CONSTANT GAUSSIAN CURVATURE SURFACES
IN THE 3-SPHERE VIA LOOP GROUPS

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In this paper we study constant positive Gauss curvature K surfaces in the 3-sphere $\mathbb{S}^3$ with $0 < K < 1$, as well as constant negative curvature surfaces. We show that the so-called *normal* Gauss map for a surface in $\mathbb{S}^3$ with Gauss curvature $K < 1$ is Lorentz harmonic with respect to the metric induced by the second fundamental form if and only if K is constant. We give a uniform loop group formulation for all such surfaces with $K \neq 0$, and use the generalized d'Alembert method to construct examples. This representation gives a natural correspondence between such surfaces with $K < 0$ and those with $0 < K < 1$.

Introduction

The study of isometric immersions from space forms into space forms is a classical and important problem of differential geometry. This subject has its origin in realizability of the hyperbolic plane geometry in Euclidean 3-space $\mathbb{E}^3$. As is well known, Hilbert [1901] proved the nonexistence of isometric immersions of the hyperbolic plane into $\mathbb{E}^3$. Analogous results hold for surfaces in the 3-sphere $\mathbb{S}^3$ and hyperbolic 3-space $\mathbb{H}^3$ as follows:

Theorem [Spivak 1979]. *There is no complete surface in $\mathbb{S}^3$ with constant curvature $K < 1$, $K \neq 0$. Moreover, the only complete immersion of constant positive curvature in $\mathbb{S}^3$ is a totally umbilic round sphere. There is no complete surface in $\mathbb{H}^3$ with constant curvature $K < -1$.*

Due to the complicated structure (nonlinearity) of the integrability condition (Gauss–Codazzi–Ricci equations) of isometric immersions between space forms, in the past decades many results on nonexistence, rather than explicit constructions of

examples, have been obtained. For this direction we refer the reader to the survey article [Borisenko 2001].

Another reason for the focus on nonexistence may be the presence of singularities. Surfaces in $\mathbb{S}^3$ with constant Gauss curvature $K < 1$ always have singularities, excepting the flat case $K = 0$ (in fact there exist infinitely many flat tori in $\mathbb{S}^3$ [Kitagawa 1988]). Recently, however, there has been some movement to broaden the class of surfaces to include those with singularities, and a number of interesting studies of the geometry of these; see, for example, [Saji et al. 2009].

On the other hand, one can see that under the asymptotic Chebyshev net parametrization, the Gauss–Codazzi equations of surfaces in $\mathbb{S}^3$ with constant curvature $K < 1$ ($K \neq 0$) are reduced to the sine-Gordon equation. The sine-Gordon equation also arises as the Gauss–Codazzi equation of pseudospherical surfaces in $\mathbb{E}^3$ (surfaces of constant negative curvature) and is associated to harmonic maps from a Lorentz surface into the 2-sphere.

By virtue of loop group techniques, an infinite-dimensional d'Alembert-type representation for solutions is available for surfaces associated to Lorentz harmonic maps. This d'Alembert-type representation is a special case of the generalized DPW method (described in [Brander and Dorfmeister 2009]): for the case of Riemannian harmonic maps the method reduces to an analogue of the Weierstrass representation; for the case of Lorentz harmonic maps, one obtains an analogue of the d'Alembert solution for the wave equation. More precisely, all solutions are given in terms of two functions, each of one variable only. This type of construction method can be traced back to [Kričever 1980]. An example of an application of this method is the solution in [Brander and Svensson 2013] of the geometric Cauchy problem for pseudospherical surfaces in $\mathbb{E}^3$, as well as for timelike constant mean curvature surfaces in Lorentz–Minkowski 3-space $\mathbb{L}^3$. The key ingredient is the generalized d'Alembert representation for Lorentz harmonic maps of Lorentz surfaces into semi-Riemannian symmetric spaces. See also [Dorfmeister 2008] and the references therein for more examples. One can expect that the approach can be adapted to other classes of isometric immersion problems.

These observations motivate us to establish a loop group method (generalized d'Alembert formula) for surfaces in $\mathbb{S}^3$ of constant curvature $K < 1$. We shall in fact give such a solution that covers all such surfaces with $K \neq 0$. The key point is to discover *which Gauss map* (there are several definitions for surfaces in $\mathbb{S}^3$) is the right one to make the connection with harmonic maps.

Outline of this article. This paper is organized as follows: After prerequisite knowledge in Sections 1 and 2, we will give a loop group formulation for surfaces in $\mathbb{S}^3$ of constant curvature $K < 1$ ($K \neq 0$) in Section 3. In particular, we will show that the Lorentz harmonicity (with respect to the conformal structure determined by the

second fundamental form) of the *normal Gauss map* of a surface with curvature $K < 1$ is equivalent to the constancy of K. The normal Gauss map is the left translation, to the Lie algebra $\mathfrak{su}(2)$, of the unit normal n to the immersion f into $\mathbb{S}^3 = \mathrm{SU}(2)$ — in symbols, $\nu = f^{-1}n$.

The harmonicity of the normal Gauss map enables us to construct constant curvature surfaces in terms of Lorentz harmonic maps. We establish a loop group theoretic d'Alembert representation for surfaces in $\mathbb{S}^3$ with constant curvature $K < 1$ ($K \neq 0$). In Section 4 we give a relation between the surfaces in $\mathbb{S}^3$ and pseudospherical surfaces in $\mathbb{E}^3$, and show how the well-known Sym formula for the latter surfaces arises naturally from our construction. Finally, we give a detailed analysis of the limiting procedure $K \to 0$.

The paper ends with some explicit examples constructed by our method. All of the images shown here were produced using a numerical implementation of the method in Matlab. The code, *ksphere*, can be found, at the time of writing, at http://davidbrander.org/software.html.

Examples. Figure 1 shows the well-known pseudospherical surface of revolution, together with a corresponding constant negative curvature surface in $\mathbb{S}^3$ obtained by a different projection from the same loop group frame. The surface in $\mathbb{S}^3$ is mapped diffeomorphically to $\mathbb{R}^3$ by the stereographic projection for rendering. See Example 4.1 below.

Figure 2 shows Amsler's pseudospherical surface in $\mathbb{E}^3$, which contains two intersecting straight lines, together with a corresponding surface of constant curvature $K = 16/25$ in $\mathbb{S}^3$ also obtained from the same loop group frame. The two straight lines correspond to two great circles. The great circles appear as straight lines in

$\mu = 1$, $K = -1$, target $\mathbb{E}^3$ $\mu = 4$, $K = -\frac{16}{25}$, target $\mathbb{S}^3$
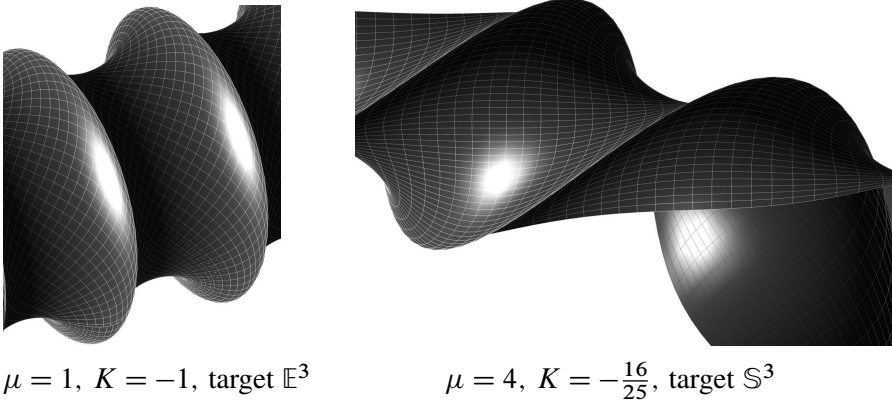
Figure 1. Left: a pseudospherical surface of revolution in $\mathbb{E}^3$. Right: a constant negative curvature analogue in $\mathbb{S}^3$. See also Figure 3.

