Lp HARMONIC 1-FORMS AND FIRST EIGENVALUE OF A
STABLE MINIMAL HYPERSURFACE

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We estimate the bottom of the spectrum of the Laplace operator on a stable minimal hypersurface in a negatively curved manifold. We also derive various vanishing theorems for $L^p$ harmonic 1-forms on minimal hypersurfaces in terms of the bottom of the spectrum of the Laplace operator. As consequences, the corresponding Liouville type theorems for harmonic functions with finite $L^p$ energy on minimal hypersurfaces in a Riemannian manifold are obtained.

1. Introduction

Hodge theory plays an important role in the topology of compact Riemannian manifolds. Unfortunately, the Hodge theory does not work anymore in noncompact manifolds. However, the $L^2$-Hodge theory works well in noncompact cases [Anderson 1988; Dodziuk 1982]. In this direction, there are various results for $L^2$ harmonic 1-forms on stable minimal hypersurfaces. Recall that a minimal hypersurface in a Riemannian manifold is called *stable* if the second variation of its volume is always nonnegative for any normal variation with compact support. More precisely, an $n$-dimensional minimal hypersurface $M$ in a Riemannian manifold $N$ is called *stable* if it holds that, for any compactly supported Lipschitz function $f$ on $M$,

$$\int_M |\nabla f|^2 - (|A|^2 + \overline{\text{Ric}}(\nu, \nu)) f^2 \, dv \geq 0,$$

where $\nu$ is the unit normal vector of $M$, $\overline{\text{Ric}}(\nu, \nu)$ denotes the Ricci curvature of $N$ in the $\nu$ direction, $|A|^2$ is the square length of the second fundamental form $A$, and $dv$ is the volume form for the induced metric on $M$.

Using the nonexistence of $L^2$ harmonic 1-forms, Palmer [1991] proved that if there exists a codimension-one cycle on a complete minimal hypersurface $M$ in Euclidean space, which does not separate $M$, $M$ is unstable. Using Bochner’s
vanishing technique, Miyaoka [1993] showed that a complete noncompact stable minimal hypersurface in a nonnegatively curved manifold has no nontrivial $L^2$ harmonic 1-forms. Pigola, Rigoli, and Setti [Pigola et al. 2005] gave general Liouville type results and the corresponding vanishing theorems on the $L^2$ cohomology of stable minimal hypersurfaces. Refer to [Carron 2002; Pigola et al. 2008] for a survey in this area. While the $L^2$ theory is quite well understood, in the case $p \neq 2$, the $L^p$ theory is less developed. See [Scott 1995] for general $L^p$ theory of differential forms on a manifold.

The purpose of this paper is twofold. Firstly, we estimate the smallest spectral value of the Laplace operator on a complete noncompact stable minimal hypersurface in a Riemannian manifold under the assumption on $L^p$ norm of the second fundamental form. Secondly, we obtain various vanishing theorems for $L^p$ harmonic 1-forms on minimal hypersurfaces.

Let $M$ be a complete noncompact Riemannian manifold and let $\Omega$ be a compact domain in $M$. Let $\lambda_1(\Omega) > 0$ denote the first eigenvalue of the Dirichlet boundary value problem

\[
\begin{cases}
\Delta f + \lambda f = 0 & \text{in } \Omega, \\
f = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where $\Delta$ denotes the Laplace operator on $M$. Then the first eigenvalue $\lambda_1(M)$ is defined by

\[\lambda_1(M) = \inf_{\Omega} \lambda_1(\Omega),\]

where the infimum is taken over all compact domains in $M$. Cheung and Leung [2001] gave the first eigenvalue estimate for an $n$-dimensional complete noncompact submanifold $M$ with the norm of its mean curvature vector bounded in the hyperbolic space. In particular, they proved that if $M$ is minimal, the first eigenvalue $\lambda_1(M)$ satisfies

\[\frac{1}{4}(n - 1)^2 \leq \lambda_1(M).
\]

Note that this inequality is sharp because equality holds if $M$ is totally geodesic [McKean 1970]. This result was extended to an $n$-dimensional complete noncompact submanifold with the norm of its mean curvature vector bounded in a complete simply connected Riemannian manifold with sectional curvature bounded above by a negative constant. More precisely, we have the following theorem.

**Theorem** [Bessa and Montenegro 2003; Seo 2012]. Let $N$ be an $n$-dimensional complete simply connected Riemannian manifold with sectional curvature $K_N$ satisfying $K_N \leq -a^2 < 0$ for a positive constant $a > 0$. Let $M$ be a an $m$-dimensional complete noncompact submanifold with bounded mean curvature vector $H$ in $N$ satisfying $|H| \leq b < (m - 1)a$. Then

\[\frac{1}{4}[(m - 1)a - b]^2 \leq \lambda_1(M).
\]
On the other hand, Candel [2007] obtained an upper bound for the bottom of the spectrum of a complete simply connected stable minimal surface in 3-dimensional hyperbolic space. With finite $L^2$ norm of the second fundamental form, one may estimate an upper bound for the bottom of the spectrum of a stable minimal hypersurface in a Riemannian manifold with pinched negative sectional curvature [Dung and Seo 2012; Seo 2011]. In Section 2, we estimate the bottom of the spectrum of the Laplace operator on stable minimal hypersurfaces under the assumption on the $L^p$ norm of the second fundamental form. Indeed, we prove the following.

**Theorem.** Let $N$ be an $(n+1)$-dimensional complete simply connected Riemannian manifold with sectional curvature satisfying $K_1 \leq K_N \leq K_2$, where $K_1, K_2$ are constants and $K_1 \leq K_2 < 0$. Let $M$ be a complete stable non-totally geodesic minimal hypersurface in $N$. Assume that, for $1 - \sqrt{2}/n < p < 1 + \sqrt{2}/n$,
\[
\lim_{R \to \infty} R^{-2} \int_{B(R)} |A|^{2p} = 0,
\]
where $B(R)$ is a geodesic ball of radius $R$ on $M$. If $|\nabla K|^2 = \sum_{i,j,k,l,m} K^2_{ijkl;m} \leq K^2_3 |A|^2$ for some constant $K_3 \geq 0$, we have
\[
-K_2 \frac{(n-1)^2}{4} \leq \lambda_1(M) \leq \frac{np^2(2K_3 - n(K_1 + K_2))}{2 - n(p-1)^2}.
\]

The author [2010] proved that if $M$ is an $n$-dimensional complete stable minimal hypersurface in hyperbolic space with $\lambda_1(M) > (2n-1)(n-1)$, there is no nontrivial $L^2$ harmonic 1-form on $M$. This result was generalized [Dung and Seo 2012] to a complete stable minimal hypersurface in a Riemannian manifold with sectional curvature bounded below by a nonpositive constant. In Section 3, we prove an extended result for $L^p$ harmonic 1-forms on a complete noncompact stable minimal hypersurface as follows.

**Theorem.** Let $N$ be an $(n+1)$-dimensional complete Riemannian manifold with sectional curvature satisfying that $K \leq K_N$ where $K \leq 0$ is a constant. Let $M$ be a complete noncompact stable minimal hypersurface in $N$. Assume that, for $0 < p < n/(n-1) + \sqrt{2n}$,
\[
\lambda_1(M) > \frac{-2n(n-1)^2 p^2 K}{2n - [(n-1)p - n]^2}.
\]
Then there is no nontrivial $L^{2p}$ harmonic 1-form on $M$.

