A VIRTUAL KAWASAKI–RIEMANN–ROCH FORMULA

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Kawasaki’s formula is a tool to compute holomorphic Euler characteristics of vector bundles on a compact orbifold \( \mathcal{X} \). Let \( \mathcal{X} \) be an orbispace with perfect obstruction theory which admits an embedding in a smooth orbifold. One can then construct the virtual structure sheaf and the virtual fundamental class of \( \mathcal{X} \). In this paper we prove that Kawasaki’s formula “behaves well” with working “virtually” on \( \mathcal{X} \) in the following sense: if we replace the structure sheaves, tangent and normal bundles in the formula by their virtual counterparts then Kawasaki’s formula stays true. Our motivation comes from studying the quantum \( K \)-theory of a complex manifold \( X \) (Givental and Tonita, 2014), with the formula applied to Kontsevich moduli spaces of genus-0 stable maps to \( X \).

1. Introduction

Given a manifold \( \mathcal{X} \) and a vector bundle \( V \) on \( \mathcal{X} \), then the Hirzebruch–Riemann–Roch formula states that

\[
\chi(\mathcal{X}, V) = \int_{\mathcal{X}} \text{ch}(V) T d(T_{\mathcal{X}}).
\]

Kawasaki [1979] generalized this formula to the case when \( \mathcal{X} \) is an orbifold. He reduces the computation of Euler characteristics on \( \mathcal{X} \) to the computation of certain cohomological integrals on the inertia orbifold \( I\mathcal{X} \):

\[
\chi(\mathcal{X}, V) = \sum_{\mu} \frac{1}{m_{\mu}} \int_{\mathcal{X}_{\mu}} T d(T_{\mathcal{X}_{\mu}}) \text{ch}\left( \frac{\text{Tr}(V)}{\text{Tr}(\Lambda^* N^*)} \right).
\]

We explain below the ingredients in the formula:

\( I\mathcal{X} \) is defined as follows: around any point \( p \in \mathcal{X} \) there is a local chart \( (\tilde{U}_p, G_p) \) such that locally \( \mathcal{X} \) is represented as the quotient of \( \tilde{U}_p \) by \( G_p \). Consider the set of conjugacy classes \( (1) = (h_p^1, h_p^2, \ldots, h_p^n) \) in \( G_p \). Define

\[
I\mathcal{X} := \{(p, (h_p^i)) \mid i = 1, 2, \ldots, n_p\}.
\]

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Pick an element $h^i_p$ in each conjugacy class. Then a local chart on $I \mathcal{X}$ is given by

$$\bigcup_{i=1}^{n_p} \tilde{U}_p^{(h^i_p)} / Z_{G_p}(h^i_p),$$

where $Z_{G_p}(h^i_p)$ is the centralizer of $h^i_p$ in $G_p$. Denote by $\mathcal{X}_\mu$ the connected components of the inertia orbifold (we’ll often refer to them as Kawasaki strata). The multiplicity $m_\mu$ associated to each $\mathcal{X}_\mu$ is given by

$$m_\mu := |\ker(Z_{G_p}(g) \to \text{Aut}(\tilde{U}_p^g))|.$$

For a vector bundle $V$ we will denote by $V^*$ the dual bundle to $V$. The restriction of $V$ to $\mathcal{X}_\mu$ decomposes in characters of the $g$ action. Let $E_{r}^{(l)}$ be the subbundle of the restriction of $E$ to $\mathcal{X}_\mu$ on which $g$ acts with eigenvalue $e^{2\pi il/r}$. Then the trace $\text{Tr}(V)$ is defined to be the orbibundle whose fiber over the point $(p, (g))$ of $\mathcal{X}_\mu$ is

$$\text{Tr}(V) := \sum_l e^{2\pi il/r} E_{r}^{(l)}.$$

Finally, $\Lambda^* N^*_{\mu}$ is the $K$-theoretic Euler class of the normal bundle $N_{\mu}$ of $\mathcal{X}_\mu$ in $\mathcal{X}$. $\text{Tr}(\Lambda^* N^*_{\mu})$ is invertible because the symmetry $g$ acts with eigenvalues different from 1 on the normal bundle to the fixed point locus. We call the terms corresponding to the identity component in the formula fake Euler characteristics:

$$\chi^f(\mathcal{X}, V) = \int_{\mathcal{X}} \text{ch}(V) T d(T\mathcal{X}).$$

In the case where $\mathcal{X}$ is a global quotient, formula (1) is the Lefschetz fixed point formula.