$\mu = 1$, $K = -1$, target $\mathbb{E}^3$ $\mu = -4$, $K = 16/25$, target $\mathbb{S}^3$
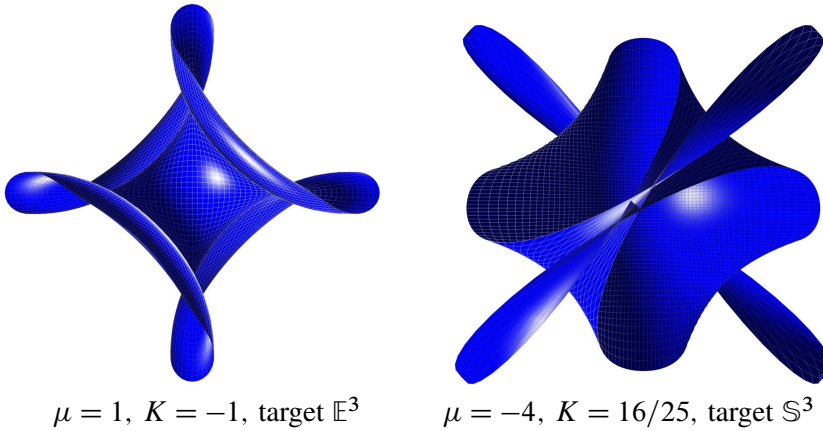
Figure 2. Left: Amsler's surface in $\mathbb{E}^3$. Right: a constant positive curvature analogue in $\mathbb{S}^3$. See also Figure 4.

the image obtained by stereographic projection to $\mathbb{R}^3$. This example shows that although the singular sets in the coordinate domain are the same for every surface in the family, the type of singularity can change. The surface obtained at $\mu = -4$ apparently has a swallowtail singularity at a point where the surfaces obtained at $\mu = 1$ and $\mu = 4$ (see Example 4.2 below) each have a cuspidal edge. This suggests that the singularities of constant curvature surfaces in $\mathbb{S}^3$ are also worth investigating.

Comparison with other methods. It should be noted that Ferus and Pedit [1996] gave a very nice loop group representation for isometric immersions of space forms $M_c^n \to \widetilde{M}_{\tilde{c}}^{n+k}$ with flat normal bundle for any $c \neq \tilde{c}$ with $c \neq 0 \neq \tilde{c}$. Finite-type solutions can be generated using the modified AKS theory described in [ibid.], and all solutions can, in principle, be constructed from curved flats using the generalized DPW method described in [Brander and Dorfmeister 2009]. For the case of surfaces, as in the present article, the construction of Ferus and Pedit is quite different from the Lorentzian harmonic map approach used here. For surfaces, the Lorentzian harmonic map representation is probably more useful, since one obtains, via the generalized d'Alembert method, all solutions from essentially arbitrary pairs of functions of one variable only; this is the key, for example, to the solution of the geometric Cauchy problem in [Brander and Svensson 2013]. If one were to use the setup in [Ferus and Pedit 1996], and the generalized DPW method of [Brander and Dorfmeister 2009], which is the analogue of generalized d'Alembert method, one instead obtains a curved flat in the Grassmannian $\mathrm{SO}(4)/(\mathrm{SO}(2) \times \mathrm{SO}(2))$ as the basic data, which is not as simple. In contrast, our basic data are essentially arbitrary functions of one variable.

Another interesting difference between the two approaches is the following: we will show below that the loop group frame corresponding to a surface of constant curvature $K < 0$ in $\mathbb{S}^3$ also corresponds to a surface with $0 < K < 1$ in $\mathbb{S}^3$, giving some kind of Lawson correspondence between two surfaces, one of which has negative curvature and the other positive. This correspondence is obtained by evaluating at a different value of the loop parameter λ. On the other hand, in [Brander 2007], the loop group maps of Ferus and Pedit are also found to produce Lawson-type correspondences between various isometric immersions of space forms by evaluating in different ranges of λ. In this case, however, one does not obtain such a correspondence between surfaces with positive and negative curvature.

Finally, we should observe that Xia [2007] has also studied isometric immersions of constant curvature surfaces in space forms via loop group methods. In that work, for surfaces in $\mathbb{S}^3$, the group SO(4) is used (as opposed to SU(2) × SU(2), used here) and a loop group representation for the surfaces is given. However, the generalized d'Alembert method to construct solutions is not given, and neither is the equivalence of this problem with Lorentz harmonic maps via the normal Gauss map. It turns out to be difficult to find a suitable loop group decomposition in the SO(4) setup used in [ibid.], which is really the setup for Lorentz harmonic maps into the Grassmannian SO(4)/(SO(2) × SO(2)). The essential problem is that the surfaces in question are not associated to arbitrary harmonic maps in the Grassmannian, but very special ones. In contrast, our use of the group SU(2) × SU(2) leads naturally to the normal Gauss map, the harmonicity of which is a basic characterization of these surfaces; this leads to a straightforward solution in terms of the known method for Lorentz harmonic maps.

1. Preliminaries

The symmetric space $\mathbb{S}^3$. Let $\mathbb{E}^4$ be the *Euclidean 4-space* with standard inner product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3.$$

We denote by $e_0 = (1, 0, 0, 0)$, $e_1 = (0, 1, 0, 0)$, $e_2 = (0, 0, 1, 0)$, $e_3 = (0, 0, 0, 1)$ the natural basis of $\mathbb{E}^4$.

The *orthogonal group* O(4) is defined by

$$O(4) = \{A \in GL(4, \mathbb{R}) \mid A^T A = I\}.$$

Here I is the identity matrix. We denote by SO(4) the identity component of O(4) (called the *rotation group*).

Let us denote by $\mathbb{S}^3$ the unit 3-sphere in $\mathbb{E}^4$ centered at the origin. The unit 3-sphere is a simply connected Riemannian space form of constant curvature 1.

The rotation group SO(4) acts isometrically and transitively on $\mathbb{S}^3$, and the isotropy subgroup at e_0 is SO(3). Hence $\mathbb{S}^3 = \mathrm{SO}(4)/\mathrm{SO}(3)$. This representation is a Riemannian symmetric space representation of $\mathbb{S}^3$ with involution $\mathrm{Ad}_{\mathrm{diag}(-1,1,1,1)}$.

The unit tangent sphere bundle. Let us denote by $U\mathbb{S}^3$ the *unit tangent sphere bundle* of $\mathbb{S}^3$. Namely, $U\mathbb{S}^3$ is the manifold of all unit tangent vectors of $\mathbb{S}^3$, and is identified with the submanifold

$$\{(\boldsymbol{x}, \boldsymbol{v}) \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 1, \ \langle \boldsymbol{x}, \boldsymbol{v} \rangle = 0\}$$

of $\mathbb{E}^4 \times \mathbb{E}^4$. The tangent space $T_{(\boldsymbol{x}, \boldsymbol{v})}U\mathbb{S}^3$ at a point $(\boldsymbol{x}, \boldsymbol{v})$ is expressed as

$$T_{(\boldsymbol{x},\boldsymbol{v})}U\mathbb{S}^3 = \{(X, V) \in \mathbb{E}^4 \times \mathbb{E}^4 \mid \langle \boldsymbol{x}, X \rangle = \langle \boldsymbol{v}, V \rangle = 0, \ \langle \boldsymbol{x}, V \rangle + \langle \boldsymbol{v}, X \rangle = 0\}.$$

Define a 1-form ω on $U\mathbb{S}^3$ by

$$\omega_{(\boldsymbol{x},\boldsymbol{v})}(X, V) = \langle X, \boldsymbol{v} \rangle = -\langle \boldsymbol{x}, V \rangle.$$

Then one can see that ω is a *contact form* on $U\mathbb{S}^3$; that is, $(\mathrm{d}\omega)^2 \wedge \omega \neq 0$. The distribution

$$\mathscr{D}_{(\boldsymbol{x},\boldsymbol{v})} := \{(X, V) \in T_{(\boldsymbol{x},\boldsymbol{v})}U\mathbb{S}^3 \mid \omega_{(\boldsymbol{x},\boldsymbol{v})}(X, V) = 0\}$$

is called the *canonical contact structure* of $U\mathbb{S}^3$.

The rotation group SO(4) acts on $U\mathbb{S}^3$ via the action $A \cdot (\boldsymbol{x}, \boldsymbol{v}) = (A\boldsymbol{x}, A\boldsymbol{v})$. It is easy to see that under this action the unit tangent sphere bundle $U\mathbb{S}^3$ is a homogeneous space of SO(4). The isotropy subgroup at $(\boldsymbol{e}_0, \boldsymbol{e}_1)$ is

$$\left\{ \begin{pmatrix} I_2 & 0 \\ 0 & b \end{pmatrix} \ \middle|\ b \in \mathrm{SO}(2) \right\}.$$

Here I_2 is the identity matrix of rank 2. Hence, $U\mathbb{S}^3 = \mathrm{SO}(4)/\mathrm{SO}(2)$. The invariant Riemannian metric induced on $U\mathbb{S}^3 = \mathrm{SO}(4)/\mathrm{SO}(2)$ is a normal homogeneous metric (and hence naturally reductive), but not Riemannian symmetric. Note that $U\mathbb{S}^3$ coincides with the *Stiefel manifold* of oriented 2-frames in $\mathbb{E}^4$.

