

*Pacific
Journal of
Mathematics*

**ERRATUM TO
“SINGULARITIES OF THE PROJECTIVE DUAL VARIETY”**

ROLAND ABUAF

ERRATUM TO “SINGULARITIES OF THE PROJECTIVE DUAL VARIETY”

ROLAND ABUAF

Volume 253:1 (2011), 1–17

We give a counterexample to a proposition claimed to be proven in an earlier paper of ours and used in the proof of its main theorem. We also show how to salvage the main result of that paper under additional hypotheses.

Let $X \subset \mathbb{P}^N$ be a nondegenerate smooth projective variety such that X^* is a hypersurface. Let $L \subset \mathbb{P}^N$ be a linear subspace such that for general $x \in X$ we have $\langle L, T_{X,x} \rangle \neq \mathbb{P}^N$. We say that L^\perp is an *unexpected equisingular space* in X^* (see Definition 3.2.1 of [Abuaf 2011], hereafter cited as [A]) if the general hyperplane containing $\langle L, T_{X,x} \rangle$ has the same multiplicity in X^* as a general hyperplane containing L . In [A], the following side-result, whose aim was to discuss a necessary hypothesis in our main theorem, was stated in Section 3 (“Open question and corollaries”):

Theorem 3.2.2 of [A]. *Let $X \subset \mathbb{P}^N$ be an irreducible, smooth, nondegenerate projective variety such that X^* is a hypersurface. Let $L \subset X$ be a linear space with $\dim(L) = \dim(X) - 1$. Assume that L^\perp is an unexpected equisingular linear space in X^* such that $\text{mult}_{L^\perp} X^* = 2$. Then X is the cubic scroll surface in \mathbb{P}^4 .*

Its proof was based on this proposition:

Proposition 3.2.3 of [A]. *Let $X \subset \mathbb{P}^N$ be a smooth, irreducible, nondegenerate projective variety such that X^* is a hypersurface. Let $[h] \in X^*$ be such that $\text{mult}_{[h]} X^* = 2$. The scheme-theoretic tangency locus of H with X is one of the following:*

- *An irreducible hyperquadric and in this case $|\mathcal{C}_{[h]}(X^*)|^* = \text{Tan}(H, X)$.*
- *The union of two (not necessarily distinct) linear spaces.*
- *A linear space with at least one embedded component.*

This proposition is false as shown by the following example.

MSC2010: primary 14B05; secondary 14N15.

Keywords: projective geometry, singularities, dual variety.

Example 1. Let V be a vector space of dimension 6 and let $W = \mathbb{G}(3, V) \subset \mathbb{P}(\wedge^3 V)$ be the Grassmannian of $\mathbb{C}^3 \subset V$ in its Plücker embedding. The dual of X is a quartic hypersurface in $\mathbb{P}(\wedge^3 V^*)$. We can decompose $\wedge^3 V^*$ as

$$\mathbb{C} \oplus U \oplus U^* \oplus \mathbb{C},$$

where U is identified with the space of 3×3 matrices (see [Landsberg and Manivel 2001, Section 5] for more details). We denote by C the determinant on U , which can be seen as a map $S^3 U \rightarrow \mathbb{C}$ or as a map $S^2 U \rightarrow U^*$. We also denote by C^* the determinant on U^* .

It is shown in (ibid.) that an equation (up to an automorphism of $\mathbb{P}(\wedge^3 V^*)$) of W^* is

$$Q(x, X, Y, y) = (3xy - \frac{1}{2}\langle X, Y \rangle)^2 + \frac{1}{3}(yC(X^{\otimes 3}) + xC^*(Y^{\otimes 3})) - \frac{1}{6}\langle C^*(Y^{\otimes 2}), C(X^{\otimes 2}) \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the standard pairing between U and U^* . The partial derivatives of Q give the equations of the variety of “stationary secants” to $W^\perp := \mathbb{G}(3, V^*) \subset \mathbb{P}(\wedge^3 V^*)$, which we denote by $\sigma_+(W^\perp)$. The Jacobian criterion shows that the variety $\sigma_+(W^\perp)$ is singular precisely along W^\perp . However, a simple Taylor expansion of Q around the point $[1, 0, \dots, 0] \in W^\perp$ shows that, contrary to what is claimed in Proposition 5.10 of (ibid.), W^\perp is not defined by all the second derivatives of Q . The orbit structure of the action of SL_6 on $\mathbb{P}(\wedge^3 V^*)$ is

$$W^\perp \subset \sigma_+(W^\perp) \subset W^* \subset \mathbb{P}(\wedge^3 V^*).$$

