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TAUT FOLIATIONS IN KNOT COMPLEMENTS

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We show that for any nontrivial knot in S^3, there is an open interval containing zero such that a Dehn surgery on any slope in this interval yields a 3-manifold with taut foliations. This generalizes a theorem of Gabai on zero frame surgery.

1. Introduction

A transversely orientable codimension-one foliation $\mathscr{F}$ of a 3-manifold M is called *taut* [Gabai 1991] if every leaf of $\mathscr{F}$ intersects some closed transverse curve. The existence of a taut foliation in a 3-manifold M provides much interesting topological information about both M and objects embedded in M. If a closed 3-manifold M contains a taut foliation, either M is finitely covered by $S^2 \times S^1$ or M is irreducible [Novikov 1965; Reeb 1952; Rosenberg 1968]. If a closed 3-manifold M contains a taut foliation, then its fundamental group is infinite [Haefliger 1962; Novikov 1965; Gabai and Oertel 1989] and acts nontrivially on interesting 1-dimensional objects (see, for example, [Thurston 1998; Calegari and Dunfield 2003; Palmeira 1978; Roberts et al. 2003]), and its universal cover is $\mathbb{R}^3$ [Palmeira 1978]. Taut foliations can be perturbed to interesting contact structures [Eliashberg and Thurston 1998; Kazez and Roberts 2014] and hence can be used to obtain Heegaard–Floer information [Ozsváth and Szabó 2004b]. In this paper we seek to add to the understanding of the existence of taut foliations by describing a new construction of taut foliations.

Let k be a nontrivial knot in S^3. In his proof of the Property R conjecture, Gabai [1987b] showed that the knot exterior $M = S^3 \setminus \operatorname{int} N(k)$ has a taut foliation whose restriction to the torus ∂M is a collection of circles of slope 0. Thus a zero frame Dehn surgery on k yields a closed 3-manifold that admits a taut foliation obtained by adding disks along the boundary circles of the taut foliation of M. In this paper, we extend Gabai's theorem from zero frame surgery to any slope in an interval that contains 0. Although we restrict attention to knots in S^3, the approach described in this paper applies more generally to manifolds $(M, \partial M)$ with boundary a nonempty

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union of tori and for which there exists a well-groomed sutured manifold hierarchy which meets each component of ∂M only in essential simple closed curves.

Theorem 1.1. *Let k be a nontrivial knot in* S^3*. Then there is an interval* $(-a, b)$*, where* $a > 0$ *and* $b > 0$*, such that for any slope* $s \in (-a, b)$*, the knot exterior* $M = S^3 \setminus \operatorname{int}(N(k))$ *has a taut foliation whose restriction to the torus* ∂M *is a collection of circles of slope s. Moreover, by attaching disks along the boundary circles, the foliation can be extended to a taut foliation in* $M(s)$*, where* $M(s)$ *is the manifold obtained by performing Dehn surgery to k with surgery slope s.*

A group G is called *left-orderable* if there is a total order on G which is invariant under left multiplication. We thank Liam Watson for calling our attention to the following results.

Corollary 1.2. *Let k be a hyperbolic knot in* S^3 *and let* $M(1/n)$ *denote the manifold obtained by* $1/n$ *Dehn filling along k. Then there is some number* $N = N(k)$ *such that* $\pi_1(M(1/n))$ *is left-orderable whenever* $|n| > N$*.*

Proof. The surgered manifold $M(1/n)$ is a homology S^3 and, by Thurston's hyperbolic Dehn surgery theorem [Thurston 1982], atoroidal when $|n|$ is sufficiently large (or, equivalently, when $1/n$ is sufficiently small). Moreover, by Theorem 1.1, $M(1/n)$ contains a transversely oriented taut foliation whenever $1/n$ is sufficiently close to 0. It therefore follows from [Calegari and Dunfield 2003, Corollary 7.6] that $\pi_1(M(1/n))$ is left-orderable. □

Ozsváth and Szabó [2004c; 2004d] defined the Heegaard–Floer homology group $\widehat{HF}(Y)$ of a 3-manifold Y. In [Ozsváth and Szabó 2005], they define L-spaces as follows.

Definition 1.3 [Ozsváth and Szabó 2005, Definition 1.1]. A closed three-manifold is called an *L-space* if $H_1(Y; \mathbb{Q}) = 0$ and $\widehat{HF}(Y)$ is a free abelian group of rank $|H_1(Y; \mathbb{Z})|$.

L-spaces are therefore the closed 3-manifolds with the simplest possible Heegaard–Floer homology groups and the following is an important open question:

Question 1.4 [Ozsváth and Szabó 2004a, Question 11]. Is there a topological characterization of L-spaces (i.e., one that makes no reference to Floer homology)?

Ozsváth and Szabó proposed the following partial answer to this question:

Conjecture 1.5 [Hedden and Levine 2012, Conjecture 1]. If Y is an irreducible homology sphere that is an L-space, then Y is homeomorphic to either S^3 or the Poincaré homology sphere.

Approaches to understanding L-spaces have included investigations into the following two questions. Are L-spaces exactly those irreducible rational homology

3-spheres which contain no transversely oriented taut foliation? Are L-spaces exactly those irreducible rational homology 3-spheres which have non-left-orderable fundamental groups? (See [Boyer et al. 2012] for a nice survey.)

Conjecture 1.6 [Boyer et al. 2012, Conjecture 1]. An irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable.

With Conjecture 1.6 in mind, we compare Corollary 1.2 with the following result, which appears in various contexts [Ozsváth and Szabó 2004b, Corollary 1.3; Ghiggini 2008, Corollary 1.5], but is stated most conveniently as [Hedden and Watson 2010, Proposition 5].

Proposition 1.7 [Ozsváth and Szabó 2004b; Hedden and Watson 2010]. *Suppose k is a nontrivial knot in S^3 and let $M(1/n)$ denote the manifold obtained by $1/n$ Dehn filling along k. If $M(1/n)$ is an L-space, then either $n = 1$ and k is the right-handed trefoil or $n = -1$ and k is the left-handed trefoil.*

It follows that Conjecture 1.5 holds for 3-manifolds obtained by surgery on knots in S^3. And it follows from Corollary 1.2 and Proposition 1.7 that Conjecture 1.6 holds for 3-manifolds obtained by $1/n$ surgery on the complement of hyperbolic knots when $|n|$ is sufficiently large.

In Theorem 1.1, the interval $(-a, b)$ depends both on the knot k and on the sutured manifold decomposition in [Gabai 1987b]. In [Roberts 2001a; 2001b], it is shown that if k is a fibered hyperbolic knot (not necessarily in S^3), then this interval can always be chosen to contain $(-1, \infty)$, $(-\infty, 1)$, or $(-\infty, \infty)$. Related results appear in [Dasbach and Li 2004; Delman and Roberts 1999; Roberts 1995]. Moreover, the values of a and b in a maximal such interval $(-a, b)$ reveal information about the pseudo-Anosov monodromy and hence the geometry of M.

