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# GENUSES OF CLUSTER QUIVERS OF FINITE MUTATION TYPE

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**In this paper, we study the distribution of the genuses of cluster quivers of finite mutation type. First, we prove that in the 11 exceptional cases, the distribution of genuses is 0 or 1. Next, we consider the relationship between the genus of an oriented surface and that of cluster quivers from this surface. It is verified that the genus of an oriented surface is an upper bound for the genuses of cluster quivers from this surface. Furthermore, for any nonnegative integer *n* and a closed oriented surface of genus *n*, we show that there always exist a set of punctures and a triangulation of this surface such that the corresponding cluster quiver from this triangulation is exactly of genus *n*.**

## 1. Introduction

Cluster quivers are a valuable notion in the theory of cluster algebras, first introduced in the famous paper [Fomin and Zelevinsky 2002]. Since then this subject has been studied extensively by many mathematicians. The original motivation was to give a combinatorial characterization of dual canonical bases in the theory of quantum groups, and for the study of total positivity for algebraic groups. Now cluster algebras are connected to various fields of mathematics such as representation theory, Poisson geometry, algebraic geometry, Lie theory, combinatorics and so on. One knows that cluster algebras are commutative algebras equipped with a distinguished set of generators, i.e., cluster variables.

Two types of cluster algebras are of special interest: those of finite type, and those of finite mutation type. The former is a special case of the latter. Cluster algebras of finite type were completely classified in [Fomin and Zelevinsky 2003], and skew-symmetric cluster algebras of finite mutation type were completely classified in [Felikson et al. 2012]. The classification of cluster algebras of finite type is identical to the Cartan–Killing classification of semisimple Lie algebras and finite root systems. For a cluster algebra of finite type, there is a one-to-one correspondence

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between the set of cluster variables and the set of almost positive roots (consisting of positive roots and negative simple roots). Additionally, the classification of skew-symmetric cluster algebras (equivalently, the classification of cluster quivers) of finite mutation type tells us that almost all skew-symmetric cluster algebras (equivalently, cluster quivers) of this type come from triangulations of surfaces, except for 11 exceptional cases.

Given an oriented 2-dimensional Riemann surface $S$ with boundary $\partial S$, let $M \subset S$ be a finite set of *marked points* such that each connected boundary component contains at least one such point. Marked points in the interior of $S$ are called *punctures*. The pair $(S, M)$ is simply called a *surface*. An *arc* [Fomin et al. 2008] is the homotopy class of a curve $\gamma$ in $S$ whose endpoints come from $M$, such that:

- $\gamma$ does not intersect itself, except that its endpoints may coincide;
- except for the endpoints, $\gamma$ is disjoint from $M$ and $\partial S$;
- $\gamma$ does not cut out an unpunctured monogon or an unpunctured digon.

An *ideal triangulation* $T$ is a maximal set of noncrossing (i.e., there are no intersections in the interior of $S$) arcs. For the details of the construction of cluster quivers from triangulations of surfaces, see Section 2B.

*In this paper, all surfaces we consider are oriented surfaces; all subgraphs and subquivers are full.*

In topological graph theory, the *genus* of a graph is the minimal genus of the surfaces where the graph can be drawn without crossings. The *genus* of a quiver is defined to be that of its underlying graph. When discussing the genus of a quiver, one only needs to consider its simple underlying graph (without multiple edges and orientation). A graph (respectively, quiver) is *planar* if it is of genus 0. It is well known that genus is a topological invariant for surfaces, as well as for topological graphs. A natural question is to find out the relation between the genus of a surface and that of a cluster quiver from this surface. As an answer, we have the main conclusion in this paper:

**Theorem 1.1.** (i) *For a triangulation $T$ of a surface $S$ with genus $g$, let $g'$ be the genus of the cluster quiver $Q$ associated with $T$. Then $g' \leq g$.*

(ii) *Furthermore, for any nonnegative integer $n$ and a closed oriented surface $S_n$ of genus $n$, there exists a set of marked points $M$ on $S_n$ and an ideal triangulation $P_n$ of $S_n$ such that the corresponding cluster quiver $T_n$ of $P_n$ has genus $n$.*

From this result, we know that the genus of a surface is in fact an upper bound for the genuses of cluster quivers from the triangulations of this surface; moreover, any nonnegative integer $n$ can be reached as the genus of some cluster quiver from surface.

The paper is organized as follows. The requisite background on cluster quivers, their mutation, and triangulations of surfaces are presented in Section 2. In Section 2A, we give the basic definitions of matrix mutation and quiver mutation. We mention the fact that skew-symmetric matrices are in bijection with cluster quivers, and also that matrix mutation and quiver mutation are compatible. In Section 2B, we recall some basic definitions and properties of triangulations of surfaces from [Fomin et al. 2008]. We recapitulate how to obtain a cluster quiver from a surface triangulation and the compatibility between quiver mutations and flips of triangulations. A cluster quiver comes from a surface if and only if it is block-decomposable. At the end of this subsection, we restate the classification of skew-symmetric cluster algebras of finite mutation type.