Yau [1976] proved that there are no nonconstant $L^p$ harmonic functions on a complete Riemannian manifold for $1 < p < \infty$. Li and Schoen [1984] proved that Yau’s result is still true for $L^p$ harmonic functions on a complete manifold of
nonnegative Ricci curvature when $0 < p < \infty$. In the case of harmonic forms, Greene and Wu [1974; 1981] announced nonexistence of nontrivial $L^p$ harmonic forms ($1 \leq p < \infty$) on complete Riemannian and Kählerian manifolds of nonnegative curvature. See also [Colding and Minicozzi 1996; 1997; 1998; Li and Tam 1987; 1992] for Liouville type theorems for harmonic functions on a complete Riemannian manifold. The Liouville property holds also for harmonic functions on minimal hypersurfaces in a Riemannian manifold. For instance, Schoen and Yau proved the Liouville type theorem on minimal hypersurfaces as follows.

**Theorem [Schoen and Yau 1976].** Let $M$ be a complete noncompact stable minimal hypersurface in a Riemannian manifold with nonnegative sectional curvature. If $f$ is a harmonic function on $M$ with finite $L^2$ energy, $f$ is constant.

Recall that a function $f$ on a Riemannian manifold $M$ has finite $L^p$ energy if $|\nabla f| \in L^p(M)$. As an application of our theorem, we immediately obtain the following, which is a generalization of Schoen and Yau’s result (see Corollary 3.10).

**Theorem.** Let $M$ be a complete noncompact stable minimal hypersurface in a Riemannian manifold with nonnegative sectional curvature with $\lambda_1(M) > 0$. Then there is no nontrivial harmonic function on $M$ with finite $L^p$ energy for $0 < p < n/(n-1) + \sqrt{2n}$.

For $n \geq 3$, it is well known [Cao et al. 1997] that an $n$-dimensional complete stable minimal hypersurface $M$ in Euclidean space cannot have more than one end. This topological result was generalized to minimal hypersurfaces with finite index in Euclidean space and stable minimal hypersurfaces in a nonnegatively curved manifold by Li and Wang [2002; 2004]. If we assume that $M$ has sufficiently small total scalar curvature instead of assuming that $M$ is stable, we can also have the same conclusion [Ni 2001; Seo 2008]. See also [Pigola and Veronelli 2012] for more general results related with $L^p$ norm of the second fundamental form. In the same spirit, Yun [2002] proved that if $M \subset \mathbb{R}^{n+1}$ is a complete minimal hypersurface with sufficiently small total scalar curvature, there is no nontrivial $L^2$ harmonic 1-form on $M$. Yun’s result was generalized [Dung and Seo 2012] to a complete noncompact stable minimal hypersurface in a complete Riemannian manifold with sectional curvature bounded below by a nonpositive constant. The corresponding vanishing theorems for $L^p$ harmonic 1-forms are obtained in Section 4.

One crucial step in the proofs of our theorems is to obtain an inequality of Simons’ type for $|\phi|^p$ rather than $|\phi|$, where $\phi$ is a geometric quantity which we want to analyze. This kind of inequalities has been used in [Deng 2008; Fu 2012; Shen and Zhu 2005]. Equipped with this Simons’ type inequality, we extend the original Bochner technique to our cases.
2. An estimate for the bottom of the spectrum of the Laplace operator

Let $M$ be an $n$-dimensional manifold immersed in an $(n+1)$-dimensional Riemannian manifold $N$. We choose a local vector field of orthonormal frames $e_1, \ldots, e_{n+1}$ in $N$ such that the vectors $e_1, \ldots, e_n$ are tangent to $M$ and the vector $e_{n+1}$ is normal to $M$. With respect to this frame field of $N$, let $K_{ijkl}$ be a curvature tensor of $N$. We denote by $K_{i jkl;m}$ the covariant derivative of $K_{ijkl}$. In this section, we follow the notation of [Schoen et al. 1975].

**Theorem 2.1.** Let $N$ be an $(n+1)$-dimensional complete simply connected Riemannian manifold with sectional curvature satisfying $K_1 \leq K_N \leq K_2$, where $K_1, K_2$ are constants and $K_1 \leq K_2 < 0$. Let $M$ be a complete stable non-totally geodesic minimal hypersurface in $N$. Assume that

$$\lim_{R \to \infty} R^{-2} \int_{B(R)} |A|^2 p = 0,$$

where $B(R)$ is a geodesic ball of radius $R$ on $M$. If $|\nabla K|^2 = \sum_{i,j,k,l,m} K^2_{ijkl;m} \leq K_3^2 |A|^2$ for some constant $K_3 \geq 0$, we have

$$-K_2 \frac{(n-1)^2}{4} \leq \lambda_1(M) \leq \frac{np^2 (2K_3 - n(K_1 + K_2))}{2 - n(p-1)^2}.$$

**Proof.** As mentioned in the introduction, one sees that the lower bound of $\lambda_1(M)$ is given as $-K_2(n - 1)^2/4$ from inequality (1) [Bessa and Montenegro 2003; Seo 2012]. Namely, the first eigenvalue of an $n$-dimensional minimal hypersurface in a complete simply connected Riemannian manifold with sectional curvature bounded above by a negative constant $K_2$ is bounded below by $-K_2(n - 1)^2/4$. Therefore, in the rest of the proof, we shall find the upper bound of the first eigenvalue $\lambda_1(M)$.

By [Schoen et al. 1975, (1.22), (1.27)], we have

$$|A| \Delta |A| + 2K_3 |A|^2 - n(2K_2 - K_1) |A|^2 + |A|^4 \geq \sum h^2_{ijk} - |\nabla |A||^2$$

at all points where $|A| \neq 0$. Because $K_2 - K_1 \geq 0$, this inequality implies

$$|A| \Delta |A| + 2K_3 |A|^2 - nK_2 |A|^2 + |A|^4 \geq \sum h^2_{ijk} - |\nabla |A||^2 = |\nabla |A||^2 - |\nabla |A||^2.$$

Applying the Kato-type inequality

$$|\nabla |A||^2 - |\nabla |A||^2 \geq \frac{2}{n} |\nabla |A||^2,$$

due to Y. L. Xin [2005], we get

$$|A| \Delta |A| + (2K_3 - nK_2) |A|^2 + |A|^4 \geq \frac{2}{n} |\nabla |A||^2.$$
For a positive number $p > 0$, we have

$$
= |A|^p \text{div}(p |A|^{p-1} \nabla |A|)
= p(p - 1)|A|^{2p-2} |\nabla |A| |^2 + p|A|^{2p-1} \Delta |A|
= \frac{p - 1}{p} |\nabla |A|^p |^2 + p|A|^{2p-2}|A| \Delta |A|.
$$

It follows from inequality (2) that

$$
|A|^p \Delta |A|^p
\geq \frac{p - 1}{p} |\nabla |A|^p |^2 + \frac{2p}{n} |A|^{2p-2} |\nabla |A| |^2 - p|A|^{2p+2} - p(2K_3 - nK_2)|A|^{2p}
= \frac{p - 1}{p} |\nabla |A|^p |^2 + \frac{2}{np} |\nabla |A|^p |^2 - p|A|^{2p+2} - p(2K_3 - nK_2)|A|^{2p}.
$$