Question 1.8. Let k be a nontrivial knot in S^3, and let $a > 0$ and $b > 0$. What is the maximal interval $(-a, b)$ such that for any slope $s \in (-a, b)$, the knot exterior $M = S^3 \setminus \text{int}(N(k))$ has a taut foliation whose restriction to the torus ∂M is a collection of circles of slope s, and the foliation can be extended to a taut foliation in $M(s)$ by attaching disks along the boundary circles, where $M(s)$ is the manifold obtained by performing Dehn surgery to k with surgery slope s?

Conjecture 1.9. Such a maximal interval will always contain $(-1, 1)$.

The proof of the main theorem uses theorems in [Li 2002; 2003] on branched surfaces to generalize the approach of [Roberts 2001a] to nonfibered knots. We first use Gabai's [1983; 1987a; 1987b] sutured manifold decomposition to construct a branched surface B. Then, after first splitting B as necessary, we add in some product disks to get a new branched surface that carries more laminations which extend to taut foliations. The key point in the construction is to add branch sectors so that the new branched surface does not contain any sink disk. By [Li 2002; 2003], this means that the branched surface carries a lamination.

2. Laminar branched surfaces

Definition 2.1. A *branched surface B* in M is a union of finitely many compact smooth surfaces, glued together to form a compact subspace (of M) locally modeled on Figure 1, left (ignore the arrows in the picture for now).

Given a branched surface B embedded in a 3-manifold M, we denote by $N(B)$ a regular neighborhood of B, as shown in Figure 1, right. One can regard $N(B)$ as an interval bundle over B. We denote by $\pi : N(B) \to B$ the projection that collapses every interval fiber to a point. As shown in Figure 1, right, the boundary of $N(B)$ consists of two parts: the horizontal boundary $\partial_h N(B)$ which is transverse to the I-fibers of $N(B)$, and the vertical boundary $\partial_v N(B)$ which is the union of subarcs of the I-fibers. The *branch locus* of B is $L = \{b \in B : b$ does not have a neighborhood in B homeomorphic to $\mathbb{R}^2\}$. We call the closure (under the path metric) of each component of $B \backslash L$ a *branch sector* of B. L is a collection of smooth immersed curves in B. Let Z be the union of double points of L. We associate with every component of $L \backslash Z$ a normal vector (in B) pointing in the direction of the cusp, as shown in Figure 1, left. We call it the *branch direction* of this arc. Let D be a disk branch sector of B. We call D a *sink disk* if the branch direction of every smooth arc in its boundary points into the disk and $D \cap \partial M = \varnothing$. We call D a *half sink disk* if $\partial D \cap \partial M \neq \varnothing$ and the branch direction of each arc in $\partial D \backslash \partial M$ points into D. Note that $\partial D \cap \partial M$ might not be connected.

Laminar branched surfaces were introduced in [Li 2002] as a branched surface with the usual properties in [Gabai and Oertel 1989] plus a condition that there is no sink disk. The notion of laminar branched surface was slightly extended to

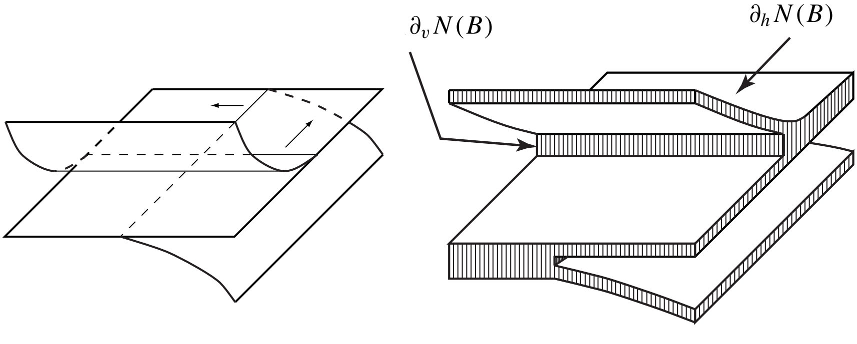

Figure 1. Left: a branched surface B. Right: a regular neighborhood $N(B)$.

branched surfaces with boundary, by adding a requirement that there is no half sink disk [Li 2003]. Note that if a branched surface has no half sink disk, then one can arbitrarily split the branched surface near its boundary train track without creating any sink disk. This plus Theorem 1 of [Li 2002] implies the following theorem from [Li 2003]. Note that the condition that there is no sink disk basically guarantees that the branched surface carries a lamination and the other conditions in [Gabai and Oertel 1989] imply that the lamination is an essential lamination.

Theorem 2.2 [Li 2003, Theorem 2.2]. *Let M be an irreducible and orientable 3-manifold whose boundary is an incompressible torus. Suppose B is a laminar branched surface and $\partial M \backslash \partial B$ is a union of bigons. Then, for any rational slope $s \in \mathbb{Q} \cup \infty$ that can be realized by the train track ∂B, if B does not carry a torus that bounds a solid torus in $M(s)$, then B fully carries a lamination $\mathcal{L}$ whose boundary consists of loops of slope s and $\mathcal{L}$ can be extended to an essential lamination in $M(s)$.*

3. Sutured manifold decompositions

Gabai [1983] introduced the notions of sutured manifold and sutured manifold decomposition. We will state basic definitions and theorems as needed for this paper but we refer the reader to [Gabai 1983; 1987a; 1987b] for a more detailed description. The papers [Altman 2012; Cantwell and Conlon 2012; Juhász 2008] and book [Candel and Conlon 2003] also provide nice descriptions of some of Gabai's sutured manifold theory. In this paper, we will use branched surfaces to describe sutured manifolds and sutured manifold decompositions.

Definition 3.1 [Gabai 1983, Definition 2.6]. A *sutured manifold* (M, γ) is a compact oriented 3-manifold M together with a set $\gamma \subset \partial M$ of pairwise disjoint annuli $A(\gamma)$ and tori $T(\gamma)$. Furthermore, the interior of each component of $A(\gamma)$ contains a *suture*, that is, a homologically nontrivial oriented simple closed curve. We denote the set of sutures by $s(\gamma)$.

Finally, every component of $R(\gamma) = \partial M \backslash \operatorname{int}(\gamma)$ is oriented. Define $R_+(\gamma)$ (or $R_-(\gamma)$) to be those components of $\partial M \backslash \operatorname{int}(\gamma)$ whose normal vectors point out of (into) M. The orientations on $R(\gamma)$ must be coherent with respect to $s(\gamma)$; that is, if δ is a component of $\partial R(\gamma)$ and is given the boundary orientation, then δ must represent the same homology class in $H_1(\gamma)$ as some suture.

Roughly speaking, a sutured manifold is a 3-manifold together with extra information about ∂M. Given a sufficiently nice surface S properly embedded in a sutured manifold (M, γ), it is important to be able to cut M open along S while keeping track of corresponding boundary information. This is captured in the following definition.