Section 3 mainly deals with the genuses of cluster quivers of finite mutation type. In Section 3A, we give the table of genus distribution of the 11 exceptional quivers by utilizing Keller's quiver mutation in Java [Keller 2006]. In Section 3B, we first prove Theorem 1.1(i) which states that the genus of a surface is an upper bound for the genuses of cluster quivers obtained by triangulations of this surface. From this result, one can easily see that genus is a mutation invariant for cluster quivers from the surface of genus 0. As another application of this result, we give a sufficient condition for two quivers not to be mutation equivalent. Part (ii) of Theorem 1.1 is proved by constructing a graph $R_n$, using topological graph theory for genus $n$ and the classification theorem of compact surfaces in algebraic topology.

## 2. Preliminaries

**2A.** ***Cluster quiver and its mutation.*** The notion of skew-symmetric matrix or equivalently of cluster quiver is crucial in the theory of cluster algebras. In the definition of cluster algebras, the most important ingredient is the so-called *seed mutation*. For our purpose in this paper, we only introduce *matrix mutation* (an important part of seed mutation) so as to understand the motivation of cluster quivers. For the details of the definitions of seed mutation and cluster algebras, we refer to [Fomin and Zelevinsky 2003].

Suppose $B = (b_{ij})$ is an $n \times n$ integer matrix. For $1 \le k \le n$, a *matrix mutation* $\mu_k$ at direction $k$ transforms $B$ into a new matrix $B' = (b'_{ij})$ where $b'_{ij}$ is defined by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \dfrac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases}$$

Here, all matrices we consider are *skew-symmetric*. It is easy to see that matrix mutation transforms a skew-symmetric matrix into another one.

Given an $n \times n$ skew-symmetric matrix $B = (b_{ij})$, we can construct a quiver $Q$ without loops and 2-cycles as follows: the vertex set is $\{1, 2, \dots, n\}$ (the set of row and column indices of the matrix $B$), and the number of arrows from $i$ to $j$ is defined to be $b_{ij}$ if $b_{ij} > 0$.

**Definition 2.1.** A quiver without loops and 2-cycles is said to be a *cluster quiver*.

There is a one-to-one correspondence between the set of skew-symmetric matrices and the set of cluster quivers. In fact, given a cluster quiver $Q$ with $n$ vertices, one can construct a skew-symmetric matrix $B = (b_{ij})$ defined by $b_{ij} = \#\{i \to j\} - \#\{j \to i\}$, where $\#\{i \to j\}$ denotes the number of arrows from $i$ to $j$. According to this one-to-one correspondence, *quiver mutation* can be deduced from matrix mutation.

**Definition 2.2.** Suppose $Q$ is a cluster quiver with vertex set $Q_0 = \{1, 2, \dots, n\}$. For $k \in Q_0$, a *quiver mutation* $\mu_k$ at vertex $k$ transforms $Q$ into $Q'$, where $Q'$ is obtained by the following three steps:

(1) For every path $i \to k \to j$, add a new arrow $i \to j$.

(2) Reverse all arrows incident with $k$.

(3) Delete all 2-cycles.

One can easily see that the resulting quiver $Q'$ is also a cluster quiver. Matrix mutation and quiver mutation are compatible in the following sense: given any $k \in \{1, 2, \dots, n\}$, $\mu_k(Q_B) = Q_{\mu_k(B)}$ and $\mu_k(B_Q) = B_{\mu_k(Q)}$.

It is easy to verify that both matrix mutation and quiver mutation are involutions, i.e., $\mu_k^2 = 1$. If $Q' = \mu_{k_1}\mu_{k_2}\dots\mu_{k_l}(Q)$ for some $k_1, k_2, \dots, k_l \in \{1, 2, \dots, n\}$, we will say that $Q$ and $Q'$ are *mutation equivalent*. Obviously, this is an equivalence relation on the set of isomorphism classes of cluster quivers with $n$ vertices. A cluster quiver (respectively, skew-symmetric cluster algebra constructed from this quiver) is said to be of *finite mutation type* if the number of quivers in its mutation-equivalence class is finite. Cluster quivers of this type were completely classified in [Felikson et al. 2012]. We will restate this classification theorem in Section 2B.

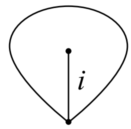
**2B.** ***Cluster quivers from surfaces.*** Given a surface $(S, M)$, the number of arcs in any triangulation of $(S, M)$ is a constant. The following lemma gives the formula to calculate the number of arcs in a triangulation.