Thus

$$
|A|^p \Delta |A|^p + p(2K_3 - nK_2)|A|^{2p} + p|A|^{2p+2} \geq \left( 1 - \frac{n - 2}{np} \right) |\nabla |A|^p |^2.
$$

Choose a Lipschitz function $f$ with compact support in a geodesic ball $B(R)$ of radius $R$ centered at a point $x \in M$. Multiplying both sides by $f^2$ and integrating over $B(R)$, we obtain

$$
\int_{B(R)} f^2 |A|^p \Delta |A|^p + p(2K_3 - nK_2)|A|^{2p} + p|A|^{2p+2}
\geq \left( 1 - \frac{n - 2}{np} \right) \int_{B(R)} f^2 |\nabla |A|^p |^2.
$$

The divergence theorem yields

$$
\int_{B(R)} f^2 |A|^p \Delta |A|^p
= \int_{B(R)} \text{div}(f^2 |A|^p \nabla |A|^p) - \int_{B(R)} f^2 |\nabla |A|^p |^2 - 2 \int_{B(R)} f|A|^p \langle \nabla f, \nabla |A|^p \rangle
= - \int_{B(R)} f^2 |\nabla |A|^p |^2 - 2 \int_{B(R)} f|A|^p \langle \nabla f, \nabla |A|^p \rangle.
$$
Therefore

\[(3)\quad \left(1 - \frac{n-2}{np}\right) \int_{B(R)} f^2 |\nabla |A|^p|^2 \]
\[\leq p(2K_3 - nK_2) \int_{B(R)} f^2 |A|^{2p} + p \int_{B(R)} f^2 |A|^{2p+2} \]
\[- \int_{B(R)} f^2 |\nabla |A|^p|^2 - 2 \int_{B(R)} f|A|^p (\nabla f, \nabla |A|^p).\]

The stability of $M$ implies that

\[(4)\quad \int_M |\nabla f|^2 - (|A|^2 + \text{Ric}(e_{n+1})) f^2 \geq 0\]

for any compactly supported Lipschitz function $f$ on $M$. From our assumption on the sectional curvature of $N$, we see that

\[nK_1 \leq \text{Ric}(e_{n+1}) = R_{n+1,1,n+1,1} + \cdots + R_{n+1,n,n+1,n} \leq nK_2.\]

Hence the stability inequality (4) gives

\[(5)\quad \int_M |\nabla f|^2 - (|A|^2 + nK_1) f^2 \geq 0\]

for any compactly supported Lipschitz function $f$ on $M$. Choose a Lipschitz function $f$ with compact support in a geodesic ball $B(R) \subset M$, as before. Replacing $f$ by $|A|^p f$ in inequality (5), we have

\[\int_M \left|\nabla (|A|^p f)\right|^2 - (|A|^{2p+2} f^2 + nK_1 |A|^{2p} f^2) \geq 0.\]

Thus

\[(6)\quad \int_{B(R)} |\nabla |A|^p|^2 f^2 + \int_{B(R)} |\nabla f|^2 |A|^{2p} + 2 \int_{B(R)} f|A|^p (\nabla f, \nabla |A|^p) \]
\[\geq \int_{B(R)} |A|^{2p+2} f^2 + nK_1 \int_{B(R)} |A|^{2p} f^2.\]

Combining the inequalities (3) and (6), we get

\[(7)\quad \left(1 - \frac{n-2}{np}\right) \int_{B(R)} f^2 |\nabla |A|^p|^2 \]
\[\leq p(2K_3 - nK_1 - nK_2) \int_{B(R)} f^2 |A|^{2p} + (p - 1) \int_{B(R)} f^2 |\nabla |A|^p|^2 \]
\[+ p \int_{B(R)} |\nabla f|^2 |A|^{2p} + 2(p - 1) \int_{B(R)} f|A|^p (\nabla f, \nabla |A|^p).\]
On the other hand, from the definition of \( \lambda_1(M) \) and the domain monotonicity of eigenvalues, it follows that

\[
\lambda_1(M) \leq \lambda_1(B(R)) \leq \frac{\int_{B(R)} |\nabla f|^2}{\int_{B(R)} f^2}
\]

for any compactly supported nonconstant Lipschitz function \( f \) on \( M \). Substituting \( |A|^p f \) for \( f \) in inequality (8), we see that

\[
\lambda_1(M) \int_{B(R)} |A|^{2p} f^2 \\
\leq \int_{B(R)} |\nabla(|A|^p f)|^2 \\
= \int_{B(R)} f^2 |\nabla|A|^p|^2 + \int_{B(R)} |A|^{2p} |\nabla f|^2 + 2 \int_{B(R)} f |A|^p \langle \nabla f, \nabla |A|^p \rangle.
\]

Plugging inequality (9) into (7), we have

\[
\left(1 - \frac{n-2}{np}\right) \int_{B(R)} f^2 |\nabla|A|^p|^2 \\
\leq \frac{p}{\lambda_1(M)} (2K_3 - nK_1 - nK_2) \left( \int_{B(R)} f^2 |\nabla|A|^p|^2 \\
+ |\nabla f|^2 |A|^{2p} + 2 \int_{B(R)} f |A|^p \langle \nabla f, \nabla |A|^p \rangle \right) \\
+ (p-1) \int_{B(R)} f^2 |\nabla|A|^p|^2 + p \int_{B(R)} |\nabla f|^2 |A|^{2p} + 2(p-1) \int_{B(R)} f |A|^p \langle \nabla f, \nabla |A|^p \rangle.
\]

Thus

\[
\left(1 - \frac{n-2}{np}\right) \int_{B(R)} f^2 |\nabla|A|^p|^2 \\
\leq \left( \frac{p}{\lambda_1(M)} (2K_3 - nK_1 - nK_2) + p - 1 \right) \int_{B(R)} f^2 |\nabla|A|^p|^2 \\
+ \left( \frac{p}{\lambda_1(M)} (2K_3 - nK_1 - nK_2) + p \right) \int_{B(R)} |\nabla f|^2 |A|^{2p} \\
+ 2 \left( \frac{p}{\lambda_1(M)} (2K_3 - nK_1 - nK_2) + p - 1 \right) \int_{B(R)} f |A|^p \langle \nabla f, \nabla |A|^p \rangle.
\]