Definition 3.2 [Gabai 1983, Definition 3.1]. Let (M, γ) be a sutured manifold and S a properly embedded surface in M such that every component λ of $S \cap \gamma$ satisfies one of these three conditions:

(1) λ is a properly embedded nonseparating arc in γ.

(2) λ is a simple closed curve in an annular component A of γ in the same homology class as $A \cap s(\gamma)$.

(3) λ is a homotopically nontrivial curve in a toral component T of γ, and if δ is another component of $T \cap S$, then λ and δ represent the same homology class in $H_1(T)$.

The surface S defines a *sutured manifold decomposition*

$$(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma'),$$

where $M' = M \backslash \operatorname{int}(N(S))$ and

$$\begin{aligned} \gamma' &= (\gamma \cap M') \cup N(S'_+ \cap R_-(\gamma)) \cup N(S'_- \cap R_+(\gamma)), \\ R'_+(\gamma') &= \big((R_+(\gamma) \cap M') \cup S'_+\big) \backslash \operatorname{int}(\gamma'), \\ R'_-(\gamma') &= \big((R_-(\gamma) \cap M') \cup S'_-\big) \backslash \operatorname{int}(\gamma'), \end{aligned}$$

where S'_+ and S'_- are those components of $\partial N(S) \cap M'$ whose normal vectors point out of and into M', respectively.

Definition 3.3 [Gabai 1987a, Definition 0.2]. A sutured manifold decomposition

$$(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$$

is called *well-groomed* if for each component V of $R(\gamma)$, $S \cap V$ is a union of parallel, coherently oriented, nonseparating closed curves and arcs.

Definition 3.4 [Gabai 1987b, Definition 3.2]. Let

$$(M, \partial M) \overset{S_1}{\rightsquigarrow} (M_1, \gamma_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n, \gamma_n)$$

be a sequence of sutured manifold decompositions where ∂M is a nonempty union of tori. Define $E_0 = \partial M$. Define E_i to be the union of those components of $E_{i-1} \backslash \operatorname{int}(N(S_i))$ which are annuli and tori (i.e., if M_i is viewed as a submanifold of M, then E_i consists of those components of γ_i which are contained in ∂M). The components of E_i are called the *boundary sutures* of γ_i.

Definition 3.5. Let (M, γ) and (N, τ) be sutured manifolds. We will call (M, γ) a *sutured submanifold* of (N, τ), and write $(M, \gamma) \subset (N, \tau)$, if M is a union of components of N and $\gamma = \tau \cap M$.

If $(M, \gamma) \subset (N, \tau)$, then we write $(N, \tau) \backslash (M, \gamma)$ to denote the sutured manifold $(N \backslash M, \tau \backslash \gamma)$.

Theorem 3.6 [Gabai 1987b, Lemmas 3.6 and 5.1]. *Let k be a knot in S^3. There is a well-groomed sutured manifold sequence*

$$(M,\gamma) \overset{S_1}{\rightsquigarrow} (M_1,\gamma_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n,\gamma_n) = (S\times I, \partial S\times I)$$

of

$$(M,\gamma) = (S^3\backslash \operatorname{int}(N(k)), \partial N(k))$$

such that $\partial S_i \cap \partial N(k)$ is a (*possibly empty*) *union of circles for each i, $1 \le i \le n$, S_1 is a minimal genus Seifert surface, and S is a compact* (*not necessarily connected*) *oriented surface.*

Sutured manifold decompositions determine branched surfaces. As described by Gabai in [1987b, Construction 4.6] (and detailed further in [Cantwell and Conlon 2012]), a sutured manifold decomposition sequence corresponds to building a (finite depth) branched surface, starting with S_1 and successively adding the S_i's. To see this, inductively construct a sequence of transversely oriented branched surfaces. Let $B_1 = S_1$. So we may view M_1 as $M\backslash\operatorname{int}(N(B_1))$, where $N(B_1)$ is a fibered neighborhood of B_1. As a sutured manifold (M_1,γ_1), its suture γ_1 is the annulus $\overline{\partial M\backslash N(B_1)}$ and the two components of $\partial_h N(B_1)$ are the plus and minus boundaries $R_+(\gamma_1)$ and $R_-(\gamma_1)$ of the sutured manifold. We may view $R_+(\gamma_1)$ and $R_-(\gamma_1)$ as lying on the plus and minus sides of S_1 respectively and we assign a normal direction for $B_1 = S_1$ pointing from the plus side to the minus side.

Suppose we have constructed a branched surface B_k using the surfaces $S_1,\dots,S_k$ in the sutured manifold decomposition, such that $M\backslash\operatorname{int}(N(B_k)) = M_k$ and the suture γ_k of (M_k,γ_k) consists of $\partial_v N(B_k)$ and a collection of annuli in the boundary torus ∂M. Now we consider the sutured manifold decomposition

$$(M_k,\gamma_k) \overset{S_{k+1}}{\rightsquigarrow} (M_{k+1},\gamma_{k+1}).$$

The surface S_{k+1} has a normal vector. Then we can deform $B_k \cup S_{k+1}$ into a branched surface B_{k+1} as follows:

(1) For each component of ∂S_{k+1} that is not totally inside $\partial_v N(B_k)$, we can deform $B_k\cup S_{k+1}$ near ∂S_{k+1} as in Figure 2, left, so that the normal directions of B_k and S_{k+1} are compatible in the newly constructed branched surface.

(2) For each component c of ∂S_{k+1} lying inside a suture $\partial_v N(B_k)$, we first slightly isotope S_{k+1} by pushing c into $R_\pm(\gamma_k) \subset \partial_h N(B_k)$, then as shown in Figure 2, right, we can deform $B_k\cup S_{k+1}$ near c into a branched surface. By the requirement of the normal directions in the sutured manifold decomposition, the normal directions of B_k and S_{k+1} are compatible in the newly constructed branched surface.

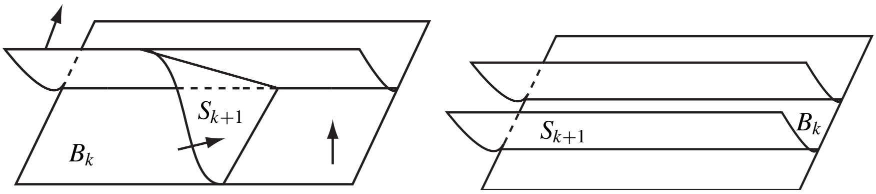

Figure 2. $B_k \cup S_{k+1}$ is deformed near ∂S_{k+1}. Left: the normal directions of B_k and S_{k+1} are compatible. Right: the neighborhoods of each suture are branched surfaces.

