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A NEW MONOTONE QUANTITY ALONG THE INVERSE MEAN CURVATURE FLOW IN $\mathbb{R}^n$

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We find a new monotone increasing quantity along smooth solutions to the inverse mean curvature flow in $\mathbb{R}^n$. As an application, we derive a sharp geometric inequality for mean convex, star-shaped hypersurfaces which relates the volume enclosed by a hypersurface to a weighted total mean curvature of the hypersurface.

1. Statement of the result

Monotone quantities along hypersurfaces evolving under the inverse mean flow have many applications in geometry and relativity. Huisken and Ilmanen [2001] applied the monotone increasing property of Hawking mass to give a proof of the Riemannian Penrose inequality. Brendle, Hung and Wang [Brendle et al. 2012] discovered a monotone decreasing quantity along the inverse mean curvature flow in anti-de Sitter–Schwarzschild manifolds and used it to establish a Minkowski-type inequality for star-shaped hypersurfaces.

In this note, we provide a new monotone increasing quantity along smooth solutions to the inverse mean curvature flow in $\mathbb{R}^n$:

Theorem 1. *Let Σ be a smooth, closed, embedded hypersurface with positive mean curvature in $\mathbb{R}^n$. Let I be an open interval and $X : \Sigma \times I \to \mathbb{R}^n$ be a smooth map satisfying*

$$\frac{\partial X}{\partial t} = \frac{1}{H}\nu, \tag{1-1}$$

where H is the mean curvature of the surface $\Sigma_t = X(\Sigma, t)$ and ν is the outward unit normal vector to Σ_t. Let Ω_t be the bounded region enclosed by Σ_t and $r = r(x)$ be the distance from x to a fixed point O. Then the function

$$Q(t) = e^{-\frac{n-2}{n-1}t}\left(n\,\mathrm{Vol}(\Omega_t) - \frac{1}{n-1}\int_{\Sigma_t} r^2 H\, d\mu\right) \tag{1-2}$$

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is monotone increasing and $Q(t)$ *is a constant function if and only if* Σ_t *is a round sphere for each* t. *Here* $\mathrm{Vol}(\Omega)$ *denotes the volume of a bounded region* Ω *and* $d\mu$ *denotes the volume form on a hypersurface.*

As an application, we derive a sharp inequality for star-shaped hypersurfaces in $\mathbb{R}^n$ which relates the volume enclosed by a hypersurface to an r^2-weighted total mean curvature of the hypersurface.

Theorem 2. *Let* Σ *be a smooth, star-shaped hypersurface with positive mean curvature in* $\mathbb{R}^n$. *Then*

$$n\,\mathrm{Vol}(\Omega) \leq \frac{1}{n-1}\int_{\Sigma} r^2 H\, d\mu \tag{1-3}$$

where $\mathrm{Vol}(\Omega)$ *is the volume of the region* Ω *enclosed by* Σ, r *is the distance to a fixed point* O *and* H *is the mean curvature of* Σ. *Furthermore, equality in* (1-3) *holds if and only if* Σ *is a sphere centered at* O.

We give some remarks about Theorem 1 and Theorem 2. The discovery of the monotonicity of $Q(t)$ in Theorem 1 is motivated by [Brendle et al. 2012, Section 5]. To prove Theorem 1, we also need a result of Ros, proved using Reilly's formula. Once we know that $Q(t)$ is monotone increasing, to prove Theorem 2, it may be tempting to ask whether $\lim_{t\to\infty} Q(t) = 0$? We do not know if this is true because both $\mathrm{Vol}(\Omega_t)$ and $\int_{\Sigma_t} r^2 H\, d\mu$ grow like $\exp\bigl(\frac{n}{n-1}\, t\bigr)$ when $\{\Sigma_t\}$ are spheres, while there is only a factor of $\exp\bigl(-\frac{n-2}{n-1}\, t\bigr)$ in (1-2). Instead, we take an alternate approach by first proving Theorem 2 for a convex hypersurface Σ. The proof in that case again makes use of Reilly's formula. When Σ is merely assumed to be mean convex and star-shaped, we prove Theorem 2 by reducing it to the convex case using solutions to the inverse mean curvature flow provided by [Gerhardt 1990] and [Urbas 1990].

