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TOTARO'S QUESTION FOR SIMPLY CONNECTED GROUPS OF LOW RANK

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Let k be a field and let G be a connected linear algebraic group over k. In a 2004 paper, Totaro asked whether a torsor X under G and over k which admits a zero cycle of degree d also admits a closed étale point of degree dividing d. We consider this question in the setting where G is a simply connected, semisimple group of rank at most 2 and k is of characteristic different from 2.

Introduction

Serre [1995, p. 233] raised the following question:

Serre's question: Let k be a field and let G be a connected linear algebraic group defined over k. Let X be a G-torsor over k. Suppose X admits a zero cycle of degree 1. Does X have a k-rational point?

An affirmative answer to Serre's question is known in a number of special cases. See, for example, [Sansuc 1981; Bayer-Fluckiger and Lenstra 1990; Black 2011a; 2011b]. Burt Totaro [2004] posed the following generalization of Serre's question:

Totaro's question: Let k be a field and let G be a connected linear algebraic group defined over k. Let X be a G-torsor over k. Suppose X admits a zero cycle of degree d. Does X have a closed étale point of degree dividing d?

An affirmative answer to Totaro's question when $G = \mathrm{PGL}_n$ is a classical result in the theory of central simple algebras. Tits [1992] associated to any absolutely simple, linear algebraic k-group G, an integer $n(G)$. The values of $n(G)$ are shown in Table 1 below, where ν denotes the 2-adic valuation. One can show that for any G-torsor X, there is a separable field extension L/k such that X has a rational point over L and $[L:k]$ divides $n(G)^2$ [Serre 1995, Section 2.3]. Thus, Tits' construction gives an affirmative answer to Totaro's question provided $n(G)^2$ divides d. Garibaldi and Hoffmann [2006] give an affirmative answer to Totaro's question for semisimple groups which are of type G_2, of reduced type F_4 or simply

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Type of group	$n(G)$
A_n	$2(n+1)$
B_n	2^n
C_n	$2^{\nu(n)+1}$
D_n $(n \neq 4)$	$2^{n+\nu(n)}$

Table 1. Values of $n(G)$ for classical groups.

connected of type ${}^1E^0_{6,6}$ or ${}^1E^{28}_{6,2}$. Their work extended previous results of Totaro [2004] which gave an affirmative answer for split, simply connected groups of type G_2, F_4 and E_6. Results in [Black 2011b] give an affirmative answer to Totaro's question in the case where G is a simply connected or adjoint, semisimple, classical group and d is prime to $n(G)$.

In this paper we show the following:

Theorem 0.1. *The answer to Totaro's question is yes if k is of characteristic different from 2 and G is a semisimple, simply connected, classical group such that rank $G_{\bar{k}} \leq 2$.*

1. Galois cohomology

Let k be a field, let k_s be a separable closure of k and let $\Gamma_k = \mathrm{Gal}(k_s/k)$ be the absolute Galois group of k. We write $H^1(k, G)$ for the first Galois cohomology set $H^1(\Gamma_k, G(k_s))$. Given any finite field extension L/k there is a canonical restriction map $H^1(k, G) \to H^1(L, G)$. If $\lambda \in H^1(k, G)$ is any element, we write λ_L for the image of λ under the restriction map $H^1(k, G) \to H^1(L, G)$.

For our convenience, we will consider the formulation of Totaro's question in Galois cohomology:

Totaro's question: Let k be a field and let G be a connected linear algebraic group defined over k. Let $\{L_i\}_{1\leq i\leq m}$ be a set of finite field extensions of k and let $d = \gcd\{[L_i : k]_{1\leq i\leq m}\}$. If $\lambda_{L_i} = 1$ for all i, is there a finite, separable field extension F of k such that $\lambda_F = 1$ and $[F : k]$ divides d?

2. Results

In this section, we consider Totaro's question for various groups G.

The case $G = \mathrm{SL}_1(A)$.

Theorem 2.1. *The answer to Totaro's question is yes if $G = \mathrm{SL}_1(A)$ for A a central simple algebra over k of prime index.*

Proof. Let $\{L_i\}_{1\leq i\leq m}$ be a set of finite field extensions of k and suppose $\lambda \in H^1(k, \mathrm{SL}_1(A))$ is an element such that $\lambda_{L_i} = 1$ for all i. Let $d = \gcd\{[L_i : k]_{1\leq i\leq m}\}$.

We will find F/k separable such that $\lambda_F = 1$ and $[F : k]$ divides d.