It follows from the definition of sutured manifold decomposition [Gabai 1983] that $M\backslash\operatorname{int}(N(B_{k+1})) = M_{k+1}$ and the suture γ_{k+1} of (M_{k+1}, γ_{k+1}) consists of $\partial_v N(B_{k+1})$ and a collection of annuli in the boundary torus ∂M. We will sometimes use the notation

$$B_{k+1} = B_{(M_{k+1},\gamma_{k+1})} = B_{\langle S_1;S_2;\dots;S_{k+1}\rangle}.$$

In summary, there is a map from the set of sutured manifold decomposition sequences to the set of properly embedded branched surfaces given by

$$(S_1, S_2, \dots, S_l) \mapsto B_{\langle S_1;S_2;\dots;S_l\rangle},$$

and a (forgetful) map from the set of properly embedded branched surfaces to the set of sutured 3-manifolds given by

$$B \mapsto (M_B, \gamma_B) = (M\backslash\operatorname{int}(N(B)), \partial_v N(B) \cup E'),$$

where $E' \subset \partial M$ satisfies $E' = E$, the set of boundary sutures, if B intersects ∂M only in longitudes. For future reference, it is useful to highlight that under this correspondence, $\partial_h N(B)$ corresponds naturally to $R_+(\gamma_B) \cup R_-(\gamma_B)$.

4. The construction

Modifying the sutured manifold hierarchy. Given a well-groomed sutured manifold hierarchy satisfying the conclusions of Theorem 3.6, we can inductively construct the sequence of branched surfaces $B_1, \dots, B_n$ corresponding to the sutured manifold decomposition. The branched surface B_n in the end has the properties that (1) $M\backslash\operatorname{int}(N(B_n))$ is a product and (2) ∂B_n is a collection of circles in ∂M of slope 0. In particular, any taut foliation carried by B_n will also necessarily meet ∂M only in simple closed curves of slope 0.

To obtain a branched surface carrying taut foliations realizing an open interval of boundary slopes about 0, it is necessary to modify the sutured manifold hierarchy, or, equivalently, the sequence of branched surfaces B_k. In this section, we describe one way of doing this. We break the process into two steps.

As a first step, we slightly modify the sutured manifold hierarchy by adding some parallel copies of the surfaces S_k. Equivalently, we modify the sequence of branched surfaces B_k by adding some parallel copies of the surfaces S_k. This operation is equivalent to a splitting of the branched surface. As a second (and final) step, we further modify the sutured manifold hierarchy by adding carefully chosen product disks.

Before giving a precise description of these steps, we introduce some terminology. Let B be a transversely oriented branched surface and let F be a component of $\partial_h N(B)$. The boundary of F has two parts: $\partial F \cap \partial M$ and $\partial F \cap \partial_v N(B)$. We call $\partial F \cap \partial_v N(B)$ the *internal boundary* of F. Let L be the branch locus of B. Let L_F be the closure of $\pi^{-1}(L) \cap \operatorname{int}(F)$, where $\pi : N(B) \to B$ is the map collapsing each interval fiber to a point. So L_F is a trivalent graph properly embedded in F. We call L_F the *projection* of the branch locus to F. Each arc in L_F has a normal direction induced from the branch direction of L.

Definition 4.1. Let F be a component of $\partial_h N(B)$ with $\partial F \cap \partial M \neq \varnothing$ and let η be an arc properly embedded in F. If F has nonempty internal boundary, we require that η connects $\partial F \cap \partial M$ to the internal boundary of F. Choose η so that it intersects L_F transversely and only at points in the interior of edges of L_F (namely, it misses all triple points). Since η is transverse to L_F, the induced branch direction of L_F gives a direction along η for each point in $\eta \cap L_F$. We say η is *good* if these induced directions are coherent along η and all point away from an endpoint of η that lies in ∂M.

We say F is *good* if F satisfies the following properties:

(1) The closure of each component D of $F \backslash L_F$ has a boundary arc with induced branch direction (from L_F) pointing out of D.

(2) If F has internal boundary, then there is a set of disjoint good arcs, denoted by Γ_F, connecting each component of $\partial F \cap \partial M$ to the internal boundary of F.

(3) If F has no internal boundary (in which case, F must be a Seifert surface of the knot exterior), then there is a properly embedded nonseparating good arc in F, which we also denote by Γ_F.

Lemma 4.2. *Let B be a branched surface. If each component of $\partial_h N(B)$ is good, then B does not contain any sink disk or half sink disk.*

Proof. Let F be a component of $\partial_h N(B)$ and let L_F be as above. Let P be the closure (under path metric) of a component of $F \backslash L_F$. So P can be viewed as a copy of a branch sector of B. It follows from part (1) of Definition 4.1 that B has no sink disk or half sink disk. □

Definition 4.3. We say the branched surface B is *good* if

(1) every component of $\partial_h N(B)$ is good, and

(2) the arc systems Γ_F as described in (2) and (3) in Definition 4.1 can be chosen so that the projections $\pi(\Gamma_F)$, as F ranges over all components of $\partial_h N(B)$, are disjoint in B.

Note that these good arcs Γ_F will be the arcs along which we will attach product disks.

Step 1: Splitting B_n. Next we will describe the first modification of a sutured manifold decomposition sequence satisfying the conclusions of Theorem 3.6.

Lemma 4.4. *Let k be a nontrivial knot in S^3 and $M = S^3 \setminus \operatorname{int}(N(k))$ the knot exterior. Let*

$$(M, \partial M) \overset{S_1}{\rightsquigarrow} (M_1, \gamma_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n, \gamma_n) = (S \times I, \partial S \times I)$$

be a well-groomed sutured manifold hierarchy that satisfies the conclusions of Theorem 3.6. Then there exists a well-groomed sutured manifold hierarchy

$$(M, \gamma) \overset{S_1}{\rightsquigarrow} (M_1', \gamma_1') \overset{R_1'}{\rightsquigarrow} (M_1'', \gamma_1'') \overset{S_2}{\rightsquigarrow} (M_2', \gamma_2') \overset{R_2'}{\rightsquigarrow} (M_2'', \gamma_2'') \overset{S_3}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n', \gamma_n')$$

which also satisfies the conclusions of Theorem 3.6. Moreover, the branched surfaces $B_l' = B_{(M_l', \gamma_l')}$, $1 \le l \le n$, satisfy the conditions:

(1) *$\partial B_l' \cap \partial M$ is a collection of simple closed curves of slope* 0 *in ∂M for each l.*

(2) *(M_l, γ_l) is a sutured submanifold of (M_l', γ_l') and $(M_l', \gamma_l') \setminus (M_l, \gamma_l)$ is a product sutured manifold for each l.*

(3) *Every branched surface B_l' is good.*

(4) *No B_l' carries a torus.*

(5) *(M_n', γ_n') is a product sutured manifold $(S' \times I, \partial S' \times I)$.*

Proof. First note that, in the sutured manifold hierarchy above, each R_i' is a parallel copy of some components of $R_+(\gamma_i') \cup R_-(\gamma_i')$.