2. Proof of the theorems

Given a compact Riemannian manifold (Ω, g) with boundary Σ, Reilly's formula [1977] asserts that

$$\begin{aligned}\int_{\Omega} |\nabla^2 u|^2 + \langle \nabla(\Delta u), \nabla u\rangle + \mathrm{Ric}(\nabla u, \nabla u)\, dV \\ = \int_{\Sigma} (\Delta u)\frac{\partial u}{\partial \nu} - II\bigl(\nabla^{\Sigma} u, \nabla^{\Sigma} u\bigr) - 2(\Delta_{\Sigma} u)\frac{\partial u}{\partial \nu} - H\left(\frac{\partial u}{\partial \nu}\right)^2 d\mu.\end{aligned} \tag{2-1}$$

Here u is a smooth function on Ω; ∇^2, Δ and ∇ denote the Hessian, the Laplacian and the gradient on Ω; Δ_{Σ} and ∇^{Σ} denote the Laplacian and the gradient on Σ; ν is the unit outward normal vector to Σ; II and H are the second fundamental form and the mean curvature of Σ with respect to ν; and Ric is the Ricci curvature of g.

To prove Theorem 1, we need a result of [Ros 1987], which was proved by choosing $\Delta u = 1$ on Ω and $u = 0$ at Σ in the above Reilly's formula.

Theorem 3 [Ros 1987]. *Let (Ω, g) be an n-dimensional compact Riemannian manifold with nonnegative Ricci curvature with boundary Σ. Suppose Σ has positive mean curvature H; then*

$$n\,\mathrm{Vol}(\Omega) \le (n-1)\int_\Sigma \frac{1}{H}\,d\mu \tag{2-2}$$

and equality holds if and only if (Ω, g) is isometric to a round ball in $\mathbb{R}^n$.

Proof of Theorem 1. We use $'$ to denote differentiation with respect to t. Some basic formulas along the inverse mean curvature flow (1-1) in $\mathbb{R}^n$ are

$$H' = -\Delta_{\Sigma_t}\left(\frac{1}{H}\right) - \frac{|II|^2}{H}, \quad d\mu' = d\mu, \quad \mathrm{Vol}(\Omega_t)' = \int_{\Sigma_t}\frac{1}{H}\,d\mu. \tag{2-3}$$

Let $u = r^2$. Then u satisfies

$$\nabla^2 u = 2g \quad \text{and} \quad \Delta u = 2n, \tag{2-4}$$

where g is the Euclidean metric. Now

$$\left(\int_{\Sigma_t} uH\,d\mu\right)' = \int_{\Sigma_t}(u'H + uH' + uH)\,d\mu. \tag{2-5}$$

Let $\langle\cdot,\cdot\rangle$ be the Euclidean inner product. By (2-3), (2-4) and the divergence theorem, we have

$$\int_{\Sigma_t} u'H\,d\mu = \int_{\Sigma_t}\left\langle \nabla u, \frac{1}{H}\nu\right\rangle H\,d\mu = \int_{\Omega_t}\Delta u\,dV = 2n\,\mathrm{Vol}(\Omega_t). \tag{2-6}$$

By (2-4), we also have

$$\Delta_{\Sigma_t} u = \Delta u - H\frac{\partial u}{\partial \nu} - \nabla^2 u(\nu,\nu) = 2(n-1) - H\frac{\partial u}{\partial \nu},$$

which together with (2-3) and (2-4) implies

$$\begin{aligned}\int_{\Sigma_t} uH'\,d\mu &= \int_{\Sigma_t}\left(-\frac{\Delta_{\Sigma_t} u}{H} - \frac{u|II|^2}{H}\right)d\mu \\ &= \int_{\Sigma_t}\left(-\frac{2(n-1)}{H} + \frac{\partial u}{\partial \nu} - \frac{u|II|^2}{H}\right)d\mu \\ &= -\int_{\Sigma_t}\frac{2(n-1)}{H}\,d\mu + 2n\,\mathrm{Vol}(\Omega_t) - \int_{\Sigma_t}\frac{u|II|^2}{H}\,d\mu.\end{aligned} \tag{2-7}$$