The space of geodesics. Next, we consider $\mathrm{Geo}(\mathbb{S}^3)$, the space of all oriented geodesics in $\mathbb{S}^3$. Take a geodesic $\gamma \in \mathrm{Geo}(\mathbb{S}^3)$; then γ is given by the intersection of $\mathbb{S}^3$ with an oriented 2-dimensional linear subspace W in $\mathbb{E}^4$. By identifying γ with W, the space $\mathrm{Geo}(\mathbb{S}^3)$ is identified with the Grassmann manifold $\mathrm{Gr}_2(\mathbb{E}^4)$ of oriented 2-planes in Euclidean 4-space. The natural projection $\pi_1 : U\mathbb{S}^3 \to \mathrm{Geo}(\mathbb{S}^3)$ is regarded as the map

$$\pi_1(\boldsymbol{x}, \boldsymbol{v}) = \text{the geodesic } \gamma \text{ satisfying the conditions } \gamma(0) = \boldsymbol{x}, \ \gamma'(0) = \boldsymbol{v}.$$

The rotation group SO(4) acts isometrically and transitively on $\mathrm{Geo}(\mathbb{S}^3)$. The isotropy subgroup at $e_0 \wedge e_1$ is $\mathrm{SO}(2) \times \mathrm{SO}(2)$.

Therefore, the tangent space $T_{e_0 \wedge e_1} \operatorname{Geo}(\mathbb{S}^3)$ is identified with the linear subspace

$$\left\{ \begin{pmatrix} 0 & 0 & -x_2 & -x_3 \\ 0 & 0 & -x_{21} & -x_{31} \\ x_2 & x_{21} & 0 & 0 \\ x_3 & x_{31} & 0 & 0 \end{pmatrix} \right\}$$

of $\mathfrak{so}(4)$. The standard invariant complex structure J on $\operatorname{Geo}(\mathbb{S}^3) = \operatorname{Gr}_2(\mathbb{E}^4)$ is given explicitly by

$$J \begin{pmatrix} 0 & 0 & -x_2 & -x_3 \\ 0 & 0 & -x_{21} & -x_{31} \\ x_2 & x_{21} & 0 & 0 \\ x_3 & x_{31} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x_{21} & x_{31} \\ 0 & 0 & -x_2 & -x_3 \\ -x_{21} & x_2 & 0 & 0 \\ -x_{31} & x_3 & 0 & 0 \end{pmatrix}.$$

One can see that $\operatorname{Gr}_2(\mathbb{E}^4)$ is a Hermitian symmetric space with Ricci tensor $2\langle \cdot, \cdot \rangle$. The Kähler form Ω is related to the contact form ω by $\pi_1^* \Omega = d\omega$.

2. Surface theory in $\mathbb{S}^3$

The Lagrangian and Legendrian Gauss maps. Let $f : M \to \mathbb{S}^3 \subset \mathbb{E}^4$ be a conformal immersion of a Riemann surface with unit normal vector field n. Then we define the (*Lagrangian*) *Gauss map* L of f by

$$L := f \wedge n : M \to \operatorname{Gr}_2(\mathbb{E}^4).$$

One can see that L is an immersion and, in addition, that it is *Lagrangian* with respect to the canonical symplectic form Ω of $\operatorname{Gr}_2(\mathbb{E}^4)$; that is, $L^* \Omega = 0$. Under the identification $\operatorname{Gr}_2(\mathbb{E}^4) = \operatorname{Geo}(\mathbb{S}^3)$, the Lagrangian Gauss map is referred as the oriented normal geodesic of f (and called the *spherical Gauss map*).

On the other hand, we have a map $\mathcal{L} := (f, n) : M \to U\mathbb{S}^3$. This map is *Legendrian* with respect to the canonical contact structure of $U\mathbb{S}^3$; that is, $\mathcal{L}^* \omega = 0$. This map $\mathcal{L}$ is called the *Legendrian Gauss map* of f.

Parallel surfaces. An oriented *geodesic congruence* in $\mathbb{S}^3$ is an immersion of a 2-manifold M into the space $\operatorname{Geo}(\mathbb{S}^3)$ of geodesics. Now, let $f : M \to \mathbb{S}^3$ be a surface with unit normal n. Then a *normal geodesic congruence* through f at a distance r is the map $f^r : M \to \mathbb{S}^3$ defined by

$$f^r := \cos r \, f + \sin r \, n.$$

If f satisfies the condition $\cos(2r) - \sin(2r)H + \sin^2(r)K \neq 0$, then f^r is an immersion. Here, H and K are the mean and Gauss curvatures of f, respectively. If f^r is an immersion, then it is called the *parallel surface* of f at the distance r. The correspondence $f \mapsto f^r$ is called the *parallel transformation*.

Legendrian lifts, frontals and fronts. The Gauss map L of an oriented surface $f : M \to \mathbb{S}^3$ with unit normal n is a Lagrangian immersion into $\mathrm{Gr}_2(\mathbb{E}^4)$. Conversely, we have:

Proposition 2.1 [Palmer 1994]. *Let $L : M \to \mathrm{Gr}_2(\mathbb{E}^4)$ be a Lagrangian immersion. Then locally L is a projection of a Legendrian immersion $\mathscr{L} : M \to \mathrm{U}\mathbb{S}^3$. The Legendrian immersion is unique up to parallel transformations.*

The Legendrian immersion $\mathscr{L}$ is called a *Lie surface* by Palmer. If $f : M \to \mathbb{S}^3$ is an immersion with unit normal n, then $\mathscr{L} := (f, n)$ is a Legendrian immersion into $\mathrm{U}\mathbb{S}^3$. However, even if $\mathscr{L}$ is a Legendrian immersion, $f := \pi_2 \circ \mathscr{L}$ need not be an immersion, although it possesses a unit normal n. Here, $\pi_2 : \mathrm{U}\mathbb{S}^3 \to \mathbb{S}^3$ is the natural projection.

Remark 2.2. A smooth map $f : M \to \mathbb{S}^3$ is called a *frontal* if for any point $p \in M$ there exists a neighborhood $\mathscr{U}$ of p and a unit vector field n along f defined on $\mathscr{U}$ such that $\langle \mathrm{d}f, n \rangle = 0$. A frontal is said to be *coorientable* if there exists a unit vector field n along f such that $\langle \mathrm{d}f, n \rangle = 0$. Namely, a coorientable frontal is a smooth map $f : M \to \mathbb{S}^3$ that has a lift $\mathscr{L} = (f, n)$ to $\mathrm{U}\mathbb{S}^3$ satisfying the *Legendrian condition* $\mathscr{L}^* \omega = \langle \mathrm{d}f, n \rangle = 0$. A coorientable frontal is called a *front* if its Legendrian lift is an immersion.

Our main interest is surfaces of constant curvature $K < 1$ in $\mathbb{S}^3$. Except in the case $K = 0$, any surface of constant Gauss curvature $K < 1$ has singularities. A theory of the singularities of fronts can be found in [Arnold 1990]. Geometric concepts such as curvature and completeness for surfaces with singularities have been defined by Saji, Umehara and Yamada in [Saji et al. 2009].

Asymptotic coordinates. Hereafter, assume that the Gaussian curvature K is less than 1. This implies that the second fundamental form II derived from n is a possibly singular Lorentzian metric on M.

Represent K as $K = 1 - \rho^2$ for some positive function ρ, and take a local *asymptotic coordinate system* (u, v) defined on a simply connected domain $\mathbb{D} \subset M$. Then the first and second fundamental forms I and II are given by (see, e.g., [Moore 1972])

$$(2\text{-}1) \qquad \mathrm{I} = A^2 \, \mathrm{d}u^2 + 2AB \cos \phi \, \mathrm{d}u \, \mathrm{d}v + B^2 \, \mathrm{d}v^2, \quad \mathrm{II} = 2\rho AB \sin \phi \, \mathrm{d}u \, \mathrm{d}v.$$

Note that asymptotic coordinates (u, v) are *conformal* with respect to the second fundamental form. We may regard M as a (singular) *Lorentz surface* [Weinstein 1996] with respect to the conformal structure determined by II (called the *second conformal structure* [Klotz 1963; Milnor 1983]). Thus, one can see that

$$C = A^2 \, \mathrm{d}u^2 + B^2 \, \mathrm{d}v^2$$

is well defined on M.