Note that Young’s inequality yields

\[
2 \int_{B(R)} f |A|^p \langle \nabla f, \nabla |A|^p \rangle \leq \varepsilon \int_{B(R)} |\nabla f|^2 |A|^{2p} + \frac{1}{\varepsilon} \int_{B(R)} f^2 |\nabla|A|^p|^2
\]
for any $\varepsilon > 0$. From inequalities (10) and (11), it follows that
\[
\left(1 - \frac{n-2}{np}\right) \int_{B(R)} f^2 |\nabla A|^p \leq \left(\frac{p}{\lambda_1(M)} (2K_3 - nK_1 - nK_2) + p - 1\right) \int_{B(R)} f^2 |\nabla A|^p + \left(\frac{p}{\lambda_1(M)} (2K_3 - nK_1 - nK_2) + p - 1\right) \int_{B(R)} |\nabla f|^2 |A|^{2p} + \left(\frac{p}{\lambda_1(M)} (2K_3 - nK_1 - nK_2) + p - 1\right) \left(\varepsilon \int_{B(R)} |\nabla f|^2 |A|^{2p} + \frac{1}{\varepsilon} \int_{B(R)} f^2 |\nabla A|^p \right),
\]
which yields that
\[
\left[1 - \frac{n-2}{np} - \left(1 + \frac{1}{\varepsilon}\right) \left(\frac{p}{\lambda_1(M)} (2K_3 - nK_1 - nK_2) + p - 1\right)\right] \int_{B(R)} f^2 |\nabla A|^p \leq \left[\left(1 + \varepsilon\right) \left(\frac{p}{\lambda_1(M)} (2K_3 - nK_1 - nK_2) + p\right) - \varepsilon\right] \int_{B(R)} |\nabla f|^2 |A|^{2p}.
\]
For a contradiction, we suppose that
\[
\lambda_1(M) > \frac{p(2K_3 - nK_1 - nK_2)}{1 - (n-2)/np - (p - 1)} = \frac{np^2(2K_3 - n(K_1 + K_2))}{2 - n(p - 1)^2}.
\]
Note the assumption that $1 - \sqrt{2/n} < p < 1 + \sqrt{2/n}$ is equivalent to
\[
2 - n(p - 1)^2 > 0.
\]
Choose a sufficiently large $\varepsilon > 0$ satisfying
\[
\left[1 - \frac{n-2}{np} - \left(1 + \frac{1}{\varepsilon}\right) \left(\frac{p}{\lambda_1(M)} (2K_3 - nK_1 - nK_2) + p - 1\right)\right] > 0.
\]
Since $|\nabla f| \leq 1/R$ by our choice of $f$, one can conclude that, by letting $R \to \infty$,
\[
\int_M |\nabla A|^p = 0,
\]
where we used the growth condition on $\int_{B(R)} |A|^{2p}$. Thus we see that $|A|$ is constant. Since the volume of $M$ is infinite [Wei 2003], we get $|A| \equiv 0$. This implies that $M$ is totally geodesic, which is impossible by our assumption. Therefore we obtain the upper bound of $\lambda_1(M)$:
\[
\lambda_1(M) \leq \frac{np^2(2K_3 - n(K_1 + K_2))}{2 - n(p - 1)^2}.
\]
Dung and the author [2012] gave an estimate of the bottom of the spectrum for the Laplace operator on a complete noncompact stable minimal hypersurface $M$ in a complete simply connected Riemannian manifold with pinched negative sectional curvature under the assumption on $L^2$-norm of the second fundamental form $A$ of $M$. In Theorem 2.1, if we take $p = 1$, we get the following.

**Corollary 2.2 [Dung and Seo 2012].** Let $N$ be an $(n + 1)$-dimensional complete simply connected Riemannian manifold with sectional curvature satisfying $K_1 \leq K_N \leq K_2$, where $K_1, K_2$ are constants and $K_1 \leq K_2 < 0$. Let $M$ be a complete stable non-totally geodesic minimal hypersurface in $N$. Assume that

$$\lim_{R \to \infty} R^{-2} \int_{B(R)} |A|^2 = 0,$$

where $B(R)$ is a geodesic ball of radius $R$ on $M$. If $|\nabla K|^2 = \sum_{i,j,k,l,m} K_{ijkl;m}^2 \leq K_3^2 |A|^2$ for some constant $K_3 > 0$, we have

$$-K_2 \left(\frac{n-1}{4}\right)^2 \leq \lambda_1(M) \leq \frac{(2K_3 - n(K_1 + K_2))n}{2}.$$

In particular, if $N$ is the $(n + 1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$, one sees that $K_1 = K_2 = -1$, and hence $|\nabla K|^2 = 0$, that is, $K_3 = 0$. Moreover, it follows from McKeans’s result [1970] that the first eigenvalue $\lambda_1(M)$ of any complete totally geodesic hypersurface $M \subset \mathbb{H}^{n+1}$ satisfies $\lambda_1(M) = (n - 1)^2/4$. Therefore we have the following consequence which is an extension of the result in [Seo 2011].

**Corollary 2.3.** Let $M$ be a complete stable minimal hypersurface in $\mathbb{H}^{n+1}$ with $\int_M |A|^{2p} \, dv < \infty$ for $1 - \sqrt{2/n} < p < 1 + \sqrt{2/n}$. Then we have

$$-K_2 \left(\frac{n-1}{4}\right)^2 \leq \lambda_1(M) \leq \frac{2n^2 p^2}{2 - np - p^2}.$$

As another application of Theorem 2.1, we have the following when $n < 8$.

**Corollary 2.4.** Let $N$ be an $(n + 1)$-dimensional complete simply connected Riemannian manifold with sectional curvature satisfying $K_1 \leq K_N \leq K_2$, where $K_1, K_2$ are constants and $K_1 \leq K_2 < 0$ for $n < 8$. Let $M$ be a complete stable non-totally geodesic minimal hypersurface in $N$. For $p = 1, 2, 3$, if $\int_M |A|^p < \infty$, we have

$$-K_2 \left(\frac{n-1}{4}\right)^2 \leq \lambda_1(M) \leq \frac{np^2(2K_3 - n(K_1 + K_2))}{2 - np - p^2}.$$

**Proof.** Since $\sqrt{2/n} > 1/2$ when $n < 8$, the conclusion can be derived from Theorem 2.1. \qed
3. Vanishing theorems on minimal hypersurfaces with $\lambda_1(M)$ bounded below

Before we prove the vanishing theorems for $L^p$ harmonic 1-forms on complete minimal hypersurface, we begin with some useful facts.

**Lemma 3.1 [Leung 1992].** Let $M$ be an $n$-dimensional complete immersed minimal hypersurface in a Riemannian manifold $N$. If all the sectional curvatures of $N$ are bounded below by a constant $K$,

$$\text{Ric} \geq (n - 1)K - \frac{n - 1}{n}|A|^2.$$

**Lemma 3.2 [Wang 2001].** Let $\omega$ be a harmonic 1-form on an $n$-dimensional Riemannian manifold $M$. Then

$$|\nabla \omega|^2 - |\omega|^2 \geq \frac{1}{n - 1}|\nabla |\omega||^2. \tag{12}$$

We also need the following well-known Sobolev inequality on a Riemannian manifold.

**Lemma 3.3 [Hoffman and Spruck 1974].** Let $M^n$ be a complete immersed minimal submanifold in a nonpositively curved manifold $N^{n+p}$, $n \geq 3$. Then, for any $\phi \in W_{0,2}^1(M)$, we have

$$\left( \int_M |\phi|^{2n/(n-2)} \, dv \right)^{(n-2)/n} \leq C_s \int_M |\nabla \phi|^2 \, dv, \tag{13}$$

where $C_s$ is the Sobolev constant which depends only on $n \geq 3$.

A complete Riemannian manifold $M$ is called nonparabolic if it admits a non-constant positive superharmonic function. Otherwise, $M$ is said to be parabolic. The following sufficient condition for parabolicity is well known.

**Theorem [Grigoryan 1983; 1985; Karp 1982; Varopoulos 1983].** Let $M$ be a complete Riemannian manifold. If, for any point $p \in M$ and a geodesic ball $B_p(r)$,

$$\int_1^\infty \frac{r}{\text{Vol}(B_p(r))} \, dr = \infty,$$

$M$ is parabolic.