Since by [Knus et al. 1998, Theorem 29.2], $H^1(k, \mathrm{GL}_1(A)) = 1$, the short exact sequence

$$1 \longrightarrow \mathrm{SL}_1(A) \longrightarrow \mathrm{GL}_1(A) \xrightarrow{\mathrm{Nrd}} G_m \longrightarrow 1$$

induces the long exact sequence

$$A^* \xrightarrow{\mathrm{Nrd}} k^* \longrightarrow H^1(k, \mathrm{SL}_1(A)) \longrightarrow 1 \tag{2.1.1}$$

in Galois cohomology, where Nrd is the reduced norm. By (2.1.1) above,

$$H^1(k, \mathrm{SL}_1(A)) \cong k^*/\,\mathrm{Nrd}(A^*),$$

and we can identify λ with the class of an element of k^* which is in $\mathrm{Nrd}(A_{L_i})$ for all i. For simplicity, we will also refer to this element as λ. Let the index of A be s and choose L contained in A a separable field extension of k of degree s which splits A [Gille and Szamuely 2006, Propositions 4.5.3 and 4.5.4]. Then $\mathrm{Nrd}(A_L) = L^*$ and λ is in $\mathrm{Nrd}(A_L)$. So if s divides d we may take $F = L$. Recall that s is prime. So if s does not divide d then $\gcd(s, d) = 1$. It is well known that $N_{L/k}(\mathrm{Nrd}(A_L)) \subseteq \mathrm{Nrd}(A)$. In particular, $\lambda^s = N_{L/k}(\lambda)$ is in $\mathrm{Nrd}(A)$. Since $\mathrm{Nrd}(A)$ is a group and $N_{L_i/k}(\lambda) \in \mathrm{Nrd}(A)$ for all i, we find that λ^d is in $\mathrm{Nrd}(A)$. In turn, λ is in $\mathrm{Nrd}(A)$ and we can take $F = k$. □

The case $G = \mathrm{SU}(A, \sigma)$.

Theorem 2.2. *The answer to Totaro's question is yes if k is of characteristic different from 2 and $G = \mathrm{SU}(A, \sigma)$ for a central simple algebra A of degree 3 over K, $k = K^\sigma$ and $[K : k] = 2$.*

Proof. Let $\{L_i\}_{1 \le i \le m}$ be a set of finite field extensions of k and suppose $\lambda \in H^1(k, \mathrm{SU}(A, \sigma))$ is an element such that $\lambda_{L_i} = 1$ for all i. Let $d = \gcd\{[L_i : k]\}$. We will find F/k separable such that $\lambda_F = 1$ and $[F : k]$ divides d.

The case where d is coprime to 2 and 3 was covered in [Black 2011b, Theorem 3.4]. If $6 \mid d$, we take L to be a separable extension of K of degree dividing 3 which splits A. Since K/k is Galois, L/k is separable of degree dividing 6. Since $H^1(K, \mathrm{SU}(A, \sigma)) = H^1(K, \mathrm{SL}_1(A))$ and L splits A, $H^1(L, \mathrm{SU}(A, \sigma)) = \{1\}$ by Hilbert's Theorem 90. Therefore, for any $\lambda \in H^1(k, \mathrm{SU}(A, \sigma))$, $\lambda_L = 1$ and we can take $F = L$. Now suppose $2 \mid d$ and $3 \nmid d$. Fix an index i such that $[L_i : k]$ is prime to 3 and $\lambda_{L_i} = 1$. Consider $L_i K$, the compositum of L_i and K. Since, by assumption, 3 is prime to $[L_i : k]$, and $[K : k] = 2$, we know that 3 is prime to $[L_i K : k]$. Therefore, 3 is prime to $[L_i K : K]$. Let L be a separable splitting field of A such that $[L : K]$ is equal to the index of A. Since $\deg_K(A) = 3$, either $[L : K] = 1$ or $[L : K] = 3$. In either case, L, $L_i K$ is a pair of field extensions of K such that $\lambda_L = 1 = \lambda_{L_i K}$ and gcd$\{[L:K],[L_iK:K]\}$ is 1. Since $H^1(K,\mathrm{SU}(A,\sigma)) = H^1(K,\mathrm{SL}_1(A))$ we have $\lambda_K = 1$ by Theorem 2.1, and we can take $F = K$. The final setting to consider is the case where $3 \mid d$ and $2 \nmid d$. Since d is odd, we can fix an index i such that $[L_i : k]$ is odd and $\lambda_{L_i} = 1$. Let $R_{K/k}G_m$ be the Weil transfer of G_m and let $R^1_{K/k}G_m$ be defined as the kernel of the norm map $N_{K/k} : R_{K/k}G_m \to G_m$. The short exact sequence

