Pacific Journal of Mathematics

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Volume 269 No. 2 June 2014

PACIFIC JOURNAL OF MATHEMATICS
Vol. 269, No. 2, 2014
dx.doi.org/10.2140/pjm.2014.269.323

VARIATIONAL REPRESENTATIONS FOR N-CYCLICALLY MONOTONE VECTOR FIELDS

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Given a convex bounded domain Ω in $\mathbb{R}^d$ and an integer $N \geq 2$, we associate to any *jointly N-monotone* $(N-1)$-tuplet $(u_1, u_2, \ldots, u_{N-1})$ of vector fields from Ω into $\mathbb{R}^d$ a Hamiltonian H on $\mathbb{R}^d \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d$ that is concave in the first variable, jointly convex in the last $N-1$ variables, and such that

$$(u_1(x), u_2(x), \ldots, u_{N-1}(x)) = \nabla_{2,\ldots,N} H(x, x, \ldots, x)$$

for almost all $x \in \Omega$. Moreover, H is N-antisymmetric in a sense made precise later, and also N-sub-antisymmetric in the sense that for all $X \in \Omega^N$ the sum $\sum_{i=0}^{N-1} H(\sigma^i(X)) \leq 0$ is nonpositive, σ being the permutation that shifts the coordinates of X leftward one slot and places the first coordinate last. This result can be seen as an extension of a theorem of E. Krauss, which associates to any monotone operator a concave-convex antisymmetric saddle function. We also give various variational characterizations of vector fields that are almost everywhere N-monotone, showing that they are dual to the class of measure-preserving N-involutions on Ω.

1. Introduction

Given a domain Ω in $\mathbb{R}^d$, recall that a single-valued map u from Ω to $\mathbb{R}^d$ is said to be *N-cyclically monotone* if for every cycle $x_1, \ldots, x_N, x_{N+1} = x_1$ of points in Ω, one has

$$\sum_{i=1}^{N} \langle u(x_i), x_i - x_{i+1} \rangle \geq 0. \tag{1}$$

A classical theorem of Rockafellar [Phelps 1993] states that a map u from Ω to $\mathbb{R}^d$

Galichon's research has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007–2013)/ ERC grant agreement n° 313699. Support from FiME, Laboratoire de Finance des Marchés de l'Énergie (http://www.fime-lab.org) is gratefully acknowledged. Ghoussoub is partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada.
MSC2010: 49J40.
Keywords: N-cyclically monotone vector fields, Krauss theorem.

is N*-cyclically monotone for every* $N \geq 2$ if and only if

$$u(x) \in \partial\phi(x) \quad \text{for all } x \in \Omega, \tag{2}$$

where $\phi : \mathbb{R}^d \to \mathbb{R}$ is a convex function. On the other hand, a result of E. Krauss [1985] yields that u is a monotone map, i.e., a 2-cyclically monotone map, if and only if

$$u(x) \in \partial_2 H(x, x) \quad \text{for all } x \in \Omega, \tag{3}$$

where H is a concave-convex antisymmetric Hamiltonian on $\mathbb{R}^d \times \mathbb{R}^d$, and $\partial_2 H$ is the subdifferential of H as a convex function in the second variable.

In this paper, we extend the result of Krauss to the class of N-cyclically monotone vector fields, where $N \geq 3$. We shall give a representation for a family of $N-1$ vector fields, which may or may not be individually N-cyclically monotone. Here is the needed concept.

Definition 1. Let $u_1, \dots, u_{N-1}$ be bounded vector fields from a domain $\Omega \subset \mathbb{R}^d$ into $\mathbb{R}^d$. We shall say that the $(N-1)$-tuple $(u_1, u_2, \dots, u_{N-1})$ is *jointly* N*-monotone* if for every cycle $x_1, \dots, x_{2N-1}$ of points in Ω such that $x_{N+i} = x_i$ for $1 \leq i \leq N-1$, one has

$$\sum_{i=1}^{N} \sum_{\ell=1}^{N-1} \langle u_l(x_i), x_i - x_{l+i} \rangle \geq 0. \tag{4}$$

Examples of jointly N-monotone families of vector fields:

- It is clear that $(u, 0, 0, \dots, 0)$ is jointly N-monotone if and only if u is N-monotone.
- More generally, if each u_ℓ is N-monotone, then the family $(u_1, u_2, \dots, u_{N-1})$ is jointly N-monotone. Actually, one only needs that for $1 \leq \ell \leq N-1$, the vector field u_ℓ be (N, ℓ)*-monotone* in the following sense: for every cycle $x_1, \dots, x_{N+\ell}$ of points in Ω such that $x_{N+i} = x_i$ for $1 \leq i \leq \ell$, we have

$$\sum_{i=1}^{N} \langle u_\ell(x_i), x_i - x_{\ell+i} \rangle \geq 0. \tag{5}$$

  This notion is sometimes weaker than N-monotonicity since if ℓ divides N, then it suffices for u to be N/ℓ -monotone in order to be an (N, ℓ)-monotone vector field. For example, if u_1 and u_3 are 4-monotone operators and u_2 is 2-monotone, then the triplet (u_1, u_2, u_3) is jointly 4-monotone.
- Another example is if (u_1, u_2, u_3) are vector fields such that u_2 is 2-monotone and

$$\langle u_1(x) - u_3(y), x - y \rangle \geq 0 \quad \text{for every } x, y \in \mathbb{R}^d.$$

In this case, the triplet (u_1, u_2, u_3) is jointly 4-monotone. In particular, if u_1 and u_2 are both 2-monotone, then the triplet (u_1, u_2, u_1) is jointly 4-monotone.

- More generally, it is easy to show that $(u, u, \ldots, u)$ is jointly N-monotone if and only if u is 2-cyclically monotone.

We shall always denote by σ the cyclic permutation on $\mathbb{R}^d \times \cdots \times \mathbb{R}^d$ defined by

$$\sigma(x_1, x_2, \ldots, x_{N-1}, x_N) = (x_2, x_3, \ldots, x_N, x_1).$$

We let

$$\mathcal{H}_N(\Omega) = \left\{ H \in C(\Omega^N) : \sum_{i=0}^{N-1} H(\sigma^i(x_1, \ldots, x_N)) = 0 \right\} \tag{6}$$

be the family of continuous Hamiltonians on Ω^N that are N*-antisymmetric*, i.e., satisfy the condition to the right of the colon in (6). We say that H is N*-sub-antisymmetric on* Ω if

$$\sum_{i=0}^{N-1} H(\sigma^i(x_1, \ldots, x_N)) \le 0 \quad \text{on } \Omega^N. \tag{7}$$

We shall also say that a function F of two variables is N*-cyclically sub-anti-symmetric on* Ω if

$$\begin{aligned} &F(x, x) = 0 \quad \text{and} \\ &\sum_{i=1}^{N} F(x_i, x_{i+1}) \le 0 \quad \text{for all cyclic families } x_1, \ldots, x_N, x_{N+1} = x_1 \text{ in } \Omega. \end{aligned} \tag{8}$$

Note that if a function $H(x_1, \ldots, x_N)$ N-sub-antisymmetric and if it only depends on the first two variables, then the function $F(x_1, x_2) := H(x_1, x_2, \ldots, x_N)$ is N-cyclically sub-antisymmetric.

We associate to any function H on Ω^N the functional given by on $\Omega \times (\mathbb{R}^d)^{N-1}$

$$L_H(x, p_1, \ldots, p_{N-1}) = \sup \left\{ \sum_{i=1}^{N-1} \langle p_i, y_i \rangle - H(x, y_1, \ldots, y_{N-1}) : y_i \in \Omega \right\}. \tag{9}$$

Note that if Ω is convex and if H is convex in the last $N-1$ variables, then L_H is nothing but the Legendre transform of $\tilde{H}$ with respect to the last $N-1$ variables, where $\tilde{H}$ is the extension of H over $(\mathbb{R}^d)^N$, defined by $\tilde{H} = H$ on Ω^N and $\tilde{H} = +\infty$ outside Ω^N. Since $H(x, \ldots, x) = 0$ for any $H \in \mathcal{H}_N(\Omega)$, we have, for any such H,

$$L_H(x, p_1, \ldots, p_{N-1}) \ge \sum_{i=1}^{N-1} \langle x, p_i \rangle, \tag{10}$$

for $x \in \Omega$ and $p_1, \ldots, p_{N-1} \in \mathbb{R}^d$. To formulate variational principles for such vector fields, we shall consider the class of σ-invariant probability measures on Ω^N, which are those $\pi \in \mathcal{P}(\Omega^N)$ such that for all $h \in L^1(\Omega^N, d\pi)$, we have

$$\int_{\Omega^N} h(x_1, \dots, x_N)\, d\pi = \int_{\Omega^N} h(\sigma(x_1, \dots, x_N))\, d\pi. \tag{11}$$