Now let $\mathcal{X}$ be a compact, complex orbispace (Deligne–Mumford stack) with a perfect obstruction theory $E^{-1} \to E^0$. This is used to define the intrinsic normal cone, which is embedded in $E_1$ — the dual bundle to $E^{-1}$ (see [Li and Tian 1998; Behrend and Fantechi 1997]). The virtual structure sheaf $\mathcal{O}_\mathcal{X}^{\text{vir}}$ was defined in [Lee 2004] as the $K$-theoretic pullback by the zero section of the structure sheaf of this cone. Let $I \mathcal{X} = \coprod \mathcal{X}_\mu$ be the inertia orbifold of $\mathcal{X}$. We denote by $i_\mu$ the inclusion of a stratum $\mathcal{X}_\mu$ in $\mathcal{X}$. For a bundle $V$ on $\mathcal{X}$, we write $i^* V = V_f^\mu \oplus V_m^\mu$ for its decomposition as the direct sum of the fixed part and the moving part under the action of the symmetry associated to $\mathcal{X}_\mu$. To avoid ugly notation we will often simply write $V^m, V^f$. The virtual normal bundle to $\mathcal{X}_\mu$ in $\mathcal{X}$ is defined as $[E_0^m] - [E_1^m]$. We will in addition assume that $\mathcal{X}$ admits an embedding $j$ in a smooth compact orbifold $\mathcal{Y}$. This is always true for the moduli spaces of genus-0 stable maps $X_{0,n,d}$ because an embedding $X \hookrightarrow \mathbb{P}^N$ induces an embedding $X_{0,n,d} \hookrightarrow (\mathbb{P}^N)_{0,n,d}$. 
Theorem 1.1. Denote by $N^\text{vir}_\mu$ the virtual normal bundle of $\mathcal{X}_\mu$ in $\mathcal{X}$. Then

$$
\chi(\mathcal{X}, j^*(V) \otimes \mathcal{O}^{\text{vir}}_{\mathcal{X}}) = \sum_{\mu} \frac{1}{m_\mu} \chi^f(\mathcal{X}_\mu, \frac{\text{Tr}(V_\mu \otimes \mathcal{O}^{\text{vir}}_{\mathcal{X}_\mu})}{\text{Tr}(\Lambda^*(N^\text{vir}_\mu)^*)}).
$$

Remark 1.2. A perfect obstruction theory $E^{-1} \to E^0$ on $\mathcal{X}$ induces canonically a perfect obstruction theory on $\mathcal{X}_\mu$ by taking the fixed part of the complex $E^{-1}_\mu \to E^0_\mu$. The proof is the same as that of Proposition 1 in [Graber and Pandharipande 1999]. This is then used to define the sheaf $\mathcal{O}^{\text{vir}}_{\mathcal{X}_\mu}$.

Remark 1.3. It is proved in [Fantechi and Göttsche 2010] that if $\mathcal{X}$ is a scheme, the Grothendieck–Riemann–Roch theorem is compatible with virtual fundamental classes and virtual fundamental sheaves, that is,

$$
\chi^f(\mathcal{X}, V \otimes \mathcal{O}^{\text{vir}}_{\mathcal{X}}) = \int_{[\mathcal{X}]} \chi(V \otimes \mathcal{O}^{\text{vir}}_{\mathcal{X}}) \cdot T d(T^{\text{vir}}),
$$

where $[\mathcal{X}]$ is the virtual fundamental class of $\mathcal{X}$ and $T^{\text{vir}}$ is its virtual tangent bundle. Their arguments carry over to the case when $\mathcal{X}$ is a stack.

Remark 1.4. The bundles $V$ to which we apply Theorem 1.1 in [Givental and Tonita 2014] are (sums and products of) cotangent line bundles $L_i$ and evaluation classes $\text{ev}_i^*(a_i)$ (where $a_i$ are $K$-theoretic classes on the target). They are pullbacks of the corresponding bundles on $(\mathbb{P}^N)_{0,n,d}$.

2. Proof of Theorem 1.1

Before proving Theorem 1.1 we recall a couple of background facts and lemmata on $K$-theory which we will use.

Let $K_0(X)$ be the Grothendieck group of coherent sheaves on $X$. Given a map $f : X \to Y$, the $K$-theoretic pullback $f^* : K_0(Y) \to K_0(X)$ is defined as the alternating sum of derived functors $\text{Tor}_i^f(\mathcal{F}, \mathcal{G}_X)$, provided that the sum is finite. This is always true for instance if $f$ is flat or if it is a regular embedding.