**Lemma 2.3** [Fomin et al. 2008]. *For a triangulation of a surface, the following formula holds:*

$$n = 6g + 3b + 3p + c - 6, \tag{1}$$

*where $n$ is the number of arcs, $g$ is the genus of the surface, $b$ is the number of connected boundary components, $p$ is the number of punctures, and $c$ is the number of marked points on the boundary.*

The arcs of an ideal triangulation cut the surface $S$ into *ideal triangles*. The three sides of an ideal triangle do not have to be distinct, i.e., we allow *self-folded* triangles, like this:



Given an ideal triangulation $T$, there is an associated signed adjacency matrix $B(T)$ (see [Fomin et al. 2008, §4]). Suppose the arcs in $T$ are labeled by the numbers $1, 2, \dots, n$, and let the rows and columns of $B(T)$ be numbered from 1 to $n$. For an arc $i$, let $\pi_T(i)$ denote the arc defined as follows: if there is a self-folded ideal triangle in $T$ folded along $i$ (see figure above), then $\pi_T(i)$ is its remaining side; otherwise, we set $\pi_T(i) = i$.

For each non-self-folded triangle $\triangle$, define the $n \times n$ integer matrix $B^{\triangle} = (b_{ij}^{\triangle})$ by setting

$$b_{ij}^{\triangle} = \begin{cases} 1 & \text{if side } \pi_T(j) \text{ immediately follows } \pi_T(i) \text{ in } \triangle \text{ going clockwise;} \\ -1 & \text{if side } \pi_T(i) \text{ immediately follows } \pi_T(j) \text{ in } \triangle \text{ going clockwise;} \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $B = B(T) = (b_{ij})$ is defined by

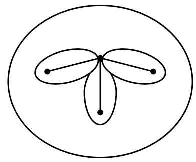
$$B = \sum_{\triangle} B^{\triangle}, \tag{2}$$

where the sum is taken over all non-self-folded triangles $\triangle$. It is easy to verify that $B(T)$ is skew-symmetric, and that all its entries are equal to 0, 1, $-1$, 2 or $-2$. Therefore, given a triangulation $T$, we can first associate a skew-symmetric matrix $B(T)$ to $T$ and then obtain a cluster quiver $Q$ corresponding to $B(T)$, just as in Section 2A. The corresponding cluster quiver $Q_B$ of $B = B(T)$ is said to *come from a surface*. Correspondingly, the cluster algebra defined by $Q_B$ is also said to *come from a surface*.

A *flip* is a transformation of an ideal triangulation $T$ into a new triangulation $T'$ obtained by replacing an arc $\gamma$ with a unique different arc $\gamma'$ and leaving other arcs unchanged. Flips of triangulation and matrix mutation are compatible in the sense of the following proposition.

**Proposition 2.4** [Fomin et al. 2008, Proposition 4.8]. *Suppose that the triangulation $\bar{T}$ is obtained from $T$ by a flip replacing an arc $k$. Then $B(\bar{T}) = \mu_k(B(T))$.*

According to [Fomin et al. 2008, Remark 4.2], all triangulations that we are interested in can be obtained by gluing together a number of puzzle pieces, except for one case: the triangulation of the 4-punctured sphere obtained by gluing three self-folded triangles to respective sides of an ordinary triangle:



There are three types of puzzle pieces:

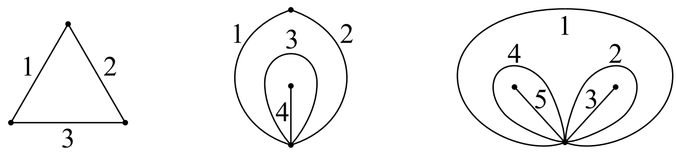
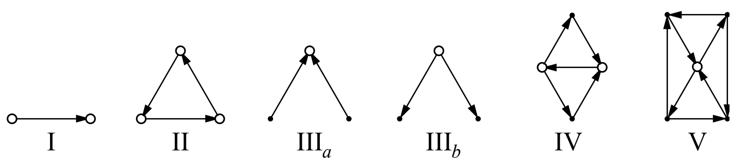



**Figure 1.** The three types of puzzle pieces.

These three types of puzzle pieces correspond to blocks of type I–V below, depending on whether the outer sides are lying on the boundary (for the details, see the proof of Theorem 13.3 in [Fomin et al. 2008]).




The vertices marked by open circles in this figure are called *outlets*.

**Definition 2.5** [Fomin et al. 2008]. A quiver is said to be *block-decomposable* if it can be obtained from a collection of disjoint blocks by the following procedure:

(1) Take a partial matching of the combined set of outlets (matching an outlet to itself or to another outlet from the same block is not allowed).

(2) Glue the outlets in each pair of the matching.

(3) Remove all 2-cycles.

According to [Fomin et al. 2008, Theorem 13.3], a cluster quiver comes from a surface if and only if it is block-decomposable.

The following theorem gives a complete classification of skew-symmetric cluster algebras of finite mutation type.

**Lemma 2.6** [Felikson et al. 2012]. *A skew-symmetric cluster algebra $\mathscr{A}$ of rank $n$ is of finite mutation type if and only if $\mathscr{A}$ comes from a surface* ($n \geq 3$), *or $n \leq 2$, or $\mathscr{A}$ is one of the* 11 *exceptional types shown in* Figure 2 (that is, $\mathscr{A}$ has a cluster quiver at one of these types).