We set

$$\mathcal{P}_{\text{sym}}(\Omega^N) = \{\pi \in \mathcal{P}(\Omega^N) : \pi\ \sigma\text{-invariant probability on } \Omega^N\}. \tag{12}$$

For a given probability measure μ on Ω, we also consider the class

$$\mathcal{P}^{\mu}_{\text{sym}}(\Omega^N) = \{\pi \in \mathcal{P}_{\text{sym}}(\Omega^N) : \text{proj}_1 \pi = \mu\}, \tag{13}$$

i.e., the set of all $\pi \in \mathcal{P}_{\text{sym}}(\Omega^N)$ with a given first marginal μ, meaning that

$$\int_{\Omega^N} f(x_1)\, d\pi(x_1, \dots, x_N) = \int_{\Omega} f(x_1)\, d\mu(x_1) \quad \text{for every } f \in L^1(\Omega, \mu). \tag{14}$$

Now consider the set $\mathcal{S}(\Omega, \mu)$ of μ-measure-preserving transformations on Ω, which can be identified with a closed subset of the sphere of $L^2(\Omega, \mathbb{R}^d)$. We shall also consider the subset of $\mathcal{S}(\Omega, \mu)$ consisting of N-involutions, that is,

$$\mathcal{S}_N(\Omega, \mu) = \{S \in \mathcal{S}(\Omega, \mu) : S^N = I\ \mu\text{-a.e.}\}.$$

2. Monotone vector fields and N-antisymmetric Hamiltonians

In this section, we establish the following extension of a theorem of Krauss.

Theorem 2. *Let $N \geq 2$ be an integer, and let $u_1, \dots, u_{N-1}$ be bounded vector fields from a convex domain $\Omega \subset \mathbb{R}^d$ into $\mathbb{R}^d$.*

1) *If the $(N-1)$-tuple $(u_1, \dots, u_{N-1})$ is jointly N-monotone, then there exists an N-sub-antisymmetric Hamiltonian H that is zero on the diagonal of Ω^N, concave in the first variable, convex in the other $N-1$ variables, and such that*

$$(u_1(x), \dots, u_{N-1}(x)) = \nabla_{2,\dots,N} H(x, x, \dots, x) \quad \textit{for a.e. } x \in \Omega. \tag{15}$$

Moreover, H is N-antisymmetric in the sense that

$$H(x_1, x_2, \dots, x_N) + H_{2,\dots,N}(x_1, x_2, \dots, x_N) = 0, \tag{16}$$

where $H_{2,\dots,N}$ is the concavification of the function $K(\boldsymbol{x}) = \sum_{i=1}^{N-1} H(\sigma^i(\boldsymbol{x}))$ with respect to the last $N-1$ variables.

Furthermore, there exists a continuous N-antisymmetric Hamiltonian $\overline{H}$ on Ω^N, such that

$$L_{\overline{H}}(x, u_1(x), u_2(x), \dots, u_{N-1}(x)) = \sum_{i=1}^{N-1} \langle u_i(x), x\rangle \quad \textit{for all } x \in \Omega. \tag{17}$$

2) *Conversely, if* $(u_1, \dots, u_{N-1})$ *satisfies* (15) *for some* N*-sub-antisymmetric Hamiltonian* H *that is zero on the diagonal of* Ω^N, *concave in the first variable, and convex in the other variables, then the* $(N-1)$*-tuple* $(u_1, \dots, u_{N-1})$ *is jointly* N*-monotone on* Ω.

Remark 3. In the case $N = 2$, $K(\boldsymbol{x}) = H(x_2, x_1)$ is concave with respect to x_2, hence $H_2(x_1, x_2) = H(x_2, x_1)$, and (16) becomes

$$H(x_1, x_2) + H(x_2, x_1) = 0;$$

thus H is antisymmetric, recovering well-known results [Krauss 1985; Ghoussoub 2009; Ghoussoub and Moameni 2013a; Millien 2011].