It immediately follows from this result that if $M$ is nonparabolic,

$$\int_1^\infty \frac{r}{\text{Vol}(B_p(r))} \, dr < \infty,$$

and hence $M$ has infinite volume. Moreover, if $\lambda_1(M) > 0$, $M$ is nonparabolic [Grigoryan 1999]. Therefore one can conclude the following.
**Proposition 3.4.** Let $M$ be an $n$-dimensional complete noncompact Riemannian manifold with $\lambda_1(M) > 0$. Then $\text{Vol}(M) = \infty$.

Note that, in the case of submanifolds, Cheung and Leung [1998] proved that the volume $\text{Vol}(B_p(r))$ of every complete noncompact submanifold $M$ in the Euclidean or hyperbolic space grows at least as a linear function of $r$ under the assumption that the mean curvature vector $H$ of $M$ is bounded in absolute value.

We are now ready to state and prove vanishing theorems for $L^p$ harmonic $1$-forms on a complete noncompact stable minimal hypersurface.

**Theorem 3.5.** Let $N$ be an $(n + 1)$-dimensional complete Riemannian manifold with sectional curvature satisfying $K \leq K_N$ where $K \leq 0$ is a constant. Let $M$ be a complete noncompact stable minimal hypersurface in $N$. Assume that,

$$\lambda_1(M) > -\frac{2n(n - 1)^2 p^2 K}{(n - 1)p - n}.$$

Then there is no nontrivial $L^{2p}$ harmonic $1$-form on $M$.

**Proof.** We consider two cases: $K < 0$ and $K = 0$.

**Case 1: $K < 0$.** Let $\omega$ be an $L^{2p}$ harmonic $1$-form on $M$, that is,

$$\Delta \omega = 0 \quad \text{and} \quad \int_M |\omega|^{2p} \, dv < \infty.$$

In an abuse of notation, we refer to both a harmonic $1$-form and its dual harmonic vector field by $\omega$. Bochner’s formula yields

$$\Delta |\omega|^2 = 2(|\nabla \omega|^2 + \text{Ric}(\omega, \omega)).$$

Moreover,

$$\Delta |\omega|^2 = 2(|\omega| \Delta |\omega| + |\nabla |\omega||^2).$$

Applying Lemma 3.1 and the Kato-type inequality (12), we see that

$$|\omega| \Delta |\omega| + \frac{n - 1}{n} |A|^2 |\omega|^2 - (n - 1)K |\omega|^2 \geq \frac{1}{n - 1} |\nabla |\omega||^2.$$  \hfill (14)

For any positive number $p$, we have

$$|\omega|^p \Delta |\omega|^p = |\omega|^p \text{div}(\nabla |\omega|^p)$$

$$= |\omega|^p \text{div}(p |\omega|^{p-1} \nabla |\omega|)$$

$$= p(p - 1) |\omega|^{2p-2} |\nabla |\omega||^2 + p |\omega|^{2p-2} \Delta |\omega|$$

$$= \frac{p - 1}{p} |\nabla |\omega||^2 + p |\omega|^{2p-2} |\omega| \Delta |\omega|. $$
Plugging inequality (14) into the above equality, we have
\[ |\omega|^p \Delta |\omega|^p + p(n-1) \left( \frac{|A|^2}{n} - K \right) |\omega|^{2p} \geq \left( 1 - \frac{1}{p} + \frac{1}{p(n-1)} \right) |\nabla |\omega|^p|^2. \]

Choose a Lipschitz function \( f \) with compact support in a geodesic ball \( B(R) \) of radius \( R \) centered at \( p \in M \). Multiplying both side by \( f^2 \) and integrating over \( B(R) \), we obtain
\[
\left( 1 - \frac{1}{p} + \frac{1}{p(n-1)} \right) \int_{B(R)} f^2 |\nabla |\omega|^p|^2 \\
\leq \int_{B(R)} f^2 |\omega|^p \Delta |\omega|^p + \frac{p(n-1)}{n} \int_{B(R)} f^2 |\nabla |A|^2|^2 |\omega|^{2p} - p(n-1)K \int_{B(R)} f^2 |\omega|^{2p}.
\]

The divergence theorem gives
\[
\int_{B(R)} f^2 |\omega|^p \Delta |\omega|^p = - \int_{B(R)} f^2 |\nabla |\omega|^p|^2 - 2 \int_{B(R)} f |\omega|^p \langle \nabla f, \nabla |\omega|^p \rangle.
\]
Thus
\[
(15) \quad \left( 1 - \frac{1}{p} + \frac{1}{p(n-1)} \right) \int_{B(R)} f^2 |\nabla |\omega|^p|^2 \\
\leq \frac{p(n-1)}{n} \int_{B(R)} f^2 |\nabla |A|^2|^2 |\omega|^{2p} - p(n-1)K \int_{B(R)} f^2 |\omega|^{2p} \\
- \int_{B(R)} f^2 |\nabla |\omega|^p|^2 - 2 \int_{B(R)} f |\omega|^p \langle \nabla f, \nabla |\omega|^p \rangle.
\]

Since \( M \) is stable,
\[
\int_M |\nabla f|^2 - (|A|^2 + \text{Ric}(e_{n+1})) f^2 \geq 0
\]
for any compactly supported Lipschitz function \( f \) on \( M \). From the assumption on the sectional curvature of \( N \), it follows that
\[
\int_M |\nabla f|^2 - (|A|^2 + nK) f^2 \geq 0
\]
for any compactly supported Lipschitz function \( f \) on \( M \). Replacing \( f \) by \( |\omega|^p f \), we have
\[
(16) \quad \int_{B(R)} f^2 |\nabla |\omega|^p|^2 + \int_{B(R)} |\nabla f|^2 |\omega|^{2p} + 2 \int_{B(R)} f |\omega|^p \langle \nabla f, \nabla |\omega|^p \rangle \\
\geq \int_{B(R)} f^2 |\nabla |A|^2|^2 |\omega|^{2p} + nK \int_{B(R)} f^2 |\omega|^{2p}.
\]
Combining the inequalities (15) and (16) gives

\[
\left(1 - \frac{1}{p} + \frac{1}{p(n-1)}\right) \int_{B(R)} f^2 |\nabla|\omega|p|^2 \\
\leq \frac{p(n-1)}{n} \left[ \int_{B(R)} f^2 |\nabla|\omega|p|^2 + \int_{B(R)} |\nabla f|^2 |\omega|^{2p} \\
+ 2 \int_{B(R)} f |\omega|^p (\nabla f, \nabla |\omega|^p) - nK \int_{B(R)} f^2 |\omega|^{2p} \right] \\
- p(n-1)K \int_{B(R)} f^2 |\omega|^{2p} - \int_{B(R)} f^2 |\nabla|\omega|p|^2 - 2 \int_{B(R)} f |\omega|^p (\nabla f, \nabla |\omega|^p). 
\]

Hence

(17) \[
\left(1 - \frac{1}{p} + \frac{1}{p(n-1)}\right) \int_{B(R)} f^2 |\nabla|\omega|p|^2 \\
\leq \left(\frac{p(n-1)}{n} - 1\right) \int_{B(R)} f^2 |\nabla|\omega|p|^2 + \frac{p(n-1)}{n} \int_{B(R)} |\nabla f|^2 |\omega|^{2p} \\
- 2p(n-1)K \int_{B(R)} f^2 |\omega|^{2p} + 2\left(\frac{p(n-1)}{n} - 1\right) \int_{B(R)} f |\omega|^p (\nabla f, \nabla |\omega|^p). 
\]

