Pacific Journal of Mathematics

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Volume 269 No. 1 May 2014

PACIFIC JOURNAL OF MATHEMATICS
Vol. 269, No. 1, 2014
dx.doi.org/10.2140/pjm.2014.269.19

THE CUP SUBALGEBRA OF A II_1 FACTOR GIVEN BY A SUBFACTOR PLANAR ALGEBRA IS MAXIMAL AMENABLE

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To every subfactor planar algebra was associated a II_1 factor with a canonical abelian subalgebra generated by the cup tangle. Using Popa's approximative orthogonality property, we show that this cup subalgebra is maximal amenable.

Introduction

The study of maximal abelian subalgebras (MASAs) was initiated by Dixmier [1954], who introduced an invariant coming from the normalizer. Other invariants were provided later, such as the Takesaki equivalence relation [1963], the Tauer length [1965], the Pukánszky invariant [1960] or the δ-invariant [Popa 1983b].

Popa [1983a] exhibited an example of a MASA $A \subset M$ in a II_1 factor that is maximal amenable.

This example answers negatively a question of Kadison asking if every abelian subalgebra of a II_1 factor (with separable predual) is included in a copy of the hyperfinite II_1 factor. We recall that a von Neumann algebra is hyperfinite if and only if it is amenable by the famous theorem of Connes [1976]. Popa introduced the notion of approximative orthogonality property (AOP) and showed that any singular MASA with the AOP is maximal amenable. Then he proved that the generator MASA in a free group factor is singular and has the AOP.

Using the same scheme of proof, Cameron et al. [2010] showed that the radial MASA in the free group factor is maximal amenable. Shen [2006], Jolissaint [2010] and Houdayer [2012] provided other examples of maximal amenable MASAs.

In this paper, we provide maximal amenable MASAs in II_1 factors using subfactor planar algebras. The theory of subfactors has been initiated by Jones [1983]. He introduced the standard invariant that has been formalized as a Popa system by Popa [1995] and as a subfactor planar algebra by Jones [1999; 2011]. Popa [1993; 1995; 2002] proved that any standard invariant comes from a subfactor. Popa and Shlyakhtenko [2003] proved that the subfactor can be realized in the infinite

Supported by ERC Starting Grant VNALG-200749 and by Région Île-de-France .
MSC2010: primary 46L10; secondary 46K15.
Keywords: planar algebra, von Neumann algebra, maximal abelian subalgebra, amenability.

free group factor $L(\mathbb{F}_\infty)$. Using planar algebras, random matrix models and free probability, Guionnet et al. [2010; 2011] (see also [Jones et al. 2010]) showed that any finite depth standard invariant can be realized as a subfactor of an interpolated free group factor. Using the same construction, Hartglass [2013] proved that any infinite depth subfactor is realized in $L(\mathbb{F}_\infty)$.

The construction in [Jones et al. 2010] associated a II_1 factor M to a subfactor planar algebra $\mathcal{P}$. This II_1 factor contains a generic MASA $A \subset M$ that we call the *cup subalgebra* (see page 22). We now state our main theorem:

Theorem 0.1. *For any nontrivial subfactor planar algebra $\mathcal{P}$, the cup subalgebra is maximal amenable.*

The construction of Jones et al. has been extended for unshaded planar algebras in [Brothier 2012; Brothier et al. 2012]. In those constructions, we have proven that the cup subalgebra is still a MASA. It seems very plausible that it is also maximal amenable. Note that the cup subalgebra is analogous to the *radial MASA* in a free group factor. We don't know if for a certain subfactor planar algebra those two subalgebras are isomorphic or not.

1. Approximative orthogonality property and maximal amenability

We briefly recall Popa's approximative orthogonality property for an abelian subalgebra $A \subset M$ and how it implies the maximal amenability of A, whenever $A \subset M$ is a singular MASA.

Definition 1.1 [Popa 1983a, Lemma 2.1]. Consider a tracial von Neumann algebra (M, tr) and a subalgebra $A \subset M$. Let ω be a free ultrafilter on $\mathbb{N}$. Then $A \subset M$ has the approximative orthogonality property if for any $x \in M^\omega \ominus A^\omega \cap A'$ and any $b \in M \ominus A$ we have $xb \perp bx$ in $L^2(M^\omega)$, that is, $\lim_{n\to\omega} \mathrm{tr}(x_n b x_n^* b^*) = 0$, where $(x_n)_n$ is a representative of x.