Substituting (2-6) and (2-7) into (2-5) yields

$$\begin{aligned}
\left(\int_{\Sigma_t} uH\,d\mu\right)' &= 4n\,\mathrm{Vol}(\Omega_t) + \int_{\Sigma_t}\left(-\frac{2(n-1)}{H} - \frac{u|II|^2}{H} + uH\right)d\mu \\
&\le 4n\,\mathrm{Vol}(\Omega_t) + \int_{\Sigma_t}\left(-\frac{2(n-1)}{H} - \frac{uH}{n-1} + uH\right)d\mu \\
&= 4n\,\mathrm{Vol}(\Omega_t) + \int_{\Sigma_t}\left(-\frac{2(n-1)}{H} + \frac{n-2}{n-1}uH\right)d\mu \\
&\le 4n\,\mathrm{Vol}(\Omega_t) - 2n\,\mathrm{Vol}(\Omega_t) + \frac{n-2}{n-1}\int_{\Sigma_t} uH\,d\mu \\
&= 2n\,\mathrm{Vol}(\Omega_t) + \frac{n-2}{n-1}\int_{\Sigma_t} uH\,d\mu,
\end{aligned} \tag{2-8}$$

where we have used $|II|^2 \ge \frac{1}{n-1}H^2$ in the second line and Theorem 3 in the fourth. On the other hand, by Theorem 3 again, we have

$$\mathrm{Vol}(\Omega_t)' = \int_{\Sigma_t} \frac{1}{H}\,d\mu \ge \frac{n}{n-1}\,\mathrm{Vol}(\Omega_t). \tag{2-9}$$

It follows from (2-8) and (2-9) that

$$\left(n(n-1)\,\mathrm{Vol}(\Omega_t) - \int_{\Sigma_t} uH\,d\mu\right)' \ge \frac{n-2}{n-1}\left(n(n-1)\,\mathrm{Vol}(\Omega_t) - \int_{\Sigma_t} uH\,d\mu\right)$$

or equivalently

$$\left[e^{-\frac{n-2}{n-1}t}\left(n\,\mathrm{Vol}(\Omega_t) - \frac{1}{n-1}\int_{\Sigma_t} r^2 H\,d\mu\right)\right]' \ge 0. \tag{2-10}$$

We conclude that $Q(t)$ is monotone increasing, moreover $Q(t)$ is a constant function if and only if equalities in (2-8) and (2-9) hold. By Theorem 3, we know these equalities hold if and only if Σ_t is a round sphere for all t. This completes the proof of Theorem 1. □

Next, we prove Theorem 2 in the case that Σ is a convex hypersurface.

Proposition 1. *Let Σ be a smooth, closed, convex hypersurface in $\mathbb{R}^n$. Then*

$$n\,\mathrm{Vol}(\Omega) \le \frac{1}{n-1}\int_{\Sigma} r^2 H\,d\mu, \tag{2-11}$$

where $\mathrm{Vol}(\Omega)$ *is the volume of the region Ω enclosed by Σ, r is the distance to a fixed point O and H is the mean curvature of Σ. Moreover, equality in* (2-11) *holds if and only if Σ is a sphere centered at O.*

Remark. Proposition 1 generalizes the first inequality in Theorem 3.2(1) of [Kwong 2012].

