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**FREE BROWNIAN MOTION AND FREE CONVOLUTION
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We consider a pair of probability measures μ, ν on the unit circle such that $\Sigma_\lambda(\eta_\nu(z)) = z/\eta_\mu(z)$. We prove that the same type of equation holds for any $t \geq 0$ when we replace ν by $\nu \boxtimes \lambda_t$ and μ by $\mathbb{M}_t(\mu)$, where λ_t is the free multiplicative analogue of the normal distribution on the unit circle of \mathbb{C} and \mathbb{M}_t is the map defined by Arizmendi and Hasebe. These equations are a multiplicative analogue of equations studied by Belinschi and Nica. In order to achieve this result, we study infinite divisibility of the measures associated with subordination functions in multiplicative free Brownian motion and multiplicative free convolution semigroups. We use the modified \mathcal{S} -transform introduced by Raj Rao and Speicher to deal with the case that ν has mean zero. The same type of the result holds for convolutions on the positive real line. In the end, we give a new proof for some Biane's results on the densities of the free multiplicative analogue of the normal distributions.

1. Introduction

Let $\mathcal{M}_{\mathbb{R}}$ be the set of probability measures on \mathbb{R} . For every $t \geq 0$, Belinschi and Nica [2008b] defined a family of maps $\mathbb{B}_t : \mathcal{M}_{\mathbb{R}} \rightarrow \mathcal{M}_{\mathbb{R}}$ by setting

$$\mathbb{B}_t(\mu) = (\mu^{\boxplus(t+1)})^{\boxplus 1/(t+1)}, \quad \mu \in \mathcal{M}_{\mathbb{R}}.$$

These maps have several remarkable properties. For any $t \geq 0$, \mathbb{B}_t is an endomorphism of $(\mathcal{M}_{\mathbb{R}^+}, \boxtimes)$, where $\mathcal{M}_{\mathbb{R}^+}$ is the set of probability measures on $[0, +\infty)$ and \boxtimes is free multiplicative convolution. $\{\mathbb{B}_t\}_{t \geq 0}$ is a semigroup and \mathbb{B}_1 is the Boolean to free Bercovici–Pata bijective map.

The maps \mathbb{B}_t have strong connections with \boxplus -infinite divisibility. They are also connected to free Brownian motion and additive free convolution semigroups. For $\mu \in \mathcal{M}_{\mathbb{R}}$, we denote by G_μ the Cauchy transform of μ and by F_μ the reciprocal Cauchy transform of μ . Given a pair of probability measures $\mu, \nu \in \mathcal{M}_{\mathbb{R}}$ such that

$$G_\nu(z) = z - F_\mu(z), \quad z \in \mathbb{C}^+,$$

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we have

$$(1-1) \quad G_{\nu \boxplus \gamma_t}(z) = z - F_{\mathbb{B}_t(\mu)}(z), \quad t > 0, z \in \mathbb{C}^+,$$

where γ_t is the semicircular distribution with variance t . This result was generalized to the multi-variable case in [Belinschi and Nica 2008a; 2009; Nica 2009]. An equivalent form of (1-1) was used to prove the superconvergence theorem in [Wang 2010]. Anshelevich [2010; 2011a; 2011b; 2012] generalized the above correspondence of $\mu \leftrightarrow \nu$ and $\mathbb{B}_t(\mu) \leftrightarrow \nu \boxplus \gamma_t$ to the context of two-state probability spaces. Motivated by these generalizations and applications, we study in this article the analogue of these equations for multiplicative free convolution.

Throughout this article, we denote by \mathbb{T} the unit circle of \mathbb{C} , by $\mathcal{M}_{\mathbb{T}}$ the set of probability measures on \mathbb{T} , and by \mathcal{M}_* the set of probability measures on \mathbb{C} with nonzero mean. We also set

$$\mathcal{M}_{\mathbb{T}}^* = \{\mu \in \mathcal{M}_{\mathbb{T}} \cap \mathcal{M}_* : \eta_{\mu}(z) \neq 0 \text{ for all } z \in \mathbb{D} \setminus \{0\}\}.$$

It was shown in [Belinschi and Bercovici 2005] that one can define multiplicative free convolution power $\mu^{\boxtimes t}$ for $\mu \in \mathcal{M}_{\mathbb{T}}^*$ and $t > 1$.

In [Arizmendi and Hasebe 2013], a family of maps \mathbb{M}_t , which is the analogue of the semigroup \mathbb{B}_t , was defined for the probability measures in $\mathcal{M}_{\mathbb{T}}^*$. The definition of \mathbb{M}_t there was more general; we only need a simpler form defined as follows. Given $\mu \in \mathcal{M}_{\mathbb{T}}^*$ having positive mean, then for $t \geq 0$, the map \mathbb{M}_t is defined by

$$\mathbb{M}_t(\mu) = (\mu^{\boxtimes(t+1)})^{\boxtimes 1/(t+1)},$$

where the convolution power $\mu^{\boxtimes(t+1)}$ and the measure $\mathbb{M}_t(\mu)$ are chosen in a way such that they have positive means.

We then state one of our main theorems.

Theorem 1.1. *Given a pair of probability measures $\mu \in \mathcal{M}_{\mathbb{T}}^*$ and $\nu \in \mathcal{M}_{\mathbb{T}}$ such that*

$$(1-2) \quad \Sigma_{\lambda}(\eta_{\nu}(z)) = \frac{z}{\eta_{\mu}(z)}, \quad z \in \mathbb{D},$$

we have

$$(1-3) \quad \Sigma_{\lambda}(\eta_{\nu \boxtimes \lambda_t}(z)) = \frac{z}{\eta_{\mathbb{M}_t(\mu)}(z)}, \quad z \in \mathbb{D},$$

where λ_t is the analogue of the normal distribution on T with $\Sigma_{\lambda_t}(z) = \exp((t/2)(1+z)/(1-z))$ and $\lambda = \lambda_1$.

In order to prove Theorem 1.1, we consider two semigroups $\nu \boxtimes \lambda_t$ and $\mu^{\boxtimes(t+1)}$ for all $t \geq 0$. It is well-known that $\eta_{\nu \boxtimes \lambda_t}$ and $\eta_{\mu^{\boxtimes(t+1)}}$ are subordinated to η_{ν} and η_{μ} respectively. We prove that the subordination functions are η -transforms of some \boxtimes -infinitely divisible probability measures on \mathbb{T} . It turns out that the equation

$\Sigma_\lambda(\eta_\nu(z)) = z/\eta_\mu(z)$ means that the subordination function of $\eta_{\nu \boxtimes \lambda_t}$ with respect to η_ν and the subordination function of $\eta_{\mu \boxtimes (t+1)}$ with respect to η_μ are the same. The proof of Theorem 1.1 will be given in Subsection 3.5.

Given $\mu \in \mathcal{M}_T$, we prove that if $\mu^{\boxtimes t}$ can be defined and $\eta_{\mu \boxtimes t}$ is subordinated to η_μ for all $t > 1$, then $\mu \in \mathcal{M}_T^*$; in addition, we prove that for nontrivial measures $\mu \in \mathcal{M}_T$ and $\nu \in \mathcal{F}\mathcal{D}(\boxtimes, T)$, the density functions of the measures $\mu \boxtimes \nu_t$ and $\mu^{\boxtimes t}$ converge to $1/2\pi$ uniformly as $t \rightarrow \infty$.

To deal with the case that $\nu \in \mathcal{M}_T \setminus \mathcal{M}_*$, we use the modified \mathcal{G} -transform [Arizmendi 2012; Rao and Speicher 2007] to study subordination functions. In this case, the subordination function of $\eta_{\nu \boxtimes \lambda_t}$ with respect to η_ν is generally not unique. However, we can prove that there exists a unique subordination function satisfying certain properties (see Theorem 3.11). Let ρ_t be the measure associated with this subordination function of $\eta_{\nu \boxtimes \lambda_t}$ with respect to η_ν , we have that $\Sigma_{\rho_t}(z) = \Sigma_{\lambda_t}(\eta_\nu(z))$.

Similar results to Theorem 1.1 for multiplicative convolution on $\mathcal{M}_{\mathbb{R}^+}$ are also valid. The proof for this case is much simpler because of the uniqueness of multiplicative convolution powers and the uniqueness of subordination functions.

Finally, we give a new proof for some results concerning the density functions of the free multiplicative analogue of the normal distributions studied by Biane [1997c], and we obtain some new results. For example, for λ_t ($t > 0$) the free multiplicative analogue of the normal distributions on T , we prove that λ_t is unimodal.

This article is organized as follows. After this introductory section, we describe some backgrounds in the additive case in Section 2. In Section 3, we consider multiplicative free and multiplicative Boolean convolution on \mathcal{M}_T , and prove our main theorems. Section 4 is devoted to studying multiplicative free and multiplicative Boolean convolution on $\mathcal{M}_{\mathbb{R}^+}$. The regularity properties of the free multiplicative analogue of the normal distributions are discussed in Section 5.

2. Background: additive case

Additive free convolution and additive Boolean convolution. For a measure $\mu \in \mathcal{M}_{\mathbb{R}}$, we define the Cauchy transform $G_\mu : \mathbb{C}^+ \rightarrow \mathbb{C}^-$ by

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{1}{z-t} d\mu(t), \quad z \in \mathbb{C}^+.$$

We set

$$F_\mu(z) = \frac{1}{g\mu(z)}, \quad z \in \mathbb{C}^+,$$

so that $F_\mu : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ is analytic.

The following result characterizes those functions which are reciprocal Cauchy transforms of probability measures.

Proposition 2.1 [Bercovici and Voiculescu 1993]. *Let $F : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ be an analytic function. The following assertions are equivalent:*

- (1) *There exists a probability measure μ on \mathbb{R} such that $F(z) = F_\mu(z)$ in \mathbb{C}^+ .*
- (2) *There exists $a \in \mathbb{R}$, and a finite positive measure ρ on \mathbb{R} such that*

$$F(z) = a + z + \int_{-\infty}^{+\infty} \frac{1 + tz}{t - z} d\rho(t) \quad \text{for all } z \in \mathbb{C}^+.$$

- (3) $\lim_{y \rightarrow +\infty} F(iy)/iy = 1$.

It was proved in [Bercovici and Voiculescu 1993] that F_μ is invertible in some domain. More precisely, for two positive numbers M and N , we set

$$\Gamma_{M,N} = \{z \in \mathbb{C}^+ : |x| < My, y > N\}.$$

Then for any $M > 0$, there exists $N > 0$ such that the left inverse F_μ^{-1} of F_μ is defined in $\Gamma_{M,N}$, and then we can define the Voiculescu transform of μ by

$$\varphi_\mu(z) = F_\mu^{-1}(z) - z,$$

for $z \in \Gamma_{M,N}$. For any two measures $\mu, \nu \in \mathcal{M}_\mathbb{R}$, we have

$$(2-1) \quad \varphi_{\mu \boxplus \nu}(z) = \varphi_\mu(z) + \varphi_\nu(z)$$

in any truncated cone $\Gamma_{M,N}$ where φ_μ, φ_ν and $\varphi_{\mu \boxplus \nu}$ are defined. This remarkable result was proved in [Voiculescu 1986] for compactly supported measures and then extended to general cases in [Bercovici and Voiculescu 1993; Maassen 1992].

Given $\nu \in \mathcal{M}_\mathbb{R}$, we say that ν is \boxplus -infinitely divisible if for every positive integer n , there exists a probability measure $\nu_{1/n} \in \mathcal{M}_\mathbb{R}$ such that

$$\nu = \underbrace{\nu_{1/n} \boxplus \nu_{1/n} \boxplus \cdots \boxplus \nu_{1/n}}_{n \text{ times}}.$$

It is known [Bercovici and Voiculescu 1993; Maassen 1992; Voiculescu 1986] that a probability measure ν on \mathbb{R} is \boxplus -infinitely divisible if and only if its Voiculescu transform φ_ν has an analytic extension defined on \mathbb{C}^+ with values in $\mathbb{C}^- \cup \mathbb{R}$. We denote by $\mathcal{ID}(\boxplus, \mathbb{R})$ the set of all \boxplus -infinitely divisible probability measures on the real line. If $\nu \in \mathcal{ID}(\boxplus, \mathbb{R})$, then for every $t > 0$, there exists a probability measure ν_t such that $\varphi_{\nu_t}(z) = t\varphi_\nu(z)$ for z in the common domain of φ_ν and φ_{ν_t} .

Proposition 2.2. *Let ν is \boxplus -infinitely divisible, and let $H(z) = z + \varphi_\nu(z)$. Then*

$$(2-2) \quad H(F_\nu(z)) = z$$

for $z \in \mathbb{C}^+$. The set $U := \{z \in \mathbb{C}^+ : \Im H(z) > 0\}$ is a simply connected domain with boundary which is a simple curve and H maps \mathbb{C}^+ conformally onto U . Moreover,

the boundary ∂U is the graph of a function and the function H is continuous up to the real axis.

Proof. The first part of the assertion appears in [Bercovici and Voiculescu 1993; Voiculescu 1986], and the second part of the assertion follows from the fact that H satisfies the conditions of [Belinschi and Bercovici 2005, Proposition 4.7]. The last part of the assertion is due to [Chistyakov and Götze 2013, Lemma 3.3; Belinschi and Bercovici 2005, Proposition 4.7]. \square

Additive Boolean convolution was introduced in [Speicher and Woroudi 1997]. For $\mu \in \mathcal{M}_{\mathbb{R}}$, we set $E_{\mu}(z) = z - F_{\mu}(z)$. For $\mu, \nu \in \mathcal{M}_{\mathbb{R}}$, the additive Boolean convolution $\mu \uplus \nu$ is characterized by the identity

$$E_{\rho}(z) = E_{\mu}(z) + E_{\nu}(z) \quad \text{for } z \in \mathbb{C}^+.$$

We can also consider the infinite divisibility with respect to additive Boolean convolution. It turns out that every $\mu \in \mathcal{M}_{\mathbb{R}}$ is \uplus -infinitely divisible; see [Speicher and Woroudi 1997]. We denote by $\mathcal{ID}(\uplus, \mathbb{R})$ the set of all \uplus -infinitely divisible probability measures on the real line.

Infinite divisibility and subordination functions. Given $\mu, \nu \in \mathcal{M}_{\mathbb{R}}$, it is known that $F_{\mu \boxplus \nu}$ is subordinated to F_{μ} and F_{ν} , and by Proposition 2.1, we can also regard these subordination functions as the reciprocal Cauchy transforms of probability measures on \mathbb{R} .

Definition 2.3. For $\mu, \nu \in \mathcal{M}_{\mathbb{R}}$, the subordination distribution [Anshelevich 2012; Lenczewski 2007; Nica 2009] $\mu \boxplus \nu$ is defined to be the unique probability measure in $\mathcal{M}_{\mathbb{R}}$ such that $F_{\mu \boxplus \nu}(z) = F_{\nu}(F_{\mu \boxplus \nu}(z))$.

Many subordination distributions in semigroups related to free convolution are infinitely divisible; see [Anshelevich 2012; Nica 2009].

Proposition 2.4. Let $\mu, \nu \in \mathcal{M}_{\mathbb{R}}$.

- (1) $\varphi_{\mu \boxplus \nu}(z) = (\varphi_{\mu} \circ F_{\nu})(z)$.
- (2) If $\mu \in \mathcal{ID}(\boxplus, \mathbb{R})$, then $\mu \boxplus \nu \in \mathcal{ID}(\boxplus, \mathbb{R})$. In particular, $\gamma_t \boxplus \nu \in \mathcal{ID}(\boxplus, \mathbb{R})$ and $\varphi_{\gamma_t \boxplus \nu}(z) = tG_{\nu}(z)$, where γ_t is the semicircular distribution with variance t .
- (3) If $\nu = \mu \boxplus \nu'$ for $\nu' \in \mathcal{M}_{\mathbb{R}}$, then $\mu \boxplus \nu \in \mathcal{ID}(\boxplus, \mathbb{R})$. In particular, $\mu \boxplus \mu \in \mathcal{ID}(\boxplus, \mathbb{R})$, and $\varphi_{\mu \boxplus \mu}(z) = z - F_{\mu}(z)$.