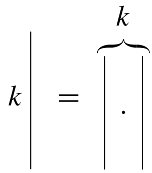
Remark 1.2. By polarization, the definition of AOP is equivalent to asking that for any $x_1, x_2 \in M^\omega \ominus A^\omega \cap A'$ and any $b_1, b_2 \in M \ominus A$ we have $x_1 b_1 \perp b_2 x_2$.

We recall the fundamental theorem of Popa that is contained in the proof of [Popa 1983a, Theorem 3.2]. A more detailed explanation of it has been given in [Cameron et al. 2010, Lemma 2.2 and Corollary 2.3].

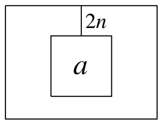
Theorem 1.3 [Popa 1983a]. *Let $A \subset M$ be a singular MASA with the AOP in a* II_1 *factor M. Then $A \subset M$ is maximal amenable.*

2. Construction of the cup subalgebra

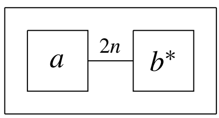
Construction of a* II_1 *factor from a subfactor planar algebra. Consider a subfactor planar algebra $\mathcal{P} = (\mathcal{P}_n)_{n\geqslant 0}$ with modulus $\delta > 1$. Let us recall the construction given in [Jones et al. 2010]. We assume that the reader is familiar with planar algebras. For more details on planar algebras, see [Jones 1999; 2011] or the introduction of [Peters 2010]. Let $\mathrm{Gr}(\mathcal{P})$ be the graded vector space equal to the algebraic direct sum $\bigoplus_{n\geqslant 0}\mathcal{P}_n$. We decorate strands in a planar tangle with natural numbers to represent cabling of that strand. For example:



An element $a\in\mathcal{P}_n$ will be represented as a box:

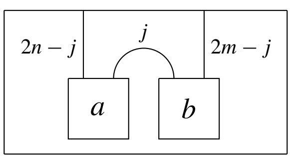
$a =$ 

We assume that the distinguished first interval is at the top left of the box. We consider the inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ on each $\mathcal{P}_n$:

$\langle a, b\rangle =$ 

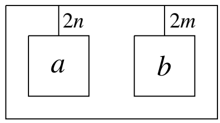
for all $a, b\in\mathcal{P}_n$.

We extend this inner product on $\mathrm{Gr}(\mathcal{P})$ in such a way that the spaces $\mathcal{P}_n$ are pairwise orthogonal. We still write $\mathcal{P}_n$ when it is considered as the n-graded part of $\mathrm{Gr}(\mathcal{P})$. Let $\mathcal{H}$ be the Hilbert space equal to the completion of $\mathrm{Gr}(\mathcal{P})$ for its pre-Hilbert structure. Note that $\mathcal{H}$ is the Hilbert space equal to the orthogonal direct sum of the spaces $\mathcal{P}_n$. We define a multiplication on $\mathrm{Gr}(\mathcal{P})$ given by the tangle

$ab = \sum_{j=0}^{\min(2n,2m)}$ 

for all $a\in\mathcal{P}_n, b\in\mathcal{P}_m$.

For a fixed $a\in\mathrm{Gr}(\mathcal{P})$, the map $b\in\mathrm{Gr}(\mathcal{P})\mapsto ab\in\mathrm{Gr}(\mathcal{P})$ is bounded for the inner product $\langle\,\cdot\,,\,\cdot\,\rangle$. This gives us a representation of the $*$-algebra $\mathrm{Gr}(\mathcal{P})$ on $\mathcal{H}$. We denote by M the von Neumann algebra equal to the bicommutant of this representation. It is a II_1 factor by [Jones et al. 2010]. We identify the graded algebra $\mathrm{Gr}(\mathcal{P})$ and its image in the von Neumann algebra M. The unique faithful normal trace tr of M is the one coming from the planar algebra structure of $\mathcal{P}$. It is equal to the formula $\mathrm{tr}(a) = \langle a, 1\rangle$, where 1 is the unity of $\mathrm{Gr}(\mathcal{P})$. Let $L^2(M)$ be the Hilbert space coming from the Gelfand–Naimark–Segal construction over the trace tr. Note that the standard representation of the von Neumann algebra M on the Hilbert space $L^2(M)$ is conjugate to the action of M on the Hilbert space $\mathcal{H}$. We will identify those two representations. Also, we identify M with its image in $L^2(M)$. The left and right actions of M on the Hilbert space $L^2(M)$ are denoted by π and ρ, so $\pi(x)\rho(y)z = xzy$, for $x, y, z \in M$. The norm of M is denoted by $\|\cdot\|$ and that of $L^2(M)$ by $\|\cdot\|_2$, or by $\|\cdot\|$ if the context is clear. We define a multiplication on $\mathrm{Gr}(\mathcal{P})$ by requiring that if $a \in \mathcal{P}_n$ and $b \in \mathcal{P}_m$, then $a \bullet b \in \mathcal{P}_{n+m}$ is given by