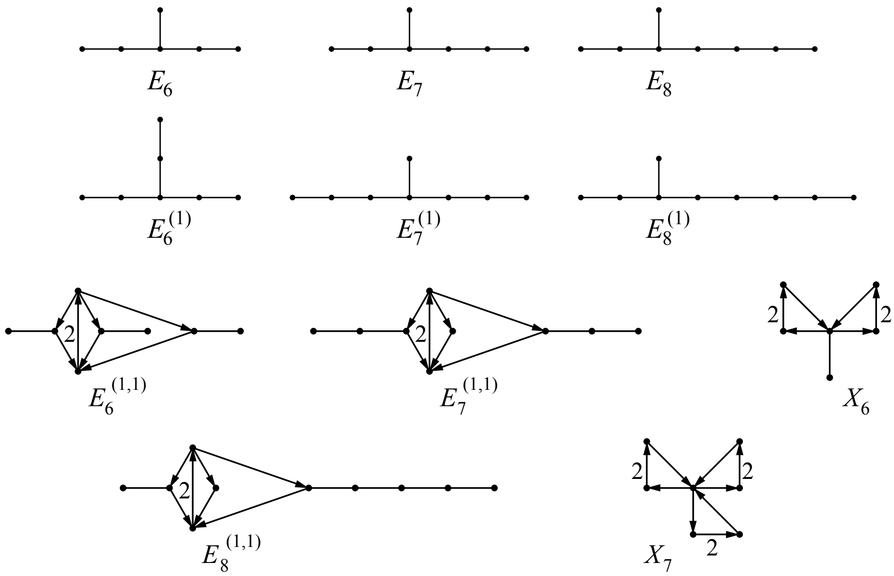



**Figure 2.** The eleven exceptional types.

## 3. Genus distribution of cluster quivers of finite mutation type

**3A.** ***Genuses of exceptional cluster quivers.*** Table 1 in this section gives the genus distribution of the 11 exceptional cluster quivers in the classification of cluster quivers of finite mutation type. Our main tool is Keller's quiver mutation in Java [Keller 2006]. To obtain the table, we note the following facts:

(1) $E_6$, $E_7$, $E_8$, $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$ are trees. According to Lemma 1.1 of [Vatne 2010], any orientations on the same tree are mutation equivalent.

(2) $E_6$, $E_7$, $E_8$ and $E_8^{(1)}$ are full subgraphs of the underlying graph of $E_8^{(1,1)}$; $E_6^{(1)}$ is a full subgraph of the underlying graph of $E_6^{(1,1)}$; $E_7^{(1)}$ is a full subgraph of the underlying graph of $E_7^{(1,1)}$. Since any quiver mutation-equivalent to a full subquiver of $Q$ must be a full subquiver of some $Q'$ that is mutation-equivalent to $Q$, we first test the mutation classes of $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ in order to see their genus distribution.

(3) To see the genus of a quiver, we only need to see its underlying graph. Hence when doing the quiver mutation in Java due to Keller [2006], we can choose the mutation class under graph isomorphism. This can greatly cut down the number of quivers in the mutation class that we have to consider.

(4) We check the quivers in the mutation classes of $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ and find they are all planar. So are the other exceptional cluster quivers of type $E$.

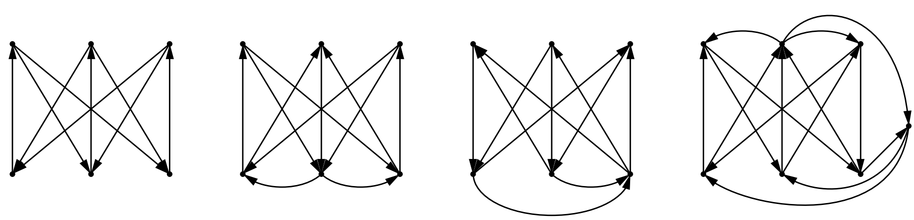
| Type | Total number | Number of genus 0 | Number of genus 1 |
|---|---|---|---|
| $E_6$ | 21 | 21 | 0 |
| $E_7$ | 112 | 112 | 0 |
| $E_8$ | 391 | 391 | 0 |
| $E_6^{(1)}$ | 52 | 52 | 0 |
| $E_7^{(1)}$ | 338 | 338 | 0 |
| $E_8^{(1)}$ | 1935 | 1935 | 0 |
| $E_6^{(1,1)}$ | 27 | 27 | 0 |
| $E_7^{(1,1)}$ | 217 | 217 | 0 |
| $E_8^{(1,1)}$ | 1886 | 1886 | 0 |
| $X_6$ | 4 | 1 | 3 |
| $X_7$ | 2 | 1 | 1 |

**Table 1.** Statistics on exceptional cluster quivers of different types.

In Table 1, the *total number* means the number of quivers in the mutation class up to quiver isomorphism, and the *number of genus* 0 (respectively, 1) means the number of quivers (up to quiver isomorphism) in the mutation class whose genus is 0 (respectively, 1).