We proceed by induction on l. Since k is nontrivial and hence S_1 has genus at least one, the branched surface $B_1' = S_1$ is easily seen to satisfy conditions (1)–(4). So suppose we have constructed

$$(M, \gamma) \overset{S_1}{\rightsquigarrow} (M_1', \gamma_1') \overset{R_1'}{\rightsquigarrow} (M_1'', \gamma_1'') \overset{S_2}{\rightsquigarrow} (M_2', \gamma_2') \overset{R_2'}{\rightsquigarrow} (M_2'', \gamma_2'') \overset{S_3}{\rightsquigarrow} \cdots \overset{S_l}{\rightsquigarrow} (M_l', \gamma_l')$$

satisfying the conclusions of Theorem 3.6 and such that the corresponding branched surfaces $B_i' = B_{(M_i', \gamma_i')}$ satisfy the conditions (1)–(4) for all i, $1 \le i \le l$.

By condition (2), (M_l, γ_l) is a sutured submanifold of (M_l', γ_l'). Let $R_+'(\gamma_l)$ and $R_-'(\gamma_l)$ be parallel copies of $R_+(\gamma_l)$ and $R_-(\gamma_l)$, chosen to be properly embedded in

$(M_l, \gamma_l) \subset (M'_l, \gamma'_l)$ and with boundary lying in $E_l \cup A(\gamma_l)$ (see Definition 3.1 and Definition 3.4). Set $R'_l = R'_+(\gamma_l) \cup R'_-(\gamma_l)$. We first consider the sutured manifold decomposition $(M'_l, \gamma'_l) \overset{R'_l}{\rightsquigarrow} (M''_l, \gamma''_l)$. By the definition of R'_l, this decomposition only adds some product complementary regions. Set $B''_l = B_{(M''_l, \gamma''_l)}$. The change from B'_l to B''_l is basically the addition of branch sectors corresponding to R'_l, and this operation creates some product complementary regions. See Figure 3, left, for a schematic picture. We may view (M_l, γ_l) as a subset of (M''_l, γ''_l), and consider the sutured manifold decompositions

$$(M'_l, \gamma'_l) \overset{R'_l}{\rightsquigarrow} (M''_l, \gamma''_l) \overset{S_{l+1}}{\rightsquigarrow} (M'_{l+1}, \gamma'_{l+1}),$$

where we now view S_{l+1} as lying in $(M_l, \gamma_l) \subset (M''_l, \gamma''_l)$. Certainly B'_{l+1} satisfies conditions (1) and (2).

Consider condition (3). We begin by considering a component F of $\partial_h N(B''_l)$. The surface F can be classified as one of the following 3 types (see Figure 3, right):

(1) F can be viewed as a component G of $\partial_h N(B'_l)$, as illustrated in Figure 3, right. Since the new branch sectors are attached to B'_l along cusp circles, L_F is obtained from L_G by adding curves parallel to curves in L_G with coherent induced branch direction, where L_G is the projection of the branch locus of

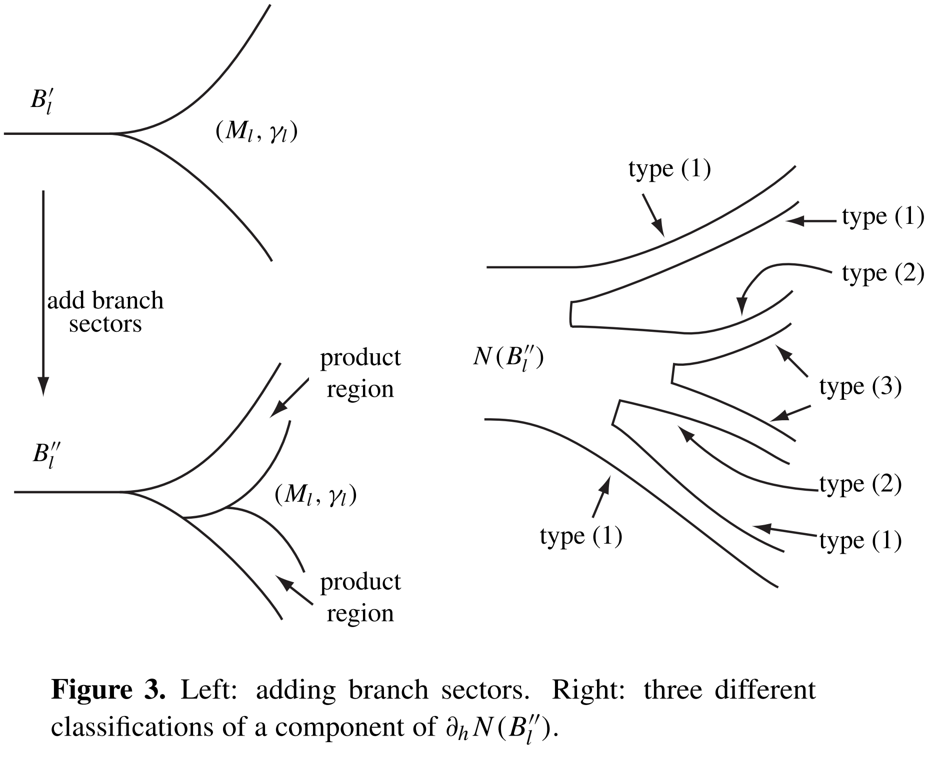

Figure 3. Left: adding branch sectors. Right: three different classifications of a component of $\partial_h N(B''_l)$.

B_l' to G. Since the branch directions are coherent, adding such parallel curves to L_G does not affect the good arcs in G. Thus in this case F is good with respect to B_l'' with the same set of good arcs as G.

(2) F is a horizontal boundary component for a newly created product complementary region and $\pi(F)$ contains part of the branch sectors added to B_l', as illustrated in Figure 3, right. In this case, each component of L_F consists of a circle C parallel to the internal boundary and with induced branch direction pointing to the internal boundary and possibly a collection of essential arcs in the annulus between C and the internal boundary.

(3) F is in the boundary of the sutured submanifold $(M_l, \gamma_l) \subset (M_l'', \gamma_l'')$. In this case, $L_F = \varnothing$.

Next consider how $\partial_h N(B_{l+1}')$ is related to $\partial_h N(B_l'')$. Let H be a component of $\partial_h N(B_{l+1}')$. Then either H can be viewed as a component of $\partial_h N(B_l'')$ or H contains a subset of one side of S_{l+1}. Our goal is to find a set of good arcs for each component H of $\partial_h N(B_{l+1}')$, so that the projections of the good arcs in B_{l+1}' are disjoint.

Case (a). H is not a component of $\partial_h N(B_l'')$

In this case, H is contained in the union of one side of S_{l+1} and $F \backslash \partial S_{l+1}$, where F is a component of $\partial_h N(B_l'')$ of type (3). By our construction, $L_F = \varnothing$. Moreover, on the other side of F, there is a corresponding component F' of $\partial_h N(B_l'')$ of type (2) such that $\pi(F) \cap \pi(F') \neq \varnothing$ in the branched surface B_l''. Adding S_{l+1} to B_l'' does not affect F', so we may also view F' as a component of $\partial_h N(B_{l+1}')$. Next we choose good arcs for both H and F'.