$$1 \to \mathrm{SU}(A,\sigma) \to U(A,\sigma) \to R^1_{K/k}G_m \to 1$$

induces the commutative diagram

$$\begin{array}{ccccc}
K^{*1} & \xrightarrow{\delta} & H^1(k,\mathrm{SU}(A,\sigma)) & \xrightarrow{j} & H^1(k,U(A,\sigma)) \\
\downarrow & & \downarrow & & \downarrow \\
(K\otimes L_i)^{*1} & \longrightarrow & H^1(L_i,\mathrm{SU}(A,\sigma)) & \longrightarrow & H^1(L_i,U(A,\sigma))
\end{array}$$

where K^{*1} and $(K\otimes L_i)^{*1}$ denote the norm-one elements in K^* and $(K\otimes L_i)^*$ respectively. By a result of Bayer-Fluckiger and Lenstra [1990, Theorem 2.1], $j(\lambda) = 1$. In particular, we can choose $\alpha \in K^{*1}$ such that $\delta(\alpha) = \lambda$. In the case where A is split, $H^1(K,\mathrm{SU}(A,\sigma)) = H^1(K,\mathrm{SL}_1(A)) = \{1\}$. Then, since K and L_i are field extensions of coprime degree with $\lambda_K = \lambda_{L_i} = 1$, the desired result holds by [Black 2011b, Theorem 4.4]. Since $\deg(A) = 3$, if A is not split, then A is a division algebra and by [Albert 1963] (see also [Knus et al. 1998, Theorem 19.14]), there is a k-subalgebra L of A such that L/k is étale of degree three. Since A is division, L is a field. Consider the diagram

$$\begin{array}{ccccccc}
U(A,\sigma)(k) & \longrightarrow & K^{*1} & \xrightarrow{\delta} & H^1(k,\mathrm{SU}(A,\sigma)) & \xrightarrow{j} & H^1(k,U(A,\sigma)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
U(A,\sigma)(L) & \longrightarrow & (K\otimes L)^{*1} & \longrightarrow & H^1(L,\mathrm{SU}(A,\sigma)) & \longrightarrow & H^1(L,U(A,\sigma))
\end{array}$$

For $x \in (K\otimes L)^{*1}$, write $x = y^{-1}\bar{y}$ for $y \in (K\otimes L)^*$ where $\bar{\ }$ denotes the nontrivial automorphism of K/k. Since $A\otimes L$ is split, y is a reduced norm from $A\otimes L$. In view of [Merkurjev 1995, Proposition 6.1], the image of $\mathrm{Nrd}(U(A,\sigma) \to (K\otimes L)^{*1})$ contains x. Thus $\lambda_L = 1$ and we may take $F = L$. □

The case G = Spin(q). The following result will be useful:

Proposition 2.3. *Let k be a field of characteristic different from 2 and let q be a quadratic form over k of dimension ≤ 5. Let $\lambda \in H^1(k,\mathrm{Spin}(q))$ be any element. Then there exists a (separable) field extension F of k such that $[F:k]$ divides 2 and $\lambda_F = 1$.*

Proof. Consider the short exact sequence

$$1 \longrightarrow \mu_2 \xrightarrow{\ i\ } \operatorname{Spin}(q) \xrightarrow{\ \pi\ } O^+(q) \longrightarrow 1,$$

which induces the exact sequence in Galois cohomology

$$H^1(k, \mu_2) \xrightarrow{\ i\ } H^1(k, \operatorname{Spin}(q)) \xrightarrow{\ \pi\ } H^1(k, O^+(q)). \tag{2.3.1}$$

The pointed set $H^1(k, O^+(q))$ classifies quadratic forms over k of the same dimension and discriminant as q. Let $q' = \pi(\lambda)$. Then $q \perp -q'$ has even dimension, trivial discriminant and trivial Clifford invariant since q' is in the image of π. Thus $q \perp -q' \in I^3(k)$.