Lemma 4. *Assume the* $(N-1)$*-tuple of bounded vector fields* $(u_1, \dots, u_{N-1})$ *on* Ω *is jointly* N*-monotone. Define*

$$f(x_1, \dots, x_N) := \sum_{l=1}^{N-1} \langle u_l(x_1), x_1 - x_{l+1} \rangle$$

and let $\tilde{f}$ *be the convexification of* f *with respect to the first variable, given by*

$$\begin{aligned}(18) \quad & \tilde{f}(x_1, x_2, \dots, x_N) \\ & = \inf\Big\{ \sum_{k=1}^{n} \lambda_k f(x_1^k, x_2, \dots, x_N) : n \in \mathbb{N},\ \lambda_k \geq 0,\ \sum_{k=1}^{n} \lambda_k = 1,\ \sum_{k=1}^{n} \lambda_k x_1^k = x_1 \Big\}.\end{aligned}$$

1) *We have* $f \geq \tilde{f}$ *on* Ω^N.
2) $\tilde{f}$ *is convex in the first variable and concave with respect to the other variables.*
3) $\tilde{f}(x, x, \dots, x) = 0$ *for each* $x \in \Omega$.
4) $\tilde{f}$ *satisfies*

$$\sum_{i=0}^{N-1} \tilde{f}(\sigma^i(x_1, \dots, x_N)) \geq 0 \quad \text{on } \Omega^N. \tag{19}$$

Proof. Since the $(N-1)$-tuple $(u_1, \dots, u_{N-1})$ is jointly N-monotone, it is easy to see that the function

$$f(x_1, \dots, x_N) := \sum_{l=1}^{N-1} \langle u_l(x_1), x_1 - x_{l+1} \rangle$$

is linear in the last $N-1$ variables, that $f(x, x, \dots, x) = 0$, and that

$$\sum_{i=0}^{N-1} f(\sigma^i(x_1, \dots, x_N)) \geq 0 \quad \text{on } \Omega^N. \tag{20}$$

It is also clear that $f \geq \tilde{f}$, that $\tilde{f}$ is convex with respect to the first variable x_1, and that it is concave with respect to the other variables $x_2, \dots, x_N$, since f itself is concave (actually linear) with respect to $x_2, \dots, x_N$. We now show that $\tilde{f}$ satisfies (19).

For that, we fix $x_1, x_2, \dots, x_N$ in Ω and consider $(x_1^k)_{k=1}^n$ in Ω, and $(\lambda_k)_k$ in $\mathbb{R}$ such that $\lambda_k \geq 0$ such that $\sum_{k=1}^n \lambda_k = 1$ and $\sum_{k=1}^n \lambda_k x_1^k = x_1$. For each k, we have

$$f(x_1^k, x_2, \dots, x_N) + f(x_2, \dots, x_N, x_1^k) + \dots + f(x_N, x_1^k, x_2, \dots, x_{N-1}) \geq 0.$$

Multiplying by λ_k, summing over k, and using that f is linear in the last $N-1$ variables, we have

$$\sum_{k=1}^n \lambda_k f(x_1^k, x_2, \dots, x_N) + f(x_2, \dots, x_N, x_1) + \dots + f(x_N, x_1, x_2, \dots, x_{N-1}) \geq 0.$$

By taking the infimum, we obtain

$$\tilde{f}(x_1, x_2, \dots, x_N) + \sum_{i=1}^{N-1} f(\sigma^i(x_1, x_2, \dots, x_N)) \geq 0.$$

Let now $n \in \mathbb{N}$, $\lambda_k \geq 0$, $x_N^k \in \Omega$ be such that $\sum_{k=1}^n \lambda_k = 1$ and $\sum_{k=1}^n \lambda_k x_2^k = x_2$. For every $1 \leq k \leq n$, we have

$$\tilde{f}(x_1, x_2^k, x_3, \dots, x_N) + f(x_2^k, x_3, \dots, x_1) + \dots + f(x_N, x_1, x_2^k, x_3, \dots, x_{N-1}) \geq 0.$$

Multiplying by λ_k, summing over k and using that $\tilde{f}$ is convex in the first variable and f is linear in the last $N-1$ variables, we obtain

$$\begin{aligned}
&\tilde{f}(x_1, x_2, x_3, \dots, x_N) + \sum_{k=1}^n \lambda_k f(x_2^k, x_3, \dots, x_1) + \dots + f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \\
&\quad \geq \sum_{k=1}^n \lambda_k \tilde{f}(x_1, x_2^k, x_3, \dots, x_N) + \sum_{k=1}^n \lambda_k f(x_2^k, x_3, \dots, x_1) \\
&\qquad\qquad + \dots + \sum_{k=1}^n \lambda_k f(x_N, x_1, x_2^k, x_3, \dots, x_{N-1}) \\
&\quad \geq 0.
\end{aligned}$$