First note that since the original sutured manifold decomposition is well-groomed, ∂S_{l+1} is homologically nontrivial in $H_1(F, \partial F)$. There is a simple closed curve η in F transverse to S_{l+1}, as shown in Figure 4 (note that the arrows in Figure 4 on ∂S_{l+1} denote the branch direction at ∂S_{l+1}), such that the algebraic intersection

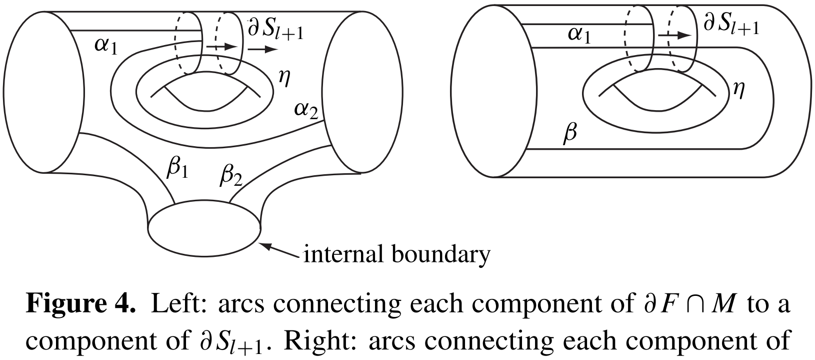

Figure 4. Left: arcs connecting each component of $\partial F \cap M$ to a component of ∂S_{l+1}. Right: arcs connecting each component of $\partial F' \cap \partial M$ to the internal boundary of F'.

number of η and ∂S_{l+1} is equal to $|\eta \cap \partial S_{l+1}|$ (this is equivalent to saying that the normal direction of ∂S_{l+1} at $\eta \cap \partial S_{l+1}$, induced from the branch direction of B'_{l+1}, are coherent along η).

Recall that H can be viewed as the union of one side of S_{l+1} and $F \backslash \partial S_{l+1}$. We first consider the components $\theta_1, \dots, \theta_p$ of $\partial H \cap \partial M$ that are not in F (i.e., each θ_i can be viewed as a component of $\partial S_{l+1} \cap \partial M$). We can find an arc γ_i connecting θ_i to the internal boundary of H such that γ_i either is totally in (one side of) S_{l+1} or consists an arc in S_{l+1} and an arc in F parallel to a subarc of η. Moreover, we can choose these arcs γ_i to be disjoint in H.

Now we consider the components of $\partial F \cap \partial M$ (which are viewed as components of $\partial H \cap \partial M$). It is easy to see from our construction that there is a collection of disjoint good arcs $\alpha_1, \dots, \alpha_q$ in F (see the arcs α_1 and α_2 in Figure 4, left), such that (1) these arcs α_j connect each component of $\partial F \cap \partial M$ to a component of ∂S_{l+1}, and (2) these arcs α_j are disjoint from the curve η describe above.

It follows from our construction that these arcs γ_i and α_j form a set of good arcs Γ_H for H.

Next we consider the component F' of $\partial_h N(B''_l)$ on the other side of F. F' is a type (2) component of $\partial_h N(B''_l)$, and we may view F' as a component of $\partial_h N(B'_{l+1})$. Moreover, we view F' as a parallel copy of F and view the curves ∂S_{l+1}, η and α_j described above as curves in F'. We have two slightly different situations. The first is that F' (and hence F) has nonempty internal boundary, and the second is that F' has no internal boundary.

If F' has nonempty internal boundary, then there are arcs $\beta_1, \dots, \beta_r$ in F' (see the arcs β_1 and β_2 in Figure 4, left), such that (1) the arcs β_k connect each component of $\partial F' \cap \partial M$ to the internal boundary of F', and (2) the arcs β_k are disjoint from η, ∂S_{l+1} and the arcs α_j. The arcs β_k form a set of good arcs $\Gamma_{F'}$ for F'. Moreover, since each β_k is disjoint from η and the arcs α_j, the projections $\pi(\Gamma_H)$ and $\pi(\Gamma_{F'})$ of the good arcs Γ_H and $\Gamma_{F'}$ for H and F' respectively are disjoint in B'_{l+1}.

If F' does not have internal boundary (in which case F' must be a Seifert surface of the knot exterior), then as shown in Figure 4, right, there is an arc β properly embedded in F' such that (1) β is disjoint from η and the arcs α_j and (2) the intersection of β with ∂S_{l+1} is minimal up to isotopy. Since the original sutured manifold is well-groomed, the requirement (2) implies that the algebraic intersection number of β and ∂S_{l+1} is equal to $|\beta \cap \partial S_{l+1}|$. Thus β is a good arc for F'. Since β is chosen to be disjoint from η and each α_j, the projections of $\pi(\beta)$ and $\pi(\Gamma_H)$ on B'_{l+1} are disjoint.

Case (b). H is a component of $\partial_h N(B''_l)$

In this case, either L_H is unchanged by the decomposition by S_{l+1} or H is the surface F' of type (2) considered in Case (a). In Case (a), we have already constructed a set of good arcs for the type (2) surface F', so we may assume that L_H is unchanged by the decomposition by S_{l+1}. Since H (viewed as a component of $\partial_h N(B_l''))$ is good in B_l'', H is good in B_{l+1}'. Furthermore, the projections of the good arcs in Case (a) and the good arcs (from the induction) of H in this case are disjoint in B_{l+1}'.

So B_{l+1}' is good. It remains to show that B_{l+1}' does not carry any torus. Since B_l' does not carry any torus and B_l'' can be obtained by splitting B_l', B_l'' does not carry any torus. Suppose B_{l+1}' carries a torus T. Then T can be expressed as the union of some copies of S_{l+1} and a surface in $N(B_l'')$ transverse to the I-fibers. Moreover, the transverse orientation of the branched surface induces a compatible normal orientation for T. Since the original sutured manifold decomposition sequence is well-groomed, $\partial S_{l+1} \cap R_\pm(\gamma_l)$ is a collection of homologically nontrivial curves in $H_1(R_\pm(\gamma_l), \partial R_\pm(\gamma_l))$. Thus there is a component F of $\partial_h N(B_l'')$, such that $T \cap F$ (with the induced orientation) is homologically nontrivial in F. However, since T is a torus in S^3, T is homologically trivial and this is impossible.

Therefore, B_{l+1}' satisfies properties (1)–(4) of the lemma and we can inductively construct the sutured manifold hierarchy and corresponding sequence of branched surfaces as claimed. □

Step 2: Adding product disks. Let B_n' be the good branched surface constructed in the proof of Lemma 4.4. It follows from the conditions on the sutured manifold hierarchy and our construction above that $\partial B_n'$ consists of circles of slope 0 in the torus ∂M. In this section, we will add some product disks and modify B_n' to get a laminar branched surface carrying more laminations.