Proof. Part (1) is Lemma 1 of [Anshelevich 2012]. Note that $\varphi_{\gamma_t}(z) = t/z$ and $(\varphi_{\mu} \circ F_{\mu})(z) = z - F_{\mu}(z)$; parts (2) and (3) follow from part 1 and Lemma 2 of [Anshelevich 2012]. See also [Chistyakov and Götze 2011, Corollary 2.3]. \square

The following result was inspired by a question of Michael Anshelevich [2012], to whom I am grateful for sending me an updated version of his paper.

Lemma 2.5. *Given $\tau, \rho \in \mathcal{M}_{\mathbb{R}}$, if $\tau \boxplus \rho \in \mathcal{F}\mathcal{D}(\boxplus, \mathbb{R})$, then $\rho \boxplus \tau^{\boxplus t}$ is defined for all $t \geq 0$ in the sense that $\varphi_{\rho} + t\varphi_{\tau}$ is the Voiculescu transform of a positive measure. Moreover, $F_{\rho \boxplus (\tau^{\boxplus t})} = F_{\rho}(F_{(\tau \boxplus \rho)^{\boxplus t}}(z))$.*

Proof. Let $\sigma = \tau \boxplus \rho$ and $\sigma_t = \sigma^{\boxplus t}$. By Proposition 2.1, there exists a unique probability measure $\mu_t \in \mathcal{M}_{\mathbb{R}}$ such that

$$F_{\mu_t} = F_{\rho}(F_{\sigma_t}(z)).$$

We claim that $\varphi_{\mu_t}(z) = \varphi_{\rho}(z) + t\varphi_{\tau}(z)$. Indeed, by Proposition 2.4, we have

$$F_{\sigma_t}^{-1}(z) - z = t \cdot \varphi_{\sigma}(z) = t \cdot \varphi_{\tau}(F_{\rho}(z)),$$

and we thus obtain

$$\begin{aligned} \varphi_{\mu_t}(F_{\rho}(z)) &= F_{\mu_t}^{-1}(F_{\rho}(z)) - F_{\rho}(z) = F_{\sigma_t}^{-1}(z) - F_{\rho}(z) = F_{\sigma_t}^{-1}(z) - z + z - F_{\rho}(z) \\ &= t \cdot \varphi_{\tau}(F_{\rho}(z)) + F_{\rho}^{-1}(F_{\rho}(z)) - F_{\rho}(z). \end{aligned}$$

By analytic continuation, we conclude that

$$\varphi_{\mu_t}(z) = t \cdot \varphi_{\tau}(z) + F_{\rho}^{-1}(z) - z = \varphi_{\rho}(z) + t \cdot \varphi_{\tau}(z),$$

which completes the proof. □

Remark. There are examples $\rho, \tau \in \mathcal{M}_{\mathbb{R}}$ such that $\tau \boxplus \rho \in \mathcal{F}\mathcal{D}(\boxplus, \mathbb{R})$ but τ does not lie in $\mathcal{F}\mathcal{D}(\boxplus, \mathbb{R})$ and is not a summand of ρ ; see [Anshelevich 2012].

Combining Proposition 2.4 and Lemma 2.5, we can reconstruct Nica–Speicher free convolution semigroups [Belinschi and Bercovici 2004; Nica and Speicher 1996] as follows.

Theorem 2.6. *Given $\mu \in \mathcal{M}_{\mathbb{R}}$, the measure $\mu^{\boxplus t} \in \mathcal{M}_{\mathbb{R}}$ is defined by $\varphi_{\mu^{\boxplus t}}(z) = t\varphi_{\mu}(z)$ for all $t > 1$. Moreover, there exists an analytic map $\omega_t : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ such that for $z \in \mathbb{C}^+$ the following conditions are satisfied:*

- $F_{\mu^{\boxplus t}}(z) = F_{\mu}(\omega_t(z))$.
- $\omega_t = F_{(\mu \boxplus \mu)^{\boxplus (t-1)}}(z)$.
- $\varphi_{(\mu \boxplus \mu)^{\boxplus (t-1)}} = (t - 1)(z - F_{\mu}(z))$ for all $t > 1$.

Let $H_t(z) = z + (t - 1)(z - F_{\mu}(z))$. By Proposition 2.2 and Theorem 2.6, we know that H_t is the left inverse of ω_t such that $H_t(\omega_t(z)) = z$ for $z \in \mathbb{C}^+$. Therefore, for $t > 1$, we can write

$$(2-3) \quad \omega_t(z) = z + \left(1 - \frac{1}{t}\right)(F_{\mu^{\boxplus t}}(z) - z), \quad z \in \mathbb{C}^+.$$

We deduce from (2-3) and the definition of ω_t in Theorem 2.6 that, for $t > 0$,

$$z - F_{(\mu \boxplus \mu)^{\boxplus t}}(z) = \left(1 - \frac{1}{t+1}\right)(z - F_{\mu^{\boxplus (t+1)}}(z)),$$

which implies that

$$(2-4) \quad (\mu \boxplus \mu)^{\boxplus t}(z) = (\mu^{\boxplus(t+1)})^{\uplus(t/(t+1))}.$$

Two formulas related to free Brownian motion. Given $\mu \in \mathcal{M}_{\mathbb{R}}$, we construct subordination functions ω_t as in Theorem 2.6. Let $\sigma_t = (\mu \boxplus \mu)^{\boxplus t} \in \mathcal{M}_{\mathbb{R}}$; then $\omega_{t+1} = F_{\sigma_t}(z)$ for $t > 0$. Given $\nu \in \mathcal{M}_{\mathbb{R}}$, let $\rho_t = \gamma_t \boxplus \nu$ and let $F_t = F_{\rho_t}(z)$ for all $t > 0$. From Proposition 2.4 and Theorem 2.6, we know that ρ_t and σ_t are \boxplus -infinitely divisible and their Voiculescu transforms are given by $\varphi_{\rho_t}(z) = tG_{\nu}$ and $\varphi_{\sigma_t}(z) = t(z - F_{\mu}(z))$. By comparing Voiculescu transform of ρ_t with Voiculescu transform of σ_t , we deduce that $F_t = \omega_{t+1}$ for some $t > 0$ if and only if $G_{\nu}(z) = z - F_{\mu}(z)$.

For $t > 0$, Belinschi and Nica [2008b] constructed a transformation $\mathbb{B}_t : \mathcal{M}_{\mathbb{R}} \rightarrow \mathcal{M}_{\mathbb{R}}$ such that

$$\mathbb{B}_t(\mu) = (\mu^{\boxplus(1+t)})^{\uplus 1/(1+t)} \quad \text{for } \mu \in \mathcal{M}_{\mathbb{R}}.$$

They also showed that \mathbb{B}_t is a semigroup and $\mathbb{B}_2 = \mathbb{B}$, where

$$\mathbb{B} : \mathcal{F}\mathcal{D}(\uplus, \mathbb{R}) \rightarrow \mathcal{F}\mathcal{D}(\boxplus, \mathbb{R})$$

is the bijective map from the \uplus -infinitely divisible distributions to the \boxplus -infinitely divisible distributions, discovered in the seminal [Bercovici and Pata 1999].

Theorem 2.7 [Belinschi and Nica 2008b]. *Let μ and ν be a pair of probability measures on the real line such that*

$$(2-5) \quad G_{\nu}(z) = z - F_{\mu}(z), \quad z \in \mathbb{C}^+.$$

Then we have

$$G_{\nu \boxplus \gamma_t}(z) = z - F_{\mathbb{B}_t(\mu)}(z), \quad t > 0, z \in \mathbb{C}^+.$$

Remark. Maassen [1992] has shown that, given $\mu, \nu \in \mathcal{M}_{\mathbb{R}}$ satisfying (2-5), μ has mean zero and variance one. Conversely, if $\mu \in \mathcal{M}_{\mathbb{R}}$ has mean zero and variance one, then there exists a unique $\nu \in \mathcal{M}_{\mathbb{R}}$ satisfying (2-5).

Given $\tau \in \mathcal{F}\mathcal{D}(\boxplus, \mathbb{R})$ and $\mu, \nu \in \mathcal{M}_{\mathbb{R}}$, we compare free Lévy process $\nu \boxplus \tau^{\boxplus t}$ and free convolution semigroup $\mu^{\boxplus(t+1)}$. If $\varphi_{\tau}(F_{\nu}(z)) = z - F_{\mu}(z)$, then $\tau \boxplus \nu = \mu \boxplus \mu$, which implies that subordination function of $F_{\nu \boxplus (\tau^{\boxplus t})}$ to F_{ν} is the same as the subordination function of $F_{\mu^{\boxplus(t+1)}}$ to F_{μ} . The following theorem generalizes Theorem 2.7. The argument is similar to the proof of Theorem 1.6 in [Belinschi and Nica 2008b] (see also the proof of Lemma 3 in [Anshelevich 2012]); therefore we omit the proof.

Theorem 2.8. *Given $\tau \in \mathcal{F}\mathcal{D}(\boxplus, \mathbb{R})$, and let μ and ν be a pair of probability measures on the real line such that*

$$\varphi_{\tau}(F_{\nu}(z)) = z - F_{\mu}(z), \quad z \in \mathbb{C}^+.$$

Then we have

$$\varphi_\tau(F_{\nu \boxplus (\tau \boxplus \mu)}(z)) = z - F_{\mathbb{B}_t(\mu)}(z), \quad t > 0, z \in \mathbb{C}^+.$$

Remark. Let $\tau = \gamma_{a,b}$ be the semicircular distribution with mean a and variance b , and let μ, ν be a pair of probability measures on the real line such that

$$\varphi_\tau(F_\nu(z)) = z - F_\mu(z).$$

We first compute

$$(2-6) \quad F_\mu(z) = z - \varphi_\tau(F_\nu(z)) = z - \left(a + \frac{b}{z}\right) \circ F_\nu(z) = z - a - bG_\nu(z).$$

Then, by Theorem 2.8,

$$(2-7) \quad \begin{aligned} F_{\mathbb{B}_t(\mu)}(z) &= z - \varphi_\tau(F_{\nu \boxplus \tau \boxplus \mu}(z)) \\ &= z - \left(\left(a + \frac{b}{z}\right) \circ F_{\nu \boxplus \gamma_{a,b}}\right)(z) = z - a - bG_{\nu \boxplus \gamma_{a,b}}(z). \end{aligned}$$

By (2-7) and the definition of Boolean convolution, we obtain

$$(2-8) \quad F_{(\mathbb{B}_t(\mu))^{\boxplus t}}(z) = z - ta - tbG_{\nu \boxplus \gamma_{a,b}}(z).$$

Equations (2-6), (2-7) and (2-8) were studied in [Anshelevich 2012]. As shown there (Proposition 1 and Example 1), we have $(\mathbb{B}_t(\mu))^{\boxplus t} \in \mathcal{F}\mathcal{D}(\boxplus, \mathbb{R})$, and $(\mathbb{B}_t(\mu))^{\boxplus t} = (\tau \boxplus \nu)^{\boxplus t} = (\mu \boxplus \mu)^{\boxplus t}$. In fact, for all $\mu \in \mathcal{M}_{\mathbb{R}}$, we can deduce from (2-4) and the identity $(\mathbb{B}_t(\mu))^{\boxplus t} = (\mu^{\boxplus(1+t)})^{\boxplus t/(1+t)}$ that $(\mathbb{B}_t(\mu))^{\boxplus t}$ is the measure associated with the subordination function of $\mu^{\boxplus(1+t)}$ with respect to μ : $(\mathbb{B}_t(\mu))^{\boxplus t} = (\mu \boxplus \mu)^{\boxplus t}$.

3. Multiplicative free convolution and multiplicative Boolean convolution on $\mathcal{M}_{\mathbb{T}}$

Given any two probability measures μ, ν on \mathbb{T} , the unit circle of \mathbb{C} , we can define their multiplicative free convolution. We first recall the calculation of the multiplicative free convolution of two measures on \mathbb{T} with nonzero means. Given $\mu \in \mathcal{M}_{\mathbb{T}}$, we define

$$\psi_\mu(z) = \int_{\mathbb{T}} \frac{tz}{1-tz} d\mu(t)$$

and set $\eta_\mu(z) = \psi_\mu(z)/(1 + \psi_\mu(z))$. The following proposition characterizes the η -transforms of probability measures on \mathbb{T} .

Proposition 3.1 [Belinschi and Bercovici 2005]. *If $\eta : \mathbb{D} \rightarrow \mathbb{C}$ is an analytic function, the following assertions are equivalent.*

- (1) *There exists a probability measure $\mu \in \mathcal{M}_{\mathbb{T}}$ such that $\eta = \eta_\mu$.*
- (2) *$\eta(0) = 0$, and $|\eta(z)| < 1$ holds for all $z \in \mathbb{D}$.*

If $\mu \in \mathcal{M}_T \cap \mathcal{M}_*$, then $\eta'_\mu(0) = \int_T t d\mu(t) \neq 0$. Therefore, the inverse η_μ^{-1} is defined in a neighborhood of zero. We set $\Sigma_\mu(z) = \eta_\mu^{-1}(z)/z$. Given $\mu, \nu \in \mathcal{M}_T \cap \mathcal{M}_*$, their multiplicative free convolution, which is denoted by $\mu \boxtimes \nu$, is the unique probability measure in $\mathcal{M}_T \cap \mathcal{M}_*$ such that

$$(3-1) \quad \Sigma_{\mu \boxtimes \nu}(z) = \Sigma_\mu(z) \Sigma_\nu(z)$$

for z in a neighborhood of zero.

It is known (see [Biane 1998; Belinschi and Bercovici 2007]) that there exist two analytic functions $\omega_1, \omega_2 : \mathbb{D} \rightarrow \mathbb{D}$ such that

- (1) $\omega_1(0) = \omega_2(0) = 0$,
- (2) $\eta_{\mu \boxtimes \nu}(z) = \eta_\mu(\omega_1(z)) = \eta_\nu(\omega_2(z))$.

A probability measure $\mu \in \mathcal{M}_T$ is said to be \boxtimes -infinitely divisible if for any positive integer n , there exists $\mu_n \in \mathcal{M}_T$ such that $\mu = (\mu_n)^{\boxtimes n} = \mu_n \boxtimes \cdots \boxtimes \mu_n$. It is shown in [Bercovici and Voiculescu 1992] that if $\mu \in \mathcal{M}_T \setminus \mathcal{M}_*$ is \boxtimes -infinitely divisible, then μ is the Haar measure on T ; and $\mu \in \mathcal{M}_T \cap \mathcal{M}_*$ is \boxtimes -infinitely divisible if and only if there exists a function

$$(3-2) \quad u(z) = \alpha i + \int_T \frac{1 + tz}{1 - tz} d\sigma(t),$$

such that $\Sigma_\mu(z) = \exp(u(z))$, where $\alpha \in \mathbb{R}$ and σ is a finite positive measure on T . Equation (3-2) is the analogue of the Lévy–Khintchine formula for multiplicative free convolution on T . The analogue of the normal distribution in this context is given by $\Sigma_{\lambda_t}(z) = \exp((t/2)(1 + z)/(1 - z))$. Denote by $\mathcal{F}\mathcal{D}(\boxtimes, T)$ the set of all \boxtimes -infinitely divisible measures on T .

Lemma 3.2. *Let $\mu \in \mathcal{M}_T \cap \mathcal{M}_*$ be \boxtimes -infinitely divisible.*

- (1) *The function $H(z) = z\Sigma_\mu(z)$ is the left inverse of $\eta_\mu(z)$, that is $H(\eta_\mu(z)) = z$ for all $z \in \mathbb{D}$.*
- (2) *The function η_μ extends to be a continuous function on $\overline{\mathbb{D}}$, and η_μ is one-to-one on $\overline{\mathbb{D}}$.*
- (3) *The set $\{z \in \mathbb{D} : |z\Sigma_\mu(z)| < 1\}$ is a simply connected domain which coincides with $\{\eta_\mu(z) : z \in \mathbb{D}\}$, and its boundary is $\eta_\mu(T)$ which is a simple closed curve.*

Proof. Observing that $H(\eta_\mu(z)) = z$ is valid in a neighborhood of zero, we obtain assertion (1) by analytic continuation.

Note that $H : \mathbb{D} \rightarrow \mathbb{C}$ satisfies the conditions in [Belinschi and Bercovici 2005, Proposition 4.5] and thus assertions (2) and (3) hold. □

Multiplicative free Brownian motion. For $\mu \in \mathcal{M}_{\mathbb{T}}$ and $t > 0$, we study the multiplicative free convolution $\mu \boxtimes \lambda_t$. We first concentrate on the case when μ has nonzero mean. The case when μ has mean zero will be studied in Subsection 3.2.

