TOTARO’S QUESTION FOR SIMPLY CONNECTED GROUPS OF LOW RANK

JO DI BL ACK AND RAMAN PARIMALA
TOTARO’S QUESTION FOR SIMPLY CONNECTED GROUPS OF LOW RANK

JODI BLACK AND RAMAN PARIMALA

Let $k$ be a field and let $G$ be a connected linear algebraic group over $k$. In a 2004 paper, Totaro asked whether a torsor $X$ under $G$ and over $k$ which admits a zero cycle of degree $d$ also admits a closed étale point of degree dividing $d$. We consider this question in the setting where $G$ is a simply connected, semisimple group of rank at most 2 and $k$ is of characteristic different from 2.

Introduction

Serre [1995, p. 233] raised the following question:

**Serre’s question:** Let $k$ be a field and let $G$ be a connected linear algebraic group defined over $k$. Let $X$ be a $G$-torsor over $k$. Suppose $X$ admits a zero cycle of degree 1. Does $X$ have a $k$-rational point?

An affirmative answer to Serre’s question is known in a number of special cases. See, for example, [Sansuc 1981; Bayer-Fluckiger and Lenstra 1990; Black 2011a; 2011b]. Burt Totaro [2004] posed the following generalization of Serre’s question:

**Totaro’s question:** Let $k$ be a field and let $G$ be a connected linear algebraic group defined over $k$. Let $X$ be a $G$-torsor over $k$. Suppose $X$ admits a zero cycle of degree $d$. Does $X$ have a closed étale point of degree dividing $d$?

An affirmative answer to Totaro’s question when $G = \text{PGL}_n$ is a classical result in the theory of central simple algebras. Tits [1992] associated to any absolutely simple, linear algebraic $k$-group $G$, an integer $n(G)$. The values of $n(G)$ are shown in Table 1 below, where $\nu$ denotes the 2-adic valuation. One can show that for any $G$-torsor $X$, there is a separable field extension $L/k$ such that $X$ has a rational point over $L$ and $[L : k]$ divides $n(G)^2$ [Serre 1995, Section 2.3]. Thus, Tits’ construction gives an affirmative answer to Totaro’s question provided $n(G)^2$ divides $d$. Garibaldi and Hoffmann [2006] give an affirmative answer to Totaro’s question for semisimple groups which are of type $G_2$, of reduced type $F_4$ or simply...
connected of type $^1E_{6,6}^0$ or $^1E_{6,2}^{28}$. Their work extended previous results of Totaro [2004] which gave an affirmative answer for split, simply connected groups of type $G_2$, $F_4$ and $E_6$. Results in [Black 2011b] give an affirmative answer to Totaro’s question in the case where $G$ is a simply connected or adjoint, semisimple, classical group and $d$ is prime to $n(G)$.

In this paper we show the following:

**Theorem 0.1.** The answer to Totaro’s question is yes if $k$ is of characteristic different from 2 and $G$ is a semisimple, simply connected, classical group such that $\text{rank } G_{\bar{k}} \leq 2$.

### 1. Galois cohomology

Let $k$ be a field, let $k_s$ be a separable closure of $k$ and let $\Gamma_k = \text{Gal}(k_s/k)$ be the absolute Galois group of $k$. We write $H^1(k, G)$ for the first Galois cohomology set $H^1(\Gamma_k, G(k_s))$. Given any finite field extension $L/k$ there is a canonical restriction map $H^1(k, G) \rightarrow H^1(L, G)$. If $\lambda \in H^1(k, G)$ is any element, we write $\lambda_L$ for the image of $\lambda$ under the restriction map $H^1(k, G) \rightarrow H^1(L, G)$.

For our convenience, we will consider the formulation of Totaro’s question in Galois cohomology:

**Totaro’s question:** Let $k$ be a field and let $G$ be a connected linear algebraic group defined over $k$. Let $\{L_i\}_{1 \leq i \leq m}$ be a set of finite field extensions of $k$ and let $d = \gcd\{[L_i:k]_{1 \leq i \leq m}\}$. If $\lambda_{L_i} = 1$ for all $i$, is there a finite, separable field extension $F$ of $k$ such that $\lambda_F = 1$ and $[F:k]$ divides $d$?

### 2. Results

In this section, we consider Totaro’s question for various groups $G$.