Proof. Apply Reilly's formula (2-1) to the Euclidean region Ω and choose $u = r^2$; we have

$$4n(n-1)\,\mathrm{Vol}(\Omega) = \int_\Sigma II\big(\nabla^\Sigma u, \nabla^\Sigma u\big) + 2(\Delta_\Sigma u)\frac{\partial u}{\partial \nu} + H\left(\frac{\partial u}{\partial \nu}\right)^2 d\mu$$

where $\Delta_\Sigma u = \Delta u - H\dfrac{\partial u}{\partial \nu} - \nabla^2 u(\nu,\nu) = 2(n-1) - H\dfrac{\partial u}{\partial \nu}$. Therefore,

$$\int_\Sigma H\left(\frac{\partial u}{\partial \nu}\right)^2 d\mu = \int_\Sigma II\big(\nabla^\Sigma u, \nabla^\Sigma u\big)\, d\mu + 4n(n-1)\,\mathrm{Vol}(\Omega). \tag{2-12}$$

Since Σ is convex, $II(\cdot\,,\cdot)$ is positive definite. Hence, (2-12) implies

$$n(n-1)\,\mathrm{Vol}(\Omega) \le \frac{1}{4}\int_\Sigma H\big\langle \nabla(r^2), \nu\big\rangle^2\, d\mu \le \int_\Sigma H r^2\, d\mu. \tag{2-13}$$

When $n(n-1)\,\mathrm{Vol}(\Omega) = \int_\Sigma H r^2\, d\mu$, we must have $II\big(\nabla^\Sigma u, \nabla^\Sigma u\big) = 0$, hence $\nabla^\Sigma u = 0$. This implies that $u = r^2$ is a constant on Σ, which shows that Σ is a sphere centered at O. □

To deform a star-shaped hypersurface to a convex hypersurface through the inverse mean curvature flow, we make use of a special case of a general result of Gerhardt and Urbas:

Theorem 4 [Gerhardt 1990; Urbas 1990]. *Let Σ be a smooth, closed hypersurface in $\mathbb{R}^n$ with positive mean curvature, given by a smooth embedding $X_0 : \mathbb{S}^{n-1} \to \mathbb{R}^n$. Suppose Σ is star-shaped with respect to a point P. Then the initial value problem*

$$\begin{cases} \dfrac{\partial X}{\partial t} = \dfrac{1}{H}\nu, \\ X(\cdot\,,0) = X_0(\cdot), \end{cases} \tag{2-14}$$

has a unique smooth solution $X : \mathbb{S}^{n-1} \times [0,\infty) \to \mathbb{R}^n$, where ν is the unit outer normal vector to $\Sigma_t = X(\mathbb{S}^{n-1}, t)$ and H is the mean curvature of Σ_t. Moreover, Σ_t is star-shaped with respect to P and the rescaled hypersurface $\widetilde{\Sigma}_t$, parametrized by $\widetilde{X}(\cdot\,,t) = e^{-t/(n-1)}X(\cdot\,,t)$, converges to a sphere centered at P in the $\mathscr{C}^\infty$ topology as $t \to \infty$.

Proof of Theorem 2. By Theorem 4, there exists a smooth solution $\{\Sigma_t\}$ to the inverse mean curvature flow with initial condition Σ. Moreover, the rescaled hypersurface $\widetilde{\Sigma}_t = \{e^{-t/(n-1)}x \mid x \in \Sigma_t\}$ converges exponentially fast in the C^∞ topology to a sphere. In particular, $\widetilde{\Sigma}_t$ and hence Σ_t, must be convex for large t.

Let T be a time when Σ_T becomes convex. By Proposition 1, we have

$$n\,\mathrm{Vol}(\Omega_T) \le \frac{1}{n-1}\int_{\Sigma_T} r^2 H\, d\mu;$$

thus $Q(T) \le 0$. By Theorem 1, we know that $Q(t)$ is monotone increasing. Hence $Q(0) \le Q(T) \le 0$, which proves (1-3).

If the equality in (1-3) holds, then $Q(0) = 0$. It follows from the monotonicity of $Q(t)$ and the fact $Q(t) \le 0$ for large t that $Q(t) = 0$ for all t. By Theorem 1, this implies that Σ_t is a sphere for each t. By Theorem 1, Σ_t is a sphere centered at O for large t. Therefore, we conclude that the initial hypersurface Σ is a sphere centered at O. $\square$

Remark. It can be shown that Theorem 2 is still true if the mean curvature is only assumed to be nonnegative. Please refer to the arXiv version of this paper (1212.1906) for details.

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