First consider the case where $\dim(q) < 4$. Then, $\dim(q \perp -q') < 8$ and by the Arason–Pfister Hauptsatz [Lam 1980, Chapter X, Hauptsatz 5.1], $q \perp -q'$ is hyperbolic. Equivalently, $q \cong q'$ and $q' = 1$ in $H^1(k, O^+(q))$. Using the exactness of (2.3.1), choose η in $H^1(k, \mu_2)$ such that $i(\eta) = \lambda$. Since $H^1(k, \mu_2) \cong k^*/k^{*2}$ we can choose F/k a field extension of degree at most 2 such that $\eta_F = 1 \in H^1(F, \mu_2)$. By commutativity of (2.3.2) below, $\lambda_F = 1$ in $H^1(F, \operatorname{Spin}(q))$.

$$\begin{array}{ccc} H^1(k, \mu_2) & \longrightarrow & H^1(k, \operatorname{Spin}(q)) \\ \downarrow & & \downarrow \\ H^1(F, \mu_2) & \longrightarrow & H^1(F, \operatorname{Spin}(q)) \end{array} \tag{2.3.2}$$

Suppose instead that $\dim(q) = 4$. Let $d = \operatorname{disc}(q)$ and write $q = a\langle 1, b, c, bcd\rangle$. By [Lam 1980, Chapter XII, Proposition 2.4], there is an element $\alpha \in k^*$ such that $q' \cong \alpha q$ and we may write $q \perp -q' \cong \langle 1, -\alpha\rangle q = a\langle 1, -\alpha\rangle\langle 1, b, c, bcd\rangle$. Let e_2 be the map from $I^2(k) \to H^2(k, \mu_2)$ induced by the Clifford invariant. Since $q \perp -q' \in I^3(k)$, $e_2(q \perp -q') = (d) \cup (\alpha) = 0 \in H^2(k, \mu_2)$ [Elman et al. 2008, 16.2] and so $\langle 1, -\alpha, -d, \alpha d\rangle$ is hyperbolic. Equivalently, $\langle 1, -\alpha\rangle d \cong \langle 1, -\alpha\rangle$ and $q \perp -q' \cong a\langle 1, -\alpha\rangle\langle 1, b, c, bc\rangle = a\langle 1, -\alpha\rangle\langle 1, b\rangle\langle 1, c\rangle$. Let $F = k(\sqrt{-b})$. Then $[F : k] \le 2$, $(q \perp -q')_F$ is hyperbolic and $q'_F = 1 \in H^1(F, O^+(q))$. Consider the diagram

$$\begin{array}{ccccccc} O^+(q)(k) & \longrightarrow & H^1(k, \mu_2) & \longrightarrow & H^1(k, \operatorname{Spin}(q)) & \longrightarrow & H^1(k, O^+(q)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ O^+(q)(F) & \xrightarrow{\ \text{sn}\ } & H^1(F, \mu_2) & \xrightarrow{\ i\ } & H^1(F, \operatorname{Spin}(q)) & \xrightarrow{\ \pi\ } & H^1(F, O^+(q)) \end{array} \tag{2.3.3}$$

By commutativity of the right rectangle, $\pi(\lambda_F) = 1$ and by the exactness of the bottom row, $\lambda_F \in \operatorname{im}(i)$. But since $q \cong a\langle 1, b, c, bcd\rangle$, q_F is isotropic. Thus, the spinor norm $\mathrm{sn}: O^+(q)(F) \to H^1(F, \mu_2)$ is onto [Baeza 1978, p. 78] and therefore, since $\lambda_F \in \mathrm{im}(i)$, $\lambda_F = 1$.

Now suppose $\dim(q) = 5$. Since $q \perp -q'$ is a rank 10 form in $I^3(k)$, it is isotropic [Lam 1980, Chapter XII, Proposition 2.8]. Therefore q and q' have a common slot and we can write $q = \langle a \rangle \perp q_1$ and $q' = \langle a \rangle \perp q_2$. Since $q_1 \perp -q_2 \in I^3 k$ is rank 8, we can proceed as in the rank 4 case and find a field extension F of k of degree at most 2 such that $(q_1 \perp -q_2)_F$ is hyperbolic and $(q_1)_F$ is isotropic. By the Arason–Pfister Hauptsatz, $(q \perp -q')_F$ is hyperbolic and thus $q_F \cong q'_F$ and $\pi(\lambda_F) = q'_F = 1 \in H^1(F, O^+(q))$. Thus λ_F is in the image of $i : H^1(F, \mu_2) \to H^1(F, \mathrm{Spin}(q))$. However, $(q_1)_F$ being isotropic, q_F is isotropic and $\mathrm{sn} : O^+(q)(F) \to H^1(F, \mu_2)$ is onto. Therefore, i is the zero map and $\lambda_F = 1$. □

Theorem 2.4. *The answer to Totaro's question is yes if k is of characteristic different from* 2 *and $G = \mathrm{Spin}(q)$ for q a quadratic form of dimension* ≤ 5.