Moreover, using the definition of the bottom of the spectrum, we see that

(18) \[
\lambda_1(M) \int_{B(R)} |\omega|^{2p} f^2 \\
\leq \int_{B(R)} |\nabla(|\omega|^{p} f)|^2 \\
= \int_{B(R)} f^2 |\nabla|\omega|p|^2 + \int_{B(R)} |\omega|^{2p} |\nabla f|^2 + 2 \int_{B(R)} f |\omega|^p (\nabla f, \nabla |\omega|^p). 
\]

From inequalities (17) and (18), it follows that

\[
\left(1 - \frac{1}{p} + \frac{1}{p(n-1)}\right) \int_{B(R)} f^2 |\nabla|\omega|p|^2 \\
\leq \left(\frac{p(n-1)}{n} - 1 - \frac{2p(n-1)K}{\lambda_1(M)} \right) \int_{B(R)} f^2 |\nabla|\omega|p|^2 \\
+ \left(\frac{p(n-1)}{n} - \frac{2p(n-1)K}{\lambda_1(M)} \right) \int_{B(R)} |\nabla f|^2 |\omega|^{2p} \\
+ 2\left(\frac{p(n-1)}{n} - 1 - \frac{2p(n-1)K}{\lambda_1(M)} \right) \int_{B(R)} f |\omega|^p (\nabla f, \nabla |\omega|^p). 
\]
Applying Young's inequality, we have
\[ 2 \int_{B(R)} f|\omega|^p \langle \nabla f, \nabla |\omega|^p \rangle \leq \varepsilon \int_{B(R)} f^2 |\nabla |\omega|^p|^2 + \frac{1}{\varepsilon} \int_{B(R)} |\nabla f|^2 |\omega|^{2p} \]
for any \( \varepsilon > 0 \). Thus
\[
\left[ 2 - \frac{1}{p} + \frac{1}{p(n-1)} + \frac{2p(n-1)K}{\lambda_1(M)} - \frac{p(n-1)}{n} - \varepsilon \left( \frac{p(n-1)}{n} - 1 - \frac{2p(n-1)K}{\lambda_1(M)} \right) \right] \\
\times \int_{B(R)} f^2 |\nabla |\omega|^p|^2 \leq \left[ \frac{p(n-1)}{n} - \frac{2p(n-1)K}{\lambda_1(M)} + \frac{1}{\varepsilon} \left( \frac{p(n-1)}{n} - 1 - \frac{2p(n-1)K}{\lambda_1(M)} \right) \right] \int_{B(R)} |\nabla f|^2 |\omega|^{2p}.
\]
Since
\[ \lambda_1(M) > \frac{-2p(n-1)K}{2 - 1/p + 1/(p(n-1))} - \frac{p(n-1)}{n} = \frac{-2n(n-1)^2 p^2 K}{2n - [(n-1)p - n]^2} \]
by the hypothesis, one can choose a sufficiently small \( \varepsilon > 0 \) satisfying that
\[
\left[ 2 - \frac{1}{p} + \frac{1}{p(n-1)} + \frac{2p(n-1)K}{\lambda_1(M)} - \frac{p(n-1)}{n} - \varepsilon \left( \frac{p(n-1)}{n} - 1 - \frac{2p(n-1)K}{\lambda_1(M)} \right) \right] > 0.
\]
Note that \( \int_M |\omega|^{2p} < \infty \), since \( \omega \) is an \( L^{2p} \) harmonic 1-form on \( M \). Letting \( R \) tend to infinity, we obtain
\[ \int_M |\nabla |\omega|^p|^2 = 0, \]
which implies that \( |\nabla |\omega| \equiv 0 \). Hence \( |\omega| \equiv \text{constant} \). From Proposition 3.4, it follows that \( |\omega| \equiv 0 \).

Case 2: \( K = 0 \). Using the inequality (17) and Young's inequality, we obtain
\[
\left[ 2 - \frac{1}{p} + \frac{1}{p(n-1)} - \frac{p(n-1)}{n} - \varepsilon \left( \frac{p(n-1)}{n} - 1 \right) \right] \int_{B(R)} f^2 |\nabla |\omega|^p|^2 \\
\leq \left[ \frac{p(n-1)}{n} + \frac{1}{\varepsilon} \left( \frac{p(n-1)}{n} - 1 \right) \right] \int_{B(R)} |\nabla f|^2 |\omega|^{2p}.
\]
Since \( 0 < p < n/(n-1) + \sqrt{2n} \), one may choose a sufficiently small \( \varepsilon > 0 \) satisfying
\[ 2 - \frac{1}{p} + \frac{1}{p(n-1)} - \frac{p(n-1)}{n} - \varepsilon \left( \frac{p(n-1)}{n} - 1 \right) > 0. \]
Letting \( R \) tend to infinity gives
\[ \int_{B(R)} |\nabla |\omega|^p|^2 = 0, \]
which implies that $|\omega| \equiv \text{constant}$. From the assumption that $\lambda_1(M) > 0$ and Proposition 3.4, it follows that $|\omega| \equiv 0$. □

As a consequence of Theorem 3.5, given a complete noncompact stable minimal hypersurface in a nonnegatively curved Riemannian manifold, one has the following result.

**Corollary 3.6.** Let $N$ be an $(n + 1)$-dimensional complete nonnegatively curved Riemannian manifold. Let $M$ be a complete noncompact stable minimal hypersurface in $N$ with $\lambda_1(M) > 0$. If $n \leq 11$, there is no nontrivial $L^p$ harmonic 1-form on $M$ for any $0 < p \leq n$.

**Proof.** For $n \leq 11$, the inequality $2(n/(n-1) + \sqrt{2n}) \geq n$ holds. □

**Corollary 3.7.** Let $N$ be an $(n + 1)$-dimensional complete nonnegatively curved Riemannian manifold. Let $M$ be a complete noncompact stable minimal hypersurface in $N$ with $\lambda_1(M) > 0$. If $n \leq 11$, there is no nontrivial $L^2$ harmonic 1-form on $M$.

In the case of $L^2$ harmonic 1-forms, Theorem 3.5 gives a generalization of [Dung and Seo 2012] as follows.

**Corollary 3.8.** Let $N$ be an $(n + 1)$-dimensional complete Riemannian manifold with sectional curvature satisfying $K \leq K_N$ where $K < 0$ is a constant. Let $M$ be a complete noncompact stable minimal hypersurface in $N$. Assume that

$$\lambda_1(M) > \frac{-2n(n - 1)^2 K}{2n - 1}.$$  

Then there are no nontrivial $L^2$ harmonic 1-forms on $M$.

In particular, if $N$ is $(n + 1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$, Corollary 3.8 improves the previous result of [Seo 2010]. Related to this result, Cavalcante, Miranda, and Vitório [Cavalcante et al. 2012] obtained the vanishing theorem for $L^2$ harmonic 1-forms on complete noncompact submanifolds in a Cartan–Hadamard manifold.

Palmer [1991] showed that if there exists a codimension-one cycle in a complete minimal hypersurface $M$ in $\mathbb{R}^{n+1}$ which does not separate $M$, $M$ is unstable. We obtain a generalization of Palmer’s result as follows.