**The case $G = \text{SL}_1(A)$.

**Theorem 2.1.** The answer to Totaro’s question is yes if $G = \text{SL}_1(A)$ for $A$ a central simple algebra over $k$ of prime index.

**Proof.** Let $\{L_i\}_{1 \leq i \leq m}$ be a set of finite field extensions of $k$ and suppose $\lambda \in H^1(k, \text{SL}_1(A))$ is an element such that $\lambda_{L_i} = 1$ for all $i$. Let $d = \gcd\{[L_i:k]_{1 \leq i \leq m}\}$. 

---

<table>
<thead>
<tr>
<th>Type of group</th>
<th>$n(G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>$2(n+1)$</td>
</tr>
<tr>
<td>$B_n$</td>
<td>$2^n$</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$2^{\nu(n)+1}$</td>
</tr>
<tr>
<td>$D_n (n \neq 4)$</td>
<td>$2^{n+\nu(n)}$</td>
</tr>
</tbody>
</table>

**Table 1.** Values of $n(G)$ for classical groups.
We will find $F/k$ separable such that $\lambda_F = 1$ and $[F : k]$ divides $d$.

Since by [Knus et al. 1998, Theorem 29.2], $H^1(k, \text{GL}_1(A)) = 1$, the short exact sequence

$$1 \longrightarrow \text{SL}_1(A) \longrightarrow \text{GL}_1(A) \xrightarrow{\text{Nrd}} \mathbb{G}_m \longrightarrow 1$$

induces the long exact sequence

$$(2.1.1) \quad A^* \xrightarrow{\text{Nrd}} k^* \longrightarrow H^1(k, \text{SL}_1(A)) \longrightarrow 1$$

in Galois cohomology, where Nrd is the reduced norm. By (2.1.1) above,

$$H^1(k, \text{SL}_1(A)) \cong k^*/\text{Nrd}(A^*),$$

and we can identify $\lambda$ with the class of an element of $k^*$ which is in $\text{Nrd}(A_{L_i})$ for all $i$. For simplicity, we will also refer to this element as $\lambda$. Let the index of $A$ be $s$ and choose $L$ contained in $A$ a separable field extension of $k$ of degree $s$ which splits $A$ [Gille and Szamuely 2006, Propositions 4.5.3 and 4.5.4]. Then $\text{Nrd}(A_L) = L^*$ and $\lambda$ is in $\text{Nrd}(A_L)$. So if $s$ divides $d$ we may take $F = L$. Recall that $s$ is prime. So if $s$ does not divide $d$ then $\text{gcd}(s, d) = 1$. It is well known that $N_{L/k}(\text{Nrd}(A_L)) \subseteq \text{Nrd}(A)$. In particular, $\lambda^s = N_{L/k}(\lambda)$ is in $\text{Nrd}(A)$. Since $\text{Nrd}(A)$ is a group and $N_{L_i/k}(\lambda) \in \text{Nrd}(A)$ for all $i$, we find that $\lambda^d$ is in $\text{Nrd}(A)$. In turn, $\lambda$ is in $\text{Nrd}(A)$ and we can take $F = k$. \qed

**The case $G = \text{SU}(A, \sigma)$.**

**Theorem 2.2.** The answer to Totaro’s question is yes if $k$ is of characteristic different from 2 and $G = \text{SU}(A, \sigma)$ for a central simple algebra $A$ of degree 3 over $K$, $k = K^\sigma$ and $[K : k] = 2$.

**Proof.** Let $\{L_i\}_{1 \leq i \leq m}$ be a set of finite field extensions of $k$ and suppose $\lambda \in H^1(k, \text{SU}(A, \sigma))$ is an element such that $\lambda_{L_i} = 1$ for all $i$. Let $d = \text{gcd}([L_i : k])$. We will find $F/k$ separable such that $\lambda_F = 1$ and $[F : k]$ divides $d$.