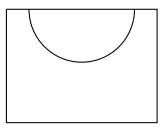
$a \bullet b =$ 

We remark that $\|a \bullet b\|_2 = \|a\|_2\|b\|_2$, for all $a \in \mathcal{P}_n$ and $b \in \mathcal{P}_m$. By the triangle inequality, the bilinear function

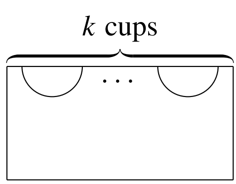
$$\mathrm{Gr}(\mathcal{P}) \times \mathrm{Gr}(\mathcal{P}) \to \mathrm{Gr}(\mathcal{P}), \quad (a, b) \mapsto a \bullet b,$$

is continuous for the norm $\|\cdot\|_2$. We extend this operation to $L^2(M) \times L^2(M)$ and still denote it by $\bullet$.

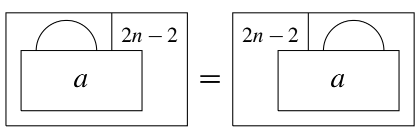
The cup subalgebra. The cup subalgebra $A \subset M$ is the abelian von Neumann algebra generated by the self-adjoint element cup:



We denote cup by the symbol $\cup$. Also we use the following notation:

$\cup^{\bullet k} =$ 

We use the convention that $0 = \cup^{\bullet k}$ for $k < 0$ and $1 = \cup^{\bullet 0}$. Let $n \geqslant 1$ and V_n be the subspace of $\mathcal{P}_n$ of elements which vanish when a cap is placed at the top right and vanish when a cap is placed at the top left, i.e.,

$V_n = \Bigg\{ a \in \mathcal{P}_n,$  $= 0 \Bigg\}.$

We denote by V the orthogonal direct sum of the V_n:

$$V = \bigoplus_{n=1}^{\infty} V_n.$$

Let $\ell^2(\mathbb{N})$ be the separable Hilbert space with orthonormal basis $\{e_n,\ n \geqslant 0\}$ and $S \in \mathbb{B}(\ell^2(\mathbb{N}))$ the unilateral shift operator.

Proposition 2.1 [Jones et al. 2010, Theorem 4.9]. *The map*

$$\Theta : L^2(M) \to \ell^2(\mathbb{N}) \oplus (\ell^2(\mathbb{N}) \otimes V \otimes \ell^2(\mathbb{N}))$$

defined by

$$\delta^{-k/2} \cup^{\bullet k} \mapsto e_k \oplus 0, \quad \delta^{-(l+r)/2} \cup^{\bullet l} \bullet v \bullet \cup^{\bullet r} \mapsto 0 \oplus e_l \otimes v \otimes e_r,$$

defines a unitary transformation, where $k, l, r \geqslant 0$, $v \in V$ and δ is the modulus of the planar algebra. We have

$$\Theta\pi\left(\frac{\cup - 1}{\delta^{1/2}}\right)\Theta^* = \begin{pmatrix} S+S^* - q_{e_0} & 0 \\ 0 & (S+S^*) \otimes 1_V \otimes 1_{\ell^2(\mathbb{N})} \end{pmatrix}$$

and

$$\Theta\,\rho\left(\frac{\cup - 1}{\delta^{1/2}}\right)\Theta^* = \begin{pmatrix} S+S^* - q_{e_0} & 0 \\ 0 & 1_{\ell^2(\mathbb{N})} \otimes 1_V \otimes (S+S^*) \end{pmatrix},$$

where q_{e_0} is the rank-one projection on $\mathbb{C}e_0$ and 1_V, $1_{\ell^2(\mathbb{N})}$ are the identity operators of the Hilbert spaces V and $\ell^2(\mathbb{N})$.