We start with the following result which is the multiplicative version of [Biane 1997b, Lemma 1].

Lemma 3.3. *Given $\mu, \nu \in \mathcal{M}_{\mathbb{T}} \cap \mathcal{M}_*$, we have*

$$\eta_{\mu}(z) = \eta_{\mu \boxtimes \nu}(z \cdot \Sigma_{\nu}(\eta_{\mu}(z)))$$

for z in a neighborhood of zero.

Proof. From (3-1), we find that

$$(3-3) \quad \frac{\eta_{\mu \boxtimes \nu}^{-1}(z)}{z} = \frac{\eta_{\mu}^{-1}(z)}{z} \cdot \frac{\eta_{\nu}^{-1}(z)}{z}$$

for z in a neighborhood of zero, which we denote by D_0 . We choose a subdomain $D_1 \subset D_0$ such that $\eta_{\mu}(D_1) \subset D_0$. Replacing z by $\eta_{\mu}(z)$ in (3-3), we obtain

$$(3-4) \quad \frac{\eta_{\mu \boxtimes \nu}^{-1}(\eta_{\mu}(z))}{\eta_{\mu}(z)} = \frac{\eta_{\mu}^{-1}(\eta_{\mu}(z))}{\eta_{\mu}(z)} \cdot \frac{\eta_{\nu}^{-1}(\eta_{\mu}(z))}{\eta_{\mu}(z)} = \frac{z}{\eta_{\mu}(z)} \cdot \frac{\eta_{\nu}^{-1}(\eta_{\mu}(z))}{\eta_{\mu}(z)}$$

for $z \in D_1$. Note that $\eta_{\nu}^{-1}(z) = z \Sigma_{\nu}(z)$ for $z \in D_0$, and we then rewrite (3-4) as

$$(3-5) \quad \eta_{\mu \boxtimes \nu}^{-1}(\eta_{\mu}(z)) = z \Sigma_{\nu}(\eta_{\mu}(z)).$$

Applying $\eta_{\mu \boxtimes \nu}$ on both sides of (3-5) yields

$$\eta_{\mu}(z) = \eta_{\mu \boxtimes \nu}(z \Sigma_{\nu}(\eta_{\mu}(z)))$$

for z in a neighborhood of zero D_1 . □

For any $t > 0$, we denote by $\eta_t : \mathbb{D} \rightarrow \mathbb{D}$ the subordination function of $\mu \boxtimes \lambda_t$ with respect to μ . Since $\eta_t : \mathbb{D} \rightarrow \mathbb{D}$ is analytic and $\eta_t(0) = 0$, Proposition 3.1 implies the existence of a probability measure ρ_t such that $\eta_{\rho_t}(z) = \eta_t(z)$.

Lemma 3.4. *The measure ρ_t is \boxtimes -infinitely divisible and its Σ -transform is*

$$\Sigma_{\rho_t}(z) = \Sigma_{\lambda_t}(\eta_{\mu}(z)).$$

Proof. Define analytic function $\Phi_t : \mathbb{D} \rightarrow \mathbb{C}$ by $\Phi_t(z) := z \Sigma_{\lambda_t}(\eta_{\mu}(z))$ for all $t > 0$. By Lemma 3.3, we have

$$\eta_{\mu}(z) = \eta_{\mu \boxtimes \lambda_t}(z \Sigma_{\lambda_t}(\eta_{\mu}(z))) = \eta_{\mu \boxtimes \lambda_t}(\Phi_t(z))$$

which implies that

$$\eta_{\mu \boxtimes \lambda_t}(z) = \eta_{\mu}(\eta_t(z)) = \eta_{\mu \boxtimes \lambda_t}(\Phi_t(\eta_t(z))).$$

Since $\eta_{\mu \boxtimes \lambda_t}$ is invertible in a neighborhood of zero, we have $\Phi_t(\eta_t(z)) = z$ in a neighborhood of zero.

We thus obtain $\eta_{\rho_t}^{-1}(z) = \eta_t^{-1}(z) = \Phi_t(z)$ holds for z in a neighborhood of zero, which yields that

$$(3-6) \quad \Sigma_{\rho_t}(z) = \frac{\eta_{\rho_t}^{-1}(z)}{z} = \Sigma_{\lambda_t}(\eta_\mu(z)).$$

By the definition of the ψ - and η -transforms, we have

$$(3-7) \quad \Sigma_{\lambda_t}(\eta_\mu(z)) = \exp\left(\frac{t}{2} \int_{\mathbb{T}} \frac{1 + \xi z}{1 - \xi z} d\mu(\xi)\right).$$

The real part of the integrand in (3-7) is positive for all $z \in \mathbb{D}$; thus the assertion follows from (3-6) and [Bercovici and Voiculescu 1992, Theorem 6.7]. \square

By (3-6), the right hand side of (3-7) is the Lévy–Khintchine representation of ρ_t . We can also write η_t in terms of λ_t and $\mu \boxtimes \lambda_t$. Replacing z by $\eta_{\mu \boxtimes \lambda_t}(z)$ in the equation

$$\frac{\eta_{\mu \boxtimes \lambda_t}^{-1}(z)}{z} = \frac{\eta_\mu^{-1}(z)}{z} \cdot \frac{\eta_{\lambda_t}^{-1}(z)}{z},$$

we obtain

$$\frac{z}{\eta_{\mu \boxtimes \lambda_t}(z)} = \frac{\eta_t(z)}{\eta_{\mu \boxtimes \lambda_t}(z)} \cdot \Sigma_{\lambda_t}(\eta_{\mu \boxtimes \lambda_t}(z)),$$

which shows that

$$(3-8) \quad \eta_t(z) = \frac{z}{\Sigma_{\lambda_t}(\eta_{\mu \boxtimes \lambda_t}(z))}.$$

Modified \mathcal{S} -transform and subordination functions. Given $\mu \in \mathcal{M}_{\mathbb{T}} \setminus \mathcal{M}_*$ and $\nu \in \mathcal{M}_{\mathbb{T}} \cap \mathcal{M}_*$, it is known from [Biane 1998] that $\eta_{\mu \boxtimes \nu}$ is subordinated to η_μ and η_ν . The subordination function for this case is generally not unique (see Example 3.5 below). However, we show that there is a nice subordination function, which we call the principal subordination function, uniquely determined by certain conditions. Using the principal subordination function, results related to subordination function in the case $\mu, \nu \in \mathcal{M}_{\mathbb{T}} \cap \mathcal{M}_*$ can be extended to the case where $\mu \in \mathcal{M}_{\mathbb{T}} \setminus \mathcal{M}_*$ and $\nu \in \mathcal{M}_{\mathbb{T}} \cap \mathcal{M}_*$.

Let us first give an example which illustrates the non-uniqueness of subordination functions.

Example 3.5. For $k \in \mathbb{N}$, and let $\lambda^{(k)} = 1/k \sum_{n=0}^{k-1} \delta_{z_n}$, where $z_n = e^{2\pi i n/k}$. We have $\psi_{\lambda^{(k)}}(z) = z^k/(1 - z^k)$ and $\eta_{\lambda^{(k)}}(z) = z^k$. Given $\nu \in \mathcal{M}_{\mathbb{T}} \cap \mathcal{M}_*$, if $\omega : \mathbb{D} \rightarrow \mathbb{D}$ is a subordination function of $\eta_{\lambda^{(k)} \boxtimes \nu}$ with respect to $\eta_{\lambda^{(k)}}$, then $\omega^{(n)}(z) := e^{2\pi i n/k} \omega(z)$ is also a subordination function of $\eta_{\lambda^{(k)} \boxtimes \nu}$ with respect to $\eta_{\lambda^{(k)}}$ for all integer $0 < n < k$.

We now introduce the modified \mathcal{S} -transform. Given two free random variables x and y in a W^* -probability space (\mathcal{A}, ϕ) , such that $\phi(x) = 0$ and $\phi(y) \neq 0$, we can not directly apply Voiculescu's \mathcal{S} -transform (Σ -transform) to calculate the distribution of xy . N. Raj Rao and R. Speicher [2007] introduce a new transform, which we call the modified \mathcal{S} -transform, to deal with this case. They apply the modified \mathcal{S} -transform to study the distribution of xy where x, y are free self-adjoint random variables such that $\phi(x) = 0, \phi(y) \neq 0$. For nonzero self-adjoint operator x , we have $\phi(x^2) \neq 0$. Assume that $\phi(x) = \dots = \phi(x^{k-1}) = 0$ and $\phi(x^k) \neq 0$. Arizmendi [2012] observe that we can calculate the distribution of xy using the idea in [Rao and Speicher 2007]. We present the details of their work for reader's convenience.

We first recall some definitions. For $\mu \in \mathcal{M}_T \cap \mathcal{M}_*$, we have $\psi_\mu(0) = 0$ and $\psi'_\mu(0) \neq 0$. It follows that there exists a function $\chi_\mu(z)$, which is analytic in a neighborhood of zero, such that

$$\psi_\mu(\chi_\mu(z)) = \chi_\mu(\psi_\mu(z)) = z$$

for sufficiently small z . The usual \mathcal{S} -transform is defined by

$$\mathcal{S}_\mu(z) = \frac{z+1}{z} \chi_\mu(z).$$

We then have

$$\Sigma_\mu(z) = \mathcal{S}_\mu\left(\frac{z}{1-z}\right), \quad \eta_\mu^{-1}(z) = \chi\left(\frac{z}{1-z}\right).$$

We set

$$\mathcal{M}_T^k = \left\{ \mu \in \mathcal{M}_T : \int_T t^n d\mu(t) = 0 \text{ for } 1 \leq n < k, \text{ and } \int_T t^k d\mu(t) \neq 0 \right\}.$$

Then, for $\mu \in \mathcal{M}_T^k$, we have

$$(3-9) \quad \begin{cases} \psi'_\mu(0) = \dots = \psi_\mu^{(k-1)}(0) = 0 = \eta'_\mu(0) = \dots = \eta_\mu^{(k-1)}(0), \\ \psi_\mu^{(k)}(0) \neq 0 \text{ and } \eta_\mu^{(k)} \neq 0. \end{cases}$$

For $\mu \in \mathcal{M}_T^k, \nu \in \mathcal{M}_T \cap \mathcal{M}_*$, from the definition of free independence, we deduce that $\mu \boxtimes \nu \in \mathcal{M}_T^k$.

We recall the following classical result in complex analysis; see, for example, [Hille 1959].

Theorem 3.6. *If $f(z)$ is holomorphic in $|z| < R$, and suppose that*

$$f(0) = f'(0) = \dots = f^{(k-1)}(0) = 0, \quad f^{(k)} \neq 0.$$

Then for small values of $w \neq 0$ the equation

$$f(z) = w$$

has k roots $z_1(w), \dots, z_k(w)$, which tend to zero when w tends to zero. Moreover, there exists a function $g(w)$, holomorphic for w sufficiently small with $g(0) = 0$ and $g'(0) \neq 0$, such that for any fixed small values $w \neq 0$,

$$z_j(w) = g(\omega^j w^{1/k}), \quad \omega = e^{2\pi i/k}, \quad 0 \leq \arg \omega^{1/k} < \frac{2\pi}{k},$$

if we put those roots in a certain order.

Remark. The converse of Theorem 3.6 is also true. More precisely, given a function $g(w)$ that is holomorphic for small enough w and satisfies $g(0) = 0$ and $g'(0) \neq 0$, if we set

$$z_j(w) = g(\omega^j w^{1/k}), \quad \omega = e^{2\pi i/k}, \quad 0 \leq \arg \omega^{1/k} < \frac{2\pi}{k}$$

for $j = 1, \dots, k$, then $z_1(w), \dots, z_k(w)$ are the roots of the equation

$$F^k(z) = w,$$

where F is a holomorphic function defined in a neighborhood of the zero such that $F(g(w)) = w$.

For $j = 1, \dots, k$, denote $D_{j,r} = \{\omega^j z : 0 \leq \arg(z) < 2\pi/k, |z| < r\}$. We record the following result for convenience.

Proposition 3.7. *Under the assumption of Theorem 3.6, we have $z_j(f(z)) = z$ for $z \in g(D_{j,r})$ for r sufficiently small.*

Given $\mu \in \mathcal{M}_{\mathbb{T}}^k$, and by Theorem 3.6, we know that there exist k functions represented by the power series in $z^{1/k}$ such that

$$(3-10) \quad \psi_{\mu}(\chi_{\mu}^{(j)}(z)) = z,$$

for z sufficiently small. Moreover, there exists a function $g_{\mu}(w)$ holomorphic in a neighborhood of the zero, such that for $j = 1, \dots, k$,

$$\chi_{\mu}^{(j)}(z) = g_{\mu}(\omega^j z^{1/k}),$$

where $\omega = e^{2\pi i/k}, 0 \leq \arg z^{1/k} < 2\pi/k$.

Definition 3.8. Given $\mu \in \mathcal{M}_{\mathbb{T}}^k$. Let $\chi_{\mu}^{(j)}$ be the inverse function of ψ_{μ} in (3-10), the modified \mathcal{G} -transform of μ is k functions $\mathcal{G}_{\mu}^{(1)}(z), \dots, \mathcal{G}_{\mu}^{(k)}(z)$, such that for $j = 1, \dots, k$,

$$\mathcal{G}_{\mu}^{(j)}(z) = \chi_{\mu}^{(j)}(z) \cdot \frac{1+z}{z}.$$

Given $\mu \in \mathcal{M}_{\mathbb{T}}^k$ and $\nu \in \mathcal{M}_{\mathbb{T}} \cap \mathcal{M}_{*}$, we set

$$\mathcal{G}^{(j)}(z) = \mathcal{G}_{\mu}^{(j)}(z) \cdot \mathcal{G}_{\nu}(z)$$

and compute

$$(3-11) \quad \begin{aligned} \chi^{(j)}(z) &= \mathcal{G}^{(j)}(z) \cdot \frac{z}{1+z} = \mathcal{G}_\mu^{(j)}(z) \cdot \mathcal{G}_\nu(z) \cdot \frac{z}{1+z} \\ &= \chi_\mu^{(j)}(z) \cdot \mathcal{G}_\nu(z) = g_\mu(\omega^j z^{1/k}) \cdot \mathcal{G}_\nu(z) = g(\omega^j z^{1/k}), \end{aligned}$$

where $g(z) = g_\mu(z) \cdot \mathcal{G}_\nu(z^k)$ is a function such that $g(0) = 0$, $g'(0) \neq 0$. From the remark after Theorem 3.6, we deduce that for different j , there exists the same left inverse ψ such that $\psi(\chi^{(j)}(z)) = z$. Therefore, we have the following proposition.

Proposition 3.9. *Given $\mu \in \mathcal{M}_T^k$ and $\nu \in \mathcal{M}_T \cap \mathcal{M}_*$, for $1 \leq j \leq k$, let*

$$\mathcal{G}^{(j)}(z) = \mathcal{G}_\mu^{(j)}(z) \cdot \mathcal{G}_\nu(z), \quad \chi^{(j)}(z) = \mathcal{G}^{(j)}(z) \cdot \frac{z}{1+z}.$$

Then there exists a unique holomorphic function ψ defined in a neighborhood of the zero such that

$$\psi((\chi^{(j)})(z)) = z.$$

Theorem 3.10 [Rao and Speicher 2007; Arizmendi 2012]. *Given $\mu \in \mathcal{M}_T^k$ and $\nu \in \mathcal{M}_T \cap \mathcal{M}_*$, we have*

$$(3-12) \quad \mathcal{G}_{\mu \boxtimes \nu}^{(j)}(z) = \mathcal{G}_\mu^{(j)}(z) \cdot \mathcal{G}_\nu(z), \quad j = 1, \dots, k,$$

where the modified \mathcal{G} -transforms are listed in a certain order.

Because of Proposition 3.9 and Theorem 3.10, for fixed $\mu \in \mathcal{M}_T^k$ and $\nu \in \mathcal{M}_T \cap \mathcal{M}_*$, we denote

$$(3-13) \quad \psi(z) = \psi_{\mu \boxtimes \nu}(z) \quad \text{and} \quad \chi^{(j)}(z) = \chi_{\mu \boxtimes \nu}^{(j)}(z),$$

and we also denote $g(z) = g_\mu(z) \cdot \mathcal{G}_\nu(z^k)$ as in (3-11).