By taking the infimum over all possible such choices, we get

$$\tilde{f}(x_1, x_2, x_3, \dots, x_N) + \tilde{f}(x_2, x_3, \dots, x_1) + \dots + f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \geq 0.$$

By repeating this procedure with $x_3, \dots, x_{N-1}$, we get

$$\sum_{i=0}^{N-2} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N)) + f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \geq 0.$$

Finally, since

$$f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \geq -\sum_{i=0}^{N-2} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N))$$

and since $\tilde{f}$ is concave in the last $N-1$ variables, the function

$$x_N \to -\sum_{i=0}^{N-2} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N))$$

for fixed $x_1, x_2, \dots, x_{N-1}$ is a convex minorant of $x_N \to f(x_N, x_1, x_2, \dots, x_{N-1})$. It follows that

$$\begin{aligned} f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) &\geq \tilde{f}(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \\ &\geq -\sum_{i=0}^{N-2} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N)), \end{aligned}$$

which yields $\sum_{i=0}^{N-1} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N)) \geq 0$. This implies that $\tilde{f}(x, x, \dots, x) \geq 0$ for $x \in \Omega$.

On the other hand, since $\tilde{f}(x, x, \dots, x) \leq f(x, x, \dots, x) = 0$, we get that $\tilde{f}(x, x, \dots, x) = 0$ for all $x \in \Omega$. □

Proof of Theorem 2. Assume the $(N-1)$-tuple of vector fields $(u_1, \dots, u_{N-1})$ is jointly N-monotone on Ω, and consider the function

$$f(x_1, \dots, x_N) := \sum_{l=1}^{N-1} \langle u_l(x_1), x_1 - x_{l+1} \rangle$$

as well as its convexification with respect to the first variable $\tilde{f}(x_1, \dots, x_N)$.

By Lemma 4, the function $\psi(x_1, \dots, x_N) := -\tilde{f}(x_1, \dots, x_N)$ satisfies the following properties:

(i) $x_1 \to \psi(x_1, \dots, x_N)$ is concave.

(ii) $(x_2, x_3, \dots, x_N) \to \psi(x_1, \dots, x_N)$ is convex.

(iii) $\psi(x_1, \dots, x_N) \geq -f(x_1, \dots, x_N) = \sum_{l=1}^{N-1} \langle u_l(x_1), x_{l+1} - x_1 \rangle$.

(iv) ψ is N-sub-antisymmetric.

Now consider the family $\overline{\mathcal{H}}$ of functions $H : \Omega^N \to \mathbb{R}$ such that

1) $H(x_1, x_2, \dots, x_N) \geq \sum_{l=1}^{N-1} \langle u_l(x_1), x_{l+1} - x_1 \rangle$ for every N-tuple $(x_1, \dots, x_N)$ in Ω^N,

2) H is concave in the first variable,

3) H is jointly convex in the last $N-1$ variables,

4) H is N-sub-antisymmetric,

5) H is zero on the diagonal of Ω^N.

Note that $\overline{\mathcal{H}} \neq \varnothing$ since ψ belongs to $\overline{\mathcal{H}}$. Note that any H satisfying conditions 1 and 4 automatically satisfies 5. Indeed, by N-sub-antisymmetry, for all $\boldsymbol{x} = (x_1, \dots, x_N) \in \Omega^N$ we have

$$H(\boldsymbol{x}) \leq -\sum_{i=1}^{N-1} H(\sigma^i(\boldsymbol{x})) \leq -\sum_{i=1}^{N-1} \psi(\sigma^i(\boldsymbol{x})). \tag{21}$$

This also yields that

$$\sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_{\ell+1} - x_1 \rangle \leq H(\boldsymbol{x}) \leq -\sum_{i=2}^{N} \sum_{\ell=1}^{N-1} \langle u_\ell(x_i), x_i - x_{i+\ell} \rangle, \tag{22}$$

where we denote $x_{i+N} := x_i$ for $i = 1, \dots, \ell$. This yields that $H(x, x, \dots, x) = 0$ for any $x \in \Omega$.

It is also easy to see that every directed family $(H_i)_i$ in $\overline{\mathcal{H}}$ has a supremum $H_\infty \in \overline{\mathcal{H}}$, meaning that $\overline{\mathcal{H}}$ is a Zorn family, and therefore has a maximal element H.