For any fiber square

$$
\begin{array}{ccc}
V' & \to & V \\
\downarrow & & \downarrow \\
B' & \to & B
\end{array}
$$

with $i$ a regular embedding one can define $K$-theoretic refined Gysin homomorphisms $i^! : K_0(V) \to K_0(V')$ (see [Lee 2004]). One way to define the map $i^!$ is the following: The class $i_*^!(\mathcal{O}_{B'}) \in K^0(B)$ has a finite resolution of vector bundles, which is exact off $B'$. We pull it back to $V$ and then cap (i.e., tensor product) with classes in $K_0(V)$, to get a class on $K_0(V)$ with homology supported on $V'$, which
we can regard as an element of $K_0(V')$, because there is a canonical isomorphism between complexes on $V$ with homology supported on $V'$ and $K_0(V')$.

In the following two lemmata, $X, Y, Y'$ are assumed DM stacks. We will use the following result:

**Lemma 2.1.** Consider the diagram:

$$
\begin{array}{ccc}
\iota^* C_{X/Y} & \longrightarrow & C_{X/Y} \\
\downarrow & & \downarrow \\
X' & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y' & \longrightarrow & Y
\end{array}
$$

with $i$ a regular embedding and $j$ an embedding, $C_{X/Y}$ is the normal cone of $X$ in $Y$ and both squares are fiber diagrams. Then

$$
i^! [\mathcal{O}_{C_{X/Y}}] = [\mathcal{O}_{C_{X'/Y'}}] \in K_0(i^* C_{X/Y}).
$$

This is stated and proved in [Lee 2004, Lemma 2]. The proof is based on a more general statement (Lemma 1 of [Lee 2004]), which has been worked out in [Kresch 1999] on the level of Chow rings. Since $K$-theoretic statements are stronger, we give below the key ingredient which allows one to carry over Kresch’s proof to $K$-theory:

**Lemma 2.2.** Let $f : X \to Y$ be a closed embedding and let $g : Y \to \mathbb{P}^1$ be a surjection such that $g \circ f$ is flat. Denote by $X_0$ and $Y_0$ the fibers over $0$ of $g \circ f$ and $g$, respectively. Moreover, assume that the restriction of $f$ to $X \setminus X_0$ is an isomorphism. Then if $i$ is the inclusion of $\{0\}$ in $\mathbb{P}^1$, we have $i^!(\mathcal{O}_Y) = \mathcal{O}_{X_0} \in K_0(Y_0)$.

**Proof.** The skyscraper sheaves at all points of $\mathbb{P}^1$ represent the same element in $K_0(\mathbb{P}^1)$, hence if we pull back a resolution of any point $P \in \mathbb{P}^1$ by $g$ we get the same elements of $K_0(Y)$. On the other hand since $f$ is an isomorphism above $\mathbb{P}^1 \setminus \{0\}$, pulling back by $g$ of the structure sheaf of a point $P \neq 0$ is the same as pulling back by $g \circ f$ followed by $f_*$. By what we said above we can replace $P$ with $0$. Now from the flatness of $g \circ f$ above $0$ the pullback of the structure sheaf of $0$ by $g \circ f$ is the structure sheaf of the fiber $X_0$. The result then follows from the definition of $i^!$. \qed