The *Gauss equation* is given by

$$\phi_{uv} - \left(\frac{\rho_v}{2\rho}\frac{B}{A}\sin\phi\right)_v - \left(\frac{\rho_u}{2\rho}\frac{A}{B}\sin\phi\right)_u + (1 - \rho^2)AB\sin\phi = 0.$$

Now, we introduce functions a and b by $a = A\rho$ and $b = B\rho$. The *Codazzi equations* are

$$(2\text{-}2) \qquad a_v - \frac{\rho_v}{2\rho}a + \frac{\rho_u}{2\rho}b\cos\phi = 0, \quad b_u - \frac{\rho_u}{2\rho}b + \frac{\rho_v}{2\rho}a\cos\phi = 0.$$

The Codazzi equations imply that if K is constant, then we have $a_v = b_u = 0$. In addition, the Gauss–Codazzi equations are invariant under the deformation

$$a \mapsto \lambda a, \; b \mapsto \lambda^{-1}b, \quad \lambda \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}.$$

Thus, there exists a one-parameter deformation $\{f_\lambda\}_{\lambda \in \mathbb{R}^*}$ of f preserving the second fundamental form and the Gauss curvature. The resulting family is called the *associated family* of f. The existence of the associated family motivates us to study constant Gauss curvature surfaces in $\mathbb{S}^3$ by loop group methods.

3. The loop group formulation

The* SU(2) × SU(2) *frame. Let us now identify $\mathbb{S}^3$ with SU(2), via

$$(z, w) \in \mathbb{S}^3 \subset \mathbb{R}^4 = \mathbb{C}^2 \quad \longleftrightarrow \quad \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \in \text{SU}(2).$$

The standard metric g on $\mathbb{S}^3$ is then given by left translating $V, W \in T_x\mathbb{S}^3$ to the tangent space at the identity, $T_e \text{SU}(2) = \mathfrak{su}(2)$; that is,

$$g(V, W) := \langle x^{-1}V, x^{-1}W \rangle,$$

where the inner product on $\mathfrak{su}(2)$ is given by $\langle X, Y \rangle = -\text{Tr}(XY)/2$. The natural basis $\{e_0, e_1, e_2, e_3\}$ of $\mathbb{E}^4$ is identified with

$$e_0 = e = I_2, \quad e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Note that $\{e_1, e_2, e_3\}$ is an orthonormal basis of $\mathfrak{su}(2)$. We have the commutators $[e_1, e_2] = 2e_3$, $[e_2, e_3] = 2e_1$ and $[e_3, e_1] = 2e_2$, so the cross product on $\mathbb{E}^3$ is given by $A \times B = \frac{1}{2}[A, B]$. Note that $\mathbb{S}^3$ is represented by $(\text{SU}(2) \times \text{SU}(2))/\text{SU}(2)$ as a Riemannian symmetric space. The natural projection is given by $(G, F) \mapsto GF^{-1}$.

Let M be a simply connected 2-manifold, and suppose given an immersion $f : M \to \mathbb{S}^3$, with global asymptotic coordinates (u, v) and first and second

fundamental forms as above at (2-1). Set $\theta = \phi/2$ and

$$\xi_1 = \cos\theta\, e_1 - \sin\theta\, e_2 = \begin{pmatrix} 0 & e^{-i\theta} \\ -e^{i\theta} & 0 \end{pmatrix},$$

$$\xi_2 = \cos\theta\, e_1 + \sin\theta\, e_2 = \begin{pmatrix} 0 & e^{i\theta} \\ -e^{-i\theta} & 0 \end{pmatrix}.$$

Then $\langle \xi_1, \xi_2 \rangle = \cos\phi$, and so we can define a map $F : M \to SU(2)$ by the equations

(3-1) $f^{-1} f_u = A\,\mathrm{Ad}_F\, \xi_1, \quad f^{-1} f_v = B\,\mathrm{Ad}_F\, \xi_2, \quad f^{-1} n = \mathrm{Ad}_F\, e_3,$

where n is the unit normal given by $n = (AB)^{-1} f(f^{-1} f_u \times f^{-1} f_v)$.

Setting $G = fF$, the map $\mathscr{F} = (F, G) : M \to SU(2) \times SU(2)$ is a lift of f, and the projection to $SU(2)$ is given by

$$f = GF^{-1}.$$

We call $\mathscr{F}$ the *coordinate frame* for f. We now want to get expressions for the Maurer–Cartan forms of F and G. Differentiating $G = fF$ and substituting in the expressions at (3-1) for $f^{-1} f_u$ and $f^{-1} f_v$, we obtain

(3-2)
$$G^{-1} G_u - F^{-1} F_u = A\xi_1,$$
$$G^{-1} G_v - F^{-1} F_v = B\xi_2.$$

Now, write $F^{-1} F_u = a_1 e_1 + a_2 e_2 + a_3 e_3$. Differentiating the expression $f^{-1} f_u = A\,\mathrm{Ad}_F\, \xi_1$, we obtain

$$f^{-1} f_{uu} = A^2 \,\mathrm{Ad}_F\, \xi_1^2 + \frac{\partial A}{\partial u} \,\mathrm{Ad}_F\, \xi_1 + A\,\mathrm{Ad}_F\, [F^{-1} F_u, \xi_1] + A\,\mathrm{Ad}_F\, \frac{\partial \xi_1}{\partial u}$$

$$= A\,\mathrm{Ad}_F\left(-A e_0 + A^{-1} \frac{\partial A}{\partial u} \xi_1 \right.$$
$$\left. + [a_1 e_1 + a_2 e_2 + a_3 e_3, \cos\theta\, e_1 - \sin\theta\, e_2] + \frac{\partial \xi_1}{\partial u} \right)$$

$$= A(-2a_1 \sin\theta - 2a_2 \cos\theta)\,\mathrm{Ad}_F\, e_3 + \mathrm{Ad}_F\, (d_0 e_0 + d_1 e_1 + d_2 e_2),$$

where we are only interested in the coefficient of $\mathrm{Ad}_F\, e_3$, that is, of $f^{-1} n$. Since the second fundamental form is assumed to be $\mathrm{II} = 2\rho AB \sin\phi\, du\, dv$, we know that $\langle f^{-1} n, f^{-1} f_{uu} \rangle = 0$. Hence the coefficient of n in the above equation is zero: $0 = A(-2a_1 \sin\theta - 2a_2 \cos\theta)$, or again

$$a_2 = -a_1 \tan\theta.$$

Next, differentiating $f^{-1} f_v = B\,\mathrm{Ad}_F\, \xi_2$ with respect to u, we deduce

$$f^{-1} f_{uv} = AB\,\mathrm{Ad}_F\, (\xi_1 \xi_2) + \mathrm{Ad}_F\left(\frac{\partial B}{\partial u} \xi_2 + B[F^{-1} F_u, \xi_2] + B \frac{\partial \xi_2}{\partial u} \right),$$

and the coefficient of $\mathrm{Ad}_F\, e_3$ on the right-hand side is

$$AB \sin 2\theta + B(2a_1 \sin\theta - 2a_2 \cos\theta).$$

Substituting in $a_2 = -a_1 \tan\theta$, the equation $\langle f^{-1}n, f^{-1}f_{uv}\rangle = \rho AB \sin\phi$ then becomes

$$\rho AB \sin 2\theta = AB \sin 2\theta + B\, 4a_1 \sin\theta.$$

Hence, $a_1 = A(\rho-1)\cos(\theta)/2$, and $a_2 = -A(\rho-1)\sin(\theta)/2$. Writing $U_0 := a_3 e_3$, we have

$$F^{-1}F_u = U_0 + \frac{\rho-1}{2}A\xi_1.$$

From the equations (3-2) we also have

$$G^{-1}G_u = U_0 + \frac{\rho+1}{2}A\xi_1.$$

Similarly, one obtains the expressions

$$F^{-1}F_v = V_0 - \frac{\rho+1}{2}B\xi_2 \quad \text{and} \quad G^{-1}G_v = V_0 - \frac{\rho-1}{2}B\xi_2,$$

where V_0 is a scalar times e_3.

The Maurer–Cartan form $\alpha = F^{-1}\mathrm{d}F$ of F thus has the expression

$$\alpha = \alpha_0 + \alpha_1 + \alpha_{-1},$$

(3-3)
$$\alpha_0 = U_0\, \mathrm{d}u + V_0\, \mathrm{d}v, \quad \alpha_1 = \frac{\rho-1}{2}A\xi_1\, \mathrm{d}u, \quad \alpha_{-1} = -\frac{\rho+1}{2}B\xi_2\, \mathrm{d}v,$$

and similarly, $G^{-1}\, \mathrm{d}G = \beta = \beta_0 + \beta_1 + \beta_{-1}$, with

(3-4) $$\beta_0 = U_0\, \mathrm{d}u + V_0 \mathrm{d}v, \quad \beta_1 = \frac{\rho+1}{2}A\xi_1\, \mathrm{d}u, \quad \beta_{-1} = -\frac{\rho-1}{2}B\xi_2\, \mathrm{d}v.$$

One can check that U_0 and V_0 are of the form $U_0 = -\theta_u e_3/2$ and $V_0 = \theta_v e_3/2$.