The case where $d$ is coprime to 2 and 3 was covered in [Black 2011b, Theorem 3.4]. If $6 \mid d$, we take $L$ to be a separable extension of $K$ of degree dividing 3 which splits $A$. Since $K/k$ is Galois, $L/k$ is separable of degree dividing 6. Since $H^1(K, \text{SU}(A, \sigma)) = H^1(K, \text{SL}_1(A))$ and $L$ splits $A$, $H^1(L, \text{SU}(A, \sigma)) = \{1\}$ by Hilbert’s Theorem 90. Therefore, for any $\lambda \in H^1(k, \text{SU}(A, \sigma))$, $\lambda_L = 1$ and we can take $F = L$. Now suppose $2 \mid d$ and $3 \nmid d$. Fix an index $i$ such that $[L_i : k]$ is prime to 3 and $\lambda_{L_i} = 1$. Consider $L_i K$, the compositum of $L_i$ and $K$. Since, by assumption, 3 is prime to $[L_i : k]$, and $[K : k] = 2$, we know that 3 is prime to $[L_i K : K]$. Therefore, 3 is prime to $[L_i K : K]$. Let $L$ be a separable splitting field of $A$ such that $[L : K]$ is equal to the index of $A$. Since $\deg_K(A) = 3$, either $[L : K] = 1$ or $[L : K] = 3$. In either case, $L, L_i K$ is a pair of field extensions of $K$ such that $\lambda_L = 1 = \lambda_{L_i K}$ and
\[ \gcd([L : K], [L_i : K]) \text{ is 1. Since } H^1(K, SU(A, \sigma)) = H^1(K, SL_1(A)) \text{ we have } \lambda_K = 1 \text{ by Theorem 2.1, and we can take } F = K. \] The final setting to consider is the case where \( 3 \mid d \) and \( 2 \nmid d \). Since \( d \) is odd, we can fix an index \( i \) such that \([L_i : k]\) is odd and \( \lambda_{L_i} = 1 \). Let \( R_{K/k}G_m \) be the Weil transfer of \( G_m \) and let \( R_{K/k}^1G_m \) be defined as the kernel of the norm map \( N_{K/k} : R_{K/k}G_m \to G_m \). The short exact sequence
\[ 1 \to SU(A, \sigma) \to U(A, \sigma) \to R_{K/k}^1G_m \to 1 \]
induces the commutative diagram
\[
\begin{array}{ccc}
K^* & \xrightarrow{\delta} & H^1(k, SU(A, \sigma)) \\
\downarrow & & \downarrow j \\
(K \otimes L_i)^* & \xrightarrow{j} & H^1(L_i, SU(A, \sigma)) \\
\end{array}
\]
where \( K^* \) and \((K \otimes L_i)^*\) denote the norm-one elements in \( K^* \) and \((K \otimes L_i)^*\) respectively. By a result of Bayer-Fluckiger and Lenstra [1990, Theorem 2.1], \( j(\lambda) = 1 \). In particular, we can choose \( \alpha \in K^* \) such that \( \delta(\alpha) = \lambda \). In the case where \( A \) is split, \( H^1(K, SU(A, \sigma)) = H^1(K, SL_1(A)) = \{1\} \). Then, since \( K \) and \( L_i \) are field extensions of coprime degree with \( \lambda_K = \lambda_{L_i} = 1 \), the desired result holds by [Black 2011b, Theorem 4.4]. Since \( \deg(A) = 3 \), if \( A \) is not split, then \( A \) is a division algebra and by [Albert 1963] (see also [Knus et al. 1998, Theorem 19.14]), there is a \( k \)-subalgebra \( L \) of \( A \) such that \( L/k \) is étale of degree three. Since \( A \) is division, \( L \) is a field. Consider the diagram
\[
\begin{array}{ccc}
U(A, \sigma)(k) & \xrightarrow{\delta} & K^* \\
\downarrow & & \downarrow j \\
U(A, \sigma)(L) & \xrightarrow{(K \otimes L)^*} & H^1(L, SU(A, \sigma)) \\
\end{array}
\]
For \( x \in (K \otimes L)^* \), write \( x = y^{-1}y \) for \( y \in (K \otimes L)^* \) where \( \bar{y} \) denotes the nontrivial automorphism of \( K/k \). Since \( A \otimes L \) is split, \( y \) is a reduced norm from \( A \otimes L \). In view of [Merkurjev 1995, Proposition 6.1], the image of \( \text{Nrd}(U(A, \sigma) \to (K \otimes L)^*) \) contains \( x \). Thus \( \lambda_L = 1 \) and we may take \( F = L \).  