**Remark 2.3.** Lemma 2.2 allows one to show Lemma 2.1: intermediately one shows, following [Kresch 1999] (notation is as in Lemma 2.1), that $[\mathcal{O}_{C_1}] = [\mathcal{O}_{C_2}]$ in $K_0(C_{X'Y \times_Y C_X Y})$, where $C_1 := C_{i^* C_{X/Y}}(C_X Y)$ and $C_2 := C_{j^* C_{Y'/Y}}(C_{Y'Y})$. 
We now go on to prove Theorem 1.1. We have
\[ \chi(\mathcal{X}, j^* V \otimes \mathcal{O}_\mathcal{X}^{\text{vir}}) = \chi(\mathcal{Y}, V \otimes j_* \mathcal{O}_\mathcal{X}^{\text{vir}}). \]
Kawasaki’s formula applied to the sheaf \( V \otimes j_* \mathcal{O}_\mathcal{X}^{\text{vir}} \) on \( \mathcal{Y} \) gives
\[ \chi(\mathcal{Y}, V \otimes j_* \mathcal{O}_\mathcal{X}^{\text{vir}}) = \sum_{\mu} \frac{1}{m_\mu} \chi^f(\mathcal{Y}_\mu, \frac{\text{Tr}(V_\mu \otimes i_{\mu*} j_\ast \mathcal{O}_\mathcal{X}^{\text{vir}})}{\text{Tr}(\Lambda^\bullet N^\ast_\mu)}). \]
From the fiber diagram
\[ \mathcal{X}_\mu \xrightarrow{i_\mu'} \mathcal{X} \]
\[ j' \downarrow \quad j \downarrow \]
\[ \mathcal{Y}_\mu \xrightarrow{i_\mu} \mathcal{Y} \]
and Theorem 6.2 in [Fulton 1998] (where this is proved for Chow rings) we have
\[ i_{\mu*} j_* \mathcal{O}_\mathcal{X}^{\text{vir}} = j'_\ast i_\mu \mathcal{O}_\mathcal{X}^{\text{vir}}. \]
Plugging this in (4) gives
\[ \chi^f(\mathcal{Y}_\mu, \frac{\text{Tr}(V_\mu \otimes i_{\mu*} j_\ast \mathcal{O}_\mathcal{X}^{\text{vir}})}{\text{Tr}(\Lambda^\bullet N^\ast_\mu)}) = \chi^f(\mathcal{Y}_\mu, \frac{\text{Tr}(V_\mu \otimes j'_{\ast} i_\mu \mathcal{O}_\mathcal{X}^{\text{vir}})}{\text{Tr}(\Lambda^\bullet N^\ast_\mu)}). \]
Let \( G_\mu \) be the cyclic group generated by one element of the conjugacy class associated to \( \mathcal{X}_\mu \). Then we will show that
\[ \text{Tr}(\frac{i_{\mu*} \mathcal{O}_\mathcal{X}^{\text{vir}}}{\Lambda^\bullet (N^\ast_\mu)}) = \text{Tr}(\frac{\mathcal{O}_\mathcal{X}^{\text{vir}}}{\Lambda^\bullet (N^{\text{vir}})^\ast_\mu}) \]
in the \( G_\mu \)-equivariant \( K \)-ring of \( \mathcal{X}_\mu \). This is essentially the computation of Section 3 in [Graber and Pandharipande 1999] carried out in \( \mathbb{C}^\ast \)-equivariant \( K \)-theory. Relation (6) then follows by embedding the group \( G_\mu \) in the torus and specializing the value of the variable \( t \) in the ground ring of \( \mathbb{C}^\ast \)-equivariant \( K \)-theory to a \( |G_\mu| \)-root of unity.

If we define a cone \( D := C_{\mathcal{X}/\mathcal{Y}} \times E_0 \), then this is a \( T_{\mathcal{Y}} \) cone (see [Behrend and Fantechi 1997]). The virtual normal cone \( D^{\text{vir}} \) is defined as \( D/T_{\mathcal{Y}} \) and \( \mathcal{O}_\mathcal{X}^{\text{vir}} \) is the pullback by the zero section of the structure sheaf of \( D^{\text{vir}} \). Alternatively there is a fiber diagram
\[ T_{\mathcal{Y}} \longrightarrow D \]
\[ \downarrow \quad \downarrow \]
\[ \mathcal{X} \longrightarrow \mathcal{0}_E \]
\[ E_1 \]
where the bottom map is the zero section of \( E_1 \). Then one can define \( \mathcal{O}_\mathcal{X}^{\text{vir}} \) as \( 0_{T_{\mathcal{Y}}} \mathcal{0}_E^{0_{E_1}} [\mathcal{O}_D] \). We’ll prove formula (6) following closely the calculation in [Graber
and Pandharipande 1999]. First, by definition of $\mathcal{O}^{\text{vir}}_{\mathcal{X}}$ and by commutativity of Gysin maps, we have

\begin{equation}
 i^!_{\mu} \mathcal{O}^{\text{vir}}_{\mathcal{X}} = i^!_{\mu} 0^*_T \mathcal{O}^1_{E_1} [\mathcal{O}_D] = 0^*_T 0^!_{E_1} i^!_{\mu} [\mathcal{O}_D].
\end{equation}

We pull back relation (3) to $(i^!_{\mu})^* D = (i^!_{\mu})^* (C_{\mathcal{X}/y} \times E_0)$ to get

\begin{equation}
 i^!_{\mu} [\mathcal{O}_D] = [\mathcal{O}_{D^\mu} \times (E^m_0)^*].
\end{equation}

In the equality above we have used the fact that $D_{\mu} = C_{\mathcal{X}/\mathcal{Y}_\mu} \times E_0^f$ and we identified the sheaf of sections of the bundle $E^m_0$ with the dual bundle $(E^m_0)^*$. Plugging (8) in (7) we get

\begin{equation}
 i^!_{\mu} \mathcal{O}^{\text{vir}}_{\mathcal{X}} = 0^*_T \mathcal{O}^1_{E_1} [\mathcal{O}_D \times (E^m_0)^*].
\end{equation}

