above theorem gives the following:

**Corollary 3.2.** Let \( B(\mathcal{H}) \) be the set of all the bounded operators on a separable Hilbert space \( \mathcal{H} \) and \( \rho \) a faithful normal state on \( B(\mathcal{H}) \). Let \( M \) be a factor of type III with separable predual and \( \varphi_1, \varphi_2, \ldots, \varphi_k \) normal states on \( M \). Then there exists a homomorphism \( \pi \) of \( B(\mathcal{H}) \) into \( M \) such that
\[
\varphi_n \circ \pi = \rho, \quad n = 1, \ldots, k.
\]

We now consider an embedding of a \( C^* \)-algebra with a faithful state into a type III factor with finitely many normal states.

**Theorem 3.3.** For a \( C^* \)-algebra \( A \) and a faithful state \( \omega \) on \( A \), the following conditions are equivalent:

(i) The Hilbert space \( \mathcal{H}_\omega \) in the GNS triple \( (\mathcal{H}_\omega, \pi_\omega, \Omega_\omega) \) of \( \omega \) is separable and \( \Omega_\omega \) is separating for \( \pi_\omega(A)' \).

(ii) There exists a representation \( (\mathcal{H}, \rho) \) of \( A \) on a separable Hilbert space \( \mathcal{H} \) and a faithful normal state \( \sigma \) on \( B(\mathcal{H}) \) with \( \omega = \sigma \circ \rho \).

(iii) For any factor \( M \) of type III with separable predual and its normal states \( \varphi_1, \ldots, \varphi_n \), there exists an injective homomorphism \( \gamma : A \to M \) with \( \varphi_j \circ \gamma = \omega \) for all \( j = 1, \ldots, n \).

**Proof.** Suppose condition (i) holds. Then \( \Omega_\omega \) is cyclic for \( \pi_\omega(A)' \). Therefore, using the separability of \( \mathcal{H}_\omega \), we have a sequence \( \{x_n\}_{n=1}^{\infty} \subset (\pi_\omega(A)')_1 \) such that \( \{x_n \Omega_\omega : n \in \mathbb{N} \} \) spans \( \mathcal{H}_\omega \). Let \( x_0 := \sqrt{1 - \sum_{n=1}^{\infty} x_n^* x_n / 2^n} \), and define a state \( \sigma \) on \( B(\mathcal{H}_\omega) \) given by the density matrix \( \sum_{n=0}^{\infty} |x_n \Omega_\omega \rangle \langle x_n \Omega_\omega| / 2^n \). This \( \sigma \) is faithful and normal. Let \( \rho = \pi_\omega \). We can check \( \sigma \circ \rho = \omega \). Hence (ii) holds.
Now suppose condition (ii) holds. We show (iii). By Theorem 3.1, we have an injective homomorphism \( \pi : K(\mathcal{H}) \to M \) such that \( \sigma \big|_{K(\mathcal{H})} = \varphi \circ \pi \). We denote the extension of \( \pi \) to \( B(\mathcal{H}) \) by \( \hat{\pi} \). Then from the way we have constructed \( \pi \), we obtain \( \sigma = \varphi \circ \hat{\pi} \). Define \( \gamma := \hat{\pi} \circ \rho : A \to M \). Then we obtain \( \varphi \circ \gamma = \varphi \circ \hat{\pi} \circ \rho = \sigma \circ \rho = \omega \).

Finally suppose condition (iii) holds, and we show this implies (i). To see this, fix a factor \( M \) of type III with a faithful normal state \( \varphi \), and let \( (\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi) \) be its GNS triple. We obtain \( \gamma \) as in (iii). Let \( K := \pi_\varphi \circ \gamma(A)\overline{\Omega_\varphi} \) and let \( \beta \) be the restriction of \( \pi_\varphi \circ \gamma \) to \( K \). Then \( (K, \beta, \Omega_\varphi) \) is the GNS triple of \( \omega \). As \( \Omega_\varphi \) is separating for \( \pi_\varphi(M) \), it is separating for \( \beta(A)' \), and (i) holds.

As an immediate corollary, we obtain the following:

**Corollary 3.4.** Let \( N \) be a von Neumann algebra with separable predual and \( \psi \) a faithful normal state on \( N \). Then for any factor \( M \) of type III with separable predual and a normal state \( \varphi \) on \( M \), there exists an injective homomorphism \( \pi : N \to M \) with \( \varphi \circ \pi = \psi \).

Another easy corollary is as follows, by a well-known result on the KMS condition [Bratteli and Robinson 1997, Corollary 5.3.9].

**Corollary 3.5.** Suppose that we have a \( C^* \)-algebra \( A \), a state \( \varphi \) on \( A \), and a one-parameter automorphism group \( \{\alpha_t\}_{t \in \mathbb{R}} \) such that these satisfy the KMS condition. Then the pair \((A, \varphi)\) satisfies the (equivalent) conditions in Theorem 3.3.

**Remark 3.6.** Note that a general faithful state on a \( C^* \)-algebra \( A \) does not satisfy condition (i) of Theorem 3.3 at all, as shown in [Takesaki 1974] by an example due to Pedersen. The \( C^* \)-algebra used by Takesaki is a very basic one, \( C(\{0, 1\}) \otimes M_2(\mathbb{C}) \).

A slight modification of the argument there also works for a simple \( C^* \)-algebra \( A_\theta \otimes M_2(\mathbb{C}) \), where \( A_\theta \) is the irrational rotation \( C^* \)-algebra.

In Theorem 3 of the same paper, Takesaki gives a sufficient condition for our condition (i) in Theorem 3.3 and calls it the quasi-KMS condition, but it seems difficult to check this condition for a given example.

**Remark 3.7.** In all the above cases, we considered embeddings into a type III factor, but actually any properly infinite von Neumann algebra with separable predual works. This is because if we have a properly infinite von Neumann algebra and normal states on it, we simply restrict the states on a type III factor which is found as a subalgebra of the original von Neumann algebra. It is easy to see that if a von Neumann algebra with separable predual has a finite direct summand, this type of embedding is impossible, so actually this embeddability characterize proper infiniteness of a von Neumann algebra with separable predual.
4. Factors of type II

The direct analogue of Theorem 2.1 for finite factors is trivially false. For example, if $M$ is of type II$_1$ with trace $\tau$ and $\rho$ is not a trace on $A$, then the conclusion of Theorem 2.1 for $\varphi_1 = \tau$ is clearly false. However, if we restrict the choice of $\omega$ in Corollary 2.2, we obtain a positive result.

**Theorem 4.1.** Let $\varphi_1, \ldots, \varphi_k$ be normal states on a factor $M$ of type II$_1$ with the unique trace $\tau$. Let $A$ be a $C^*$-subalgebra of $M$ isomorphic to $M_m(\mathbb{C})$ with $1 \in A$. Then there exists a unitary operator $u \in M$ satisfying $\varphi_i \circ \text{Ad} u|_A = \tau|_A$ for $i = 1, \ldots, k$.

**Proof.** We may assume that $\varphi_1 = \tau$ is the unique trace on $M$. We proceed as in the proof of Theorem 2.1. The only difference is that we take $\tau(e_{ii}) = 1/m$ instead of the proof of Lemma 2.4. □

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