Now consider the function

$$\overline{H}(\boldsymbol{x}) = \frac{1}{N}\left((N-1)H(\boldsymbol{x}) - \sum_{i=1}^{N-1} H(\sigma^i(\boldsymbol{x}))\right).$$

(i) $\overline{H}$ is N-antisymmetric, since $\overline{H}(\boldsymbol{x}) = \frac{1}{N}\sum_{i=1}^{N-1}[H(\boldsymbol{x}) - H(\sigma^i(\boldsymbol{x}))]$, and each summand is N-antisymmetric.

(ii) $\overline{H} \geq H$ on Ω^N, since $N[\overline{H}(\boldsymbol{x}) - H(\boldsymbol{x})] = -\sum_{i=0}^{N-1} H(\sigma^i(\boldsymbol{x})) \geq 0$ (because H itself is N-sub-antisymmetric).

The maximality of H would have implied that $H = \overline{H}$ is N-antisymmetric if only $\overline{H}$ was jointly convex in the last $N-1$ variables, but since this is not necessarily the case, we consider for $\boldsymbol{x} = (x_1, x_2, \dots, x_N)$ the function

$$K(x_1, x_2, \dots, x_N) = K(\boldsymbol{x}) := -\sum_{i=1}^{N-1} H(\sigma^i(\boldsymbol{x})),$$

which is already concave in the first variable x_1. Its convexification in the last $N-1$ variables, that is,

$$K^{2,\dots,N}(\boldsymbol{x}) = \inf\left\{\sum_{i=1}^{n} \lambda_i K(x_1, x_2^i, \dots, x_N^i) : \lambda_i \geq 0,\ \sum_{i=1}^{n} \lambda_i (x_2^i, \dots, x_N^i, 1) = (x_2, \dots, x_N, 1)\right\},$$

is still concave in the first variable, but is now convex in the last $N-1$ variables. Moreover,

$$H \le K^{2,\dots,N} \le K = -\sum_{i=1}^{N-1} H \circ \sigma^i. \tag{23}$$

Indeed, $K^{2,\dots,N} \le K$ from the definition of $K^{2,\dots,N}$, while $H \le K^{2,\dots,N}$ because $H \le K$ and H is already convex in the last $N-1$ variables. It follows that

$$H \le \frac{(N-1)H + K^{2,\dots,N}}{N} \le \frac{(N-1)H + K}{N} = \frac{1}{N}\left((N-1)H - \sum_{i=1}^{N-1} H \circ \sigma^i\right) = \overline{H}.$$

The function $H' = ((N-1)H + K^{2,\dots,N})/N$ belongs to the family $\overline{\mathcal{H}}$ and therefore $H = H'$ by the maximality of H.

This finally yields that H is N-sub-antisymmetric, that $H(x,\dots,x) = 0$ for all $x \in \Omega$ and that

$$H(\boldsymbol{x}) + H_{2,\dots,N}(\boldsymbol{x}) = 0 \quad \text{for every } \boldsymbol{x} \in \Omega^N,$$

where $H_{2,\dots,N} = -K^{2,\dots,N}$, which for a fixed x_1 is nothing but the concavification of $(x_2,\dots,x_N) \to \sum_{i=1}^{N-1} H(\sigma^i(x_1,x_2,\dots,x_N))$.

Note now that since for any $x_1,\dots,x_N$ in Ω

$$H(x_1,x_2,\dots x_N) \ge \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_{\ell+1} - x_1\rangle, \tag{24}$$

and

$$H(x_1,x_1,\dots,x_1) = 0, \tag{25}$$

we have

$$H(x_1,x_2,\dots,x_N) - H(x_1,\dots,x_1) \ge \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_{\ell+1} - x_1\rangle. \tag{26}$$

Since H is convex in the last $N-1$ variables, this means that for all $x \in \Omega$, we have

$$(u_1(x), u_2(x), \dots, u_{N-1}(x)) \in \partial_{2,\dots,N} H(x,x,\dots,x), \tag{27}$$

as claimed in (15). This also yields

$$L_H(x,u_1(x),\dots,u_{N-1}(x)) + H(x,x,\dots,x) = \sum_{\ell=1}^{N-1} \langle u_\ell(x), x\rangle \quad \text{for all } x \in \Omega.$$

In other words, $L_H(x,u_1(x),\dots,u_{N-1}(x)) = \sum_{\ell=1}^{N-1} \langle u_\ell(x), x\rangle$ for all $x \in \Omega$. As above, consider

$$\overline{H}(\boldsymbol{x}) = \frac{1}{N}\Big((N-1)H(\boldsymbol{x}) - \sum_{i=1}^{N-1} H(\sigma^i(\boldsymbol{x}))\Big).$$