Given $\mu \in \mathcal{M}_T^k$, $\nu \in \mathcal{M}_T \cap \mathcal{M}_*$, we set $\iota_\mu^{(j)}(z) = \chi_\mu^{(j)}(z/(1-z))$ and $\iota_\nu(z) = \chi_\nu(z/(1-z))$. Theorem 3.10 implies that

$$(3-14) \quad \chi_{\mu \boxtimes \nu}^{(j)}(z) = \chi_\mu^{(j)}(z) \cdot \chi_\nu(z) \cdot \frac{1+z}{z}.$$

We also have $\chi_\mu^{(j)}(z) = g_\mu(\omega^j z^{1/k})$ and $\chi_{\mu \boxtimes \nu}^{(j)}(z) = g_\mu(\omega^j z^{1/k}) \cdot \mathcal{G}_\nu(z) = g(\omega^j z^{1/k})$. Substituting z by $\psi_{\mu \boxtimes \nu}(z)$ in (3-14), and applying Proposition 3.7, we find that

$$z = \chi_{\mu \boxtimes \nu}^{(j)}(\psi_{\mu \boxtimes \nu}(z)) = \chi_\mu^{(j)}(\psi_{\mu \boxtimes \nu}(z)) \cdot \chi_\nu(\psi_{\mu \boxtimes \nu}(z)) \cdot \frac{1 + \psi_{\mu \boxtimes \nu}(z)}{\psi_{\mu \boxtimes \nu}(z)},$$

where $z \in g(D_{j,r})$ for r sufficiently small. Thus

$$(3-15) \quad z\eta_{\mu \boxtimes \nu} = \iota_\mu^{(j)}(\eta_{\mu \boxtimes \nu}) \cdot \iota_\nu(\eta_{\mu \boxtimes \nu})$$

holds in the same domain.

We can now utilize the argument in [Belinschi and Bercovici 2007] to prove the existence of subordination function of $\eta_{\mu \boxtimes \nu}$ with respect to η_μ for $\mu \in \mathcal{M}_T^k, \nu \in \mathcal{M}_T \cap \mathcal{M}_*$. Note that part of the following result is known in [Biane 1998].

Theorem 3.11. *Given $\mu \in \mathcal{M}_T^k, \nu \in \mathcal{M}_T \cap \mathcal{M}_*$, there exists two unique analytic functions $\omega_1, \omega_2 : \mathbb{D} \rightarrow \mathbb{D}$ such that*

- (1) $\omega_1(0) = \omega_2(0) = 0$;
- (2) $\eta_{\mu \boxtimes \nu}(z) = \eta_\mu(\omega_1(z)) = \eta_\nu(\omega_2(z))$,
- (3) $\omega_1(z)\omega_2(z) = z\eta_{\mu \boxtimes \nu}(z)$ for all $z \in \mathbb{D}$.

Proof. Since $\eta_\mu(0) = 0, \eta_\nu(0) = 0$, we can write $\eta_\mu(z) = zf_1(z), \eta_\nu(z) = zf_2(z)$ for two analytic functions $f_1, f_2 : \mathbb{D} \rightarrow \mathbb{D}$. Fix $1 \leq j \leq k$, set $\omega_1(z) = \iota_\mu^{(j)}(\eta_{\mu \boxtimes \nu}(z)), \omega_2(z) = \iota_\nu(\eta_{\mu \boxtimes \nu}(z))$ defined in $g(D_{j,r})$ for r sufficiently small.

By (3-15), we have

$$z\eta_{\mu \boxtimes \nu}(z) = \omega_1(z)\omega_2(z)$$

for $z \in g(D_{j,r})$. We thus obtain

$$\omega_1(z) = \frac{z\eta_{\mu \boxtimes \nu}}{\omega_2(z)} = \frac{z\eta_\nu(\omega_2(z))}{\omega_2(z)} = zf_2(\omega_2(z)).$$

Similarly, we have $\omega_2(z) = zf_1(\omega_1(z))$ for $z \in g(D_{j,r})$. Regarding $\omega_1(z), \omega_2(z)$ as Denjoy–Wolff points, the same argument as in [Belinschi and Bercovici 2007] implies ω_1, ω_2 can be extended analytically to \mathbb{D} . By the uniqueness of Denjoy–Wolff points, ω_1, ω_2 does not depend on the choice of j .

By the definitions of $\iota_\mu^{(j)}, \iota_\nu$, we have $\eta_\mu(\omega_1(z)) = \eta_\nu(\omega_2(z)) = \eta_{\mu \boxtimes \nu}$ for z in $g(D_{j,r})$. Thus (2) and (3) hold by analytic continuation. Since $\eta'_\nu(0) \neq 0, \eta_\nu$ is locally invertible near the origin and therefore ω_2 is unique. Finally (3) implies the uniqueness of ω_1 . □

Since $\mu \in \mathcal{M}_T^k$ and $\mu \boxtimes \nu \in \mathcal{M}_T^k$, we have $\omega'_1(0) \neq 0$, where ω_1 is given in Theorem 3.11.

Definition 3.12. For $\mu \in \mathcal{M}_T^k, \nu \in \mathcal{M}_T \cap \mathcal{M}_*$, the subordination ω_1 satisfying the relations (1), (2) and (3) in Theorem 3.11 is called the *principal* subordination function of $\eta_{\mu \boxtimes \nu}$ with respect to η_μ . The measure $\rho \in \mathcal{M}_T \cap \mathcal{M}_*$ satisfying $\eta_\rho(z) = \omega_1(z)$ is called the *principal* subordination distribution of $\eta_{\mu \boxtimes \nu}$ with respect to η_μ .

Note that for $\mu, \nu \in \mathcal{M}_T \cap \mathcal{M}_*$, the principal subordination function of $\eta_{\mu \boxtimes \nu}$ with respect to η_μ is the usual subordination function.

The following result might be obtained by approximation. We provide a direct proof.

Corollary 3.13. *Given $\mu \in \mathcal{M}_T^k$, $\nu \in \mathcal{M}_T \cap \mathcal{M}_*$, let ρ be the principal subordination distribution of $\eta_{\mu \boxtimes \nu}$ with respect to η_μ , we have*

$$\Sigma_\rho(z) = \Sigma_\nu(\eta_\mu(z)).$$

In particular, if $\nu \in \mathcal{F}\mathcal{D}(\boxtimes, T)$, we have $\rho \in \mathcal{F}\mathcal{D}(\boxtimes, T)$.

Proof. By choosing a sequence $\mu_n \in \mathcal{M}_T \cap \mathcal{M}_*$ such that μ_n converges to μ weakly, Lemma 3.3 implies

$$\eta_\mu(z) = \eta_{\mu \boxtimes \nu}(z \Sigma_\nu(\eta_\mu(z)))$$

for z in a neighborhood of zero.

Set $\Phi(z) = z \Sigma_\nu(\eta_\mu(z)) = z \cdot \mathcal{S}_\nu(\psi_\mu(z))$, and we thus have

$$(3-16) \quad \eta_{\mu \boxtimes \nu}(z) = \eta_\mu(\omega_1(z)) = \eta_{\mu \boxtimes \nu}(\Phi(\omega_1(z))) = \eta_{\mu \boxtimes \nu}(\Phi(\eta_\rho(z))).$$

Fix $1 \leq j \leq k$, we claim that if $z \in g(D_{j,r})$, then $\Phi(\omega_1(z)) = z$. Indeed, for $0 \leq \arg(w^{1/k}) < 2\pi/k$ and $z = g(\omega^j w^{1/k}) = \chi^{(j)}(w)$, using the construction of ω_1 in Theorem 3.11, we have

$$(3-17) \quad \omega_1(z) = \iota_\mu^{(j)}(\eta_{\mu \boxtimes \nu}(z)) = \chi_\mu^{(j)}(\psi_{\mu \boxtimes \nu}(z)).$$

From (3-11) and (3-13), we have

$$(3-18) \quad \chi^{(j)}(w) = g(\omega^j w^{1/k}) \quad \text{and} \quad \psi_{\mu \boxtimes \nu}(\chi^{(j)}(w)) = w.$$

Equations (3-17) and (3-18) imply that $\omega_1(g(\omega^j w^{1/k})) = \chi_\mu^{(j)}(w)$.

Note that $\psi_\mu(\omega_1(g(\omega^j w^{1/k}))) = \psi_{\mu \boxtimes \nu}(g(\omega^j w^{1/k})) = w$. Thus we obtain

$$\Phi(\omega_1(z)) = \Phi(\omega_1(g(\omega^j w^{1/k}))) = \chi_\mu^{(j)}(w) \mathcal{S}_\nu(w) = \chi^{(j)}(w) = z.$$

The above claim, (3-16) and Proposition 3.7 imply

$$z = \Phi(\omega_1(z)) = \Phi(\eta_\rho(z))$$

for $z \in g(D_{j,r})$. We conclude that $\Sigma_\rho(z) = \Phi(z)/z = \Sigma_\nu(\eta_\mu(z))$ for z in a small neighborhood of zero by applying the above argument for all $1 \leq j \leq k$.

If $\nu \in \mathcal{F}\mathcal{D}(\boxtimes, T)$, then by [Bercovici and Voiculescu 1993, Theorem 6.7], there exists an analytic function $u(z)$ defined in \mathbb{D} such that $\Sigma_\nu(z) = \exp(u(z))$ and $\Re u(z) \geq 0$ for all $z \in \mathbb{D}$. Thus $\Sigma_\nu(\eta_\mu(z)) = \exp(u(\eta_\mu(z)))$ and $\Re(u(\eta_\mu(z))) \geq 0$ for all $z \in \mathbb{D}$, and then the second assertion follows from [Bercovici and Voiculescu 1993, Theorem 6.7]. □

Remark. If $k = 1$, noticing that $\mathcal{M}_T^k = \mathcal{M}_T \cap \mathcal{M}_*$, the modified \mathcal{S} -transform is the usual \mathcal{S} -transform. We see that Corollary 3.14 holds when $\mu, \nu \in \mathcal{M}_T \cap \mathcal{M}_*$.

The following result is the multiplicative analogue of Lemma 2.6.

Proposition 3.14. *Given $\rho, \tau \in \mathcal{M}_{\mathbb{T}} \cap \mathcal{M}_{*}$, let σ be a measure in $\mathcal{M}_{\mathbb{T}} \cap \mathcal{M}_{*}$ such that $\eta_{\rho \boxtimes \tau}(z) = \eta_{\rho}(\eta_{\sigma}(z))$. If $\sigma \in \mathcal{I}^{\mathbb{D}}(\boxtimes, \mathbb{T})$, then $\rho \boxtimes \tau^{\boxtimes t}$ can be defined for all $t \geq 0$ in the sense that $\Sigma_{\rho \boxtimes (\tau^{\boxtimes t})}(z) = \Sigma_{\rho}(z)(\Sigma_{\tau}(z))^t$.*

Proof. For $t > 0$, there exists $\mu_t \in \mathcal{M}_{\mathbb{T}} \cap \mathcal{M}_{*}$ such that

$$\eta_{\mu_t}(z) = \eta_{\rho}(\eta_{\sigma^{\boxtimes t}}).$$

Using a similar argument as in the proof of Lemma 2.6 and applying Corollary 3.13, we can find that $\Sigma_{\mu_t}(z) = \Sigma_{\rho}(z)(\Sigma_{\tau}(z))^t$. □

Semigroups related to multiplicative free convolution. Recall that

$$\mathcal{M}_{\mathbb{T}}^* = \{\mu \in \mathcal{M}_{\mathbb{T}} \cap \mathcal{M}_{*} : \eta_{\mu}(z) \neq 0 \text{ for all } z \in \mathbb{D} \setminus \{0\}\}.$$

Let $\mu \in \mathcal{M}_{\mathbb{T}}^*$ and $t > 1$ be given, and let u be an analytic function such that $z/(\eta_{\mu}(z)) = e^{u(z)}$ for z in a neighborhood of zero. Set $H_t(z) = ze^{(t-1)u(z)} = z[z/(\eta_{\mu}(z))]^{t-1}$. It is shown in [Belinschi and Bercovici 2005] that H_t has a right inverse $\omega_t : \mathbb{D} \rightarrow \mathbb{D}$ such that $H_t(\omega_t(z)) = z$, and there exists a probability measure $\mu^{\boxtimes t} \in \mathcal{M}_{\mathbb{T}}^*$ such that

- (1) $\eta_{\mu^{\boxtimes t}}(z) = \eta_{\mu}(\omega_t(z))$ and $\Sigma_{\mu^{\boxtimes t}}(z) = (\Sigma_{\mu}(z))^t$,
- (2) $\omega_t(z) = \eta_{\mu^{\boxtimes t}}(z)[z/\eta_{\mu^{\boxtimes t}}(z)]^{1/t}$ for $z \in \mathbb{D}$, where the power is chosen so that the equation holds.

Observe that for each $t > 0$, by Proposition 3.1, there exists a probability measure $\sigma_t \in \mathcal{M}_{\mathbb{T}}$ such that $\eta_{\sigma_t}(z) = \omega_{t+1}(z)$. It turns out that σ_t is \boxtimes -infinitely divisible and its Σ -transform is $\Sigma_{\sigma_t}(z) = [z/\eta_{\mu}(z)]^t$, which can be obtained by applying the same argument as in the proof of Lemma 3.4.

The following result is a partial converse of [Belinschi and Bercovici 2005, Theorem 3.5].

Theorem 3.15. *Given $\mu \in \mathcal{M}_{\mathbb{T}} \cap \mathcal{M}_{*}$, assume that for any $t > 1$, there exists a probability measure $\mu_t \in \mathcal{M}_{\mathbb{T}}$ such that*

$$(3-19) \quad \Sigma_{\mu_t}(z) = (\Sigma_{\mu}(z))^t.$$

Assume in addition that μ_t is subordinated with respect to μ for all $t > 1$. Then $\eta_{\mu}(z) \neq 0$ for all $z \in \mathbb{D} \setminus \{0\}$, that is $\mu \in \mathcal{M}_{\mathbb{T}}^$.*

Proof. For each $t > 1$, we denote by ω_t the subordination function of μ_t to μ . Observing that $\mu_t \in \mathcal{M}_{*}$ and $\omega_t'(0) \neq 0$, for each $t > 1$, there exists a probability measure $\sigma_{t-1} \in \mathcal{M}_{\mathbb{T}} \cap \mathcal{M}_{*}$ such that $\eta_{\sigma_{t-1}}(z) = \omega_t(z)$. We rewrite (3-19) as

$$(3-20) \quad \frac{\eta_{\mu_t}^{-1}(z)}{z} = \left[\frac{\eta_{\mu}^{-1}(z)}{z} \right]^t$$

for z in a neighborhood of zero.

Note that $\omega_t^{-1}(z) = \eta_{\mu_t}^{-1}(\eta_\mu(z))$ for z in a neighborhood of zero. Replacing z by $\eta_\mu(z)$ in (3-20), we obtain

$$\frac{\omega_t^{-1}(z)}{\eta_\mu(z)} = \frac{\eta_{\mu_t}^{-1}(\eta_\mu(z))}{\eta_\mu(z)} = \left[\frac{\eta_\mu^{-1}(\eta_\mu(z))}{\eta_\mu(z)} \right]^t = \left[\frac{z}{\eta_\mu(z)} \right]^t,$$

which implies

$$\frac{\omega_t^{-1}(z)}{z} = \left[\frac{z}{\eta_\mu(z)} \right]^{t-1}.$$

Given $t > 0$, we thus have $\Sigma_{\sigma_t}(z) = [z/\eta_\mu(z)]^t$ for z in a neighborhood of zero. Therefore σ_t is \boxtimes -infinitely divisible.

By [Bercovici and Voiculescu 1992, Theorem 6.7], there exists an analytic function $u(z)$ in \mathbb{D} such that $\Re u(z) \geq 0$ if $z \in \mathbb{D}$ and $\Sigma_{\sigma_t}(z) = \exp(u(z))$. We thus obtain $z/\eta_\mu(z) = \exp(u(z))$, which implies that $\eta_\mu(z) \neq 0$ for all $z \in \mathbb{D}$. \square

It was pointed out in [Belinschi and Bercovici 2005] that $\mu^{\boxtimes t}$ is only determined up to a rotation by a multiple of $2\pi t$. Note that ω_t and σ_t are determined by the choice of $\mu^{\boxtimes t}$.