Ruh–Vilms property. Now we investigate Lorentz harmonicity, with respect to the second conformal structure, of the *normal Gauss map* $\nu = f^{-1}n$ of f. By definition, ν takes values in the unit 2-sphere $\mathbb{S}^2 = \mathrm{Ad}_{\mathrm{SU}(2)}\, e_3$ in the Lie algebra $\mathfrak{su}(2)$. Since f and ν are given by $f = GF^{-1}$, $\nu = \mathrm{Ad}_F\, e_3$, we have

$$\nu_u = \mathrm{Ad}_F[U, e_3], \quad \nu_v = \mathrm{Ad}_F[V, e_3],$$

where $U = F^{-1}F_u$ and $V = F^{-1}F_v$. From these we have

$$\frac{\partial}{\partial v}\nu_u = \mathrm{Ad}_F\left(\left[\frac{\partial U}{\partial v}, e_3\right] + [V, [U, e_3]]\right),$$

$$\frac{\partial}{\partial u}\nu_v = \mathrm{Ad}_F\left(\left[\frac{\partial V}{\partial u}, e_3\right] + [U, [V, e_3]]\right).$$

Next, we have

$$\left[\frac{\partial U}{\partial v}, e_3\right] = \{A(\rho - 1)\sin\theta\}_v e_1 + \{A(\rho - 1)\cos\theta\}_v e_2,$$

$$\left[\frac{\partial V}{\partial v}, e_3\right] = -\{B(\rho + 1)\sin\theta\}_u e_1 + \{B(\rho + 1)\cos\theta\}_u e_2,$$

$$[V, [U, e_3]] = -A(\rho - 1)\theta_v(\cos(\theta)e_1 - \sin(\theta)e_2) + \tfrac{1}{2}AB(\rho^2 - 1)\cos(2\theta)e_3,$$

$$[U, [V, e_3]] = B(\rho + 1)\theta_u(\cos(\theta)e_1 + \sin(\theta)e_2) + \tfrac{1}{2}AB(\rho^2 - 1)\cos(2\theta)e_3.$$

Here we recall that a smooth map $\nu : M \to \mathbb{S}^2 \subset \mathbb{E}^3$ of a Lorentz surface M into the 2-sphere is said to be a *Lorentz harmonic map* (or *wave map*) if its tension field with respect to any Lorentzian metric in the conformal class vanishes. This is equivalent to the existence of a function k such that

$$\nu_{uv} = k\nu$$

for any conformal coordinates (u, v).

First, $(\nu_u)_v$ is parallel to ν if and only if

$$A_v(\rho - 1) + A\rho_v = 0.$$

Inserting the Codazzi equation (2-2) into this, we get

(3-5) $$a(\rho + 1)\rho_v - b(\rho - 1)(\cos\phi)\rho_u = 0.$$

Analogously, $(\nu_v)_u$ is parallel to ν if and only if

$$B_u(1 + \rho) + B\rho_u = 0.$$

Inserting the Codazzi equation again, we get

(3-6) $$b(\rho - 1)\rho_v - a(\rho + 1)(\cos\phi)\rho_v = 0.$$

Thus ν is Lorentz harmonic if and only if (3-5) and (3-6) hold. The system (3-5)–(3-6) can be written in matrix form as

$$\begin{pmatrix} b(\rho - 1) & -a(\rho + 1)\cos\phi \\ -b(\rho - 1)\cos\phi & a(\rho + 1) \end{pmatrix} \begin{pmatrix} \rho_u \\ \rho_v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of the coefficient matrix is $ab(\rho^2 - 1)\sin^2\phi$. Thus, under the condition $\rho \neq 1$, i.e., $K \neq 0$, we have ν is Lorentz harmonic if and only if K is constant.

In case $\rho = 1$, we have $U = -\theta_u e_3/2$ and so $\nu_u = \mathrm{Ad}_F[U, e_3] = 0$. Hence $\nu_{uv} = 0$. Thus g is Lorentz harmonic.
   goodbreak

Theorem 3.1. *Let $f : M \to \mathbb{S}^3$ be an isometric immersion of Gauss curvature $K < 1$. Then the normal Gauss map ν is Lorentz harmonic with respect to the conformal structure determined by the second fundamental form if and only if K is constant.*

This characterization is referred as the *Ruh–Vilms property* for constant curvature surfaces in $\mathbb{S}^3$ with $K < 1$.

Remark 3.2. Under the identification $\mathbb{S}^3 = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathrm{SU}(2)$, the space $\mathrm{Geo}(\mathbb{S}^3)$ is identified with the Riemannian product

$$\mathbb{S}^2 \times \mathbb{S}^2 = (\mathrm{SU}(2) \times \mathrm{SU}(2))/(\mathrm{U}(1) \times \mathrm{U}(1)).$$

The Lagrangian Gauss map L corresponds to the map

$$L \longleftrightarrow (nf^{-1}, f^{-1}n) = (\mathrm{Ad}_G\, e_3, \mathrm{Ad}_F\, e_3)$$

(see [Aiyama and Akutagawa 2000; Kitagawa 1995]). Thus the Ruh–Vilms property can be rephrased as follows:

Corollary 3.3. *Let $f : M \to \mathbb{S}^3$ be an isometric immersion of Gauss curvature $K < 1$. Then the Lagrangian Gauss map L is Lorentz harmonic with respect to the conformal structure determined by the second fundamental form if and only if K is constant.*

The Legendrian Gauss map has the formula $\mathscr{L} = (f, n) = (GF^{-1}, Ge_3F^{-1})$.

Remark 3.4. Consider an oriented minimal surface $f : M \to \mathbb{S}^3$ with unit normal n. Then its Lagrangian Gauss map $L = f \wedge n$ is a *harmonic map* with respect to the conformal structure determined by the *first fundamental form*. Hence L is a minimal Lagrangian surface in the Grassmannian [Palmer 1994, Proposition 3.1]; see also [Castro and Urbano 2007].

The loop group formulation for constant curvature surfaces. Let α and β be as defined above at (3-3) and (3-4). Let us now define the family of 1-forms

$$\alpha^\lambda = \alpha_0 + \lambda\alpha_1 + \lambda^{-1}\alpha_{-1},$$

where λ is a (nonzero) complex parameter. The integrability conditions for the 1-forms α and β are $d\alpha + \alpha \wedge \alpha = 0$ and $d\beta + \beta \wedge \beta = 0$. Using these two equations, which must already be satisfied, it is straightforward to deduce that α^λ is integrable for all λ if and only if ρ is *constant*; in other words, if and only if the immersion f has *constant* curvature $1 - \rho^2$. In this case we have, of course, $\alpha = \alpha^1$, but also

(3-7)
$$\beta = \alpha^\mu, \quad \text{where } \mu = \frac{\rho + 1}{\rho - 1}.$$

From now on, we assume that ρ is constant, so $d\alpha^\lambda + \alpha^\lambda \wedge \alpha^\lambda = 0$ for all nonzero

complex values of λ. Let us choose coordinates for the ambient space such that

$$(3\text{-}8) \qquad\qquad F(u_0, v_0) = f(u_0, v_0) = G(u_0, v_0) = I$$

at some base point (u_0, v_0). We further assume that M is simply connected. Then we can integrate the equations

$$\widehat{F}^{-1}\, d\widehat{F} = \alpha^\lambda, \quad \widehat{F}(u_0, v_0) = I,$$

to obtain a map $\widehat{F} : M \to \mathcal{G} = \Lambda\mathrm{SL}(2, \mathbb{C})_{\sigma\tau}$. Here the *twisted loop group* $\Lambda\mathrm{SL}(2, \mathbb{C})_{\sigma\tau}$ is the fixed point subgroup of the free loop group $\Lambda\mathrm{SL}(2, \mathbb{C})$ by the involutions σ and τ, which are defined as

$$\sigma x(\lambda) := \mathrm{Ad}_{\mathrm{diag}(1,-1)}(x(-\lambda)), \quad \tau x(\lambda) := \overline{x((\bar{\lambda}))^T}^{-1}.$$

Elements of $\mathcal{G}$ take values in $\mathrm{SU}(2)$ for real values of λ.

By definition, we have $F = \widehat{F}|_{\lambda=1}$, and, moreover, from (3-7) and the initial condition (3-8) we also have $G = \widehat{F}|_{\lambda=\mu}$. Thus, $\widehat{F}$ can be thought of as a lift of the coordinate frame $\mathscr{F} = (F, G)$, with the projections $(F, G) = (\widehat{F}|_{\lambda=1}, \widehat{F}|_{\lambda=\mu})$ and

$$(3\text{-}9) \qquad\qquad f = \widehat{F}\big|_{\lambda=\mu}\, \widehat{F}^{-1}\big|_{\lambda=1}.$$

Thus we may call the map $\widehat{F}$ the *extended coordinate frame* for f.