Corollary 2.2. *The cup subalgebra is a singular MASA.*

Proof. The A-bimodule $L^2(M) \ominus L^2(A)$ is isomorphic to an infinite direct sum of the coarse bimodule $L^2(A) \otimes L^2(A)$. This implies that $A \subset M$ is maximal abelian. See [Jones et al. 2010] for more details. Suppose that there exists a unitary u in the normalizer of A inside M which is orthogonal to A. It generates a A-subbimodule

$$\mathcal{K} \subset \bigoplus_{j=0}^{\infty} L^2(A) \otimes L^2(A). \tag{1}$$

We have the inclusion (1) if and only if the automorphism $a \in A \mapsto uau^*$ is trivial. This implies that $u \in A' \cap M$. Hence $u \in A$, a contradiction. Therefore, $A \subset M$ is singular. □

Basic facts on the unilateral shift operator. Consider the semicircular measure

$$d\nu(t) = \frac{\sqrt{4-t^2}}{2\pi}\, dt$$

defined on the interval $[-2; 2]$. Let $P_i \in \mathbb{R}[X]$ be the family of polynomials such that

$$P_0(X) = 1, \qquad P_1(X) = X, \qquad P_i(X) = X P_{i-1}(X) - P_{i-2}(X) \quad \text{for } i \geqslant 2. \tag{2}$$

By [Voiculescu et al. 1992, Example 3.4.2], the map

$$\Psi : \ell^2(\mathbb{N}) \to L^2([-2;2],\nu), \quad e_i \mapsto P_i, \tag{3}$$

defines a unitary transformation. Further, for any continuous function $f \in \mathscr{C}([-2;2])$ we have $(\Psi^* f(S+S^*)\Psi)(t) = tf(t)$ for almost every $t \in [-2;2]$.

Lemma 2.3. *For $I \geqslant 0$, let $R_I : [-2;2] \to \mathbb{R}$ be given by $R_I(t) = \sum_{i=0}^{I} P_i(t)^2$. The sequence $(R_I)_{I\geqslant 0}$ converges uniformly to $+\infty$.*

Proof. Let us prove the simple convergence to $+\infty$. Suppose there exists $t_0 \in [-2;2]$ such that the sequence $(R_I(t_0))_k$ does not converge to $+\infty$. The polynomials P_i have real coefficient. Hence, for any $t \in [-2;2]$, $P_i(t)$ is real; thus, $(R_I(t_0))_k$ is an increasing sequence in $\mathbb{R}$. If this sequence does not diverge, then it is bounded. Then, the sequence $(P_i(t_0))_i$ is square summable. In particular we have $\lim_{i\to\infty} P_i(t_0) = 0$. We put $\varepsilon_i = P_i(t_0)$. We have that $\varepsilon_{i+1} = t_0\varepsilon_i - \varepsilon_{i-1}$ and $\lim_{i\to\infty} \varepsilon_i = 0$. There is only one sequence that satisfies those axioms and it is the sequence equal to zero. Since $0 \neq 1 = P_0(t_0) = \varepsilon_0$, we arrive at a contradiction and thus, $\lim_{I\to\infty} S_I(t) = +\infty$ for any $t \in [-2;2]$. To conclude we use the following well known result due to Dini: Let $(f_I)_I$ be a sequence of continuous functions from a compact topological space K to $\mathbb{R}$ such that $f_I \leqslant f_{I+1}$. If for any $t \in K$, $\lim_{I\to\infty} f_I(t) = +\infty$, then the sequence $(f_I)_I$ converges uniformly to $+\infty$. □

Proof of Theorem 0.1. According to Theorem 1.3 and Corollary 2.2, it is sufficient to show that the cup subalgebra has the AOP. Fix $x \in M^\omega \ominus A^\omega \cap A'$ and $b \in M \ominus A$. Let us show that $xb \perp bx$. By the Kaplansky density theorem we can assume that there exists $J \geqslant 1$ such that $b \in \bigoplus_{j=0}^{J} \mathscr{P}_j$. Suppose that $\|x\| \leqslant 1$ and fix a sequence $x_n \in M$ which is a representative of x such that $x_n \in M \ominus A$ and $\|x_n\| \leqslant 1$ for all $n \geqslant 0$.