Multiplicative Boolean convolution and the Bercovici–Pata bijection. Multiplicative Boolean convolution on \mathbb{T} was studied by Franz [2008]. Let $\mu \in \mathcal{M}_{\mathbb{T}}$, and we set $k_\mu(z) = z/\eta_\mu(z)$. Given two probability measures $\mu, \nu \in \mathcal{M}_{\mathbb{T}}$, their multiplicative Boolean convolution $\mu \boxtimes \nu$ is a probability measure on \mathbb{T} such that

$$k_{\mu \boxtimes \nu}(z) = k_\mu(z)k_\nu(z)$$

for all $z \in \mathbb{D}$.

A probability $\mu \in \mathcal{M}_{\mathbb{T}}$ is said to be \boxtimes -infinitely divisible, if for any positive integer n , there exists $\mu_n \in \mathcal{M}_{\mathbb{T}}$ such that $\mu = (\mu_n)^{\boxtimes n}$. Let P_0 be the Haar measure. It is shown in [Franz 2008] that $\mu \in \mathcal{M}_{\mathbb{T}} \setminus \{P_0\}$ is \boxtimes -infinitely divisible if and only if $\eta'_\mu(0) \neq 0$ and $\eta_\mu \neq 0$ for all $z \in \mathbb{D} \setminus \{0\}$, that is $\mu \in \mathcal{M}_{\mathbb{T}}^*$, which is equivalent to

$$(3-21) \quad k_\mu(z) = \exp\left(bi + \int_{\mathbb{T}} \frac{1 + \xi z}{1 - \xi z} d\tau_\mu(\xi) \right),$$

where $b \in \mathbb{R}$ and τ_μ is a finite measure on \mathbb{T} . Equation (3-21) is the analogue of the Lévy–Khintchine formula in this context.

The multiplicative Bercovici–Pata bijection from \boxtimes to \boxtimes was studied in [Wang 2008]. Denote the set of all \boxtimes -infinitely divisible measures on \mathbb{T} by $\mathcal{F}\mathcal{D}(\boxtimes, \mathbb{T})$, and the multiplicative Bercovici–Pata bijection from \boxtimes to \boxtimes by \mathbb{M} . Then we have $k_\mu(z) = \Sigma_{\mathbb{M}(\mu)}(z)$.

Given $\mu \in \mathcal{F}\mathcal{D}(\boxtimes, \mathbb{T}) \setminus P_0 = \mathcal{M}_{\mathbb{T}}^*$, let ω_2 be the subordination function of $\mu^{\boxtimes 2}$ with respect to μ , and let σ be the probability measure on \mathbb{T} such that $\eta_\sigma(z) = \omega_2(z)$.

Then σ is \boxtimes -infinitely divisible and its Σ -transform is $\Sigma_\sigma(z) = z/\eta_\mu(z) = k_\mu(z)$. Therefore, σ is the same as $\mathbb{M}(\mu)$. Since $P_0 \boxtimes P_0 = P_0$ and $\eta_{P_0} = z$, the subordination function of $P_0 \boxtimes P_0$ with respect to P_0 is the identity map z , and the measure associated with the identity map z is P_0 . To summarize:

Corollary 3.16. *Given $\mu \in \mathcal{F}\mathcal{D}(\boxtimes, \mathbb{T})$, let ω_2 be the subordination function of $\mu^{\boxtimes 2}$ with respect to μ , and let σ be the probability measure on \mathbb{T} such that $\eta_\sigma(z) = \omega_2(z)$. Then $\sigma = \mathbb{M}(\mu)$, where \mathbb{M} is the multiplicative Bercovici–Pata bijection from \boxtimes to \boxtimes .*

Proposition 3.17. *If $\mu \in \mathcal{M}_\mathbb{T}$, the following are equivalent.*

- (1) $\mu \in \mathcal{F}\mathcal{D}(\mathbb{T}, \boxtimes)$.
- (2) $\delta_\beta \wr \mu \in \mathcal{F}\mathcal{D}(\mathbb{T}, \boxtimes)$ for any $\beta \in \mathbb{T}$.

Proof. It is enough to prove that (1) implies (2) for $\mu \in \mathcal{M}_\mathbb{T} \cap \mathcal{M}_*$. Observing that $\eta_{\delta_\beta \wr \mu}(z) = \beta \cdot \eta_\mu(z)$, we thus have

$$\Sigma_{\delta_\beta \wr \mu}(z) = \frac{\eta_{\delta_\beta \wr \mu}^{-1}(z)}{z} = \frac{\eta_\mu^{-1}(\bar{\beta}z)}{z} = \bar{\beta} \cdot \Sigma_\mu(\bar{\beta}z).$$

The result follows from the Lévy–Khintchine formula for the multiplicative free convolution on \mathbb{T} . □

An analogue of equations studied by Belinschi and Nica. In this subsection, we prove our Theorem 1.1. Recall that λ_t is the free multiplicative analogue of the normal distribution on \mathbb{T} , the unit circle of \mathbb{C} , with $\Sigma_{\lambda_t}(z) = \exp((t/2)(1+z)/(1-z))$ and we set $\lambda = \lambda_1$. For $\mu \in \mathcal{M}_\mathbb{T}$, we denote $m_1(\mu) = \int_\mathbb{T} \xi d\mu(\xi)$.

Proposition 3.18. *Given $\mu \in \mathcal{M}_\mathbb{T}^*$ and a finite measure τ on \mathbb{T} , define an analytic map u by*

$$(3-22) \quad u(z) = bi + \int_\mathbb{T} \frac{1 + \xi z}{1 - \xi z} d\tau(\xi), \quad z \in \mathbb{D},$$

where $b \in [0, 2\pi)$. If $k_\mu(z) = z/\eta_\mu(z) = \exp u(z)$, then $b = \arg 1/m_1(\mu) \in [0, 2\pi)$ and $\tau(\mathbb{T}) = \ln |1/m_1(\mu)|$. In particular, there exists a probability measure $\nu \in \mathcal{M}_\mathbb{T}$ such that $k_\mu(z) = \Sigma_\lambda(\eta_\nu(z))$ if and only if $m_1(\mu) = e^{-1/2}$.

Proof. By definition, we have

$$(3-23) \quad k_\mu(0) = \lim_{z \rightarrow 0} \frac{z}{\eta_\mu(z)} = \frac{1}{\eta'_\mu(0)} = \frac{1}{m_1(\mu)}.$$

Since $u(0) = bi + \tau(\mathbb{T})$, we obtain

$$(3-24) \quad b = \arg\left(\frac{1}{m_1(\mu)}\right) \quad \text{and} \quad \tau(\mathbb{T}) = \ln\left|\frac{1}{m_1(\mu)}\right|.$$

The first assertion follows.

By (3-7), we have

$$\Sigma_\lambda(\eta_\nu(z)) = \exp\left(\frac{1}{2} \int_{\mathbb{T}} \frac{1 + \xi z}{1 - \xi z} d\nu(\xi)\right).$$

Noticing that k_μ has the Herglotz representation as (3-21), we conclude that $k_\mu(z)$ can be written in the form of $\Sigma_\lambda(\eta_\nu(z))$ for a probability measure ν on \mathbb{T} if and only if $\ln(1/m_1(\mu)) = 1/2$. This implies the second half of the assertion. \square

For $\mu \in \mathcal{P}\mathcal{D}(\otimes, \mathbb{T}) \setminus P_0 = \mathcal{M}_\mathbb{T}^*$ with $m_1(\mu) > 0$, let $u(z)$ be the analytic function satisfying $k_\mu(z) = \exp(u(z))$ and $u(0) > 0$. Given $t > 1$, let $H_t(z) = z \exp((t - 1)u(z))$, and denote its right inverse by $\omega_t : \mathbb{D} \rightarrow \mathbb{D}$ with $\omega_t(0) = 0$. We define (see [Belinschi and Bercovici 2005]) $\mu^{\boxtimes t}$ by the relation

$$(3-25) \quad \eta_{\mu^{\boxtimes t}}(z) = \eta_\mu(\omega_t(z)).$$

Then we see that $H'_t(0) > 0$, $\omega'_t(0) > 0$ and that

$$m_1(\mu^{\boxtimes t}) = \eta'_{\mu^{\boxtimes t}}(0) > 0.$$

For $t > 0$, we also define $\mu^{\boxdot t}$ by the relation

$$(3-26) \quad k_{\mu^{\boxdot t}}(z) = \exp(tu(z)).$$

For this choice of the Boolean convolution power, we have

$$m_1(\mu^{\boxdot t}) > 0.$$

Definition 3.19. Given $\mu \in \mathcal{M}_\mathbb{T}^*$ such that $m_1(\mu) > 0$, we define a family of maps $\{\mathbb{M}_t\}_{t \neq 0}$ by

$$\mathbb{M}_t(\mu) = (\mu^{\boxtimes(t+1)})^{\boxdot 1/(t+1)},$$

where we choose $\mu^{\boxtimes(t+1)}$ and $\mathbb{M}_t(\mu)$ in a way such that they have positive means.

The next result is a special case of [Arizmendi and Hasebe 2013, Theorem 4.4].

Lemma 3.20. Given $\mu \in \mathcal{M}_\mathbb{T}^*$ with $m_1(\mu) > 0$, the following assertions are true.

- (1) $\mathbb{M}_{t+s}(\mu) = \mathbb{M}_t(\mathbb{M}_s(\mu))$ for all $t, s \geq 0$.
- (2) $\mathbb{M}_1(\mu) = \mathbb{M}(\mu)$.

Proof of Theorem 1.1. We set

$$(3-27) \quad u(z) = \frac{1}{2} \int_{\mathbb{T}} \frac{1 + \xi z}{1 - \xi z} d\nu(\xi).$$

By (3-7) and the assumption (1-2), we have

$$(3-28) \quad \frac{z}{\eta_\mu(z)} = \Sigma_\lambda(\eta_\nu(z)) = \exp(u(z)).$$

By Proposition 3.18, we see that $m_1(\mu) > 0$. We therefore can choose the multiplicative convolution power $\mu^{\boxtimes(t+1)}$ such that $m_1(\mu^{\boxtimes(t+1)}) > 0$.

Let η_t be the principal subordination function of $\nu \boxtimes \lambda_t$ with respect to ν and ω_{t+1} be the subordination function of $\mu^{\boxtimes(t+1)}$ with respect to μ . Let $\rho_t, \sigma_t \in \mathcal{M}_T$ such that $\eta_{\rho_t} = \eta_t$ and $\eta_{\sigma_t} = \omega_{t+1}$.

By Corollary 3.13, (1-2) implies that $\Sigma_{\rho_t}(z) = \Sigma_{\lambda_t}(\eta_{\mu}(z)) = \exp(tu(z))$. From the choice of $\mu^{\boxtimes(t+1)}$, the function $H_{t+1}(z) := z \exp(tu(z))$ is the left inverse of ω_{t+1} such that $H_{t+1}(\omega_{t+1}(z)) = z$ for all $z \in \mathbb{D}$, which implies that

$$(3-29) \quad \Sigma_{\sigma_t}(z) = \exp(tu(z)).$$

We thus obtain that $\rho_t = \sigma_t$ and $\eta_t = \omega_{t+1}$.

Replacing z by η_t in (1-2), we obtain

$$(3-30) \quad \begin{aligned} \Sigma_{\lambda}(\eta_{\nu \boxtimes \lambda_t}(z)) &= \Sigma_{\lambda}(\eta_{\nu}(\eta_t(z))) = \frac{\eta_t(z)}{\eta_{\mu}(\eta_t(z))} \\ &= \frac{\omega_{t+1}(z)}{\eta_{\mu}(\omega_{t+1}(z))} = \frac{\omega_{t+1}(z)}{\eta_{\mu^{\boxtimes(t+1)}}(z)} = \left(\frac{z}{\eta_{\mu^{\boxtimes(t+1)}}(z)} \right)^{1/t+1}. \end{aligned}$$

On the other hand, by the definition of \mathbb{M}_t , we have

$$\frac{z}{\eta_{\mathbb{M}_t(\mu)}(z)} = \frac{z}{\eta_{(\mu^{\boxtimes(t+1)}) \boxtimes (1/(t+1))}(z)} = \left(\frac{z}{\eta_{\mu^{\boxtimes(t+1)}}(z)} \right)^{1/t+1},$$

completing the proof of Theorem 1.1. □

Some examples and applications. We start with the multiplicative analogues of examples studied in [Anshelevich 2010; 2012; Arizmendi and Hasebe 2013; Belinschi and Nica 2008b]. We define the set

$$(A) = \{\mu \in \mathcal{M}_T^* : m_1(\mu) = e^{-1/2}\}.$$

By (3-22), the set \mathcal{M}_T is in one-to-one correspondence with the set (A) via the bijection $\nu \leftrightarrow \mu$, such that $\Sigma_{\lambda}(\eta_{\nu}(z)) = z/\eta_{\mu}(z)$.

Definition 3.21. The bijective map $\Lambda : \mathcal{M}_T \rightarrow (A)$ is defined by

$$\Sigma_{\lambda}(\eta_{\nu}(z)) = \frac{z}{\eta_{\Lambda[\nu]}(z)} \quad \text{for all } \nu \in \mathcal{M}_T.$$

Using the Λ notation, Theorem 1.1 implies that

$$\Lambda[\nu \boxtimes \lambda_t] = \mathbb{M}_t[\Lambda(\nu)] \quad \text{for all } \nu \in \mathcal{M}_T.$$

Example 3.22. Let δ_1 be the Dirac measure at 1, and let $\mu = \Lambda[\delta_1]$, we have $z/\eta_\mu(z) = \Sigma_\lambda(\eta_{\delta_1}(z)) = \exp(\frac{1}{2}(1+z)/(1-z))$. For $t \geq 0$, Theorem 1.1 implies that

$$\frac{z}{\eta_{\mathbb{M}_t(\mu)}(z)} = \Sigma_\lambda(\eta_{\delta_1 \boxtimes \lambda_t}(z)) = \Sigma_\lambda(\eta_{\lambda_t}(z)).$$

In particular, when $t = 1$,

$$\eta_{\mathbb{M}_1(\mu)}(z) = \frac{z}{\Sigma_\lambda(\eta_{\lambda_1}(z))} = \eta_{\lambda_1}(z),$$

where we used the equality $((z\Sigma_\lambda) \circ \eta_\lambda)(z) = z$ and $\lambda = \lambda_1$. Therefore, $\mathbb{M}_1(\mu)$ is the free multiplicative analogue of the normal distribution on \mathbb{T} .

Example 3.23. More generally, we consider $\lambda_{b,t} = \delta_b \boxtimes \lambda_t$ and $\mu_{b,t} = \Lambda[\lambda_{b,t}]$. Then

$$\Sigma_\lambda(\eta_{\lambda_{b,t}}(z)) = \frac{z}{\eta_{\mu_{b,t}}(z)} \quad \text{for } t \neq 0.$$

On the other hand, Theorem 1.1 implies that

$$\Sigma_\lambda(\eta_{\lambda_{b,t_1+t_2}}(z)) = \Sigma_\lambda(\eta_{\lambda_{b,t_1} \boxtimes \lambda_{t_2}}(z)) = \frac{z}{\eta_{\mathbb{M}_{t_2}(\mu_{b,t_1})}(z)} \quad \text{for } t_1, t_2 \geq 0,$$

which yields that $\mu_{b,t_1+t_2} = \mathbb{M}_{t_2}(\mu_{b,t_1})$ for $t_1, t_2 \geq 0$.

We would like to provide another example which covers part of [Arizmendi and Hasebe 2013, Example 4.10].

Example 3.24. Let P_0 be the Haar measure on \mathbb{T} . Then by the free independence, $P_0 \boxtimes \lambda_t = P_0$. We set $\mu = \Lambda[P_0]$, and we have

$$\Sigma_\lambda(\eta_{P_0 \boxtimes \lambda_t}(z)) = \Sigma_\lambda(\eta_{P_0}(z)) = \frac{z}{\eta_\mu(z)},$$

which implies that $\mathbb{M}_t(\mu) = \mu$ for all $t \geq 0$. To calculate the distribution of μ , we note that $\eta_{P_0} \equiv 0$, which shows that $\eta_\mu = e^{-1}z$, and thus $\psi_\mu(z) = z/e - z$. Using the identity

$$\frac{1}{\pi} \left(\psi_\mu(z) + \frac{1}{2} \right) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(e^{-it}),$$

and Stieltjes's inversion formula, we obtain

$$\mu(dt) = \frac{1}{2\pi} \frac{1 - e^{-2}}{1 + e^{-2} - 2e^{-1} \cos(t)} dt, \quad 0 \leq t \leq 2\pi.$$

We then give some applications of results concerning infinity divisibility of the measures associated with subordination functions. For $\mu \in \mathcal{M}_\mathbb{T}$, we say μ is nontrivial if it is not a Dirac measure at a point on \mathbb{T} .