Proof. Let $\{L_i\}_{1 \leq i \leq m}$ be a set of finite field extensions of k and suppose $\lambda \in H^1(k, \mathrm{Spin}(q))$ is an element such that $\lambda_{L_i} = 1$ for all i. Let $d = \gcd\{[L_i : k]\}$. We want to find F/k separable such that $\lambda_F = 1$ and $[F : k]$ divides d. If d is odd we are done by [Black 2011b, Theorem 3.7] and can take $F = k$. If d is even, by Proposition 2.3, there is a separable extension F/k of degree at most 2 such that $\lambda_F = 1$. □

Theorem 2.5. *The answer to Totaro's question is yes if k is of characteristic different from* 2 *and $G = \mathrm{Sp}(A, \sigma)$ where A is a central simple algebra with symplectic involution and* $\deg(A)$ *is* 2 *or* 4.

Proof. Let q be a quadratic form of dimension 3 (resp. 5) with trivial discriminant. Then the even Clifford algebra $A = C_0(V, q)$ is a central simple algebra of degree 2 (resp. 4) and the canonical involution on the Clifford algebra is symplectic and $\mathrm{Spin}(q) \cong \mathrm{Sp}(A, \sigma)$ [Knus et al. 1998, Section 15.C]. Moreover, every algebra A of degree 2 or 4 with a symplectic involution arises in this way. Thus, a positive answer to Totaro's question for $\mathrm{Sp}(A, \sigma)$ follows from Proposition 2.3. □

The case $G = \mathrm{Spin}(A, \sigma)$.

Theorem 2.6. *The answer to Totaro's question is yes if k is of characteristic different from* 2 *and $G = \mathrm{Spin}(A, \sigma)$, where A is a central simple algebra of degree* 4 *over k and σ is an orthogonal involution on A.*

Proof. Let $\{L_i\}_{1 \leq i \leq m}$ be a set of finite field extensions of k and suppose $\lambda \in H^1(k, \mathrm{Spin}(A, \sigma))$ is an element such that $\lambda_{L_i} = 1$ for all i. Let $d = \gcd\{[L_i : k]\}$. We will find F/k separable such that $\lambda_F = 1$ and $[F : k]$ divides d.

By [Black 2011b, Theorem 3.7], when d is odd we may take $F = k$. So we may suppose that d is even. Suppose (A, σ) has trivial discriminant. Then

$(A,\sigma)\cong(Q_1\otimes Q_2,\tau_1\otimes\tau_2)$ [Knus et al. 1998, Corollary 15.12], where Q_1 and Q_2 are quaternion algebras with the symplectic involution given by conjugation. In turn $\mathrm{Spin}(A,\sigma)\cong\mathrm{SL}_1(Q_1)\times\mathrm{SL}_1(Q_2)$ [Knus et al. 1998, Corollary 15.13]. There exist $\lambda_1,\lambda_2\in k^*$ such that $\lambda=(\bar\lambda_1,\bar\lambda_2)$ with $\bar\lambda_i\in k^*/\mathrm{Nrd}(Q_i)\cong H^1(k,\mathrm{SL}_1(Q_i))$ for $i=1,2$. In the case $4\mid d$, let F_1, F_2 be extensions of k of degree at most 2 which split Q_1 and Q_2 respectively. Then $\lambda_{F_1F_2}=1$ and $[F_1F_2:k]$ divides 4. In the case $2\mid d$ and $4\nmid d$, we can fix an L_j/k such that $[L_j:k]=2m$, where m is odd and $\lambda_{L_j}=1$. Following arguments as in [Garibaldi and Hoffmann 2006, Lemma 1.5] we suppose without loss of generality that $k\subseteq L\subseteq L_j$ with $[L:k]$ odd, $[L_j:L]=2$ and $\lambda_{L_j}=1$. Let N_{Q_1}, N_{Q_2} be the norm forms for the quaternion algebras Q_1, Q_2 respectively and let $\phi_1=\langle 1,-\lambda_1\rangle N_{Q_1}$ and $\phi_2=\langle 1,-\lambda_2\rangle N_{Q_2}$. The fact that $\lambda_{L_j}=1$ implies that ϕ_1,ϕ_2 are hyperbolic over L_j. Then by [Garibaldi and Hoffmann 2006, Lemma 1.4] there exists $\mu\in k^*$ such that $\phi_1\cong\langle 1,\mu\rangle\tilde\phi_1$ and $\phi_2\cong\langle 1,\mu\rangle\tilde\phi_2$, where $\tilde\phi_1,\tilde\phi_2$ are 2-fold Pfister forms. Let $F=k(\sqrt{-\mu})$. Then ϕ_1,ϕ_2 are hyperbolic over F and thus $\lambda_1\in\mathrm{Nrd}(Q_{1_F})$ and $\lambda_2\in\mathrm{Nrd}(Q_{2_F})$. That is, $\lambda_F=1$. Also, F/k is separable and degree at most 2 by construction.