**Corollary 3.9.** Let $N$ be an $(n + 1)$-dimensional complete Riemannian manifold with sectional curvature satisfying $K \leq K_N$ where $K \leq 0$ is a constant. Let $M$ be a complete noncompact minimal hypersurface in $N$. Assume that

$$\lambda_1(M) > \frac{-2n(n - 1)^2 K}{2n - 1}.$$
Suppose that there exists a codimension-one cycle in $M$ which does not separate $M$. Then $M$ cannot be stable.

**Proof.** Suppose that $M$ is stable in $N$. From [Dodziuk 1982], there exists a nontrivial $L^2$ harmonic 1-form on $M$, which is a contradiction to Corollary 3.8. □

Let $M$ be a complete Riemannian manifold and let $f$ be a harmonic function on $M$ with finite $L^p$ energy. Then the total differential $df$ is obviously an $L^p$ harmonic 1-form on $M$. As another application of Theorem 3.5, we prove the following Liouville type theorem for harmonic functions with finite $L^p$ energy on a complete noncompact stable minimal hypersurface, which is a generalization of Schoen and Yau’s result [1976], as mentioned in the introduction.

**Corollary 3.10.** Let $N$ be an $(n+1)$-dimensional complete Riemannian manifold with sectional curvature satisfying $K \leq K_N$ where $K \leq 0$ is a constant. Let $M$ be a complete noncompact stable minimal hypersurface in $N$. Assume that, for $0 < p < n/(n-1) + \sqrt{2n}$,

$$
\lambda_1(M) > \frac{-2n(n-1)^2 p^2 K}{2n - [(n-1)p - n]^2}.
$$

Then there is no nontrivial harmonic function on $M$ with finite $L^p$ energy.

So far, we have assumed that $\lambda_1(M) > 0$ for a complete noncompact stable minimal hypersurface $M$ in a nonnegatively curved Riemannian manifold. However, we do not know whether the assumption that $\lambda_1(M) > 0$ is necessary or not. It would be interesting to remove the condition in these results.

**4. Vanishing theorems on minimal hypersurfaces with small $L^n$ or $L^\infty$ norm of the second fundamental form**

In the following, we prove a vanishing theorem for $L^p$ harmonic 1-forms on a complete stable minimal hypersurface $M$, assuming that $M$ has sufficiently small total scalar curvature instead of assuming that $M$ is stable.

**Theorem 4.1.** Let $N$ be an $(n+1)$-dimensional complete simply connected Riemannian manifold with sectional curvature $K_N$ satisfying that $K_1 \leq K_N \leq K_2 < 0$, where $K_1, K_2$ are constants and $n \geq 3$. Let $M$ be a complete minimal hypersurface in $N$. Assume that $K := K_2/K_1$ satisfies

$$
K > \frac{4(n-2)}{(n-1)^2}.
$$
For

\[
\frac{(n-1)K}{4} - \frac{1}{2} \sqrt{\frac{(n-1)^2 K^2}{4} - (n-2)K} < p < \frac{(n-1)K}{4} + \frac{1}{2} \sqrt{\frac{(n-1)^2 K^2}{4} - (n-2)K},
\]

assume that

\[
\left( \int_M \left| A \right|^n \right)^{2/n} < \frac{n(2p(n-1) - n + 2 - 4p^2 K)}{p^2(n-1)^2 C_s},
\]

where \(C_s\) is the Sobolev constant in [Hoffman and Spruck 1974]. Then there are no nontrivial \(L^2p\) harmonic 1-forms on \(M\).

**Proof.** A similar argument as in the proof of Theorem 3.5 shows

\[
|\omega|^p \Delta |\omega|^p + p(n-1) \left( \frac{|A|^2}{n} - K_1 \right) |\omega|^{2p} \geq \left( 1 - \frac{1}{p} + \frac{1}{p(n-1)} \right) |\nabla |\omega|^p|^2
\]

for any Lipschitz function \(f\) with compact support in a geodesic ball \(B(R)\) of radius \(R\) centered at a point \(p \in M\). Multiplying both sides by \(f^2\), integrating over \(B(R)\), and applying the divergence theorem, we see that

(19) \[
\left( 1 - \frac{1}{p} + \frac{1}{p(n-1)} \right) \int_{B(R)} f^2 |\nabla |\omega|^p|^2
\]

\[
\leq \frac{p(n-1)}{n} \int_{B(R)} f^2 |A|^2 |\omega|^{2p} - p(n-1)K_1 \int_{B(R)} f^2 |\omega|^{2p}
\]

\[
- \int_{B(R)} f^2 |\nabla |\omega|^p|^2 - 2 \int_{B(R)} f |\omega|^p \langle \nabla f, \nabla |\omega|^p \rangle.
\]

On the other hand, the Sobolev inequality (13) implies that

\[
\int_{B(R)} f^2 |A|^2 |\omega|^{2p} \leq \left( \int_M |A|^n \right)^{2/n} \left( \int_M (|\omega|^p f)^{(2n)/n-2} \right)^{(n-2)/n}
\]

\[
\leq C_s \left( \int_M |A|^n \right)^{2/n} \int_M |\nabla (|\omega|^p f)|^2
\]

\[
\leq C_s \left( \int_M |A|^n \right)^{2/n} \left( \int_{B(R)} f^2 |\nabla |\omega|^p|^2 \right) + \int_{B(R)} |\nabla f|^2 |\omega|^{2p}
\]

\[
+ 2 \int_{B(R)} f |\omega|^p \langle \nabla f, \nabla |\omega|^p \rangle.
\]
Plugging this inequality into (19) gives

\begin{equation}
(20) \quad \left(1 - \frac{1}{p} + \frac{1}{p(n-1)}\right) \int_{B(R)} f^2 |\nabla|^{2p} \leq \frac{p(n-1)C_s}{n} \left(\int_M |A|^n\right)^{2/n} \int_{B(R)} |\nabla f|^2 |\omega|^{2p} \\
\quad + \left(\frac{p(n-1)C_s}{n} \left(\int_M |A|^n\right)^{2/n} - 1\right) \int_{B(R)} f^2 |\nabla|^{2p} \\
\quad + 2 \left(\frac{p(n-1)C_s}{n} \left(\int_M |A|^n\right)^{2/n} - 1\right) \int_{B(R)} |\omega|^p \langle \nabla f, \nabla |\omega|^p \rangle \\
\quad - p(n-1) K_1 \int_{B(R)} f^2 |\omega|^{2p}.
\end{equation}

An estimate (1) for the bottom of the spectrum yields

\[-\frac{K_2(n-1)^2}{4} \leq \lambda_1(M) \leq \frac{\int_{B(R)} |\nabla (|\omega|^p f)|^2}{\int_{B(R)} (|\omega|^p f)^2},\]

which gives

\begin{equation}
(21) \quad \int_{B(R)} (|\omega|^p f)^2 \leq -\frac{4}{K_2(n-1)^2} \left(\int_{B(R)} f^2 |\nabla|^{2p} + \int_{B(R)} u |\nabla f|^2 |\omega|^{2p} \right) \\
\quad + 2 \int_{B(R)} |\omega|^p \langle \nabla f, \nabla |\omega|^p \rangle.
\end{equation}