From the table, one can easily see that the genus of the quiver of type $E$ is invariant under quiver mutation, but the genus of the quiver of type $X$ will vary under quiver mutation.

**Proposition 3.1.** *There are exactly four nonplanar cluster quivers of exceptional finite mutation types that have genus* 1:

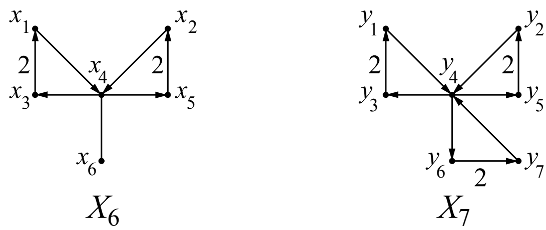


(1) (2) (3) (4)

*Quivers* (1), (2), *and* (3) *are in the mutation-equivalence class of* $X_6$, *and quiver* (4) *is in the mutation-equivalence class of* $X_7$.

*Proof.*

- Quiver (1) is obtained from $X_6$ by mutation on the vertices $x_4$ and $x_6$, the vertex labeling being as shown on the top of the next page.
- Quiver (2) is obtained from $X_6$ by mutation on the vertex $x_4$.
- Quiver (3) is obtained from $X_6$ by mutation on the vertices $x_4$ and $x_3$.

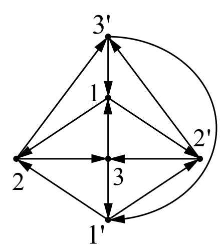



- Quiver (4) is obtained from $X_7$ by mutation on the vertex $y_4$. □

**3B.** ***Proof of the main conclusion.*** We will begin by proving the first part of the theorem, i.e., that the genuses of cluster quivers obtained from the triangulations of a surface are not greater than that of the surface.

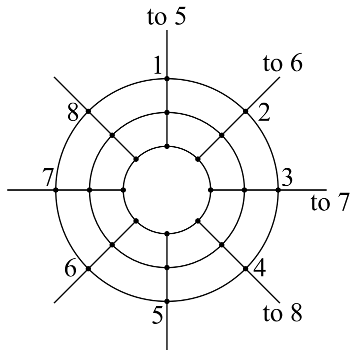
*Proof of Theorem 1.1*(i). By the correspondence of puzzle pieces and blocks, each puzzle piece corresponds to a block of type I–V. For each puzzle piece, we put its corresponding block into the face bounded by it. If two puzzle pieces have a common edge, then we glue the two vertices corresponding to the common edge between these two blocks. Hence we obtain the quiver $Q$ of $T$ in this way, and moreover the underlying graph of $Q$ can be drawn without self-crossings on the surface $S$. We then have $g' \le g$ by definition of the genus of a quiver.

To complete the proof of the theorem, we should consider the only exceptional case the triangulation of which cannot be obtained by gluing the puzzle pieces. Let $T$ be the triangulation of the 4-punctured sphere obtained by gluing three self-folded triangles to respective sides of an ordinary triangle. The corresponding cluster quiver of $T$ can be obtained by gluing four blocks of type II, as follows:



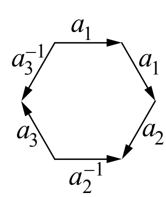

In this figure, for $i = 1, 2$ and 3, $i$ and $i'$ denote the corresponding vertices of two arcs in the same self-folded triangles. Obviously it is a planar quiver, and hence in this case $g' = g = 0$. This completes the proof. □

To prove Theorem 1.1(ii), we need some preliminaries. First, we borrow from [Gross and Tucker 1987, Example 3.4.2] a class of graphs with arbitrary large genus. For each positive integer $n$, the graph $R_n$ is constructed by taking $n + 1$ concentric cycles consisting of $4n$ edges each, together with $4n^2$ inner edges connecting the $n + 1$ cycles to each other and $2n$ outer edges adjoining antipodal vertices on the outermost cycle. Here is the graph $R_2$:




It was shown in [Gross and Tucker 1987] that $R_n$ is of genus $n$.

Secondly, recall that the classification theorem for compact (or closed) surfaces (see, for example, [Massey 1977, Chapter 1, Theorem 5.1]) asserts that any compact surface is homeomorphic to a sphere, a connected sum of tori, or a connected sum of projective planes. Any compact surface can be considered as the quotient space of a polygon with directed edges identified in pairs. There is a convenient way to indicate which paired edges are to be identified in such a polygon. We give a letter (for example, $a, b, c, \dots$) to each pair of edges, different pairs receiving different letters. Starting at a definite vertex, we traverse the boundary of the polygon either clockwise or counterclockwise. If the arrow on an edge points in the same traversing direction, we put no exponent (or the exponent $+1$) on the letter for that edge; otherwise, we write the letter for that edge with the exponent $-1$. For example, the string $a_1a_1a_2a_2^{-1}a_3a_3^{-1}$ indicates the same identifications as this figure:




The various surfaces can then be described by the following strings (see [Massey 1977, §5]):

(1) The sphere: $aa^{-1}$.