**The case \( G = \text{Spin}(q) \).** The following result will be useful:

**Proposition 2.3.** Let \( k \) be a field of characteristic different from 2 and let \( q \) be a quadratic form over \( k \) of dimension \( \leq 5 \). Let \( \lambda \in H^1(k, \text{Spin}(q)) \) be any element. Then there exists a (separable) field extension \( F \) of \( k \) such that \([F : k]\) divides 2 and \( \lambda_F = 1 \).
Proof. Consider the short exact sequence

$$1 \longrightarrow \mu_2 \overset{i}{\longrightarrow} \text{Spin}(q) \overset{\pi}{\longrightarrow} O^+(q) \longrightarrow 1,$$

which induces the exact sequence in Galois cohomology

$$H^1(k, \mu_2) \overset{i}{\longrightarrow} H^1(k, \text{Spin}(q)) \overset{\pi}{\longrightarrow} H^1(k, O^+(q)). \quad (2.3.1)$$

The pointed set $H^1(k, O^+(q))$ classifies quadratic forms over $k$ of the same dimension and discriminant as $q$. Let $q' = \pi(\lambda)$. Then $q \perp -q'$ has even dimension, trivial discriminant and trivial Clifford invariant since $q'$ is in the image of $\pi$. Thus $q \perp -q' \in I^3(k)$.

First consider the case where $\text{dim}(q) < 4$. Then, $\text{dim}(q \perp -q') < 8$ and by the Arason–Pfister Hauptsatz [Lam 1980, Chapter X, Hauptsatz 5.1], $q \perp -q'$ is hyperbolic. Equivalently, $q \cong q'$ and $q' = 1$ in $H^1(k, O^+(q))$. Using the exactness of (2.3.1), choose $\eta$ in $H^1(k, \mu_2)$ such that $i(\eta) = \lambda$. Since $H^1(k, \mu_2) \cong k^*/k^{*2}$ we can choose $F/k$ a field extension of degree at most 2 such that $\eta_F = 1 \in H^1(F, \mu_2)$. By commutativity of (2.3.2) below, $\lambda_F = 1$ in $H^1(F, \text{Spin}(q))$.

$$H^1(k, \mu_2) \longrightarrow H^1(k, \text{Spin}(q)) \longrightarrow H^1(F, \mu_2) \longrightarrow H^1(F, \text{Spin}(q)). \quad (2.3.2)$$

Suppose instead that $\text{dim}(q) = 4$. Let $d = \text{disc}(q)$ and write $q = a(1, b, c, bcd)$. By [Lam 1980, Chapter XII, Proposition 2.4], there is an element $\alpha \in k^*$ such that $q' \cong \alpha q$ and we may write $q \perp -q' \cong (1, -\alpha)q = a(1, -\alpha)(1, b, c, bcd)$. Let $e_2$ be the map from $I^2(k) \rightarrow H^2(k, \mu_2)$ induced by the Clifford invariant. Since $q \perp -q' \in I^3(k)$, $e_2(q \perp -q') = (d) \cup (\alpha) = 0 \in H^2(k, \mu_2)$ [Elman et al. 2008, 16.2] and so $(1, -\alpha, -d, \alpha d)$ is hyperbolic. Equivalently, $(1, -\alpha)d \cong (1, -\alpha)$ and $q \perp -q' \cong a(1, -\alpha)(1, b, c, bc) = a(1, -\alpha)(1, b)(1, c)$. Let $F = k(\sqrt{-b})$. Then $[F : k] \leq 2$, $(q \perp -q')_F$ is hyperbolic and $q'_F = 1 \in H^1(F, O^+(q))$. Consider the diagram

$$O^+(q)(k) \longrightarrow H^1(k, \mu_2) \longrightarrow H^1(k, \text{Spin}(q)) \longrightarrow H^1(k, O^+(q)) \quad (2.3.3)$$

$$O^+(q)(F) \overset{\text{sn}}{\longrightarrow} H^1(F, \mu_2) \overset{i}{\longrightarrow} H^1(F, \text{Spin}(q)) \overset{\pi}{\longrightarrow} H^1(F, O^+(q))$$

By commutativity of the right rectangle, $\pi(\lambda_F) = 1$ and by the exactness of the bottom row, $\lambda_F \in \text{im}(i)$. But since $q \cong a(1, b, c, bcd)$, $q_F$ is isotropic. Thus, the
spinor norm $sn : O^+(q)(F) \to H^1(F, \mu_2)$ is onto [Baeza 1978, p. 78] and therefore, since $\lambda_F \in \text{im}(i)$, $\lambda_F = 1$.