As $M \setminus \text{int}(N(B_n'))$ is a product, we may suppose $M \setminus \text{int}(N(B_n')) = S \times I$, where S is a compact and possibly disconnected surface. Let $S_+ = S \times \{0\}$ and $S_- = S \times \{1\}$. So $\partial_h N(B_n') = S_+ \cup S_-$. It is possible to decompose $S \times I$ as the disjoint union

$$S \times I = (F \times I) \cup (G \times I),$$

where F is the union of the components of S without internal boundary. Thus $\partial F \subset \partial M$ and each component of G has nonempty internal boundary. Moreover, each component of F must be a Seifert surface in the knot exterior. Note that, since we take parallel copies of surfaces in the horizontal boundary in each step of the sutured manifold decompositions (see Lemma 4.4), $F \neq \varnothing$. Furthermore, $G = \varnothing$ only if k is fibered.

Let $m = |\partial S_\pm \cap \partial M|$ be the number of components of the noninternal boundary $S_\pm \cap \partial M$. Since B_n' is good, there is a collection of pairwise disjoint good arcs in S_+, denoted by $\eta_1, \dots, \eta_m$, and a collection of pairwise disjoint good arcs in S_-, denoted by $\delta_1, \dots, \delta_m$, such that $\pi\left(\bigcup_i \eta_i\right) \cap \pi\left(\bigcup_i \delta_i\right) = \varnothing$ (in B_n') and each component of $\partial S_\pm \cap \partial M$ has exactly one incident good arc η_i and one incident good arc δ_i attached to it. After relabeling as necessary, we may assume that for $1 \le i \le r$, η_i and δ_i lie in $F \times \{0, 1\}$, while for $r + 1 \le i \le m$, η_i and δ_i lie in $G \times \{0, 1\}$. It follows that each η_i and each δ_i, $1 \le i \le r$, has both endpoints lying on ∂M while each η_i and δ_i, $r + 1 \le i \le m$, has exactly one endpoint lying on ∂M.

Consider first $F \times [0, 1]$. Recall that each component of F is a Seifert surface of the knot exterior. Let F_1 be any component of F and relabel as necessary so that $\eta_1 \subset F_1 \times \{0\}$ and $\delta_1 \subset F_1 \times \{1\}$. By [Roberts 2001a, Lemma 4.4], there is a sequence of simple arcs

$$\alpha_0 = \eta_1, \alpha_1, \dots, \alpha_l = \delta_1$$

such that $\alpha_i \cap \alpha_{i+1} = \varnothing$ and a regular neighborhood of $\alpha_i \cup \alpha_{i+1} \cup \partial F_1$ in F_1 is a twice-punctured torus for each i, $1 \le i \le l$. For $1 \le i \le l$, let F_1 induce a consistent orientation on each $F_1 \times \left\{\frac{i}{l+1}\right\}$ and orient the disks $\alpha_i \times \left[\frac{i}{l+1}, \frac{i+1}{l+1}\right]$ arbitrarily. Add branch sectors to B'_n as prescribed by the following sequence of sutured manifold decompositions:

$$(M'_n, \gamma'_n) \overset{A}{\rightsquigarrow} (M'_{n+1}, \gamma'_{n+1}) \overset{B}{\rightsquigarrow} (M_{F_1}, \gamma_{F_1}),$$

where

$$A = F_1 \times \left\{\tfrac{1}{l+1}, \dots, \tfrac{l}{l+1}\right\} \quad \text{and} \quad B = \bigcup_i \left(\alpha_i \times \left[\tfrac{i}{l+1}, \tfrac{i+1}{l+1}\right]\right).$$

Repeat for each remaining component of F and let (M_F, γ_F) denote the resulting sutured manifold. Set $B_F = B_{(M_F, \gamma_F)}$. Notice that the conditions satisfied by the arcs α_i guarantee that B_F is laminar.

Now consider $G \times I$. Let G_1 be a component of G and let $p = |\partial G_1 \cap \partial M|$. Let $\{C_1, \dots, C_p\}$ be a listing of the components of $G_1 \cap \partial M$. After relabeling as necessary, we may assume $\eta_{r+1}, \dots, \eta_{r+p}$ lie in $G_1 \times \{0\}$ and $\delta_{r+1}, \dots, \delta_{r+p}$ lie in $G_1 \times \{1\}$, with $\{\eta_{r+i}(0), \delta_{r+i}(0)\} \subset C_i$ for each $1 \le i \le p$.

Lemma 4.5. *Let $\{\alpha_1, \dots, \alpha_p\}$ and $\{\beta_1, \dots, \beta_p\}$ each be a set of pairwise disjoint arcs properly embedded in G_1 with $\{\alpha_i(0), \beta_i(0)\} \subset C_i$ and $\{\alpha_i(1), \beta_i(1)\} \subset \partial G \backslash \{C_1, \dots, C_p\}$, the internal boundary of G_1. Let $s = \left|\bigcup_i \alpha_i \cap \bigcup_i \beta_i\right|$. Then either $s = 0$ or there is a set $\{\gamma_1, \dots, \gamma_p\}$ of pairwise disjoint arcs properly embedded in G_1 with $\gamma_i(0) \in C_i$, $\gamma_i(1) \in \partial G_1 \backslash \{C_1, \dots, C_p\}$, such that*

$$\max\left\{\left|\bigcup_i \alpha_i \cap \bigcup_i \gamma_i\right|, \left|\bigcup_i \beta_i \cap \bigcup_i \gamma_i\right|\right\} < s.$$

Proof. Suppose $s \neq 0$. Relabeling as necessary, we may assume that α_1 and $\bigcup_i \beta_i$ intersect. Choose z to be the point in $\alpha_1 \cap \bigcup_i \beta_i$ that is furthest along α_1. So there are j, t_0, t_1 such that $z = \alpha_1(t_0) = \beta_j(t_1)$ and $\alpha_1(t_0, 1] \cap \bigcup_i \beta_i = \varnothing$. Let γ_j be the concatenation of the two arcs $\beta_j[0, t_1]$ and $\alpha_1[t_0, 1]$, perturbed slightly so that it intersects α_1 transversely and minimally. For $i \neq j$, set $\gamma_i = \beta_i$. Then $\left|\bigcup_i \alpha_i \cap \bigcup_i \gamma_i\right| < \left|\bigcup_i \alpha_i \cap \bigcup_i \beta_i\right|$ and $\left|\bigcup_i \gamma_i \cap \bigcup_i \beta_i\right| = 0$. □

The next corollary follows immediately.

Corollary 4.6. *There are sets of arcs* $\mathscr{A}_i = \{\alpha_1^i, \dots, \alpha_p^i\}$, $1 \le i \le q$, *such that*

(1) *for each* i, *the arcs in* $\mathscr{A}_i$ *are pairwise disjoint and properly embedded in* G_1, $\alpha_j^i(0) \in C_j$, *and* $\alpha_j^i(1) \in \partial G_1 \setminus \{C_1, \dots, C_p\}$, $j = 1, \dots, p$,

(2) $\mathscr{A}_0 = \{\eta_{r+1}, \dots, \eta_{r+p}\}$ *and* $\mathscr{A}_{q+1} = \{\delta_{r+1}, \dots, \delta_{r+p}\}$, *and*

(3) $\bigcup_j \alpha_j^i \cap \bigcup_j \alpha_j^{i+1} = \varnothing$ *for each* i.