Suppose instead that (A,σ) has nontrivial discriminant. One can associate to (A,σ) its Clifford algebra Q, which is a quaternion algebra with center $K=k(\sqrt{\delta})$, where $\delta=\mathrm{disc}(A,\sigma)$ [Knus et al. 1998, Theorem 15.7]. Then $\mathrm{Spin}(A,\sigma)=R_{K/k}\,\mathrm{SL}_1(Q)$ [Knus et al. 1998, Proposition 15.10] and $H^1(k,\mathrm{Spin}(A,\sigma))=H^1(K,\mathrm{SL}_1(Q))$. If Q is split, $\lambda=1$ and we take $F=k$. So suppose Q is not split. If $4\mid d$ we can take F a splitting field of Q such that F/K is a separable extension of degree 2. Since

$$H^1(F,\mathrm{Spin}(A,\sigma))=H^1(K\otimes F,\mathrm{SL}_1(Q))\cong H^1(F\times F,\mathrm{SL}_1(Q))=\{1\},$$

we obtain $\lambda_F=1$. Further $[F:k]=4$, and since F/K and K/k are separable, F/k is separable. We are left to consider the case where (A,σ) has nontrivial discriminant and $4\nmid d$ and $2\mid d$.

Consider the short exact sequence

$$1\to R_{K/k}\,\mathrm{SL}_1(Q)\to R_{K/k}\,\mathrm{GL}_1(Q)\to R_{K/k}G_m\to 1,$$

which induces

$$\mathrm{GL}_1(Q)(K)\xrightarrow{\mathrm{Nrd}}K^*\longrightarrow H^1(K,\mathrm{SL}_1(Q))\longrightarrow 1.$$

Choose $\lambda\in H^1(K,\mathrm{SL}_1(Q))$ such that $\lambda_{L_i}=1$ for all i and let $\beta\in K^*$ satisfy $\delta(\beta)=\lambda$. Following [Garibaldi and Hoffmann 2006, Lemma 1.5], we may suppose that $\lambda_{L_j}=1$ where $k\subseteq L\subseteq L_j$ and $[L_j:L]=2$.

$$\begin{array}{ccccccc} \mathrm{GL}_1(Q)(K) & \xrightarrow{\mathrm{Nrd}} & K^* & \longrightarrow & H^1(K, \mathrm{SL}_1(Q)) & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{GL}_1(Q)(K \otimes_k L_j) & \xrightarrow{\mathrm{Nrd}} & (K \otimes_k L_j)^* & \longrightarrow & H^1(K \otimes_k L_j, \mathrm{SL}_1(Q)) & \longrightarrow & 1 \end{array} \tag{2.6.1}$$

Write $L_j = L(\sqrt{a})$ for $a \in L^*/L^{*2}$. Let f be the norm form on Q and let f^0 denote the norm form restricted to the traceless elements of Q, which we denote by Q^0. Since $\lambda_{L_j} = 0$, choose $x_0, y_0 \in Q \otimes L$ such that

$$\beta = f(x_0 + y_0\sqrt{a}). \tag{2.6.2}$$

If $y_0 = 0$ we have $\beta \in \mathrm{Nrd}(Q \otimes L)$, and, L/K being of odd degree, this implies $\beta \in \mathrm{Nrd}(Q)$. We take $F = k$. So suppose $y_0 \neq 0$. Since Q is a division algebra, $f(y_0) \neq 0$ and

$$\beta = f(x_0) + af(y_0). \tag{2.6.3}$$

If we let b_f denote the adjoint bilinear form, we have

$$b_f(x_0, y_0) = 0 \tag{2.6.4}$$

and

$$\beta f(y_0^{-1}) = f(x_0 y_0^{-1}) + a, \tag{2.6.5}$$

where the reduced trace $\mathrm{trd}(x_0 y_0^{-1})$ vanishes by (2.6.4). Therefore,