Since W^\perp is the deepest strata in W^* and all the second derivatives of the equation of W^* do not vanish on W^\perp , we conclude that there are no point of multiplicity bigger than 2 in W^* . However one can prove (see (ibid.) for instance) that a point in W^\perp is tangent to W along a cone over $\mathbb{P}^2 \times \mathbb{P}^2$. This gives a counterexample to the above proposition. Note that an easy computation shows that if $p = (p_0, P_0, P_1, p_1) \in \mathbb{P}(\wedge^3 V^*)$ is a generic point then the cubic hypersurface (which we denote by $\mathcal{P}(Q, p)$) defined by the equation $p_0 \frac{\partial Q}{\partial x} + P_0 \frac{\partial Q}{\partial X} + P_1 \frac{\partial Q}{\partial Y} + p_1 \frac{\partial Q}{\partial y}$ is smooth. Moreover the polar $\mathcal{P}(W^*, p) := W^* \cap \mathcal{P}(Q, p)$ has multiplicity 3 along W^\perp .

In [A] I claim that I “prove” Proposition 3.2.3 in the appendix. This proof relies on the following statement:

Lemma A.3 of [A]. *Let $Z \subset \mathbb{P}^N$ be an irreducible and reduced hypersurface, whose defining equation is denoted by f_Z . Let $z \in Z$ and let $k \in \{-1, \dots, N - 2\}$. Then one of the following holds for general $D \in \mathbb{G}(k, N)$:*

• $z \notin P(Z, D)$.

• $\text{mult}_z P(Z, D) = \text{mult}_z Z \cdot \text{mult}_z P(f_Z, D)$

if $\dim(Z_{\text{sing}}^{(z)}) < \dim P(Z, D)$, where $Z_{\text{sing}}^{(z)}$ is an irreducible component of Z_{sing} of maximal dimension passing through z .

• $\text{mult}_z P(Z, D) < \text{mult}_z Z \cdot \text{mult}_z P(f_Z, D)$

if $\dim(Z_{\text{sing}}^{(z)}) \geq \dim P(Z, D)$, where $Z_{\text{sing}}^{(z)}$ is an irreducible component of Z_{sing} of maximal dimension passing through z .

This lemma is also false as shown by [Example 1](#). Indeed the hypersurface $\mathcal{P}(Q, p)$ is smooth for generic p , the hypersurface W^* has multiplicity 2 along W^\perp , but the polar $\mathcal{P}(W^*, p) := W^* \cap \mathcal{P}(Q, p)$ has multiplicity 3 along W^\perp . The mistake in the proof of the lemma can be easily found. On line 5, page 14 of [\[A\]](#), I write “Let $(Z_i)_{i \in I}$ be a stratification of Z such that Z_i is smooth and Z is normally flat along Z_i for all $i \in I$. Such a stratification exists, due to the open nature of normal flatness [...]. Consider the Gauss map $G : Z \rightarrow (\mathbb{P}^N)^*$. It restricts to a map $G_i : Z_i \rightarrow (\mathbb{P}^N)^* \dots$ ” This last sentence is nonsense since the Gauss map is not defined on the singular locus of Z .

I used [Lemma A.3](#) of [\[A\]](#) in the form of the following corollary:

Corollary A.4 of [A]. *Let $Z \subset \mathbb{P}^N$ be an hypersurface and let $z \in Z$ such that $\text{mult}_z Z = 2$ and let $k \in \{-1, \dots, N - 2\}$. Then, for generic $D \in \mathbb{G}(k, N)$, we have $\text{mult}_z \mathcal{P}(Z, D) \leq 2$.*

This corollary is again false as shown in [Example 1](#), but it seems natural to use its conclusion as an hypothesis. Indeed the rest of the proof of Proposition 3.2.3 of [\[A\]](#) is correct, and thus we get the following result:

Proposition 2 (replacement for Proposition 3.2.3 of [\[A\]](#)). *Let $X \subset \mathbb{P}^N$ be a smooth, irreducible, nondegenerate projective variety such that X^* is a hypersurface. Let $[h] \in X^*$ be such that $\text{mult}_{[h]} X^* = 2$ and that for all $k \in \{-1, \dots, N - 2\}$ and generic $D \in \mathbb{G}(k, N)$, we have $\text{mult}_{[h]} \mathcal{P}(X^*, D) \leq 2$. The scheme theoretic tangency locus of H with X is one of the following:*

- *An irreducible hyperquadric and in this case $|\mathcal{C}_{[h]}(X^*)|^* = \text{Tan}(H, X)$.*
- *The union of two (not necessarily distinct) linear spaces.*
- *A linear space with at least one embedded component.*