Now suppose $\dim(q) = 5$. Since $q \perp -q'$ is a rank 10 form in $I^3(k)$, it is isotropic [Lam 1980, Chapter XII, Proposition 2.8]. Therefore $q$ and $q'$ have a common slot and we can write $q = \langle a \rangle \perp q_1$ and $q' = \langle a \rangle \perp q_2$. Since $q_1 \perp -q_2 \in I^3k$ is rank 8, we can proceed as in the rank 4 case and find a field extension $F$ of $k$ of degree at most 2 such that $(q_1 \perp -q_2)_F$ is hyperbolic and $(q_1)_F$ is isotropic. By the Arason–Pfister Hauptsatz, $(q \perp -q')_F$ is hyperbolic and thus $q_F \cong q'_F$ and $\pi(\lambda_F) = q'_F = 1 \in H^1(F, O^+(q))$. Thus $\lambda_F$ is in the image of $i : H^1(F, \mu_2) \to H^1(F, \text{Spin}(q))$. However, $(q_1)_F$ being isotropic, $q_F$ is isotropic and $sn : O^+(q)(F) \to H^1(F, \mu_2)$ is onto. Therefore, $i$ is the zero map and $\lambda_F = 1$. \qed

**Theorem 2.4.** The answer to Totaro’s question is yes if $k$ is of characteristic different from 2 and $G = \text{Spin}(q)$ for $q$ a quadratic form of dimension $\leq 5$.

**Proof.** Let $\{L_i\}_{1 \leq i \leq m}$ be a set of finite field extensions of $k$ and suppose $\lambda \in H^1(k, \text{Spin}(q))$ is an element such that $\lambda_{L_i} = 1$ for all $i$. Let $d = \gcd([L_i : k])$. We want to find $F/k$ separable such that $\lambda_F = 1$ and $[F : k]$ divides $d$. If $d$ is odd we are done by [Black 2011b, Theorem 3.7] and can take $F = k$. If $d$ is even, by Proposition 2.3, there is a separable extension $F/k$ of degree at most 2 such that $\lambda_F = 1$. \qed

**Theorem 2.5.** The answer to Totaro’s question is yes if $k$ is of characteristic different from 2 and $G = \text{Sp}(A, \sigma)$ where $A$ is a central simple algebra with symplectic involution and $\deg(A) = 2$ or 4.

**Proof.** Let $q$ be a quadratic form of dimension 3 (resp. 5) with trivial discriminant. Then the even Clifford algebra $A = C_0(V, q)$ is a central simple algebra of degree 2 (resp. 4) and the canonical involution on the Clifford algebra is symplectic and $\text{Spin}(q) \cong \text{Sp}(A, \sigma)$ [Knus et al. 1998, Section 15.C]. Moreover, every algebra $A$ of degree 2 or 4 with a symplectic involution arises in this way. Thus, a positive answer to Totaro’s question for $\text{Sp}(A, \sigma)$ follows from Proposition 2.3. \qed

**The case $G = \text{Spin}(A, \sigma)$.**

**Theorem 2.6.** The answer to Totaro’s question is yes if $k$ is of characteristic different from 2 and $G = \text{Spin}(A, \sigma)$, where $A$ is a central simple algebra of degree 4 over $k$ and $\sigma$ is an orthogonal involution on $A$.

**Proof.** Let $\{L_i\}_{1 \leq i \leq m}$ be a set of finite field extensions of $k$ and suppose $\lambda \in H^1(k, \text{Spin}(A, \sigma))$ is an element such that $\lambda_{L_i} = 1$ for all $i$. Let $d = \gcd([L_i : k])$. We will find $F/k$ separable such that $\lambda_F = 1$ and $[F : k]$ divides $d$.

By [Black 2011b, Theorem 3.7], when $d$ is odd we may take $F = k$. So we may suppose that $d$ is even. Suppose $(A, \sigma)$ has trivial discriminant. Then
we obtain

\[
\delta_k Q \quad \text{which split}
\]

\[
\phi \quad \text{respectively and let}
\]