Lemma 3.25. *Given $\sigma \in \mathcal{F}\mathcal{D}(\boxtimes, \mathbb{T})$ which is non-trivial, and $0 < \epsilon < 1$, there exists a positive number $n(\epsilon)$ such that*

$$\eta_{\sigma_t}(\bar{\mathbb{D}}) \subset \mathbb{D}_\epsilon = \{z = re^{i\theta} : 0 \leq r \leq \epsilon, 0 \leq \theta < 2\pi\}$$

for any $t > n(\epsilon)$, where $\sigma_t = \sigma^{\boxtimes t}$.

Proof. If $\sigma = P_0$, the Haar measure on \mathbb{T} , the result is trivial. If $\sigma \neq P_0$ is nontrivial, then by [Bercovici and Voiculescu 1992, Theorem 6.7], there exists a finite positive measure ν on \mathbb{T} with $\nu(\mathbb{T}) > 0$, $\alpha \in \mathbb{R}$, and an analytic function u defined by

$$u(z) = i\alpha + \int_{\mathbb{T}} \frac{1 + \xi z}{1 - \xi z} d\nu(\xi) \quad \text{for } z \in \mathbb{D},$$

such that $\Sigma_\sigma(z) = \exp(u(z))$. We choose $\sigma_t \in \mathcal{M}_{\mathbb{T}}$ satisfying $\Sigma_{\sigma_t}(z) = \exp(tu(z))$. Noticing that other choices of the multiplicative free convolution power of σ can be obtained from σ_t by a rotation of a multiple of $2\pi t$, it is enough to prove the assertion for σ_t .

We set $\Phi_{\sigma_t} = z\Sigma_{\sigma_t}(z)$; then, by Lemma 3.2, we have

$$\Phi_{\sigma_t}^{-1}(\mathbb{D}) = \eta_{\sigma_t}(\mathbb{D}).$$

For $z = re^{i\theta} \in \mathbb{D}$, we calculate

$$\begin{aligned} (3-31) \quad |\Phi_{\sigma_t}(z)| &= r \exp\left(t \int_{\mathbb{T}} \frac{1-r^2}{|1-\xi z|^2} d\nu(\xi)\right) \\ &\geq r \exp\left(t \int_{\mathbb{T}} \frac{1-r^2}{|1+r|^2} d\nu(\xi)\right) = r \exp\left(t \cdot \nu(\mathbb{T}) \frac{1-r}{1+r}\right). \end{aligned}$$

Since

$$\lim_{t \rightarrow \infty} r \exp\left(t \cdot \nu(\mathbb{T}) \frac{1-r}{1+r}\right) = \infty,$$

we deduce that for any $0 < \epsilon < 1$, there exists a positive number $n(\epsilon)$ such that, for all $t > n(\epsilon)$, we have

$$|\Phi_{\sigma_t}(z)| > 1 \quad \text{for } |z| = \epsilon.$$

By Lemma 3.2, $\Phi_{\sigma_t}(\mathbb{D})$ is a simply connected domain which contains zero, which implies that

$$\eta_{\sigma_t}(\mathbb{D}) = \Phi_{\sigma_t}^{-1}(\mathbb{D}) \subset \mathbb{D}_\epsilon, \quad \text{for } t > n(\epsilon).$$

The assertion follows because η_{σ_t} extends to a continuous function on $\bar{\mathbb{D}}$. □

For $\mu \in \mathcal{M}_{\mathbb{T}}$, we have

$$\frac{1}{2\pi} \left(\frac{1 + \eta_\mu(z)}{1 - \eta_\mu(z)} \right) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(e^{-i\theta}), \quad z \in \mathbb{D}.$$

The real part of this function is the Poisson integral of the measure $d\mu(e^{-i\theta})$, we can recover μ by Stieltjes's inversion formula. The functions

$$(3-32) \quad \frac{1}{2\pi} \Re \left(\frac{1 + \eta_\mu(re^{i\theta})}{1 - \eta_\mu(re^{i\theta})} \right) = \frac{1}{2\pi} \frac{1 - |\eta_\mu(re^{i\theta})|^2}{|1 - \eta_\mu(re^{i\theta})|^2}$$

converge to the density of $\mu(e^{-i\theta})$ a.e. relative to Lebesgue measure, and they converge to infinity a.e. relative to the singular part of this measure.

Proposition 3.26. *Given $\mu \in \mathcal{M}_T$ and $\sigma \in \mathcal{F}\mathcal{D}(\mathbb{R}, T)$ which is nontrivial, let μ_t be the unique probability measure on T such that*

$$\eta_{\mu_t}(z) = \eta_\mu(\eta_{\sigma_t}(z)).$$

Then we have

$$\lim_{t \rightarrow \infty} \sup_{\theta \in [0, 2\pi]} \left| \frac{d\mu_t(e^{i\theta})}{d\theta} - \frac{1}{2\pi} \right| = 0,$$

where $d\mu_t(e^{i\theta})/d\theta$ is the density function of μ_t at $e^{i\theta}$ with respect to Lebesgue measure.

Proof. Given $0 < \epsilon < 1$, by Lemma 3.25, there exists $n(\epsilon) > 0$ such that $\eta_{\sigma_t}(e^{i\theta}) < \epsilon$ for $t \geq n(\epsilon)$, which yields that $\eta_{\mu_t}(z)$ extends continuously to \mathbb{D} . Thus

$$|\eta_{\mu_t}(e^{i\theta})| = |\eta_\mu(\eta_{\sigma_t}(e^{i\theta}))| \leq |\eta_{\sigma_t}(e^{i\theta})| < \epsilon,$$

which implies that

$$(3-33) \quad \frac{1 - \epsilon}{1 + \epsilon} = \frac{1 - \epsilon^2}{|1 + \epsilon|^2} \leq \frac{1 - |\eta_{\mu_t}(e^{i\theta})|^2}{|1 - \eta_{\mu_t}(e^{i\theta})|^2} \leq \frac{1}{|1 - \epsilon|^2}.$$

Since ϵ is arbitrary, combining (3-32) with (3-33), we prove our assertion. □

Corollary 3.27. *Given $\mu \in \mathcal{M}_T$ and a nontrivial measure $\nu \in \mathcal{F}\mathcal{D}(\mathbb{R}, T)$, the density functions of the measures $\mu \boxtimes \nu_t$ converge to $1/2\pi$ uniformly as $t \rightarrow \infty$; if $\mu \in \mathcal{M}_T$ is nontrivial, the density functions of the measures $\mu^{\boxtimes t}$ converge to $1/2\pi$ uniformly as $t \rightarrow \infty$.*

Proof. Noticing Corollary 3.13, Propositions 3.14, 3.26 and Subsection 3.3, we only need to prove the case of $\mu^{\boxtimes t}$ for $\mu \notin \mathcal{M}_T^*$. We point out that the measures are nontrivial imply that the subordination distributions involved are nontrivial.

For $\mu \in \mathcal{M}_T \setminus \mathcal{M}_*$, we have $\mu^{\boxtimes n} = P_0$, where P_0 is the Haar measure on T . Thus the assertion is true for this case. For $\mu \in \mathcal{M}_T \cap \mathcal{M}_*$, but $\mu \notin \mathcal{M}_T^*$, it is shown in [Belinschi and Bercovici 2005] that $\mu \boxtimes \mu \in \mathcal{M}_T^*$; thus this case reduces to the case when $\mu \in \mathcal{M}_T^*$. This finishes the proof. □

4. Multiplicative convolution on $\mathcal{M}_{\mathbb{R}^+}$

Multiplicative free convolution on $\mathcal{M}_{\mathbb{R}^+}$. We are interested in the probability measures on the positive real line \mathbb{R}^+ , which are different from the Dirac measure at zero, we thus set

$$\mathcal{M}_{\mathbb{R}^+}^* = \mathcal{M}_{\mathbb{R}^+} \setminus \{\delta_0\}.$$

Given $\mu \in \mathcal{M}_{\mathbb{R}^+}^*$, we define

$$\psi_\mu(z) = \int_0^{+\infty} \frac{tz}{1-tz} d\mu(t),$$

and $\eta_\mu(z) = \psi_\mu(z)/(1 + \psi_\mu(z))$. The transform η_μ is characterized by the following proposition; see [Belinschi and Bercovici 2005].

Proposition 4.1. *Let $\eta : \mathbb{C} \setminus \mathbb{R}^+ \rightarrow \mathbb{C}$ be an analytic function such that $\eta(\bar{z}) = \overline{\eta(z)}$ for all $z \in \mathbb{C} \setminus \mathbb{R}^+$. Then the following two conditions are equivalent.*

- (1) $\eta = \eta_\mu$ for some $\mu \in \mathcal{M}_{\mathbb{R}^+}^*$.
- (2) $\eta(0-) = 0$ and $\arg(\eta(z)) \in [\arg z, \pi)$ for all $z \in \mathbb{C}^+$.

It can be shown that η_μ is invertible in some neighborhood of $(-\infty, 0)$, and we set $\Sigma_\mu(z) = \eta_\mu^{-1}(z)/z$ where η_μ^{-1} is defined in some neighborhood of $(\alpha, 0)$. Given two measures $\mu, \nu \in \mathcal{M}_{\mathbb{R}^+}^*$, the multiplicative free convolution of μ and ν is the probability measure $\mu \boxtimes \nu$ in $\mathcal{M}_{\mathbb{R}^+}^*$ such that

$$\Sigma_{\mu \boxtimes \nu}(z) = \Sigma_\mu(z) \Sigma_\nu(z)$$

in some neighborhood of $(\alpha, 0)$, where these functions are defined.

It is known from [Belinschi and Bercovici 2007; Biane 1998] that there exist two analytic functions $\omega_1, \omega_2 : \mathbb{C} \setminus \mathbb{R}^+ \rightarrow \mathbb{C} \setminus \mathbb{R}^+$ such that

- (1) $\omega_j(0-) = 0$ for $j = 1, 2$,
- (2) for any $\lambda \in \mathbb{C}^+$, we have $\omega_j(\bar{\lambda}) = \overline{\omega_j(\lambda)}$ for $j = 1, 2$,
- (3) $\eta_{\mu \boxtimes \nu}(z) = \eta_\mu(\omega_1(z)) = \eta_\nu(\omega_2(z))$ for $z \in \mathbb{C} \setminus \mathbb{R}^+$.

For simplicity, we say that ω_1 (resp. ω_2) is the subordination function of $\mu \boxtimes \nu$ with respect to μ (resp. ν), and $\mu \boxtimes \nu$ is subordinated to μ and ν .

The analogy of the Lévy–Khintchine in this setting was proved in [Bercovici and Voiculescu 1992; 1993]. A measure $\mu \in \mathcal{M}_{\mathbb{R}^+}$ is \boxtimes -infinitely divisible if and only if $\Sigma_\mu(z) = \exp(u(z))$, with

$$u(z) = a - bz + \int_0^{+\infty} \frac{1+tz}{z-t} d\sigma(t),$$

where $b \in \mathbb{R}$ and σ is a finite positive measure on \mathbb{R}^+ . The analogue of the normal distribution in this context is given by $\Sigma_{\lambda_t}(z) = \exp((t/2)(z + 1)/(z - 1))$.

Lemma 4.2. *If $\mu, \nu \in \mathcal{M}_{\mathbb{R}^+}^*$, we have*

$$\eta_\mu(z) = \eta_{\mu \boxtimes \nu}(z \Sigma_\nu(\eta_\mu(z)))$$

in some neighborhood of interval $(\alpha, 0)$.

The proof of Lemma 4.2 is identical to the proof of Lemma 3.3, therefore we omit the details.

For any $t > 0$, assume that $\eta_t : \mathbb{D} \rightarrow \mathbb{D}$ is the subordination function of $\mu \boxtimes \lambda_t$ with respect to μ , by Lemma 4.2 and the characterization of η -transform, there exists a probability measure ρ_t in $\mathcal{M}_{\mathbb{R}^+}^*$ such that $\eta_{\rho_t}(z) = \eta_t(z)$. The argument in the proof of Lemma 3.4 implies the following result.

Proposition 4.3. *The measure ρ_t is \boxtimes -infinitely divisible and its Σ -transform is $\Sigma_{\rho_t}(z) = \Sigma_{\lambda_t}(\eta_\mu(z))$, and*

$$\Sigma_{\lambda_t}(\eta_\mu(z)) = \exp\left(\frac{t}{2} \int_0^{+\infty} \frac{1 + \xi z}{\xi z - 1} d\mu(\xi)\right).$$

We now discuss free convolution semigroups. Given $t > 1$, it is proved in [Belinschi and Bercovici 2005] that one can define $\mu^{\boxtimes t} \in \mathcal{M}_{\mathbb{R}^+}^*$ such that $\Sigma_{\mu^{\boxtimes t}}(z) = (\Sigma_\mu(z))^t$ for $z < 0$ sufficiently close to zero. Similar to the case of \mathcal{M}_T , $\mu^{\boxtimes t}$ is subordinated with respect to μ and we denote the subordination function by ω_t . By [Belinschi and Bercovici 2005, Theorem 2.6] and the characterization of η -transform, there exists a probability $\sigma_t \in \mathcal{M}_{\mathbb{R}^+}^*$ such that $\eta_{\sigma_t}(z) = \omega_{t+1}$ for all $t > 0$. Moreover, σ_t is \boxtimes -infinitely divisible and its Σ -transform is $\Sigma_{\sigma_t}(z) = [z/\eta_\mu(z)]^t$.

Multiplicative Boolean convolution on $\mathcal{M}_{\mathbb{R}^+}$ and the semigroup \mathbb{M}_t . Bercovici [2006] proved that the multiplicative Boolean convolution does not preserve $\mathcal{M}_{\mathbb{R}^+}$. But we can still define $\mu^{\boxtimes t}$ for $\mu \in \mathcal{M}_{\mathbb{R}^+}$ and $0 \leq t \leq 1$ as follows. Let $k_\mu(z) = z/\eta_\mu(z)$, the Boolean convolution power $\mu^{\boxtimes t}$ is defined by

$$k_{\mu^{\boxtimes t}}(z) = (k_\mu(z))^t.$$

Definition 4.4 [Arizmendi and Hasebe 2013]. A family of maps from $\mathcal{M}_{\mathbb{R}^+}$ to itself is defined by

$$\mathbb{M}_t(\mu) = (\mu^{\boxtimes(t+1)})^{\boxtimes(1/(t+1))}.$$

It is also shown in [Arizmendi and Hasebe 2013] that $\mathbb{M}_{t+s} = \mathbb{M}_t \circ \mathbb{M}_s$ for $t, s \geq 0$.

Analogous equations. Given a pair of probability measures $\nu, \mu \in \mathcal{M}_{\mathbb{R}^+}$, we also consider, as in the case \mathcal{M}_T , the semigroups $\nu \boxtimes \lambda_t$ and $\mu^{\boxtimes(t+1)}$, the subordination functions η_t and ω_{t+1} , and their associated probability measures ρ_t, σ_t for all $t > 0$. Since $\Sigma_{\rho_t}(z) = \Sigma_{\lambda_t}(\eta_\nu(z))$ and $\Sigma_{\sigma_t}(z) = [z/\eta_\mu(z)]^t$, we deduce that $\eta_t = \omega_{t+1}$ if and only if

$$\Sigma_\lambda(\eta_\nu(z)) = \frac{z}{\eta_\mu(z)}.$$

Applying the same argument as in the proof of the Theorem 1.1, we obtain the following result.

Theorem 4.5. *Given a pair of probability measures $\mu, \nu \in \mathcal{M}_{\mathbb{R}^+}$ such that*

$$(4-1) \quad \Sigma_\lambda(\eta_\nu(z)) = \frac{z}{\eta_\mu(z)}, \quad z \in \mathbb{C}^+,$$

we have $\Sigma_\lambda(\eta_{\nu \boxtimes \lambda_t}(z)) = \frac{z}{\eta_{\mathbb{M}_t(\mu)}(z)}$, $z \in \mathbb{C}^+$.