Let us now consider a general map into the twisted loop group $\mathcal{G}$ that has a similar Maurer–Cartan form to α^λ; first, let K be the diagonal subgroup of $\mathrm{SU}(2)$ and $\mathfrak{su}(2) = \mathfrak{k} + \mathfrak{p}$ be the symmetric space decomposition induced by $\mathbb{S}^2 = G/K = \mathrm{SU}(2)/\mathrm{U}(1)$ of the Lie algebra; that is,

$$\mathfrak{k} = \mathrm{span}(e_3) \quad \text{and} \quad \mathfrak{p} = \mathrm{span}(e_1, e_2).$$

Definition 3.5. Let M be a simply connected subset of $\mathbb{R}^2$ with coordinates (u, v). An *admissible frame* is a smooth map $\widehat{F} : M \to \mathcal{G}$, the Maurer–Cartan form of which has the Fourier expansion

$$\widehat{F}^{-1}\, d\widehat{F} = \alpha_0 + \lambda B_1\, du + \lambda^{-1} B_{-1}\, dv, \qquad \alpha_0 \in \mathfrak{k} \otimes \Omega^1(M),\ B_{\pm 1}(u, v) \in \mathfrak{p}.$$

The admissible frame is *regular at a point* p if $B_1(p)$ and $B_{-1}(p)$ are linearly independent, and $\widehat{F}$ is called *regular* if it is regular at every point.

Note that the extended coordinate frame for a constant curvature $1-\rho^2$ immersion, defined above, is a regular admissible frame on M. Conversely:

Lemma 3.6. *Let* $\widehat{F} : M \to \mathcal{G}$ *be a regular admissible frame. Let* μ *be any real number not equal to* 1 *or* 0. *Then the map* $f : M \to \mathbb{S}^3 = \mathrm{SU}(2)$ *defined by the projection* (3-9) *is an immersion of constant curvature*

$$K_\mu = 1 - \rho^2, \quad \text{where } \rho := \frac{\mu+1}{\mu-1}.$$

The first and second fundamental form are given by

$$(3\text{-}10) \qquad \mathrm{I} = A^2 \, du^2 + 2AB \cos\phi \, du\,dv + B^2 \, dv^2, \qquad \mathrm{II} = 2\rho AB \sin\phi \, du\,dv,$$

where $A = (\mu - 1)|B_1|$, $B = (\mu^{-1} - 1)|B_{-1}|$, and ϕ is the angle between B_1 and B_{-1}.

Proof. Set

$$F := \widehat{F}\big|_{\lambda=1}, \quad G := \widehat{F}\big|_{\lambda=\mu},$$

so that $f = GF^{-1}$. Differentiating this formula and using the expressions $F^{-1}dF = \alpha_0 + B_1 \, du + B_{-1} \, dv$ and $G^{-1}dG = \alpha_0 + \mu B_1 \, du + \mu^{-1} B_{-1} \, dv$, we obtain

$$(3\text{-}11) \qquad f^{-1}f_u = (\mu - 1)\,\mathrm{Ad}_F\, B_1, \quad f^{-1}f_v = (\mu^{-1} - 1)\,\mathrm{Ad}_F\, B_{-1}.$$

Thus, since $B_{\pm 1}$ are linearly independent for a regular admissible frame, the map f is an immersion and the first fundamental form is given by

$$(\mu-1)^2|B_1|^2 \, du^2 + 2(\mu-1)\Big(\frac{1}{\mu}-1\Big)\cos\phi\,|B_1||B_{-1}|\,du\,dv + \Big(\frac{1}{\mu}-1\Big)^2|B_{-1}|^2 \, dv^2,$$

where ϕ is the angle between B_1 and B_{-1}. This gives formula $(3\text{-}10)_1$ for the first fundamental form.

It remains to show formula $(3\text{-}10)_2$ for the *second* fundamental form, from which it will follow that the intrinsic curvature is $1 - \rho^2$. Since $B_{\pm 1}$ take values in $\mathfrak{p}$, and e_3 is perpendicular to $\mathfrak{p}$, it follows from equations $(3\text{-}11)$ that a choice of unit normal is given by

$$n = f\,\mathrm{Ad}_F\, e_3.$$

Differentiating equations $(3\text{-}11)$ then leads to

$$\langle f^{-1}f_{uu}, f^{-1}n\rangle = \langle f^{-1}f_{vv}, f^{-1}n\rangle = 0,$$

$$\langle f^{-1}f_{uv}, f^{-1}n\rangle = (1-\mu)(1+\mu^{-1})|B_1|\,|B_{-1}|\sin\phi$$
$$= \rho\,(\mu-1)(\mu^{-1}-1)|B_1|\,|B_{-1}|\sin\phi,$$

which gives the formula at $(3\text{-}11)$ for II. $\qquad\square$

Note that

$$K_\mu \in (0, 1] \quad \text{and} \quad K_{-1} = 1 \qquad \text{for } \mu < 0,$$
$$K_\mu < 0 \qquad \text{and} \quad \lim_{\mu \to 1} K_\mu = -\infty \quad \text{for } \mu > 0.$$

The Legendrian Gauss map and Lagrangian Gauss map of $f = \widehat{F}_{\lambda=\mu}\,\widehat{F}_{\lambda=1}^{-1}$ are given respectively by

$$\mathscr{L} = (\widehat{F}_{\lambda=\mu}\,\widehat{F}_{\lambda=1}^{-1}, \widehat{F}_{\lambda=\mu}e_3\,\widehat{F}_{\lambda=1}^{-1}), \quad L = (\mathrm{Ad}_{\widehat{F}_{\lambda=\mu}}\,e_3, \mathrm{Ad}_{\widehat{F}_{\lambda=1}}\,e_3).$$

The generalized d'Alembert representation. As we have shown, the problem of finding a nonflat constant curvature immersion $f : M \to \mathbb{S}^3$ with $K < 1$ is equivalent to finding an admissible frame. As a matter of fact, Definition 3.5 of an admissible frame is identical to the extended SU(2) frame for a pseudospherical surface in the Euclidean space $\mathbb{E}^3$ (see, for example, [Brander and Svensson 2013; Dorfmeister and Sterling 2002; Toda 2005]). The surfaces in $\mathbb{E}^3$ are obtained from the same frame, not by the projection (3-9) but by the so-called *Sym formula*. We will explain the connection between these problems in the next section, but the point we are making here is that the problem of constructing these admissible frames by the generalized d'Alembert representation has already been solved in [Toda 2005].

A presentation of the method, using similar definitions to those found here, can be found in [Brander and Svensson 2013]. The building blocks of any admissible frame are these:

Definition 3.7 [Brander and Svensson 2013, Definition 5.1]. Let I_u and I_v be two real intervals, with coordinates u and v respectively. A potential pair (η_+, η_-) is a pair of smooth $\Lambda\mathfrak{sl}(2, \mathbb{C})_{\sigma\tau}$-valued 1-forms on I_u and I_v respectively with Fourier expansions in λ

$$\eta_+ = \sum_{j=-\infty}^{1} (\eta_+)_j \lambda^j \, du, \quad \eta_- = \sum_{j=-1}^{\infty} (\eta_-)_j \lambda^j \, dv.$$

The potential pair is called *regular* if $[(\eta_+)_1]_{12} \neq 0$ and $[(\eta_-)_{-1}]_{12} \neq 0$.

The admissible frame $\widehat{F}$ is then obtained by solving $F_\pm^{-1} \, dF_\pm = \eta_\pm$ with initial conditions $F_\pm(0) = I$, and thereafter performing at each (u, v) a *Birkhoff decomposition* [Pressley and Segal 1986]

$$F_+^{-1}(u) F_-(v) = H_-(u, v) H_+(u, v), \quad \text{with } H_\pm(u, v) \in \Lambda^\pm \mathrm{SL}(2, \mathbb{C}),$$

and then setting $\widehat{F}(u, v) = F_+(u) H_-(u, v)$.

Example solutions, using a numerical implementation of this method, are computed below.

4. Limiting cases: pseudospherical surfaces in Euclidean space and flat surfaces in the 3-sphere

In this section we discuss the interpretation of admissible frames at degenerate values of the loop parameter μ, namely, the case $\mu = 1$, which was excluded from the above construction, and the limit $\mu \to 0$ or $\mu \to \infty$.