(2) The connected sum of $n$ tori: $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_nb_na_n^{-1}b_n^{-1}$.

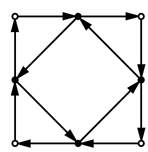
(3) The connected sum of $n$ projective planes: $a_1a_1a_2a_2\dots a_na_n$.

Given a polygon, if the letter designating a certain pair of edges occurs with both exponents $+1$ and $-1$ in the symbol, then this pair of edges is said to be of the *first kind*; otherwise the pair is said to be of the *second kind*. From the proof of Theorem 5.1 in [Massey 1977], we know that if all the pairs of edges are of the first kind, then the resulting surface is oriented; if there exists a pair of edges of the second kind, then the resulting surface is nonoriented. Moreover, since the pair of adjacent edges of the first kind can be eliminated, the resulting surface of a $4n$-gon with pairs all of the first kind is an oriented surface with genus at most $n$.

To prepare for the proof of Theorem 1.1(ii), we first prove a lemma.

**Lemma 3.2.** *For an arbitrary nonnegative integer $n$, there always exists a block-decomposable cluster quiver $T_n$ such that the genus $g(T_n)$ of $T_n$ satisfies $g(T_n) \geq n$.*

*Proof.* Given a graph $R_n$ as above, label the $n+1$ cycles from innermost to outermost by 1 to $n+1$. For each $i \in \{1, 2, \dots, n\}$, there are $4n$ rectangles between the $i$-th cycle and the $(i+1)$-st cycle. For the outermost cycle, there exist $2n$ rectangles between the $(n+1)$-st cycle and itself. Two rectangles are said to be *neighbors* if they share a common edge; otherwise, they are said to be *distant*. It is easy to observe that there are $4n^2 + 2n$ rectangles in $R_n$. Given any rectangle $A$ in $R_n$, we first choose four rectangles distant from $A$ but having a common vertex with $A$. We repeat this process for each of these four rectangles; continuing this process, we will obtain a maximal set of mutually distant rectangles. This is denoted by $\mathscr{S}$. This set contains $2n^2 + n$ rectangles. The other $2n^2 + n$ rectangles form another maximal set of mutually distant rectangles. This is denoted by $\mathscr{T}$. Trivially, the two sets $\mathscr{S}$ and $\mathscr{T}$ are independent of the choice of the original rectangle $A$. Consider the set $\mathscr{S}$: each rectangle in $\mathscr{S}$ can be obtained by gluing four blocks of type II as shown on the right.



For the innermost cycle, there are $2n$ edges which do not lie in any rectangles of $\mathscr{S}$. We can then substitute one block of type IV for each such edge. For all these $2n$ edges, we need $2n$ blocks of type IV.

In summary, we obtain a quiver $T_n$ by gluing $8n^2 + 4n$ blocks of type II and $2n$ blocks of type IV. According to the construction of $T_n$, obviously, $R_n$ is a subgraph of the underlying graph of $T_n$. Therefore, the genus $g(T_n)$ is at least $g(R_n) = n$. Figure 3 on the next page illustrates the case $n = 2$. □

*Proof of* Theorem 1.1(ii). We will use the fact that the quiver $T_n$ given in the proof of Lemma 3.2 can be obtained from a closed surface of genus $n$. By Lemma 3.2, $g(T_n) \geq n$. It is easy to check that $T_n$ is a uniquely block-decomposable quiver and hence $T_n$ can be uniquely encoded by its corresponding triangulation, that is, blocks of type II are encoded by puzzle pieces of the first type (see the left graph in Figure 1) and blocks of type IV are encoded by puzzle pieces of the second type (see the middle graph in Figure 1). In order to draw $T_n$, we first draw a planar quiver $T'_n$ which has $4n$ unglued outlets. After gluing these $4n$ outlets in pairs, one obtains $T_n$, where each pair consists of one outlet and its opposite one. See Figure 3 for an