Thus, from inequalities (20) and (21), it follows that

\[
\left(1 - \frac{1}{p} + \frac{1}{p(n-1)}\right) \int_{B(R)} f^2 |\nabla|^{2p} \leq B \int_{B(R)} |\nabla f|^2 |\omega|^{2p} + (B-1) \int_{B(R)} f^2 |\nabla|^{2p} + 2(B-1) \int_{B(R)} |\omega|^p \langle \nabla f, \nabla |\omega|^p \rangle,
\]

where

\[
B = \frac{p(n-1)C_s}{n} \left(\int_M |A|^n\right)^{2/n} + \frac{4p}{(n-1) K}.
\]

Applying Young's inequality

\[
2 \int_{B(R)} |\omega|^p \langle \nabla f, \nabla |\omega|^p \rangle \leq \varepsilon \int_{B(R)} f^2 |\nabla|^{2p} + \frac{1}{\varepsilon} \int_{B(R)} |\nabla f|^2 |\omega|^{2p}
\]
for any $\varepsilon > 0$, we see that
\[
\left(2 - \frac{1}{p} + \frac{1}{p(n-1)} - B - \varepsilon(B - 1)\right)\int_{B(R)} f^2|\nabla|\omega|^p|^2
\leq \left(B + \frac{1}{\varepsilon}(B - 1)\right)\int_{B(R)} |\nabla f|^2|\omega|^{2p}.
\]
From the assumption on the total curvature of $M$, one can make
\[
\left(2 - \frac{1}{p} + \frac{1}{p(n-1)} - B - \varepsilon(B - 1)\right) > 0
\]
by choosing a sufficiently small $\varepsilon > 0$. Letting $R \to \infty$ and using that $\omega$ is an $L^{2p}$ harmonic 1-form, we conclude that
\[
\int_M |\nabla|\omega|^p|^2 = 0.
\]
The same argument as before shows that $|\omega| \equiv 0$. □

**Corollary 4.2.** Let $M$ be a complete minimal hypersurface in $\mathbb{H}^{n+1}$ satisfying
\[
\left(\int_M |A|^n\right)^{2/n} < \frac{n(-4p^2 + 2p(n-1) - n + 2)}{p^2(n-1)^2C_s}
\]
for $1/2 < p < n/2 - 1$. Then there are no nontrivial $L^{2p}$ harmonic 1-forms on $M$.

**Corollary 4.3.** Under the same conditions as in Theorem 4.1, there is no nontrivial harmonic function on $M$ with finite $L^p$ energy.

When the $L^\infty$ norm of the second fundamental form of a complete minimal hypersurface is bounded, the following vanishing theorem holds.

**Theorem 4.4.** Let $N$ be an $(n+1)$-dimensional complete simply connected Riemannian manifold with sectional curvature $K_N$ satisfying $K_1 \leq K_N \leq K_2 < 0$, where $K_1$, $K_2$ are constants and $n \geq 3$. Let $M$ be a complete noncompact minimal hypersurface in $N$. Assume that $K := K_2/K_1 > 4(n-2)/(n-1)^2$ and the second fundamental form $A$ satisfies
\[
|A|^2 \leq C < \frac{4p^2K_1 - (2p(n-1) - n + 2)K_2}{4p^2}
\]
for
\[
\frac{(n-1)K}{4} - \frac{1}{2}\sqrt{\frac{(n-1)^2K^2}{4} - (n-2)K}
\]
for
\[
< p < \frac{(n-1)K}{4} + \frac{1}{2}\sqrt{\frac{(n-1)^2K^2}{4} - (n-2)K}.
\]
Then there are no nontrivial $L^{2p}$ harmonic 1-forms on $M$.

**Proof.** A similar argument as before shows

$$
\left(1 - \frac{1}{p} + \frac{1}{p(n-1)}\right) \int_{B(R)} f^2 |\nabla|\omega|^p|^2
\leq \frac{p(n-1)}{n} \int_{B(R)} f^2 |A|^2 |\omega|^{2p} - p(n-1)K_1 \int_{B(R)} f^2 |\omega|^{2p}
- \int_{B(R)} f^2 |\nabla|\omega|^p|^2 - 2 \int_{B(R)} f |\omega|^p \langle \nabla f, \nabla|\omega|^p \rangle.
$$

Since $|A|^2 \leq C$,

$$
\left(2 - \frac{1}{p} + \frac{1}{p(n-1)}\right) \int_{B(R)} f^2 |\nabla|\omega|^p|^2
\leq \left(\frac{p(n-1)C}{n} - p(n-1)K_1\right) \int_{B(R)} f^2 |\omega|^{2p} - 2 \int_{B(R)} f |\omega|^p \langle \nabla f, \nabla|\omega|^p \rangle.
$$

Using an estimate for the bottom of the spectrum and Young’s inequality again, we have

$$
\left(2 - \frac{1}{p} + \frac{1}{p(n-1)} - D - \varepsilon(D - 1)\right) \int_{B(R)} f^2 |\nabla|\omega|^p|^2
\leq \left(D + \frac{1}{\varepsilon}(D - 1)\right) \int_{B(R)} |\nabla f|^2 |\omega|^2p,
$$

where

$$
D = \frac{-4}{(n-1)^2 K_2} \left(\frac{p(n-1)C}{n} - p(n-1)K_1\right).
$$

Since

$$
C < \frac{4p^2 K_1 - (2p(n-1) - n + 2)K_2}{4p^2},
$$

by our assumption, we may choose a sufficiently small $\varepsilon > 0$ satisfying

$$
\left(2 - \frac{1}{p} + \frac{1}{p(n-1)} - D - \varepsilon(D - 1)\right) > 0.
$$

Thus we get

$$
\int_{B(R)} |\nabla|\omega|^p|^2 = 0
$$

by letting $R$ tend to infinity. Hence $\omega \equiv 0$. 

□
Corollary 4.5. Let $M$ be a complete minimal hypersurface in $\mathbb{H}^{n+1}$ with the second fundamental form $A$ satisfying

$$|A|^2 \leq C < \frac{-4p^2 + 2p(n - 1) - n + 2}{4p^2}$$

for $1/2 < p < n/2 - 1$. Then there are no nontrivial $L^{2p}$ harmonic 1-forms on $M$.

Corollary 4.6. Under the same conditions as in Theorem 4.4, there is no nontrivial harmonic function on $M$ with finite $L^p$ energy.

We remark that there are lots of examples of minimal hypersurfaces with finite $L^n$ or $L^\infty$ norm of the second fundamental form in $\mathbb{H}^{n+1}$ [do Carmo and Dajczer 1983; Mori 1981; Ripoll 1989; Seo 2011].

References


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Controlled connectivity for semidirect products acting on locally finite trees

KEITH JONES
An indispensable classification of monomial curves in $\mathbb{A}^4(\mathbb{K})$

ANARGYROS KATSABEKIS and IGNACIO OJEDA
Contracting an axially symmetric torus by its harmonic mean curvature

CHRISTOPHER KIM
Composition operators on strictly pseudoconvex domains with smooth symbol

HYUNGWOON KOO and SONG-YING LI
The Alexandrov problem in a quotient space of $\mathbb{H}^2 \times \mathbb{R}$

ANA MENEZES
Twisted quantum Drinfeld Hecke algebras

DEEPAK NAIDU
$L^p$ harmonic 1-forms and first eigenvalue of a stable minimal hypersurface

KEOMKYO SEO
Reconstruction from Koszul homology and applications to module and derived categories

RYO TAKAHASHI
A virtual Kawasaki–Riemann–Roch formula

VALENTIN TONITA