For $1 \le i \le q$, let G_1 induce a consistent orientation on each $G_1 \times \left\{\frac{i}{q+1}\right\}$. Orient the disks $\alpha_j^i \times \left[\frac{i}{q+1}, \frac{i+1}{q+1}\right]$ so that the orientation induced on their boundaries agrees with the orientation of α_j^i (which is the orientation from its starting point in ∂M to its ending point in the internal boundary). Add branch sectors to B_F as given by the following sequence of sutured manifold decompositions:

$$(M_F, \gamma_F) \overset{A}{\rightsquigarrow} (M_F', \gamma_F') \overset{B}{\rightsquigarrow} (M_{G_1}, \gamma_{G_1}),$$

where

$$A = G_1 \times \left\{\tfrac{1}{q+1}, \dots, \tfrac{q}{q+1}\right\} \quad \text{and} \quad B = \bigcup_{i,j} \left(\alpha_j^i \times \left[\tfrac{i}{q+1}, \tfrac{i+1}{q+1}\right]\right).$$

Repeat for each remaining component of G and let (M_G, γ_G) denote the resulting sutured manifold. Set $B_G = B_{(M_G, \gamma_G)}$. Notice that the conditions satisfied by the arcs α_j^i guarantee that B_G is laminar.

By Lemma 4.4, B_n' does not carry any torus. Therefore, any branched surface obtained by splitting B_n' also cannot carry a torus. And finally, any (closed) torus carried by B_G but not by this splitting of B_n' would necessarily pass through one of the added disk branches and hence would necessarily have nonempty boundary. Thus B_G does not carry a torus.

Noting that for each product disk in the above construction, its two normal directions give two ways of deforming it into a branched surface, let B_G' denote the branched surface obtained from B_G by reversing the orientations of the disks $\alpha_j^i \times \left[\frac{i}{q+1}, \frac{i+1}{q+1}\right]$. Notice that B_G' is also laminar, has only product complementary regions, and does not carry a torus.

Hence we have laminar branched surfaces B_G and B_G' with only product complementary regions and which do not carry a torus. We may therefore apply Theorem 2.2 to conclude the existence of taut foliations realizing any boundary slope carried by $B_G \cap \partial M$ or $B_G' \cap \partial M$. It remains to compute these boundary slopes.

The boundary train tracks. Let τ denote the train track $B_G \cap \partial M$ and let τ' denote the train track $B_G' \cap \partial M$.

Lemma 4.7. *Together,* τ *and* τ' *realize all slopes in* $(-a, b)$ *for some* $a, b > 0$.

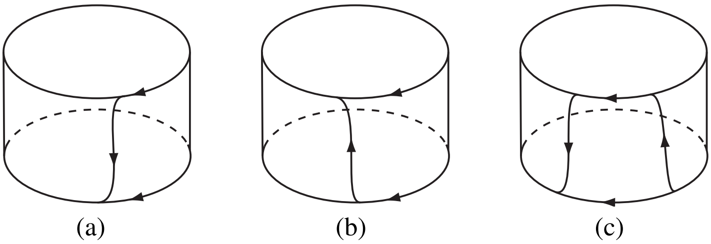

Figure 5. Train tracks that realize all slopes in $(-a, b)$ for $a, b > 0$.

Proof. Consider an annular component A_G of $\partial G_1 \times \left[\frac{i}{q+1}, \frac{i+1}{q+1}\right]$. The train tracks τ and τ' restricted to A_G have the form indicated in parts (a) and (b), respectively, of Figure 5. Similarly, consider an annular component A_F of $\partial F_1 \times \left[\frac{i}{l+1}, \frac{i+1}{l+1}\right]$. Recall that each $F_1 \times \left\{\frac{i}{l+1}\right\}$ is a Seifert surface and the good arc for F_1 has both endpoints on the circle ∂F_1. Thus both τ and τ' restricted to A_F have the form indicated in Figure 5(c). Call all such nonlongitudinal branches of τ or τ' *vertical.*

Since all vertical branches of τ (or τ', respectively) are of one of the three types shown in Figure 5, it follows that τ (or τ') is a train track obtained by concatenating pieces of the types of Figure 5(a) or (c) (or (b) or (c), respectively). Examples are shown in Figure 6. Notice that τ and τ' are orientable and measurable; namely, they admit a transverse measure [Hatcher 1988, page 66; Penner and Harer 1992, page 86]. Assign weights x, y, and $x + y$ to the vertical branches of τ and τ' as indicated in Figure 6; namely, vertical branches in $G \times I$ regions are weighted x, the compatibly oriented branches in $F \times I$ regions are weighted $x + y$, and the remaining branches in $F \times I$ regions are weighted y. Then assign weights from $\{1, 1 + x, 1 + y, 1 + x + y\}$ to the remaining branches of τ and τ' to obtain a measure μ on τ and a measure μ' on τ'.

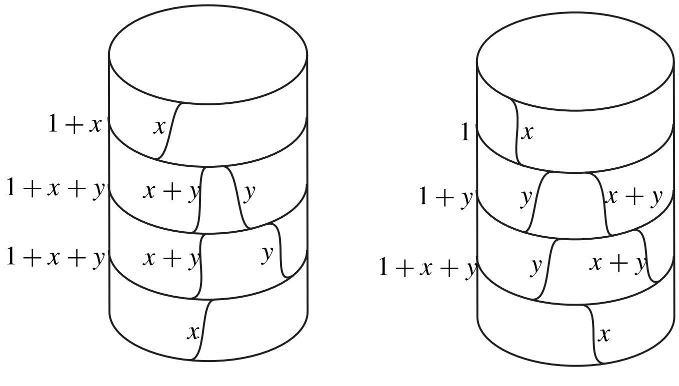

Figure 6. Examples of train tracks.

Recall that if γ is a simple closed curve in a torus, then the slope of γ is given in standard coordinates by

$$\operatorname{slope}(\gamma) = \frac{\langle \lambda, \gamma \rangle}{\langle \gamma, m \rangle}, \tag{1}$$

where $\langle\ ,\ \rangle$ denotes algebraic intersection number and λ is the longitude and m is the meridian of the knot k in S^3.

Applying (1) to the measured train tracks (τ, μ) and (τ', μ') while letting x, y range over all values $0 < y \ll x$, we see that (τ, μ) and (τ', μ') together carry all boundary slopes in some open interval $(-a, b)$ about 0. ☐

By Theorem 2.2, if τ (or τ') fully carries a curve of slope s, then B_G (or B'_G, respectively) fully carries an essential lamination whose boundary consists of loops of slope s in ∂M. Moreover, this lamination extends to an essential lamination in $M(s)$. Since $M \backslash \operatorname{int}(N(B_G))$ and $M \backslash \operatorname{int}(N(B'_G))$ consist of product regions, such essential laminations can be extended to taut foliations. This proves Theorem 1.1.

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