Finally, we can formulate a version of Theorem 3.2.2 of [\[A\]](#), whose proof relies on the above proposition:

Theorem 3 (replacement for Theorem 3.2.2 of [A]). *Let $X \subset \mathbb{P}^N$ be an irreducible, smooth, nondegenerate projective variety such that X^* is a hypersurface. Let $L \subset X$ be a linear space with $\dim(L) = \dim(X) - 1$. Assume that L^\perp is an unexpected equisingular linear space in X^* such that $\text{mult}_{L^\perp}(X^*) = 2$. Assume moreover that for all $[h] \in L^\perp$, for all $k \in \{-1, \dots, N - 2\}$ and generic $D \in \mathbb{G}(k, N)$, we have $\text{mult}_{[h]} \mathcal{P}(X^*, D) \leq 2$. Then X is the cubic scroll surface in \mathbb{P}^4 .*

References

- [Abuaf 2011] R. Abuaf, “Singularities of the projective dual variety”, *Pacific J. Math.* **253**:1 (2011), 1–17. [MR 2869431](#) [Zbl 1252.14035](#)
- [Landsberg and Manivel 2001] J. M. Landsberg and L. Manivel, “The projective geometry of Freudenthal’s magic square”, *J. Algebra* **239**:2 (2001), 477–512. [MR 2002g:14070](#) [Zbl 1064.14053](#)

Received September 11, 2012.

ROLAND ABUAF
INSTITUT FOURIER
100 RUE DES MATHS, BP 74
38402 SAINT-MARTIN D’HÈRES CEDEX
FRANCE
abuaf@ujf-grenoble.fr

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

EDITORS

Don Blasius (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI
CALIFORNIA INST. OF TECHNOLOGY
INST. DE MATEMÁTICA PURA E APLICADA
KEIO UNIVERSITY
MATH. SCIENCES RESEARCH INSTITUTE
NEW MEXICO STATE UNIV.
OREGON STATE UNIV.

STANFORD UNIVERSITY
UNIV. OF BRITISH COLUMBIA
UNIV. OF CALIFORNIA, BERKELEY
UNIV. OF CALIFORNIA, DAVIS
UNIV. OF CALIFORNIA, LOS ANGELES
UNIV. OF CALIFORNIA, RIVERSIDE
UNIV. OF CALIFORNIA, SAN DIEGO
UNIV. OF CALIF., SANTA BARBARA

UNIV. OF CALIF., SANTA CRUZ
UNIV. OF MONTANA
UNIV. OF OREGON
UNIV. OF SOUTHERN CALIFORNIA
UNIV. OF UTAH
UNIV. OF WASHINGTON
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.


See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2014 is US \$410/year for the electronic version, and \$535/year for print and electronic. Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by [Mathematical Reviews](#), [Zentralblatt MATH](#), [PASCAL CNRS Index](#), [Referativnyi Zhurnal](#), [Current Mathematical Publications](#) and [Web of Knowledge \(Science Citation Index\)](#).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW[®] from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2014 Mathematical Sciences Publishers

PACIFIC JOURNAL OF MATHEMATICS

Volume 268 No. 2 April 2014

In memoriam: Jonathan Rogawski	257
DON BLASIUS, DINAKAR RAMAKRISHNAN and V. S. VARADARAJAN	
Formes modulaires sur la \mathbb{Z}_p -extension cyclotomique de \mathbb{Q}	259
LAURENT CLOZEL	
Weight zero Eisenstein cohomology of Shimura varieties via Berkovich spaces	275
MICHAEL HARRIS	
Λ -adic Barsotti–Tate groups	283
HARUZO HIDA	
Le flot géodésique des quotients géométriquement finis des géométries de Hilbert	313
MICKAËL CRAMPON and LUDOVIC MARQUIS	
Nonplanarity of unit graphs and classification of the toroidal ones	371
A. K. DAS, H. R. MAIMANI, M. R. POURNAKI and S. YASSEMI	
Discrete semiclassical orthogonal polynomials of class one	389
DIEGO DOMINICI and FRANCISCO MARCELLÁN	
A note on conformal Ricci flow	413
PENG LU, JIE QING and YU ZHENG	
On representations of $GL_{2n}(F)$ with a symplectic period	435
ARNAB MITRA	
Linked triples of quaternion algebras	465
ALEXANDER S. SIVATSKI	
Finite nonsolvable groups with many distinct character degrees	477
HUNG P. TONG-VIET	
Errata to “Dynamics of asymptotically hyperbolic manifolds”	493
JULIE ROWLETT	
Erratum to “Singularities of the projective dual variety”	507
ROLAND ABUAF	