Consider the closed subspaces of $L^2(M)$ given by

$$Y_L = \overline{\operatorname{span}}\{\cup^{\bullet l} \bullet v \bullet \cup^{\bullet r},\ l, r \leqslant L,\ v \in V\},$$
$$Z_L = \overline{\operatorname{span}}\{\cup^{\bullet l} \bullet v \bullet \cup^{\bullet r},\ l \text{ or } r \leqslant L,\ v \in V\},$$

for all $L \geqslant 0$. Note that b is in Y_{J-1}.

We claim that for any $z \in M$ which is orthogonal to A and Z_{J-1} we have

$$zb \perp bz. \tag{4}$$

The element z is a weak limit of finite linear combinations of $\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j}$, where $i, j \geqslant J$ and $v \in V$. The element b is a finite linear combination of $\cup^{\bullet k} \bullet \tilde{v} \bullet \cup^{\bullet r}$, where $k, r \leqslant J-1$ and $\tilde{v} \in V$. We have

$$\begin{aligned}(\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j})(\cup^{\bullet k} \bullet \tilde{v} \bullet \cup^{\bullet r}) \\ = (\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j+k} \bullet \tilde{v} \bullet \cup^{\bullet r}) + (\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j+k-1} \bullet \tilde{v} \bullet \cup^{\bullet r}) + \cdots \\ + \delta^k (\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j-k} \bullet \tilde{v} \bullet \cup^{\bullet r}) + \delta^k (\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j-k-1} \bullet \tilde{v} \bullet \cup^{\bullet r}),\end{aligned}$$

for any $i, j \geqslant J$ and $k, r \leqslant J-1$. It is easy to see that $v \bullet \cup^{\bullet n} \bullet \tilde{v}$ is an element of V for any n. Hence, the product $(\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j})(\cup^{\bullet k} \bullet \tilde{v} \bullet \cup^{\bullet r})$ is in the vector space

$$\overline{\operatorname{span}}\{\cup^{\bullet l} \bullet w \bullet \cup^{\bullet r}, l \geqslant J, w \in V, r \leqslant J-1\}$$

and so is zb. A similar computation shows that bz is in the closed vector space

$$\overline{\operatorname{span}}\{\cup^{\bullet l} \bullet v \bullet \cup^{\bullet r}, l \leqslant J-1, w \in V, r \geqslant J\}.$$

Therefore, we have $zb \perp bz$. This proves (4). Hence, if we show that x is in the orthogonal of Z^{ω}_{J-1} then we would have proven that xb is orthogonal to bx. Consider $Q_J : L^2(M) \to Z_{J-1}$, the orthogonal projection of range Z_{J-1}. We remark that

$$\Theta Q_J \Theta^* = \bigoplus_{j=0}^{J-1} \big((q_{e_j} \otimes 1_V \otimes 1_{\ell^2(\mathbb{N})}) \oplus (1_{\ell^2(\mathbb{N})} \otimes 1_V \otimes q_{e_j})\big),$$

where Θ is the unitary transformation defined in Proposition 2.1 and 1_V, $1_{\ell^2(\mathbb{N})}$ are the identity operators of V and $\ell^2(\mathbb{N})$. By symmetry, it is sufficient to show that

$$\lim_{n\to\omega} \|(q_{e_j} \otimes 1_V \otimes 1_{\ell^2(\mathbb{N})})\xi_n\| = 0 \quad \text{for any } j \geqslant 0, \tag{5}$$

where $\xi_n := \Theta(x_n)$. We know that $x \in M^\omega \cap A'$. Hence by conjugation by Θ we obtain the equation

$$\lim_{n\to\omega} \|((S+S^*) \otimes 1_V \otimes 1_{\ell^2(\mathbb{N})} - 1_{\ell^2(\mathbb{N})} \otimes 1_V \otimes (S+S^*))\xi_n\| = 0. \tag{6}$$

We will show that (6) implies (5).