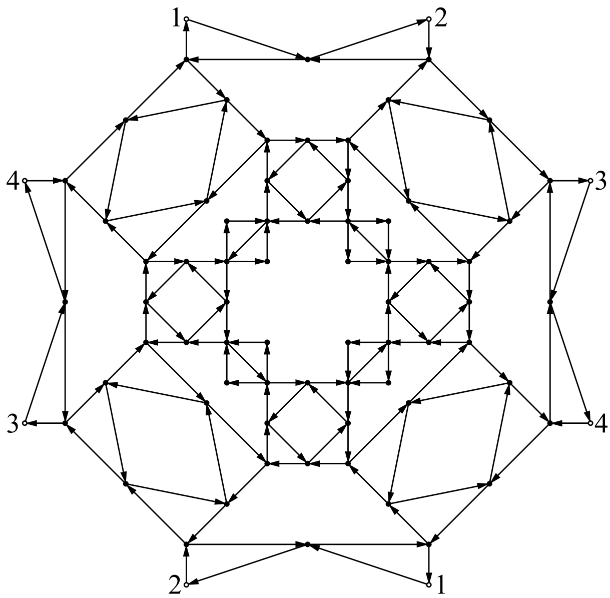
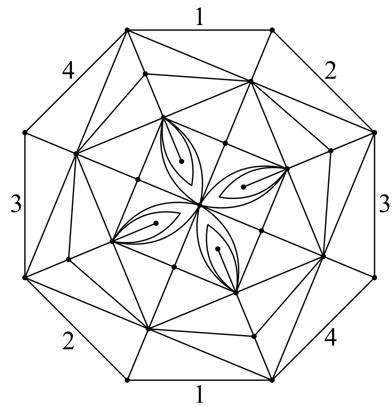



**Figure 3.** Quiver corresponding to the graph $T_2$. Vertices labeled by the same numbers should be glued together.

illustration of the case $n = 2$. Now we will construct a closed surface $S_n$ of genus $n$ and a triangulation $P_n$ of $S_n$ such that the corresponding cluster quiver is $T_n$.

We will chase $T_n$ from innermost to outermost. Blocks of type II and type IV are encoded by puzzle pieces of the first and second types, respectively. For the outermost $4n$ oriented triangles in $T_n'$, we let each of them correspond to a puzzle piece of the first type. Thus we obtain a $4n$-gon with a triangulation. Denote this $4n$-gon with triangulation by $S_n'$. Then we can obtain a closed oriented surface $S_n$ by identifying the edges of $S_n'$ in pairs and gluing all outermost vertices into one, and then obtain a triangulation $P_n$ of $S_n$ such that its corresponding quiver is exactly $T_n$. For the case $T_2$, its corresponding $S_2'$ is given on the right.




To obtain $S_2$ and $P_2$, one only needs to glue the edges labeled by the same number in pairs and to glue all 8 outermost vertices into one.

By the proof of the classification theorem of compact surfaces in [Massey 1977], the genus of $S_n$ is at most $n$.

Since $T_n$ is obtained from a triangulation of $S_n$, by Theorem 1.1(i), $g(T_n) \leq n$.

On the other hand, by Lemma 3.2, $g(T_n) \geq n$. Hence, $g(T_n) = n$.

For the genus $g(S_n)$ of $S_n$, since $n = g(T_n) \leq g(S_n) \leq n$, we also have $g(S_n) = n$.

Then Theorem 1.1(ii) easily follows from the fact that all closed oriented surfaces with the same genus are homeomorphic. □

**3C.** ***Applications and further problems.*** As an application of Theorem 1.1(i), we give two corollaries.

**Corollary 3.3.** *Let $S$ be a surface of genus* 0 *and $M$ a set of marked points of $S$. Given any triangulation $T$ of $(S, M)$, suppose $Q$ is the associated cluster quiver. Then all quivers in the mutation-equivalence class of $Q$ are of genus* 0*.*

Besides the cluster quivers of type E in Section 3A, this corollary gives another class of cluster quivers of finite mutation type whose genuses are invariant under mutation.

**Corollary 3.4.** *Let $S$ be a surface of genus $g$, with $M$ its set of marked points. For any triangulation $T$ of $(S, M)$, let $Q$ be its corresponding quiver and let $Q'$ be another cluster quiver of genus $g'$ such that $g' > g$. Then $Q$ and $Q'$ are not mutation equivalent.*

*Proof.* According to Proposition 12.3 in [Fomin et al. 2008], all quivers in the mutation-equivalence class of $Q$ are the corresponding quivers of some triangulations of $(S, M)$. Hence, by Theorem 1.1(i), the genuses of these quivers are not greater than $g$. Hence $Q'$ is not in the mutation-equivalence class of $Q$, that is, $Q$ and $Q'$ are not mutation equivalent. □

This corollary gives us a necessary condition for two quivers with the same number of vertices, one coming from a triangulation of a surface and the other nonplanar, to be mutation equivalent.

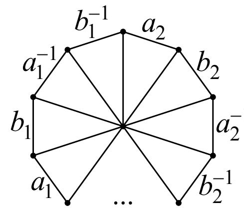
**Remark 3.5.** An easy calculation shows that the number of marked points on the closed surface $S_n$ in the proof of Theorem 1.1(ii) is $4n^2 + 2n + 2$. For example, in the case $n = 2$, one can easily see that there are 22 marked points on $S_2$; here the outermost 8 marked points in $S_2'$ (see figure at the bottom of page 144) are glued into one.

Theorem 1.1(ii) tells us that, given a closed surface $S$ of genus $n$, the upper bound of genuses of quivers from triangulations of $S$ given in part (i) of Theorem 1.1 can be reached.

On the other hand, the lower bound 0 of genuses can also be reached; that is, given any closed oriented surface $S$ with genus $n$, there always exists a triangulation $T$ of $S$ such that the corresponding cluster quiver $Q$ of $T$ is planar.