\[
\lambda \text{ if } 4 \mid d, \text{ let } F_1, F_2 \text{ be extensions of } k \text{ of degree at most } 2 \text{ which split } Q_1 \text{ and } Q_2 \text{ respectively. Then } \lambda_{F_1F_2} = 1 \text{ and } [F_1 F_2 : k] \text{ divides } 4. \text{ In the case } 2 \mid d \text{ and } 4 \nmid d, \text{ we can fix an } L_j/k \text{ such that } [L_j : k] = 2m, \text{ where } m \text{ is odd and } \lambda_{L_j} = 1. \text{ Following arguments as in [Garibaldi and Hoffmann 2006, Lemma 1.5] we suppose without loss of generality that } k \subseteq L \subseteq L_j \text{ with } [L : k] \text{ odd, } [L_j : L] = 2 \text{ and } \lambda_{L_j} = 1. \text{ Let } N_{Q_1}, N_{Q_2} \text{ be the norm forms for the quaternion algebras } Q_1, Q_2 \text{ respectively and let } \phi_1 = (1, -\lambda_1) N_{Q_1} \text{ and } \phi_2 = (1, -\lambda_2) N_{Q_2}. \text{ The fact that } \lambda_{L_j} = 1 \text{ implies that } \phi_1, \phi_2 \text{ are hyperbolic over } L_j. \text{ Then by [Garibaldi and Hoffmann 2006, Lemma 1.4] there exists } \mu \in k^* \text{ such that } \phi_1 \cong (1, \mu) \tilde{\phi}_1 \text{ and } \phi_2 \cong (1, \mu) \tilde{\phi}_2, \text{ where } \tilde{\phi}_1, \tilde{\phi}_2 \text{ are } 2\text{-fold Pfister forms. Let } F = k(\sqrt{-\mu}). \text{ Then } \phi_1, \phi_2 \text{ are hyperbolic over } F \text{ and thus } \lambda_1 \in \Nrd(Q_{1_f}) \text{ and } \lambda_2 \in \Nrd(Q_{2_f}). \text{ That is, } \lambda_F = 1. \text{ Also, } F/k \text{ is separable and degree at most } 2 \text{ by construction.}

Suppose instead that \((A, \sigma)\) has nontrivial discriminant. One can associate to \((A, \sigma)\) its Clifford algebra \(Q\), which is a quaternion algebra with center \(K = k(\sqrt{\delta})\), where \(\delta = \disc(A, \sigma)\) [Knus et al. 1998, Theorem 15.7]. Then \(\Spin(A, \sigma) = R_{K/k} SL_1(Q)\) [Knus et al. 1998, Proposition 15.10] and \(H^1(k, \Spin(A, \sigma)) = H^1(K, SL_1(Q))\). If \(Q\) is split, \(\lambda = 1\) and we take \(F = k\). So suppose \(Q\) is not split. If \(4 \mid d\) we can take \(F\) a splitting field of \(Q\) such that \(F/K\) is a separable extension of degree 2. Since

\[
H^1(F, \Spin(A, \sigma)) = H^1(K \otimes F, SL_1(Q)) \cong H^1(F \times F, SL_1(Q)) = \{1\},
\]

we obtain \(\lambda_F = 1\). Further \([F : k] = 4\), and since \(F/K\) and \(K/k\) are separable, \(F/k\) is separable. We are left to consider the case where \((A, \sigma)\) has nontrivial discriminant and \(4 \nmid d\) and \(2 \mid d\).

Consider the short exact sequence

\[
1 \to R_{K/k} SL_1(Q) \to R_{K/k} GL_1(Q) \to R_{K/k} G_m \to 1,
\]

which induces

\[
\GL_1(Q)(K) \xrightarrow{\Nrd} K^\star \xrightarrow{} H^1(K, SL_1(Q)) \to 1.
\]

Choose \(\lambda \in H^1(K, SL_1(Q))\) such that \(\lambda_{L_i} = 1\) for all \(i\) and let \(\beta \in K^\star\) satisfy \(\delta(\beta) = \lambda\). Following [Garibaldi and Hoffmann 2006, Lemma 1.5], we may suppose that \(\lambda_{L_j} = 1\) where \(k \subseteq L \subseteq L_j\) and \([L_j : L] = 2\).
Write \( L_j = L(\sqrt{a}) \) for \( a \in L^*/L^{*2} \). Let \( f \) be the norm form on \( Q \) and let \( f^0 \) denote the norm form restricted to the traceless elements of \( Q \), which we denote by \( Q^0 \). Since \( \lambda_{L_j} = 0 \), choose \( x_0, y_0 \in Q \otimes L \) such that

\[
(2.6.2) \quad \beta = f(x_0 + y_0\sqrt{a}).
\]