All the operators involved in our context act trivially on the factor V. For simplicity of the notations we stop writing the extra "$\otimes 1_V \otimes$" in the formula and denote the identity operator $1_{\ell^2(\mathbb{N})}$ by 1. Therefore, we assume that ξ_n is a vector of $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$. Equations (5) and (6) become

$$\lim_{n\to\omega} \|(q_{e_i} \otimes 1)\xi_n\| = 0 \quad \text{for any } i \geqslant 0 \tag{7}$$

and

$$\lim_{n\to\omega} \|((S+S^*) \otimes 1 - 1 \otimes (S+S^*))\xi_n\| = 0. \tag{8}$$

Consider the partial isometry $v_i \in \mathbb{B}(\ell^2(\mathbb{N}))$ such that $v_i^* v_i = q_{e_i}$ and $v_i v_i^* = q_{e_0}$. We claim that for all $i \geqslant 0$ we have

$$\lim_{n\to\omega} \|((v_i \otimes 1) - (q_{e_0} \otimes P_i(S+S^*)))\xi_n\| = 0, \tag{9}$$

where $\{P_i\}_i$ is the family of polynomials defined in (2). For all $k \geqslant 2$ we have

$$\begin{aligned}(S+S^*)^k &\otimes 1 - 1 \otimes (S+S^*)^k \\ &= ((S+S^*)\otimes 1 - 1\otimes (S+S^*)) \circ \left(\sum_{j=0}^{k-1} (S+S^*)^j \otimes (S+S^*)^{k-1-j}\right).\end{aligned}$$

Therefore, (8) implies that

$$\lim_{n\to\omega} \|(P(S+S^*)\otimes 1 - 1 \otimes P(S+S^*))\xi_n\| = 0 \quad \text{for all polynomials } P.$$

In particular,

$$\lim_{n\to\omega} \|(P_i(S+S^*)\otimes 1 - 1 \otimes P_i(S+S^*))\xi_n\| = 0 \quad \text{for all } i \geqslant 0.$$

Note that $P_i(S+S^*)(e_0) = e_i$ for all $i \geqslant 0$. Furthermore, P_i has real coefficients. Therefore, the operator $P_i(S+S^*)$ is self-adjoint. We have

$$\begin{aligned}\langle q_{e_0} \circ P_i(S+S^*)e_l, e_r\rangle &= \langle P_i(S+S^*)e_l, q_{e_0} e_r\rangle = \delta_{r,0}\langle P_i(S+S^*)e_l, e_0\rangle \\ &= \delta_{r,0}\langle e_l, P_i(S+S^*)e_0\rangle = \delta_{r,0}\delta_{l,i},\end{aligned}$$

where $i, l, r \geqslant 0$ and $\delta_{n,m}$ is the Kronecker symbol. Hence $q_{e_0} \circ P_i(S+S^*) = v_i$, for all $i \geqslant 0$. We have

$$\lim_{n\to\omega} \|(q_{e_0}\otimes 1)\circ (P_i(S+S^*)\otimes 1 - 1 \otimes P_i(S+S^*))\xi_n\| = 0.$$

Therefore, we have

$$\lim_{n\to\omega} \|(v_i \otimes 1 - q_{e_0} \otimes P_i(S+S^*))\xi_n\| = 0.$$

This proves the claim. We have

$$\lim_{n\to\omega} \|(q_{e_i} \otimes 1 - v_i^* q_{e_0} \otimes P_i(S+S^*))\xi_n\| = 0.$$

This means that

$$\lim_{n\to\omega} \|(q_{e_i}\otimes 1)\xi_n - (v_i^* \otimes P_i(S+S^*))\circ (q_{e_0}\otimes 1)\xi_n\| = 0.$$

Hence, we have

$$\begin{aligned}\lim_{n\to\omega} \|(q_{e_i}\otimes 1)\xi_n\| &\leqslant \lim_{n\to\omega} \|(v_i^* \otimes P_i(S+S^*))\circ (q_{e_0}\otimes 1)\xi_n\| \\ &\leqslant \|v_i^* \otimes P_i(S+S^*)\| \lim_{n\to\omega} \|(q_{e_0}\otimes 1)\xi_n\|.\end{aligned}$$