Relation to pseudospherical surfaces in Euclidean space $\mathbb{E}^3$. As alluded to above, in addition to the constant Gauss curvature $K = 1 - \rho^2$ surfaces in $\mathbb{S}^3$ of Lemma 3.6,

one also obtains from a regular admissible frame $\widehat{F}$ a constant negative curvature -1 surface in $\mathbb{E}^3$ by the *Sym formula*

$$(4\text{-}1) \qquad \check{f} = 2\frac{\partial \widehat{F}}{\partial \lambda}\,\widehat{F}^{-1}\big|_{\lambda=1}.$$

Here we explain how this formula arises naturally from the construction of surfaces in $\mathbb{S}^3 = \mathrm{SU}(2)$.

Obviously the projection formula

$$f_\mu = \widehat{F}\big|_{\lambda=\mu}\,\widehat{F}^{-1}\big|_{\lambda=1}$$

for the surface in $\mathbb{S}^3$ degenerates to a constant map for $\mu = 1$. On the other hand, we can see that $K_\mu = 1 - (\mu+1)^2/(\mu-1)^2$ approaches $-\infty$ when μ approaches 1. This suggests that we multiply our projection formula by some factor, allowing the size of the sphere to vary, so that K approaches some finite limit instead, in order to have an interpretation for the map at $\mu = 1$. Set

$$\check{f}_\mu = \frac{2}{1-\mu}\,(f_\mu - e_0).$$

Note that $e_0 = (1,0,0,0)$ under our identification $\mathbb{E}^4 = \mathfrak{su}(2) + \mathrm{span}(e_0)$. Now, for $\mu \neq 1$ the function f_μ is a constant curvature K_μ surface in $\mathbb{S}^3$, and $\check{f}_\mu$ is obtained by a constant dilation of $\mathbb{E}^4$ by the factor $2(1-\mu)^{-1}$, plus a constant translation which has no geometric significance. It follows that $\check{f}_\mu$ is a surface in a (translated) sphere of radius $2(1-\mu)^{-1}$, and that $\check{f}_\mu$ has constant curvature

$$(4\text{-}2) \qquad \check{K}_\mu = (1/4)(1-\mu)^2 K_\mu.$$

Now, consider the function $g : M \times (1-\varepsilon, 1+\varepsilon) \to \mathbb{E}^4$, for some small positive real number ε, given by

$$g(u,v,\lambda) = 2\big(\widehat{F}(u,v)\big|_{\lambda=\lambda}\,\widehat{F}(u,v)^{-1}\big|_{\lambda=1} - e_0\big).$$

This function is differentiable in all arguments, and

$$\frac{\partial g}{\partial \lambda}\bigg|_{\lambda=1} = 2\lim_{\mu\to 1}\frac{F\big|_{\lambda=\mu}\,F^{-1}\big|_{\lambda=1} - e_0}{1-\mu} = \lim_{\mu\to 1}\check{f}_\mu.$$

Hence the limit on the right-hand side exists and is a smooth function $M \to \mathbb{E}^4$. On the other hand, differentiating the definition of g, we obtain the right-hand side of the Sym formula (4-1). Note that since F is $\mathrm{SU}(2)$-valued, this expression takes values in the Lie algebra $\mathfrak{su}(2) = \mathrm{span}(e_1, e_2, e_2)$, which in our representation of $\mathbb{E}^4$ is the hyperplane $x_0 = 0$. In other words, $\lim_{\mu\to 1}\check{f}_\mu$ takes values in $\mathbb{E}^3 \subset \mathbb{E}^4$. Assuming that our surface in $\mathbb{S}^3$ is regular, then one can verify that the regularity

assumption on the frame F implies that this map is an immersion, and it is clear from expression (4-2) that this surface has constant curvature -1.

Example 4.1. In Figure 3, various projections of the same admissible frame are plotted. These are computed using the generalized d'Alembert method (see [Toda 2005]), using the potential pair

$$\eta_+ = A \, du, \quad \eta_- = A \, dv, \quad A = \begin{pmatrix} 0 & -\lambda^{-1} + i\lambda \\ \lambda^{-1} + i\lambda & 0 \end{pmatrix}.$$

The first image, the surface in $\mathbb{E}^3$ obtained via the Sym formula (4-1), is part of a hyperbolic surface of revolution (a plot of a larger region is shown in Figure 1). The two cuspidal edges that can be seen in this image also appear in the other surfaces at the same places in the coordinate domain, because the condition on the admissible frame for the surface to be regular is independent of μ. The surfaces in $\mathbb{S}^3$ are of course distorted by the stereographic projection, which is taken from the south

$\mu = 1$, $K = -1$, target $\mathbb{E}^3$ $\mu = 4$, $K = -\frac{16}{9}$, target $\mathbb{S}^3$

$\mu = -4$, $K = \frac{16}{25}$, target $\mathbb{S}^3$ $\mu = -1$, $K = 1$, target $\mathbb{S}^3$
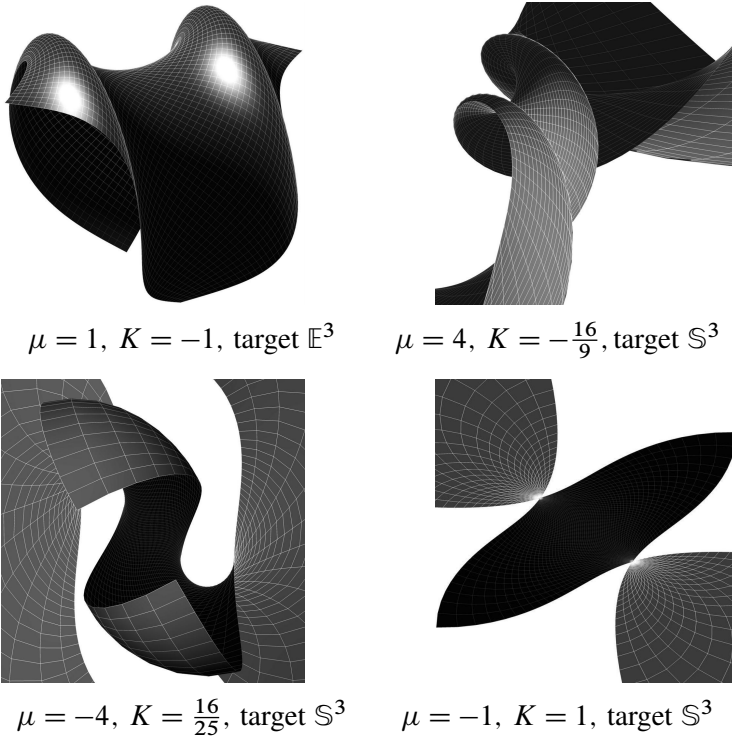
Figure 3. Surfaces obtained from one admissible frame evaluated at different values of μ. All images are of the same coordinate patch. The first image is obtained via the Sym formula, and the others are in $\mathbb{S}^3$, stereographically projected to $\mathbb{R}^3$ for plotting.
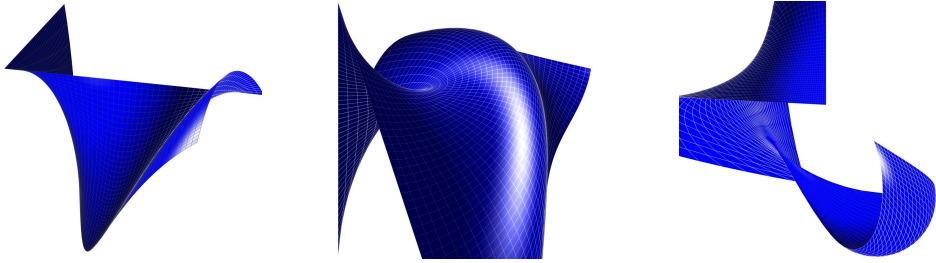
$\mu=1$, $K=-1$, target $\mathbb{E}^3$ $\mu=4$, $K=-\frac{16}{9}$, target $\mathbb{S}^3$ $\mu=-4$, $K=\frac{16}{25}$, target $\mathbb{S}^3$

Figure 4. Amsler's surface and generalizations in the 3-sphere. The surfaces are obtained from one admissible frame evaluated at different values of μ. All images are of the same coordinate patch.

pole $(-1, 0, 0, 0) \in \mathbb{E}^4$; the north pole $(1, 0, 0, 0)$ is at the center of the coordinate domain plotted. The last image is in fact planar, the projection of a part of a totally geodesic hypersphere $\mathbb{S}^2 \subset \mathbb{S}^3$. In this case, each of the two singular curves in the coordinate domain maps to a single point in the surface.