We have $\overline{H} \in \overline{\mathcal{H}}_N(\Omega)$ and $\overline{H} \geq H$, and therefore $L_{\overline{H}} \leq L_H$. On the other hand, for all $x \in \Omega$ we have

$$\begin{aligned} L_{\overline{H}}(x, u_1(x), \dots, u_{N-1}(x)) &= L_{\overline{H}}(x, u_1(x), \dots, u_{N-1}(x)) + \overline{H}(x, x, \dots, x) \\ &\geq \sum_{\ell=1}^{N-1} \langle u_\ell(x), x\rangle. \end{aligned}$$

To prove (17), we use the appendix in [Ghoussoub and Moameni 2013b] to deduce that for $i = 2, \dots, N$, the gradients $\nabla_i H(x, x, \dots, x)$ actually exist for a.e. x in Ω.

The converse is straightforward since if (27) holds, then (26) does, and since we also have (25), then the property that $(u_1, \dots, u_{N-1})$ is jointly N-monotone follows from (24) and the sub-antisymmetry of H. □

In the case of a single N-monotone vector field, we can obviously apply the above theorem to the $(N-1)$-tuple $(u, 0, \dots, 0)$, which is then N-monotone, to find an N-sub-antisymmetric Hamiltonian H, which is concave in the first variable and convex in the last $N-1$ variables such that

(28) $$(-u(x), u(x), 0, \dots, 0) = \nabla H(x, x, \dots, x) \quad \text{for a.e. } x \in \Omega.$$

However, in this case we can restrict ourselves to N-cyclically sub-antisymmetric functions of two variables and establish the following extension of the theorem of Krauss.

Theorem 5. *If u is N-cyclically monotone on Ω, then there exists a concave-convex function of two variables F that is N-cyclically sub-antisymmetric and zero on the diagonal, such that*

(29) $$(-u(x), u(x)) \in \partial F(x, x) \quad \textit{for all } x \in \Omega,$$

where ∂H is the subdifferential of H as a concave-convex function [Rockafellar 1970]. *Moreover,*

(30) $$u(x) = \nabla_2 F(x, x) \quad \textit{for a.e. } x \in \Omega.$$

Proof. Let $f(x, y) = \langle u(x), x - y\rangle$ and let $f^1(x, y)$ be its convexification in x for fixed y, that is,

(31) $$f^1(x, y) = \inf\Big\{\sum_{k=1}^{n} \lambda_k f(x_k, y) : \lambda_k \geq 0, \sum_{k=1}^{n} \lambda_k = 1, \sum_{k=1}^{n} \lambda_k x_k = x\Big\}.$$

Since $f(x, x) = 0$, f is linear in y, and $\sum_{i=1}^{N} f(x_i, x_{i+1}) \geq 0$ for any cyclic family

$x_1, \dots, x_N, x_{N+1} = x_1$ in Ω, it is easy to show that $f \geq f^1$ on Ω, f^1 is convex in the first variable and concave with respect to the second, $f^1(x, x) = 0$ for each $x \in \Omega$, and that f^1 is N-cyclically supersymmetric in the sense that for any cyclic family $x_1, \dots, x_N, x_{N+1} = x_1$ in Ω, we have $\sum_{i=1}^N f^1(x_i, x_{i+1}) \geq 0$.

Now consider $F(x, y) = -f^1(x, y)$ and note that $x \to F(x, y)$ is concave, $y \to F(x, y)$ is convex, $F(x, y) \geq -f(x, y) = \langle u(x), y - x \rangle$ and F is N-cyclically sub-antisymmetric. By the antisymmetry, we have

$$\langle u(x_1), x_2 - x_1 \rangle \leq F(x_1, x_2) \leq \langle u(x_2), x_2 - x_1 \rangle, \tag{32}$$

which yields that $(-u(x), u(x)) \in \partial F(x, x)$ for all $x \in \Omega$.