$$\beta f(y_0^{-1}) = f^0(x_0 y_0^{-1}) + a. \tag{2.6.6}$$

Let $f = f_1 + \sqrt{\delta} f_2$ with f_1 and f_2 quadratic forms on Q with values in k. Further let $f^0 = f_1^0 + \sqrt{\delta} f_2^0$ where f_1^0, f_2^0 are quadratic forms on Q^0 with values in k. Setting $z_0 = y_0^{-1}$ and $w_0 = x_0 y_0^{-1}$, we have

$$a = \beta_1 f_1(z_0) + \beta_2 \delta f_2(z_0) - f_1^0(w_0), \tag{2.6.7}$$

$$0 = \beta_1 f_2(z_0) + \beta_2 f_1(z_0) - f_2^0(w_0), \tag{2.6.8}$$

with $z_0 \in Q \otimes L$ and $w_0 \in Q^0 \otimes L$. Define k-quadratic forms $q_1 : Q \oplus Q^0 \to k$ and $q_2 : Q \oplus Q^0 \to k$ by

$$q_1(z, w) = \beta_1 f_1(z) + \beta_2 \delta f_2(z) - f_1^0(w), \tag{2.6.9}$$

$$q_2(z, w) = \beta_1 f_2(z) + \beta_2 f_1(z) - f_2^0(w), \tag{2.6.10}$$

for $z \in Q$ and $w \in Q_0$. Since $y_0 \neq 0$, $z_0 = y_0^{-1} \neq 0$ and (z_0, w_0) is a nontrivial zero of q_2 over L. Then by Springer's theorem [1952], q_2 has a nontrivial zero (z_1, w_1)

over k. By a general position argument, we may assume that $z_1 \neq 0$. Let

$$\alpha = \beta_1 f_1(z_1) + \beta_2 \delta f_2(z_1) - f_1^0(w_1). \tag{2.6.11}$$

We have

$$0 = \beta_1 f_2(z_0) + \beta_2 f_1(z_0) - f_2^0(w_1). \tag{2.6.12}$$

Adding these two equations, we find

$$\alpha = \beta f(z_1) - f^0(w_1), \tag{2.6.13}$$

or, equivalently,

$$\beta f(z_1) = \alpha + f^0(w_1). \tag{2.6.14}$$

Let $F = k(\sqrt{\alpha})$. Then $[F:k] \le 2$, $(\sqrt{\alpha}+w_1)z_1^{-1} \in Q_F$ and $\beta = \mathrm{Nrd}((\sqrt{\alpha}+w_1)z_1^{-1})$. Thus, $\lambda_F = 1$. □

Theorem 2.7. *The answer to Totaro's question is yes if k is of characteristic different from* 2 *and $G = \mathrm{SU}(A, \sigma)$ where A is a quaternion algebra with unitary involution σ.*

Proof. The norm algebra $N_{K/k}(A, \sigma)$ equals (B, τ) for B a central simple algebra of degree 4 and τ an orthogonal involution on B. Since $\mathrm{Spin}(B, \tau) \cong \mathrm{SU}(A, \sigma)$, that Totaro's question has an affirmative answer in this case is a consequence of Theorem 2.6. □

3. Conclusion

Theorem 3.1. *The answer to Totaro's question is yes for k a field of characteristic different from* 2 *and G a simply connected, semisimple, classical group of rank* ≤ 2.

Proof. We suppose in all cases that G is simply connected and semisimple and that the rank of $G_{\bar{k}} \le 2$. If G is of type ${}^1A_1$ or ${}^1A_2$ then G is of the form $\mathrm{SL}_1(A)$ for A a central simple algebra of degree 2 or 3 [Knus et al. 1998, Theorem 26.9]. A positive answer to Totaro's question for a group of this form was shown in Theorem 2.1. If G is of type ${}^2A_1$ then $G = \mathrm{SU}(A, \sigma)$ for A a central simple algebra of degree 2 with unitary involution σ. The proof for this case was given in Theorem 2.7. If G is of type ${}^2A_2$ then G is of the form $\mathrm{SU}(A, \sigma)$, where A is a central simple algebra of degree 3 with unitary involution σ [Knus et al. 1998, Theorem 26.9]. Thus an affirmative answer to Totaro's question for a group of type ${}^2A_2$ follows from Theorem 2.2 above. If G is of type B_1 or B_2, then $G = \mathrm{Spin}(q)$ for q a quadratic form of dimension 3 or 5 [Knus et al. 1998, Theorem 26.12] and the desired result was proven in Theorem 2.4. If G is of type C_1 or C_2, then $G = \mathrm{Sp}(A, \sigma)$, where A is a central simple algebra of degree 2 or 4 and σ is a symplectic involution on A. The proof of our result in this case was covered in Theorem 2.5. If G is of type D_2 then either $G = \mathrm{Spin}(q)$ for q a quadratic form of dimension 2 or 4 or G is of the form $\mathrm{Spin}(A, \sigma)$ for A a central simple algebra over k of degree 4 and σ an orthogonal involution on A [Knus et al. 1998, Theorem 26.15]. In the first case the desired results follows from Theorem 2.4 and in the latter it follows from Theorem 2.6. □

Remark 3.2. Since Garibaldi and Hoffman [2006] have given a proof in the case G is of type G_2, Totaro's question has a positive answer for any simply connected, semisimple group of rank ≤ 2.