Example 4.2. Amsler's surface in $\mathbb{E}^3$ can be computed by the generalized d'Alembert method, using the potential pair

$$\eta_+ = \begin{pmatrix} 0 & i\lambda \\ i\lambda & 0 \end{pmatrix} du, \quad \eta_- = \begin{pmatrix} 0 & -\lambda^{-1} \\ \lambda^{-1} & 0 \end{pmatrix} dv.$$

The image of a rectangle $[0, a] \times [0, b]$ in the positive quadrant of the uv-plane is plotted in Figure 4, evaluated at 3 different values of μ. The coordinate axes correspond to straight lines for the surface in $\mathbb{E}^3$, and to great circles for the surfaces in $\mathbb{S}^3$, which project to straight lines under the stereographic projection from the south pole. The north pole $(1, 0, 0, 0)$ corresponds to $(u, v) = (0, 0)$.

The singular set in the coordinate patch corresponds to a cuspidal edge in each of the first two images, but contains a swallowtail singularity in the third. See also Figure 2.

Relation to flat surfaces in the 3-sphere. We have considered above the surfaces f_μ, obtained by the projection

(4-3) $$\widehat{F}|_{\lambda=\mu} \widehat{F}^{-1}|_{\lambda=1},$$

for all nonzero real values of μ. We now consider the limit as μ approaches 0 or ∞. From the formula $K_\mu = 1 - (\mu + 1)^2 / (\mu - 1)^2$, it is clear that the limiting surface, if it exists, will be flat. We discuss the case $\mu \to 0$ here.

Observe that the admissible frame $\widehat{F}$ has a pole at $\lambda = 0$, so we cannot evaluate (4-3) at $\mu = 0$. However, in the Maurer–Cartan form of $\widehat{F}$, the factor λ^{-1} appears

only as a coefficient of dv. Hence a change of coordinates could remove the pole in λ. For $\mu > 0$, we set $\tilde{u} = u$ and $\tilde{v} = v/\mu$, so that

$$f_\mu(u, v) = f_\mu(\tilde{u}, \mu\tilde{v}) =: g_\mu(\tilde{u}, \tilde{v}).$$

For simplicity, let us assume that M is a rectangle $(a, b) \times (c, d) \subset \mathbb{R}^2$, containing the origin $(0, 0)$ and with coordinates (u, v). We denote by M_μ the same rectangle in the coordinates $(\tilde{u}, \tilde{v})$, that is, $M_\mu = (a, b) \times (c/\mu, d/\mu)$, and we define $M_0 := (a, b) \times (-\infty, \infty)$.

We have already seen that for $\mu > 0$, the map $g_\mu : M_\mu \to \mathbb{S}^3$ is an immersion of constant curvature $K_\mu = 1 - (\mu + 1)^2/(\mu - 1)^2$, since this is just the same map as f_μ in different coordinates. For fixed $\mu_0 \in (0, 1)$, if $0 < \mu < \mu_0$ then $M_\mu \supset M_{\mu_0}$, and so we can restrict g_μ to M_{μ_0} and talk about a family of maps $g_\mu : M_{\mu_0} \to \mathbb{S}^3$ with a fixed domain.

Lemma 4.3. *For any fixed $\mu_0 \in (0, 1)$, the family of maps $g_\mu : M_{\mu_0} \to \mathbb{S}^3$ extends real analytically in μ to $\mu = 0$. Moreover, the map $g_0 : M_{\mu_0} \to \mathbb{S}^3$ extends to the whole of $M_0 = (a, b) \times (-\infty, \infty)$, and is an immersion of zero Gaussian curvature.*

Proof. Write $\widehat{G}_\mu(\tilde{u}, \tilde{v}) = \widehat{F}(\tilde{u}, \mu\tilde{v}) = \widehat{F}(u, v)$, so $\widehat{G}_\mu : M_\mu \to \mathcal{G}$. Then

$$g_\mu(\tilde{u}, \tilde{v}) = H_\mu(\tilde{u}, \tilde{v}) K_\mu^{-1}(\tilde{u}, \tilde{v}), \quad \text{where } H_\mu := \widehat{G}_\mu\big|_{\lambda = \mu}, \ K_\mu := \widehat{G}_\mu\big|_{\lambda = 1}.$$

Since $\widehat{F}$ is an admissible frame, we can write

$$\begin{aligned}
\widehat{F}^{-1} \, d\widehat{F} &= (U_0 + \lambda U_1) \, du + (V_0 + \lambda^{-1} V_1) \, dv \\
&= \big(U_0(\tilde{u}, \mu\tilde{v}) + \lambda U_1(\tilde{u}, \mu\tilde{v})\big) \, d\tilde{u} + \big(\mu V_0(\tilde{u}, \mu\tilde{v}) + \mu\lambda^{-1} V_1(\tilde{u}, \mu\tilde{v})\big) \, d\tilde{v},
\end{aligned}$$

and thus

$$H_\mu^{-1} \, dH_\mu = (U_0 + \mu U_1) \, d\tilde{u} + (\mu V_0 + V_1) \, d\tilde{v},$$

so

$$H_0^{-1} \, dH_0 = U_0(\tilde{u}, 0) \, d\tilde{u} + V_1(\tilde{u}, 0) \, d\tilde{v},$$

and

$$K_\mu^{-1} \, dK_\mu = (U_0 + U_1) \, d\tilde{u} + (\mu V_0 + \mu V_1) \, d\tilde{v},$$

so

$$K_0^{-1} \, dK_0 = (U_0(\tilde{u}, 0) + U_1(\tilde{u}, 0)) \, d\tilde{u}.$$

Since H_μ and K_μ are both obviously real analytic in μ in a neighborhood of $\mu = 0$, so also is g_μ. Finally, the 1-forms $\gamma = H_0^{-1} \, dH_0$ and $\delta = K_0^{-1} \, dK_0$ are both integrable on $M_{\mu_0} = (a, b) \times (c/\mu_0, d/\mu_0)$ for any fixed μ_0. But, since the coefficients of the 1-forms are constant in $\tilde{v}$, this means that they are in fact integrable on the whole of $(a, b) \times (-\infty, \infty)$. This implies the claim. $\qquad\square$
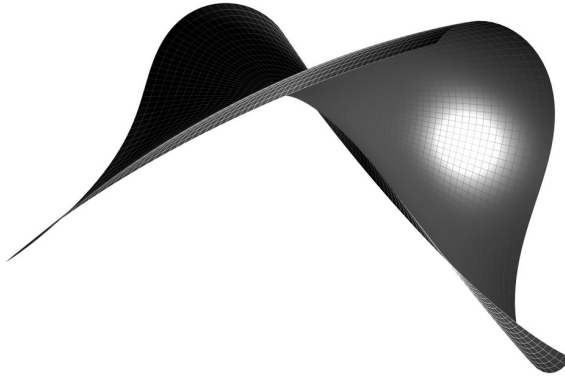
Figure 5. The surface g_μ for $\mu = 10^{-9}$, obtained from the same admissible frame used in Figure 3.

Using the expressions γ and δ above, we obtain the formula

$$g_0^{-1} \, \mathrm{d}g_0 = \mathrm{Ad}_{K_0}(-U_1(\tilde{u}, 0) \, \mathrm{d}\tilde{u} + V_1(\tilde{u}, 0) \, \mathrm{d}\tilde{v}),$$

from which we have the following expression for the first fundamental form of g_0:

$$I(\tilde{u}, \tilde{v}) = (|B_1|^2 \, \mathrm{d}\tilde{u}^2 - 2\cos\phi \, |B_1||B_{-1}| \, \mathrm{d}\tilde{u} \, \mathrm{d}\tilde{v} + |B_{-1}|^2 \, \mathrm{d}\tilde{v}^2)\big|_{(\tilde{u}, 0)}.$$

Letting $\mu \to 0$ in expression (3-10), we conclude that the second fundamental form of g_0 is

$$\mathrm{II} = 2|B_1(\tilde{u}, 0)| \, |B_{-1}(\tilde{u}, 0)| \sin(\phi(\tilde{u}, 0)) \, \mathrm{d}\tilde{u} \, \mathrm{d}\tilde{v}.$$

Example 4.4. In Figure 5 is shown the surface g_μ for $\mu = 10^{-9}$, obtained from the same admissible frame $\widehat{F}_\mu$ used in Example 4.1. A square region in the $(\tilde{u}, \tilde{v})$-plane is plotted, approximately equal to the region $(a, b) \times (c/\mu, d/\mu)$ in the uv-plane, where $(a, b) \times (c, d)$ is the region plotted in Example 4.1. The region plotted here is actually slightly larger, in order to make the singular set visible. As μ approaches zero, the cuspidal edges, which, in the nonflat surface, were approximated by $v = \pm u + \text{constant}$, are now approaching curves of the form $\tilde{v} = \text{constant}$.

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