Therefore, to prove (7) it is sufficient to show that

$$\lim_{n\to\omega} \|(q_{e_0}\otimes 1)\xi_n\| = 0.$$

Let us fix $\varepsilon > 0$; we have to find an element of the ultrafilter $E \in \omega$ such that $\|(q_{e_0}\otimes 1)\xi_n\| < \varepsilon$ for any $n \in E$. By the triangle inequality, we have

$$\|(q_{e_0}\otimes P_i(S+S^*))\xi_n\| \leqslant \|(q_{e_0}\otimes P_i(S+S^*))\xi_n - (v_i\otimes 1)\xi_n\| + \|(v_i\otimes 1)\xi_n\|,$$

for all $i \geqslant 0$. We have $\|(v_i\otimes 1)\xi_n\| \leqslant \|\xi_n\| \leqslant 1$; thus,

$$\begin{aligned}\|(v_i\otimes 1)\xi_n\|^2 \geqslant \|(q_{e_0}\otimes P_i(S+S^*))\xi_n\|^2 - \|(q_{e_0}\otimes P_i(S+S^*))\xi_n-(v_i\otimes 1)\xi_n\|^2 \\ - 2\|(q_{e_0}\otimes P_i(S+S^*))\xi_n - (v_i\otimes 1)\xi_n\|.\end{aligned} \tag{10}$$

By Lemma 2.3, there exists an integer $I\in\mathbb{N}$ such that $\inf\limits_{t\in[-2;2]} S_I(t) > \dfrac{2}{\varepsilon}$. We have

$$\begin{aligned}\sum_{i=0}^{I} \|(q_{e_0}\otimes P_i(S+S^*))\xi_n\|^2 &= \sum_{i=0}^{I} \|(1\otimes P_i(S+S^*))\circ(q_{e_0}\otimes 1)\xi_n\|^2 \\ &= \sum_{i=0}^{I}\int_{[-2;2]} \|P_i(t)((q_{e_0}\otimes\Psi)\xi_n)(t)\|^2\, d\nu(t) \\ &= \int_{[-2;2]} \|((q_{e_0}\otimes\Psi)\xi_n)(t)\|^2 \sum_{i=0}^{I} P_i(t)^2\, d\nu(t) \\ &\geqslant \frac{2}{\varepsilon}\|(q_{e_0}\otimes\Psi)\xi_n\|^2 = \frac{2}{\varepsilon}\|(q_{e_0}\otimes 1)\xi_n\|^2,\end{aligned} \tag{11}$$

where Ψ is the unitary transformation defined in (3).

By (9), there exists an element of the ultrafilter $E\in\omega$ such that for any $n\in E$ and $i\in\{0,\dots,I\}$ we have

$$\|((q_{e_0}\otimes P_i(S+S^*)) - (v_i\otimes 1))\xi_n\| < \tfrac{1}{4}. \tag{12}$$

By Pythagoras' theorem and the inequalities (10), (11) and (12) we have

$$\begin{aligned}1 &\geqslant \|\xi_n\|^2 = \sum_{i\geqslant 0}\|(q_{e_i}\otimes 1)\xi_n\|^2 \geqslant \sum_{i=0}^{I}\|(q_{e_i}\otimes 1)\xi_n\|^2 = \sum_{i=0}^{I}\|(v_i\otimes 1)\xi_n\|^2 \\ &\geqslant \sum_{i=0}^{I}\|(q_{e_0}\otimes P_i(S+S^*))\xi_n\|^2 - (I+1)\Big(\frac{1}{4^2} + 2\cdot\frac{1}{4}\Big) \\ &\geqslant \frac{2(I+1)}{\varepsilon}\|(q_{e_0}\otimes 1)\xi_n\| - (I+1).\end{aligned}$$

This implies

$$\|(q_{e_0}\otimes 1)\xi_n\| \leqslant \varepsilon \quad \text{for all } n\in E.$$

We have proved that
$$\lim_{n\to\omega} \|(q_{e_0}\otimes 1)\xi_n\|_2 = 0.$$
Therefore, $\lim_{n\to\omega}\|Q_J(x_n)\| = 0$ which implies that x is orthogonal to Z^{ω}_{J-1}. The equality (4) implies that $xb \perp bx$. Thus, the cup subalgebra $A \subset M$ has the AOP. By Corollary 2.2, $A \subset M$ is a singular MASA. Hence, by Theorem 1.3, the cup subalgebra is maximal amenable.

Acknowledgments

I would like to thank the Fondation Sciences Mathématiques de Paris, which provided me extra support for my stay at UC Berkeley during the spring of 2009 when the greater part of this work was done. I am happy to thank Melanie MacTavish and Vaughan Jones for making my stay in California very pleasant.

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Received October 17, 2012. Revised May 3, 2013.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 269 No. 1 May 2014