If \( y_0 = 0 \) we have \( \beta \in \text{Nrd}(Q \otimes L) \), and, \( L/K \) being of odd degree, this implies \( \beta \in \text{Nrd}(Q) \). We take \( F = k \). So suppose \( y_0 \neq 0 \). Since \( Q \) is a division algebra, \( f(y_0) \neq 0 \) and

\[
(2.6.3) \quad \beta = f(x_0) + af(y_0).
\]

If we let \( b_f \) denote the adjoint bilinear form, we have

\[
(2.6.4) \quad b_f(x_0, y_0) = 0
\]

and

\[
(2.6.5) \quad \beta f(y_0^{-1}) = f(x_0y_0^{-1}) + a,
\]

where the reduced trace \( \text{trd}(x_0y_0^{-1}) \) vanishes by (2.6.4). Therefore,

\[
(2.6.6) \quad \beta f(y_0^{-1}) = f^0(x_0y_0^{-1}) + a.
\]

Let \( f = f_1 + \sqrt{\delta} f_2 \) with \( f_1 \) and \( f_2 \) quadratic forms on \( Q \) with values in \( k \). Further let \( f^0 = f_1^0 + \sqrt{\delta} f_2^0 \) where \( f_1^0, f_2^0 \) are quadratic forms on \( Q^0 \) with values in \( k \). Setting \( z_0 = y_0^{-1} \) and \( w_0 = x_0y_0^{-1} \), we have

\[
(2.6.7) \quad a = \beta_1 f_1(z_0) + \beta_2 \delta f_2(z_0) - f_1^0(w_0),
\]

\[
(2.6.8) \quad 0 = \beta_1 f_2(z_0) + \beta_2 f_1(z_0) - f_2^0(w_0),
\]

with \( z_0 \in Q \otimes L \) and \( w_0 \in Q^0 \otimes L \). Define \( k \)-quadratic forms \( q_1 : Q \oplus Q^0 \to k \) and \( q_2 : Q \oplus Q^0 \to k \) by

\[
(2.6.9) \quad q_1(z, w) = \beta_1 f_1(z) + \beta_2 \delta f_2(z) - f_1^0(w),
\]

\[
(2.6.10) \quad q_2(z, w) = \beta_1 f_2(z) + \beta_2 f_1(z) - f_2^0(w),
\]

for \( z \in Q \) and \( w \in Q_0 \). Since \( y_0 \neq 0 \), \( z_0 = y_0^{-1} \neq 0 \) and \( (z_0, w_0) \) is a nontrivial zero of \( q_2 \) over \( L \). Then by Springer’s theorem [1952], \( q_2 \) has a nontrivial zero \( (z_1, w_1) \).
over $k$. By a general position argument, we may assume that $z_1 \neq 0$. Let
\begin{equation}
2.6.11 \quad \alpha = \beta_1 f_1(z_1) + \beta_2 \delta f_2(z_1) - f_1^0(w_1).
\end{equation}
We have
\begin{equation}
2.6.12 \quad 0 = \beta_1 f_2(z_0) + \beta_2 f_1(z_0) - f_2^0(w_1).
\end{equation}

Adding these two equations, we find
\begin{equation}
2.6.13 \quad \alpha = \beta f(z_1) - f^0(w_1),
\end{equation}
or, equivalently,
\begin{equation}
2.6.14 \quad \beta f(z_1) = \alpha + f^0(w_1).
\end{equation}

Let $F = k(\sqrt{\alpha})$. Then $[F : k] \leq 2$, $(\sqrt{\alpha} + w_1)z_1^{-1} \in Q_F$ and $\beta = \text{Nrd}((\sqrt{\alpha} + w_1)z_1^{-1})$. Thus, $\lambda_F = 1$.

**Theorem 2.7.** The answer to Totaro’s question is yes if $k$ is of characteristic different from 2 and $G = \text{SU}(A, \sigma)$ where $A$ is a quaternion algebra with unitary involution $\sigma$.

**Proof.** The norm algebra $N_{K/k}(A, \sigma)$ equals $(B, \tau)$ for $B$ a central simple algebra of degree 4 and $\tau$ an orthogonal involution on $B$. Since $\text{Spin}(B, \tau) \cong \text{SU}(A, \sigma)$, that Totaro’s question has an affirmative answer in this case is a consequence of Theorem 2.6. □

3. Conclusion

**Theorem 3.1.** The answer to Totaro’s question is yes for $k$ a field of characteristic different from 2 and $G$ a simply connected, semisimple, classical group of rank $\leq 2$.