5. A description of the analogue of the normal distribution

Biane [1997a; 1997c] studied free Brownian motion and proved many important results. In this section, we give a new proof for the density functions of the free multiplicative analogue of the normal distributions, which was first obtained in [Biane 1997c] (See also [Demni and Hmidi 2012] for a different approach). Some results are new. For example, we show that λ_t is unimodal for the circle case; and we show that $\Phi_\lambda^{-1}(\mathbb{C}^+)$ contains infinitely many connected components where λ is the free multiplicative analogue of the normal distribution on the positive half line with $\Sigma_\lambda(z) = \exp((z + 1)/(z - 1))$. We also give a description of the boundaries Ω_t, Ω (defined below), we observe that $\partial\Omega_t$ can be parametrized by θ and $\partial\Omega$ can be parametrized by r .

The circle case. Let $\lambda_t \in \mathcal{M}_{\mathbb{T}}$ be the analogue of the normal distribution such that $\Sigma_{\lambda_t}(z) = \exp((t/2)(1 + z)/(1 - z))$. We set $\Phi_t(z) = z\Sigma_{\lambda_t}(z)$, and let $\Omega_t = \{z \in \mathbb{D} : |\Phi_t(z)| < 1\}$. By Lemma 3.2, η_{λ_t} extends continuously to the unit circle \mathbb{T} , Ω_t is simply connected and bounded by a simple closed curve, and we have $\partial\Omega_t = \eta_{\lambda_t}(\mathbb{T})$.

Observe that for $t \neq 4$, Φ_t has zeros of order one at $z_1(t) = (2 - t + \sqrt{t^2 - 4t})/2$ and $z_2(t) = (2 - t - \sqrt{t^2 - 4t})/2$. Φ_4 has a zero of order two at -1 ; and for all t , Φ_t has an essential singularity at 1 , and no other zeros and singularities. For $0 < t < 4$, $z_1(t), z_2(t) \in \mathbb{T}$ and $z_2(t) = \overline{z_1(t)}$, we let $\theta_1(t) \in (0, \pi)$ and $\theta_2(t) \in (\pi, 2\pi)$ such that $z_1(t) = e^{i\theta_1(t)}$ and $z_2(t) = e^{i\theta_2(t)}$. We have $z_1(4) = z_2(4) = -1$ and for $t > 4$, $z_1(t) \in (-1, 0)$ and $z_2(t) \in (-\infty, -1)$.

We define

$$g_t(r, \theta) = r \exp\left(\frac{t}{2} \frac{1 - r^2}{1 - 2r \cos \theta + r^2}\right) = |\Phi_t(z)|$$

for $z = re^{i\theta}$. The unit circle is parametrized by $\mathbb{T} = \{e^{i\theta} : 0 \leq \theta < 2\pi\}$.

Lemma 5.1. *For $0 < t < 4$, $\partial\Omega_t = \{z = e^{i\theta} : \theta_1(t) \leq \theta \leq \theta_2(t)\} \cup \mathcal{L}_{1,t} \cup \mathcal{L}_{2,t}$, where $\mathcal{L}_{1,t}$ is an analytic curve, and $\mathcal{L}_{1,t}$ is in $\mathbb{D} \cap \mathbb{C}^+$ except one of its endpoints, and $\mathcal{L}_{2,t}$ is the reflection of $\mathcal{L}_{1,t}$ about x -axis. $\mathcal{L}_{1,t}$ can be parametrized by $\gamma_t(u)$ ($0 \leq u \leq 1$) such that $\gamma_t(0) \in \mathbb{R}$, $\gamma_t(1) = z_1(t)$ and $\gamma_t(u) \subset \mathbb{D} \cap \mathbb{C}^+$ for $0 < u < 1$. Moreover, $|\gamma_t(u)|$ is an increasing function of u on the interval $[0, 1]$.*

Proof. Observing that $\Phi_t(\bar{z}) = \overline{\Phi_t(z)}$, we see that $\partial\Omega_t$ is symmetric with respect to x -axis. Since Ω_t is simply connected and $\partial\Omega_t$ is a simple closed curve, $\partial\Omega_t$ intersects x -axis at two points.

Restricting Φ_t to real numbers, we find that $\Phi_t(\mathbb{R}) \subset \mathbb{R}$, and that Φ_t is an increasing function on $(-1, 1)$ since $\Phi'_t(z)$ is positive for $z \in (-1, 1)$. From $\Phi_t(-1) = -1$ and $\lim_{z \rightarrow 1^-} \Phi_t(z) = +\infty$, we deduce that

$$(5-1) \quad \Phi_t^{-1}((-1, 1)) = (-1, x(t)),$$

where $x(t)$ is the unique solution of the equation $\Phi_t(z) = 1$ for $z \in (-1, 1)$. The fact that $\Phi'_t(z) \neq 0$ for $z \neq z_1(t), z_2(t)$ implies that Φ_t is locally invertible for $z \neq z_1(t), z_2(t)$. Combining the fact that $\Phi_t(\mathbb{T} \setminus \{1\}) \subset \mathbb{T}$, we obtain that

$$\{e^{i\theta} : \theta_1(t) \leq \theta \leq \theta_2(t)\} \subset \partial\Omega_t$$

and $\partial\Omega_t$ has corners of opening $\pi/2$ at $z_1(t)$ and $z_2(t)$.

Since Φ_t is a conformal mapping from Ω_t to \mathbb{D} , by the symmetry $\Phi_t(\bar{z}) = \overline{\Phi_t(z)}$ and (5-1), noticing that $\Phi'_t(0) = 1$, we thus deduce that $\Phi_t(\Omega_t \cap \mathbb{C}^+) \subset \overline{\mathbb{D}} \cap \mathbb{C}^+$. Since $\partial\Omega_t$ is a simple closed curve, $z_1(t)$ and $x(t)$ are connected by $\partial\Omega_t$. It is clear that $\partial\Omega_t \setminus \{e^{i\theta} : \theta_1(t) \leq \theta \leq \theta_2(t)\}$ does not intersect with \mathbb{T} , we thus assume the curve $\gamma_t = \{\gamma_t(u) : 0 \leq u \leq 1\}$ is the part of Ω_t which connects $z_1(t)$ and $x(t)$ such that $\gamma_t(0) = x(t)$, $\gamma_t(1) = z_1(t)$ and $\gamma_t(u) \in \mathbb{D}$ for $0 < u < 1$.

We claim that $|\gamma_t(u)|$ is an increasing function of u on the interval $[0, 1]$. For given $0 < r < 1$, we define the function of θ by

$$g_{t,r}(\theta) = g_t(r, \theta) = |\Phi_t(re^{i\theta})|.$$

Then $g_{t,r}$ is a strictly decreasing function of θ on the interval $[0, \pi]$. From the fact that Ω_t is simply connected, we deduce that, for $z_0 \in \overline{\Omega_t} \cap \mathbb{D} \cap \mathbb{C}^+$, the arc

$$(5-2) \quad \{re^{i\theta} : |r| = |z_0|, \arg z_0 < \theta \leq \pi\} \subset \Omega_t.$$

Given $0 < u_1 < u_2 < 1$, we need to prove that $|\gamma_t(u_1)| < |\gamma_t(u_2)|$. Since $[0, x(t)] \subset \overline{\Omega_t}$, we obtain from (5-2) that

$$(5-3) \quad \{re^{i\theta} : 0 \leq r \leq x(t), 0 < \theta \leq \pi\} \subset \Omega_t,$$

which shows that $|\gamma_t(u_1)| > x(t)$. Suppose that $|\gamma_t(u_1)| \geq |\gamma_t(u_2)|$. There exists $0 < u'_1 \leq u_1$ such that $|\gamma_t(u'_1)| = |\gamma_t(u_2)|$. If $\arg(\gamma_t(u'_1)) > \arg(\gamma_t(u_2))$, then by (5-2), $\gamma_t(u_2) \in \Omega_t$ and thus $\gamma_t(u_2) \notin \partial\Omega_t$; if $\arg(\gamma_t(u'_1)) < \arg(\gamma_t(u_2))$, then $\gamma_t(u'_1) \in \Omega_t$ and thus $\gamma_t(u'_1) \notin \partial\Omega_t$. For both cases, we obtain a contradiction. Thus $|\gamma_t(u_1)| < |\gamma_t(u_2)|$ and our claim is proved. \square

For $t > 0$, we let $x_1(t) \in (0, 1)$ be the unique solution of the equation $\Phi_t(z) = 1$ for $z \in (0, 1)$. For $0 < t \leq 4$ we let $x_2(t) = -1$; for $t > 4$, we let $x_2(t) \in (-1, 0)$ be the unique solution of the equation $\Phi_t(z) = -1$ for $z \in (-1, 0)$.

Lemma 5.2. *For $t \geq 4$, $\partial\Omega_t = \mathcal{L}_{1,t} \cup \mathcal{L}_{2,t}$, where $\mathcal{L}_{1,t}$ is an analytic curve, and $\mathcal{L}_{1,t}$ is in $\mathbb{D} \cap \mathbb{C}^+$ except its endpoints, and $\mathcal{L}_{2,t}$ is the reflection of $\mathcal{L}_{1,t}$ about x -axis. $\mathcal{L}_{1,t}$ can be parametrized by $\gamma_t(u)$ ($0 \leq u \leq 1$) such that $\gamma_t(0) = x_1(t)$, $\gamma_t(1) = x_2(t)$ and $\gamma_t(u) \in \mathbb{D} \cap \mathbb{C}^+$ for $0 < u < 1$. Moreover, $|\gamma_t(u)|$ is an increasing function of u on the interval $[0, 1]$.*

Proof. Recall that Φ_4 has a zero of order two at -1 . For all $t > 4$, $z_2(t) < -1$ and $z_1 \in (-1, 0)$. The assertion follows from the similar arguments in the proof Lemma 5.1. □

From the proof of Lemmas 5.1 and 5.2, for $t > 0$, we have $\Phi_t^{-1}((-1, 1)) = (x_2(t), x_1(t))$. Moreover, $x_1(t) = \min\{|z| : z \in \partial\Omega_t\}$ and $-x_2(t) = \max\{|z| : z \in \partial\Omega_t\}$.

Remark. In fact, for any $t > 0$, from the equation

$$g_t(r, \theta) = 0, \quad 0 < r < 1, \quad 0 \leq \theta \leq \pi,$$

we can prove that $dr/d\theta > 0$ for $0 < \theta < \pi$, which implies that if $z \in \partial\Omega_t$, the entire radius $\{rz : 0 \leq r < 1\}$ is contained in Ω_t . Therefore, $\partial\Omega_t$ can be parametrized by θ .

Lemma 5.3. *Using the same notations in Lemmas 5.1 and 5.2, for $t > 0$, the function $|1 - \gamma_t(u)|$ is an increasing function of u on $[0, 1]$.*

Proof. We only prove the case when $0 < t < 4$, the proof for other cases are similar. Noticing that $|1 - re^{i\theta}|^2 = 1 - 2\cos\theta + r^2$, since $|\gamma_t(u)|$ is an increasing function of u , to prove the assertion, we only need to prove that for the implicit function $r \exp((t/2)(1 - r^2)/h) = 1$ of r and h , the value of h increases when r increases on $(0, 1)$. From this equation, we have $h = h(r) = -(t/2)(1 - r^2)/(\ln r)$. One can check that $h'(r) > 0$ for $0 < r < 1$, therefore h is an increasing function of r . □

Theorem 5.4. *Denote by A_t the support of λ_t .*

- (1) *For $t > 0$, the measure λ_t has no singular part, and its density function is an analytic function. $A_{t_1} \subset A_{t_2}$ if $t_1 < t_2 < 4$. $A_t \subsetneq \mathbb{T}$ for $0 < t < 4$ and $A_t = \mathbb{T}$ for $t \geq 4$.*
- (2) *The measure λ_t is unimodal for all $t > 0$ and its density is maximal at $z = 1$ and is minimal at $z = -1$.*
- (3) *The density function $d\lambda_t/d\theta$ converges uniformly to $1/(2\pi)$ as $t \rightarrow \infty$.*

Proof. Since $z = 1$ is not in the closure of $\Omega_t = \eta_{\lambda_t}(\mathbb{D})$, the singular part of λ_t vanishes. From the analyticity of Φ_t or a general theorem in [Belinschi and Bercovici 2005], the density function is analytic.

For $0 < t < 4$, set $a_1(t) = \Phi_t(z_1(t))$, $a_2(t) = \Phi_t(z_2(t))$. Note that $\eta_{\lambda_t}(\Phi_t(z)) = z$ for $z \in \overline{\Omega}_t$. From (3-32) we see that A_t is the closed arc on \mathbb{T} with endpoints $a_1(t)$, $a_2(t)$ which contains 1. Thus, to prove that $A_{t_1} \subset A_{t_2}$, it is enough to prove that $\arg(a_1(t))$ is an increasing function of t . A direct computation shows that $|z_1(t) - 1|^2 = t$ and $\arg(\Sigma_{\lambda_t}(z_1(t))) = \Im z_1(t) = \sqrt{t(4-t)}/2$. We thus have

$$\arg(a_1(t)) = \Im z_1(t) + \arg(z_1(t)) = \sin(\theta_1(t)) + \theta_1(t).$$

From $z_1(t) = (2 - t + \sqrt{t^2 - 4t})/2$ we see that $\theta_1(t)$ is an increasing function of t . The function $\theta \rightarrow \sin(\theta) + \theta$ is an increasing function on $(0, \pi)$. Thus $\arg(a_1(t))$ is an increasing function of t and (1) is proved.

To prove (2), recall that a probability measure is unimodal if its density with respect to Lebesgue measure has a unique local maximum. η_{λ_t} extends continuously to \mathbb{T} , we thus have

$$(5-4) \quad \frac{d\lambda_t(e^{-i\theta})}{d\theta} = \frac{1}{2\pi} \frac{1 - |\eta_{\lambda_t}(e^{i\theta})|^2}{|1 - \eta_{\lambda_t}(e^{i\theta})|^2}.$$

We first prove the case when $0 < t < 4$. From $\eta_{\lambda_t}(\Phi_t(z)) = z$ for $z \in \overline{\Omega}_t$ and $\eta_{\lambda_t}(1) = x(t)$, to prove λ_t is unimodal, by the boundary correspondence, it is enough to show that the function f_t of u defined by

$$f_t(u) := \frac{1 - |\gamma_t(u)|^2}{|1 - \gamma_t(u)|^2},$$

is a decreasing function on $[0, 1]$ and is maximal at 0. Since $\gamma_t(u) \in \partial\Omega_t$, we have $|\Phi_t(\gamma_t(u))| = 1$. In other words, we have

$$(5-5) \quad |\gamma_t(u)| \exp\left(\frac{t}{2} f_t(u)\right) = |\gamma_t(u)| \exp\left(\frac{t}{2} \frac{1 - |\gamma_t(u)|^2}{|1 - \gamma_t(u)|^2}\right) = 1.$$

As we shown in Lemma 5.1 that the function $|\gamma_t(u)|$ is an increasing function of u , from (5-5), we deduce that f_t is a decreasing function of u and $\max\{f_t\} = f_t(0)$. By the symmetric property of the function Φ_t in Lemma 5.1, the density function is symmetric with respect to x -axis as well. Thus the density of λ_t has only one local maximum at $\Phi_t(\gamma_t(0)) = \Phi_t(x_1(t)) = 1$.

The proof for the case $t \geq 4$ is similar. In this case $A_t = \mathbb{T}$ and $\max\{f_t\} = f_t(0)$ and $\min\{f_t\} = f_t(1)$. Part (3) is a consequence of Corollary 3.27. □

Remark. From the proof of Theorem 5.4, we see that, for $t < 4$,

$$\arg(a_1(t)) = \theta_1(t) + \sin(\theta_1(t)) = \frac{1}{2}\sqrt{t(4-t)} + \arccos\left(1 - \frac{t}{2}\right),$$

which implies a known result in [Biane 1997c], namely

$$A_t = \left\{ e^{i\theta} : -\frac{1}{2}\sqrt{t(4-t)} - \arccos\left(1 - \frac{t}{2}\right) \leq \theta \leq \frac{1}{2}\sqrt{t(4-t)} + \arccos\left(1 - \frac{t}{2}\right) \right\}.$$

The positive half line case. Let $\lambda \in \mathcal{M}_{\mathbb{R}^+}$ be the analogue of the normal distribution such that $\Sigma_\lambda(z) = \exp((z + 1)/(z - 1))$.