Notice that the action of $T_{\mathcal{Y}_\mu}$ leaves $D_{\mu} \times (E^m_0)^*$ invariant (it acts trivially on $(E^m_0)^*$). Now we can write $0^*_T = 0^*_T \mathcal{O}^1_{E_1} \times 0^*_T \mathcal{O}^1_{\mathcal{Y}_\mu}$ and since $D^\text{vir}_{\mu} = D_{\mu}/T_{\mathcal{Y}_\mu}$ we rewrite (9) as

\begin{equation}
 i^!_{\mu} \mathcal{O}^{\text{vir}}_{\mathcal{X}} = 0^*_T \mathcal{O}^1_{E_1} [\mathcal{O}_{D^\text{vir}_{\mu}} \times (E^m_0)^*].
\end{equation}

The proof of Lemma 1 in [Graber and Pandharipande 1999] works in our set-up as well: it uses excess intersection formula which holds in $K$-theory. It shows that the following relation holds in the $\mathbb{C}^*$-equivariant $K$-ring of $\mathcal{X}_\mu$:

\begin{equation}
 0^*_T \mathcal{O}^1_{E_1} [\mathcal{O}_{D^\text{vir}_{\mu}} \times (E^m_0)^*] = 0^*_E (0^!_{E_1} \mathcal{O}^1_{D^\text{vir}_{\mu}} \times (E^m_0)^*) \cdot \Lambda^*(T_{\mathcal{Y}_\mu})^*. \tag{11}
\end{equation}

The class $0^!_{E_1} \mathcal{O}^1_{D^\text{vir}_{\mu}} \times (E^m_0)^*$ lives in the $\mathbb{C}^*$-equivariant $K$-ring of $E^m_0$. The class doesn’t depend on the bundle map $E^m_0 \to E^m_1$ so we can assume this map to be 0. Then by excess intersection formula and the definition of $\mathcal{O}^{\text{vir}}_{\mathcal{X}_\mu}$ we get

\begin{equation}
 0^*_E (0^!_{E_1} \mathcal{O}^1_{D^\text{vir}_{\mu}} \times (E^m_0)^*) = \mathcal{O}^{\text{vir}}_{\mathcal{X}_\mu} \cdot \Lambda^*(E^m_1)^*. \tag{12}
\end{equation}

Formula (12) holds because $D^\text{vir}_{\mu} \times (E^m_0) \subset E^f_1 \times E^m_0$ and $0^!_{E_1}$ acts as $0^!_{E_1} \times 0^!_{E_1}$ on factors. $0^!_{E_1} \mathcal{O}^1_{D^\text{vir}_{\mu}} = \mathcal{O}^{\text{vir}}_{\mathcal{X}_\mu}$ by definition of $\mathcal{O}^{\text{vir}}_{\mathcal{X}_\mu}$. By excess intersection formula applied to the fiber square

\[
\begin{array}{ccc}
E^m_0 & \longrightarrow & E^m_0 \\
\downarrow \pi & & \downarrow \\
\mathcal{X}_\mu & \longrightarrow & E^m_1
\end{array}
\]

we have $0^*_E 0^!_{E_1} [(E^m_0)^*] = 0^*_E \pi^* \Lambda^*(E^m_1)^* = \Lambda^*(E^m_1)^*$. Plugging formula (12) in (11) (note that $N_{\mu} = T_{\mathcal{Y}_\mu}$ and $N^\text{vir}_{\mu} = [E^m_0] - [E^m_1]$) and taking traces proves (6).
We now plug (6) in (5) and then pull back to $\mathcal{X}_\mu$ to get

$$
\chi^f \left( \mathcal{Y}_\mu, \frac{\text{Tr}(V_\mu \otimes j_\mu^* \mathcal{N}_\mu^{\text{vir}})}{\text{Tr}(\Lambda^* \mathcal{N}_\mu^*)} \right) = \chi^f \left( \mathcal{Y}_\mu, \frac{\text{Tr}(\mathcal{N}_\mu^{\text{vir}})}{\text{Tr}(\Lambda^* \mathcal{N}_\mu^{\text{vir}})^*)} \right)
$$

$$
= \chi^f \left( \mathcal{X}_\mu, \frac{\text{Tr}(V_\mu \otimes \mathcal{N}_\mu^{\text{vir}})}{\text{Tr}(\Lambda^* \mathcal{N}_\mu^{\text{vir}})^*)} \right).
$$

\[\square\]

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