**Proof.** We suppose in all cases that $G$ is simply connected and semisimple and that the rank of $G_{\overline{k}} \leq 2$. If $G$ is of type $^1A_1$ or $^1A_2$ then $G$ is of the form $\text{SL}_1(A)$ for $A$ a central simple algebra of degree 2 or 3 [Knus et al. 1998, Theorem 26.9]. A positive answer to Totaro’s question for a group of this form was shown in Theorem 2.1. If $G$ is of type $^2A_1$ then $G = \text{SU}(A, \sigma)$ for $A$ a central simple algebra of degree 2 with unitary involution $\sigma$. The proof for this case was given in Theorem 2.7. If $G$ is of type $^2A_2$ then $G$ is of the form $\text{SU}(A, \sigma)$, where $A$ is a central simple algebra of degree 3 with unitary involution $\sigma$ [Knus et al. 1998, Theorem 26.9]. Thus an affirmative answer to Totaro’s question for a group of type $^2A_2$ follows from Theorem 2.2 above. If $G$ is of type $B_1$ or $B_2$, then $G = \text{Spin}(q)$ for $q$ a quadratic form of dimension 3 or 5 [Knus et al. 1998, Theorem 26.12] and the desired result was proven in Theorem 2.4. If $G$ is of type $C_1$ or $C_2$, then $G = \text{Sp}(A, \sigma)$, where $A$ is a central simple algebra of degree 2 or 4 and $\sigma$ is a symplectic involution.
on $A$. The proof of our result in this case was covered in Theorem 2.5. If $G$ is of type $D_2$ then either $G = \text{Spin}(q)$ for $q$ a quadratic form of dimension 2 or 4 or $G$ is of the form $\text{Spin}(A, \sigma)$ for $A$ a central simple algebra over $k$ of degree 4 and $\sigma$ an orthogonal involution on $A$ [Knus et al. 1998, Theorem 26.15]. In the first case the desired results follows from Theorem 2.4 and in the latter it follows from Theorem 2.6.

\[ \square \]

**Remark 3.2.** Since Garibaldi and Hoffman [2006] have given a proof in the case $G$ is of type $G_2$, Totaro’s question has a positive answer for any simply connected, semisimple group of rank $\leq 2$.

**References**


Received November 28, 2012. Revised February 19, 2013.

JODI BLACK
DEPARTMENT OF MATHEMATICS
BUCKNELL UNIVERSITY
LEWISBURG, PA 17837
UNITED STATES
jodi.black@bucknell.edu

RAMAN PARIMALA
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
EMORY UNIVERSITY
400 DOWNMAN DRIVE W401
ATLANTA, GA 30322
UNITED STATES
parimala@mathcs.emory.edu
Totaro’s question for simply connected groups of low rank
JODI BLACK and RAMAN PARIMALA

Uniform hyperbolicity of the curve graphs
BRIAN H. BOWDITCH

Constant Gaussian curvature surfaces in the 3-sphere via loop groups
DAVID BRANDER, JUN-ICHI INOGUCHI and SHIMPEI KOBAYASHI

On embeddings into compactly generated groups
PIERRE-EMMANUEL CAPRACE and YVES CORNULIER

Variational representations for N-cyclically monotone vector fields
ALFRED GALICHON and NASSIF GHOUSSOUB

Restricted successive minima
MARTIN HENK and CARSTEN THIEL

Radial solutions of non-Archimedean pseudodifferential equations
ANATOLY N. KOCHUBEI

A Jantzen sum formula for restricted Verma modules over affine Kac–Moody algebras at the critical level
JOHANNES KÜBEL

Notes on the extension of the mean curvature flow
YAN LENG, ENTAO ZHAO and HAORAN ZHAO

Hypersurfaces with prescribed angle function
HENRIQUE F. DE LIMA, ERALDO A. LIMA JR. and ULISSES L. PARENTE

Existence of nonparametric solutions for a capillary problem in warped products
JORGE H. LIRA and GABRIELA A. WANDERLEY

A counterexample to the simple loop conjecture for PSL(2, ℝ)
KATHRYN MANN

Twisted Alexander polynomials of 2-bridge knots for parabolic representations
TAKAYUKI MORIFUJI and ANH T. TRAN

Schwarzian differential equations associated to Shimura curves of genus zero
FANG-TING TU

Polynomial invariants of Weyl groups for Kac–Moody groups
ZHAO XU-AN and JIN CHUNHUA