We restate [Bercovici and Voiculescu 1993, Proposition 6.14] in terms of η and Σ transforms as follows.

Lemma 5.5. *Let μ be a \boxtimes -infinitely divisible measure on \mathbb{R}^+ , and set $\Phi_\mu(z) := z\Sigma_\mu(z)$.*

- (1) *We have $\Phi_\mu(\eta_\mu(z)) = z$ for every $z \in \mathbb{C}^+$.*
- (2) *The set $\{\eta_\mu(z) : z \in \mathbb{C}^+\} = \Omega$, where Ω is the component of the set $\{z \in \mathbb{C}^+ : \Im(\Phi_\mu(z)) > 0\}$ whose boundary contains the left half line $(-\infty, 0)$. Moreover, $\eta_\mu(\Phi_\mu(z)) = z$ for $z \in \Omega$.*

We set $\Phi_\lambda(z) = z \exp((z + 1)/(z - 1))$. The following lemma is elementary.

Lemma 5.6. *Φ_λ has zero of order one at $2 - \sqrt{3}$ and $2 + \sqrt{3}$, and Φ_λ has an essential singularity at 1. These are the only zeros and singularities of Φ_λ .*

Theorem 5.7. *The measure λ has no singular part. The support of this measure is the closure of its interior, and this interior has only one connected component.*

Proof. By [Bercovici and Voiculescu 1992, Theorem 7.5], the measure λ has compact support on \mathbb{R}^+ .

Let Ω be the component of $\{z \in \mathbb{C}^+ : \Im(\Phi_\lambda(z)) > 0\}$ whose boundary contains $(-\infty, 0)$. By Lemma 5.5, $\eta_\lambda : \mathbb{C}^+ \rightarrow \Omega$ is a conformal map and Φ_λ is its inverse map; thus Ω is simply connected. By Lemma 5.6, $\partial\Omega$ is locally analytic. A general theorem in complex analysis tells us that η_λ extends continuously to $\mathbb{C}^+ \cup \mathbb{R}$ and it establishes a homeomorphism between the real axis and $\partial\Omega$. We continue to denote by η_λ and Φ_λ their extensions.

We claim that

$$\partial\Omega = (-\infty, 2 - \sqrt{3}] \cup [2 + \sqrt{3}, +\infty) \cup \mathcal{L},$$

where \mathcal{L} is an analytic and open curve in \mathbb{C}^+ with endpoints $2 - \sqrt{3}$ and $2 + \sqrt{3}$. We denote $\gamma(t) = \eta_\lambda(t)$, $t \in \mathbb{R}$ be a parametrization of $\partial\Omega$. Set $t_1 = \Phi_\lambda(2 - \sqrt{3}) > 0$ and $t_2 = \Phi_\lambda(2 + \sqrt{3}) > 0$. Then $\eta_\lambda(t_1) = 2 - \sqrt{3}$ and $\eta_\lambda(t_2) = 2 + \sqrt{3}$, and $\mathcal{L} = \{\gamma(t)\}_{t_1 < t < t_2}$. Note that

- (1) $(-\infty, 0) \subset \partial\Omega$,
- (2) $\Phi'_\lambda(x) > 0$ for all $x \in (-\infty, 2 - \sqrt{3})$.

From this we deduce that $(-\infty, 2 - \sqrt{3}) \subset \partial\Omega$. Lemma 5.6 tells us Φ_λ has a zero of order one at $2 - \sqrt{3}$, therefore $\partial\Omega$ has a corner of opening $\pi/2$ at $2 - \sqrt{3}$. Note that $\Phi'_\lambda(x) > 0$ for all $x \in (2 + \sqrt{3}, +\infty)$, thus $(2 + \sqrt{3}, +\infty) \subset \partial\Omega$, and $\partial\Omega$ has a corner of opening $\pi/2$ at $2 + \sqrt{3}$.

It remains to prove that $\mathcal{L} \cap \mathbb{R} = \emptyset$. First we show $1 \notin \mathcal{L}$. Suppose that is the case, and suppose $\gamma(t_0) = 1$ where $t_1 < t_0 < t_2$, by continuity, we have

$$\gamma(t) \exp\left(\frac{\gamma(t) + 1}{\gamma(t) - 1}\right) = \Phi_\lambda(\gamma(t)) = \Phi_\lambda(\eta_\lambda(t)) = t$$

for all $t \in \mathbb{R}$. Therefore in a small neighborhood of t_0 , we have

$$\frac{\gamma(t) + 1}{\gamma(t) - 1} = \ln \frac{t}{\gamma(t)}.$$

The left side of the above equation blows up, while the right hand side is bounded. This contradiction tells us that $1 \notin \mathcal{L}$. Now suppose \mathcal{L} touches the real axis at $x_0 \in (2 - \sqrt{3}, 1) \cup (1, 2 + \sqrt{3})$. Since Ω is connected, it is not hard to see that x_0 must be a critical point of Φ_λ . This is not possible by Lemma 5.6. We therefore proved that $\mathcal{L} \subset \mathbb{C}^+$ and the claim.

From the definitions of the Cauchy transform and η -transform, one can easily check that

$$G_\lambda\left(\frac{1}{z}\right) = \frac{z}{1 - \eta_\lambda(z)}.$$

From the above equation we know that G_λ extends to be a continuous function on $\mathbb{C} \cup \mathbb{R}$, and $\{x \in \mathbb{R} : \Im(G_\mu(x)) > 0\} = (1/t_2, 1/t_1)$. By the Stieltjes inverse formula, we deduce that the support of λ is $(1/t_2, 1/t_1)$. From the analyticity of the curve $\mathcal{L} \subset \mathbb{C}^+$, we conclude that λ has positive and analytic density in the interior of its support. □

We are interested in the level curves of the function

$$(5-6) \quad f(r, \theta) = \theta - \frac{2r \sin \theta}{1 - 2r \cos \theta + r^2} = \arg(\Phi_\lambda(z)),$$

where $z = r^{i\theta} \in \mathbb{C}^+$. For $t \leq 0$, set $\gamma_t = \{z = r e^{i\theta} \in \mathbb{C}^+ : f(r, \theta) = t\}$.

Proposition 5.8. (A) γ_0 is a simple open curve with endpoints $2 - \sqrt{3}, 2 + \sqrt{3}$ and $\gamma_0 = \mathcal{L}$.

(B) γ_t is a simple open curve which starts at $z = 1$ and ends at $z = 1$ as well for all $t < 0$.

Denote by Ω_0 the open domain bounded $\gamma_0 \cup [2 - \sqrt{3}, 2 + \sqrt{3}]$. For all $t < 0$, denote by Ω_t the open domain bounded $\gamma_t \cup \{1\}$.

(C) For $t_1 < t_2 \leq 0$, we have $\Omega_{t_1} \subset \Omega_{t_2}$; and for all $t_0 \leq 0$, $\Omega_{t_0} = \cup_{t < t_0} \Omega_t$.

Proof. Given $\theta \in (0, \pi)$, we define a function of r by $f_\theta(r) = f(r, \theta)$ for $r \in (0, +\infty)$. We first note that $f(r, \theta) < \theta < \pi$ and observe that

$$\lim_{r \rightarrow +\infty} f_\theta(r) = \theta.$$

Thus $\{z = re^{i\theta} : f(r, \theta) > 0, 0 < \theta < \pi\} \subset \Phi^{-1}(\mathbb{C}^+)$.

Given $\theta \in (0, \pi)$ and $t \leq 0$, the equation $f(r, \theta) = t$ is equivalent to the quadratic equation

$$(5-7) \quad h_\theta(r) := (\theta - t)r^2 - (2(\theta - t) \cos \theta + 2 \sin \theta)r + \theta - t = 0$$

with discriminant $d(\theta, t) = [2(\theta - t) \cos \theta + 2 \sin \theta]^2 - 4(\theta - t)^2$. We then rewrite $d(\theta, t)$ as follows.

$$(5-8) \quad d(\theta, t) = 4(1 - \cos^2(\theta)) \left[\frac{\sin \theta}{1 + \cos \theta} + \theta - t \right] \cdot \left[\frac{\sin \theta}{1 - \cos \theta} - \theta + t \right],$$

The first two factors in (5-8) are never zero for $\theta \in (0, \pi)$; thus only the last factor in (5-8) matters to determine the sign of $d(\theta, t)$. We consider the function k by $k(\theta) = \sin \theta / (1 - \cos \theta) - \theta$ for $\theta \in (0, \pi)$, and calculate

$$(5-9) \quad k'(\theta) = \frac{1}{\cos \theta - 1} - 1 < 0,$$

which implies that k is a decreasing function of θ . For $t \leq 0$, we now set $d_t(\theta) := d(\theta, t)$. We then deduce that $d_t(\theta) = 0$ has exactly one solution, which we denote by θ_t , and $d_t(\theta) > 0$ if and only if $0 < \theta < \theta_t$. Therefore, the half line $r = \theta$ intersects with γ_t at two points if and only if $0 < \theta < \theta_t$ and the half line $r = \theta_t$ is tangent to γ_t . Moreover, $\theta_{t_1} < \theta_{t_2}$ if $t_1 < t_2 \leq 0$.

For the solutions of the equation $f(r, \theta) = 0$, one can check as $\theta \rightarrow 0$, r satisfying the equation $r^2 - 4r + 1$. Given $t < 0$, for the solutions of the equation $f(r, \theta) = t$, we can easily see that r tend to 1 as $\theta \rightarrow 0$. Now (A) and (B) follow from this observation.

Given $\theta \in (0, \pi)$, from (5-6), we see that the function $f_\theta(r)$ defined by $f_\theta(r) = f(r, \theta)$ has exactly one local minimum at $r = 1$. $f_\theta(r)$ is a decreasing function of r on $(0, 1)$ and an increasing function of r on $(1, \infty)$. Therefore, if the half line $r = \theta$ intersects γ_t at two points, one of them is inside the unit circle of \mathbb{C} and the other one is outside the unit circle. We conclude that (C) is valid. \square

It is interesting to compare the next result with Proposition 2.2 and Lemma 3.2.

Corollary 5.9. *We have $\Phi_\lambda^{-1}(\mathbb{C}^+) = \Omega \cup_{k=1}^\infty (\Omega_{(2k-1)\pi} \setminus \Omega_{(2k-2)\pi})$. Moreover, Ω and $\Omega_{(2k-1)\pi} \setminus \Omega_{(2k-2)\pi}$ ($k = 1, 2, \dots$) are all connected components of Φ_λ . In particular, $\Phi_\lambda^{-1}(\mathbb{C}^+)$ has infinitely many connected components.*

We would like to point out that for $z = re^{i\theta} \in \mathcal{L} = \gamma_0$, the curve \mathcal{L} can be parametrized by r . Noticing (5-7) and (5-8), we first observe the following equivalences:

$$(5-10) \quad d(\theta, 0) = 0 \iff \theta \cos \theta + \sin \theta = \theta \iff r = 1.$$

By (5-9), we see that (5-10) has exactly one solution θ_0 for $\theta \in (0, \pi)$. By differentiating the equation $f(r, \theta) = 0$, we obtain

$$(5-11) \quad \frac{d\theta}{dr} = \frac{2\theta \cos \theta + 2 \sin \theta - 2\theta r}{r^2 + 2\theta \sin \theta - 4 \cos \theta + 1}.$$

Thus, $d\theta/dr = 0$ if and only if $r = (\theta \cos \theta + \sin \theta)/\theta$. Fix θ , the equation $f_\theta(r) = 0$ is equivalent to the quadratic equation $\theta r^2 - (2\theta \cos \theta + 2 \sin \theta)r + \theta = 0$, from which we deduce that $r = (\theta \cos \theta + \sin \theta)/\theta$ if and only if $d(r, 0) = 0$. From (5-11) and continuity of $d\theta/dr$, we see that $d\theta/dr > 0$ for $0 < \theta < \theta_0, r < 1$ and $d\theta/dr < 0$ for $0 < \theta < \theta_0, r > 1$. Therefore, for the solutions of the equation $f(r, \theta) = 0$, θ is a function of r and the curve \mathcal{L} can be parametrized by r .

Denote by g the density function of λ . From the equation $G_\lambda(1/x) = x/(1 - \eta_\lambda(x))$, we obtain the following formula for the density function of λ .

Proposition 5.10. *Given $z = re^{i\theta} \in \gamma_0 = \mathcal{L}$, we have*

$$g(1/x) = \theta \Phi_\lambda(z) = r\theta \exp\left(\frac{r^2 - 1}{1 - 2r \cos \theta + r^2}\right),$$

where $x = \Phi_\lambda(z)$.

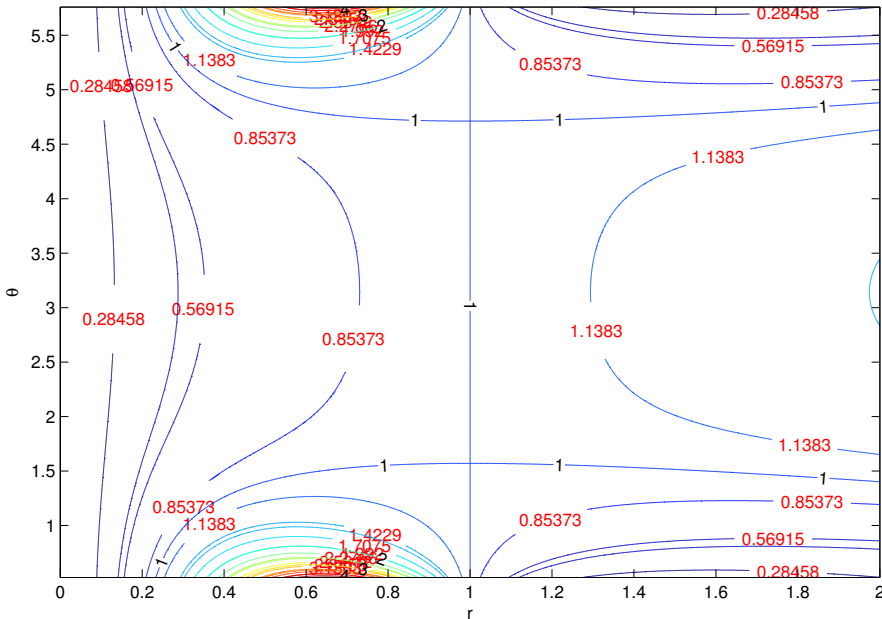
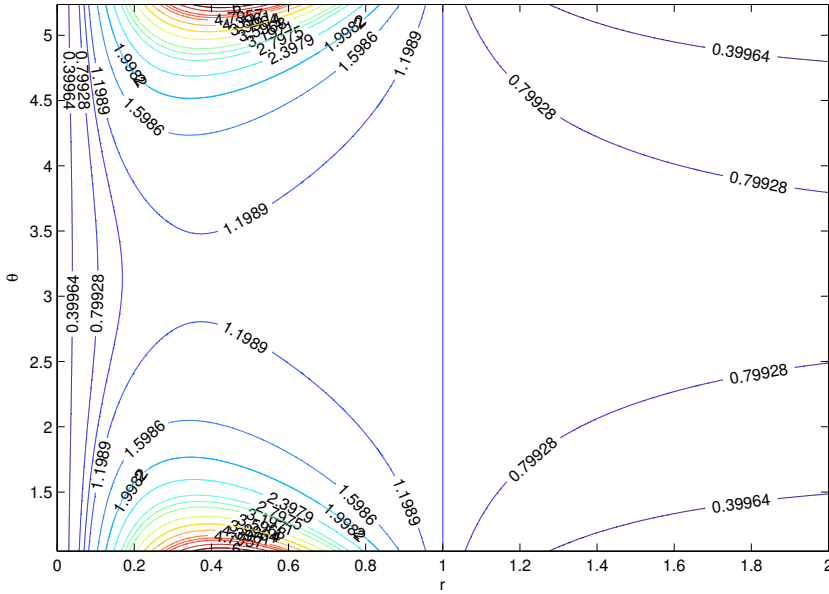


Figure 1. Level curves of $g_2(r, \theta) = |\Phi_2(re^{i\theta})|$. The vertical axis indicates θ , and the horizontal axis indicates r .



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
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