References

[Albert 1963] A. A. Albert, "On involutorial associative division algebras", *Scripta Math.* **26** (1963), 309–316. MR 31 #3451 Zbl 0147.28702

[Baeza 1978] R. Baeza, *Quadratic forms over semilocal rings*, Lecture Notes in Mathematics **655**, Springer, Berlin, 1978. MR 58 #10972 Zbl 0382.10014

[Bayer-Fluckiger and Lenstra 1990] E. Bayer-Fluckiger and H. W. Lenstra, Jr., "Forms in odd degree extensions and self-dual normal bases", *Amer. J. Math.* **112**:3 (1990), 359–373. MR 91h:11030 Zbl 0729.12006

[Black 2011a] J. Black, "Implications of the Hasse principle for zero cycles of degree one on principal homogeneous spaces", *Proc. Amer. Math. Soc.* **139**:12 (2011), 4163–4171. MR 2012g:11071 Zbl 1257.11038

[Black 2011b] J. Black, "Zero cycles of degree one on principal homogeneous spaces", *J. Algebra* **334** (2011), 232–246. MR 2012g:12008 Zbl 05990153

[Elman et al. 2008] R. Elman, N. Karpenko, and A. S. Merkurjev, *The algebraic and geometric theory of quadratic forms*, American Mathematical Society Colloquium Publications **56**, American Mathematical Society, Providence, RI, 2008. MR 2009d:11062 Zbl 1165.11042

[Garibaldi and Hoffmann 2006] S. Garibaldi and D. W. Hoffmann, "Totaro's question on zero-cycles on G_2, F_4 and E_6 torsors", *J. London Math. Soc.* (2) **73**:2 (2006), 325–338. MR 2007g:11049 Zbl 1092.11021

[Gille and Szamuely 2006] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Studies in Advanced Mathematics **101**, Cambridge University Press, Cambridge, 2006. MR 2007k:16033 Zbl 1137.12001

[Knus et al. 1998] M.-A. Knus, A. S. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, American Mathematical Society Colloquium Publications **44**, American Mathematical Society, Providence, RI, 1998. MR 2000a:16031 Zbl 0955.16001

[Lam 1980] T. Y. Lam, *The algebraic theory of quadratic forms*, Benjamin/Cummings, Reading, MA, 1980. MR 83d:10022 Zbl 0437.10006

[Merkurjev 1995] A. S. Merkurjev, "Норменный принцип для алгебраических групп", *Algebra i Analiz* **7**:2 (1995), 77–105. Translated as "The norm principle for algebraic groups", *St. Petersburg Mathematical Journal* **7**:2 (1996), 243–264. MR 96k:20088 Zbl 0859.20039

[Sansuc 1981] J.-J. Sansuc, "Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres", *J. Reine Angew. Math.* **327** (1981), 12–80. MR 83d:12010 Zbl 0468.14007

[Serre 1995] J.-P. Serre, *Cohomologie Galoisienne: progrès et problèmes*, pp. 229–257, Astérisque **227**, Société Mathématique de France, Paris, 1995, Available at http://eudml.org/doc/110186. Séminaire Bourbaki, Vol. 1993/94, Exp. No. **783**:4. MR 97d:11063 Zbl 0837.12003

[Springer 1952] T. A. Springer, "Sur les formes quadratiques d'indice zéro", *C. R. Acad. Sci. Paris* **234** (1952), 1517–1519. MR 13,815j Zbl 0046.24303

[Tits 1992] J. Tits, "Sur les degrés des extensions de corps déployant les groupes algébriques simples", *C. R. Acad. Sci. Paris Sér. I Math.* **315**:11 (1992), 1131–1138. MR 93m:20059 Zbl 0823.20042

[Totaro 2004] B. Totaro, "Splitting fields for E_8-torsors", *Duke Math. J.* **121**:3 (2004), 425–455. MR 2005h:11081 Zbl 1048.11031

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