In fact, if the closed surface is a sphere, this obviously holds by Corollary 3.3; whereas if the closed surface $S$ is of genus $n \geq 1$, it is homeomorphic to the connected sum of $n$ tori. In this case, the symbol of the corresponding polygon is

$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_nb_na_n^{-1}b_n^{-1}$. A triangulation $T$ of $S$ with two punctures is shown on the right. For this triangulation, the outer $4n$ vertices in fact come from the same puncture and the only inner vertex is the other puncture. One can easily check that the corresponding cluster quiver $Q$ of $T$ is planar.




Restricting the discussion to the torus, we reach the following conclusion:

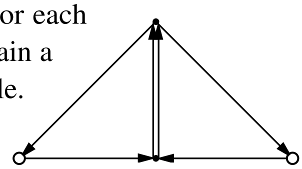
**Proposition 3.6.** *For a given cluster quiver $Q$ from the torus $S$ with $p$ punctures, there exists at least one planar quiver in the mutation-equivalence class of $Q$.*

*Proof.* According to Proposition 12.3 in [Fomin et al. 2008], the corresponding quivers from all triangulations of $S$ are mutually mutation equivalent. Hence, we only need to find a triangulation $T$ of $S$ such that its corresponding quiver is planar.

For the convenience of describing the desired triangulation, we first restate how a torus is constructed. Given two circles $C$ and $C'$, assume the radius of $C$ is greater than that of $C'$. Let the center of $C'$ run along $C$ for one round; then a torus is built. The circle $C$ is called a *basic circle* for this torus.

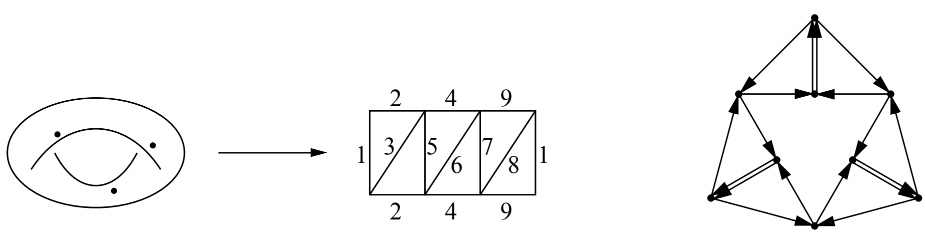
For the torus $S$ with $p$ punctures, we construct a triangulation $T$ as follows:

For each puncture, construct a closed arc on $S$ perpendicular to the basic circle such that its two endpoints coincide at the puncture; we have $p$ such arcs. These $p$ arcs cut down the torus into $p$ pieces of cylinders. For each cylinder, drawing an arc between two punctures, we obtain a rectangle. Moreover, we draw a diagonal in this rectangle. The corresponding quiver from such a rectangle with its diagonal is shown on the right.



All $p$ such rectangles with diagonal are arranged continuously together to form a graph. The quiver $Q$ of $T$ is obtained by gluing $p$ pieces of such quivers along the outlets. Obviously, it is a planar quiver. $\square$

For example, in the case $p = 3$, the triangulation and the corresponding cluster quiver are as follows, where the numbers $1, \dots, 9$ label the arcs:



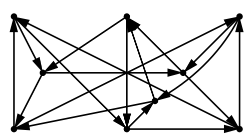

Since both the upper and lower bounds for genuses of cluster quivers from closed surfaces can be attained, based on Theorem 1.1 and Proposition 3.6 we propose these further interesting problems:

**Problem 3.7.** For any closed surface $S$ with genus $n$ and $0 \leq i \leq n$, does there exist a certain number of punctures and an ideal triangulation $T^{(i)}$ of $S$ such that the corresponding cluster quiver $Q_i$ from $T^{(i)}$ is of genus $i$?

**Problem 3.8.** Given a closed surface $S$ of genus $n$, find the minimal number of punctures on $S$ with the property that there exists an ideal triangulation $T$ of $S$ such that the corresponding cluster quiver $Q_n$ of $T$ is of genus exactly $n$.

For the case of the torus, we know at least one planar quiver in each mutation-equivalence class according to Proposition 3.6. Hence, for a given number of punctures we can check the corresponding mutation-equivalence class of this planar quiver by Keller's quiver mutation in Java [Keller 2006]. Since the genus of a quiver has nothing to do with the orientations of the arrows, we can choose the mutation-equivalence class under graph isomorphism when doing quiver mutation in Java.

For the cases $p = 1$ and $p = 2$, all quivers in their two mutation-equivalence classes are planar. When $p = 3$, there exists exactly one quiver of genus 1 in the mutation class:



Therefore, the answer to Problem 3.8 for the case of the torus is $p = 3$, which is much smaller than the number $4 \times 1^2 + 2 \times 1 + 2 = 8$ of punctures given in Remark 3.5 when constructing $T_1$ from the torus.

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