Since F is antisymmetric and concave-convex, the possibly multivalued map $x \to \partial_2 F(x, x)$ is monotone on Ω, and therefore single-valued and differentiable almost everywhere [Phelps 1993]. This completes the proof. □

Remark 6. We cannot expect to have a function F such that $\sum_{i=1}^N F(x_i, x_{i+1}) = 0$ for all cyclic families $x_1, \dots, x_N, x_{N+1} = x_1$ in Ω. Actually, we believe that the only function satisfying such an N-antisymmetry for $N \geq 3$ must be of the form $F(x, y) = f(x) - f(y)$. This is why one needs to consider functions of N variables in order to get N-antisymmetry. In other words, the function defined by

$$H(x_1, x_2, \dots, x_N) := \frac{1}{N}\left((N-1)F(x_1, x_2) - \sum_{i=2}^{N-1} F(x_i, x_{i+1})\right) \tag{33}$$

is N-antisymmetric in the sense of (6) and $H(x_1, x_2, \dots, x_N) \geq F(x_1, x_2)$ for all $(x_1, x_2, \dots, x_N)$ in Ω^N.

3. Variational characterization of monotone vector fields

In order to simplify the exposition, we shall always assume in the sequel that $d\mu$ is Lebesgue measure dx normalized to be a probability on Ω. We shall also assume that Ω is convex and that its boundary has measure zero.

Theorem 7. *Let $u_1, \dots, u_{N-1} : \Omega \to \mathbb{R}^d$ be bounded measurable vector fields. The following properties are then equivalent*:

1) *The $(N-1)$-tuple $(u_1, \dots, u_{N-1})$ is jointly N-monotone a.e., that is, there exists a measure-zero set Ω_0 such that $(u_1, \dots, u_{N-1})$ is jointly N-monotone on $\Omega \setminus \Omega_0$.*
2) *The infimum of the Monge–Kantorovich problem*

$$\inf\left\{\int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_1 - x_{\ell+1} \rangle \, d\pi(x_1, x_2, \dots, x_N)) : \pi \in \mathcal{P}^\mu_{\mathrm{sym}}(\Omega^N)\right\} \tag{34}$$

is equal to zero, and is therefore attained by the push-forward of μ *by the map* $x \to (x, x, \ldots, x)$.

3) $(u_1, \ldots, u_{N-1})$ *is in the polar of* $\mathcal{S}_N(\Omega, \mu)$ *in the following sense*:

$$\inf\left\{\int_\Omega \sum_{\ell=1}^{N-1} \langle u_\ell(x), x - S^\ell x\rangle \, d\mu : S \in \mathcal{S}_N(\Omega, \mu)\right\} = 0. \tag{35}$$

4) *The following holds*:

$$\inf\left\{\int_\Omega \sum_{\ell=1}^{N-1} |u_\ell(x) - S^\ell x|^2 \, d\mu : S \in \mathcal{S}_N(\Omega, \mu)\right\} = \sum_{\ell=1}^{N-1} \int_\Omega |u_\ell(x) - x|^2 \, d\mu. \tag{36}$$

5) *There exists an* N*-sub-antisymmetric Hamiltonian* H *which is concave in the first variable, convex in the last* $N-1$ *variables, and vanishing on the diagonal such that*

$$(u_1(x), \ldots, u_{N-1}(x)) = \nabla_{2,\ldots,N} H(x, x, \ldots, x) \quad \text{for a.e. } x \in \Omega. \tag{37}$$

Moreover, H *is* N*-symmetric in the sense of* (16).

6) *The following duality holds*:

$$\inf\left\{\int_\Omega L_H(x, u_1(x), \ldots, u_{N-1}(x)) \, d\mu : H \in \mathcal{H}_N(\Omega)\right\} = \sup\left\{\int_\Omega \sum_{\ell=1}^{N-1} \langle u_\ell(x), S^\ell x\rangle \, d\mu : S \in \mathcal{S}_N(\Omega, \mu)\right\}$$

and the latter is attained at the identity map.

We start with the following lemma, which identifies those probabilities in $\mathcal{P}^\mu_{\text{sym}}(\Omega^N)$ that are carried by graphs of functions from Ω to Ω^N.

Lemma 8. *Let* $S : \Omega \to \Omega$ *be a* μ*-measurable map. The following properties are equivalent*:

1) *The image of* μ *by the map* $x \to (x, Sx, \ldots, S^{N-1}x)$ *belongs to* $\mathcal{P}^\mu_{\text{sym}}(\Omega^N)$.
2) S *is* μ*-measure-preserving and* $S^N(x) = x$ μ*-a.e.*
3) *For any bounded Borel measurable* N*-antisymmetric* H *on* Ω^N, *we have* $\int_\Omega H(x, Sx, \ldots, S^{N-1}x) \, d\mu = 0$.

Proof. Clearly 1) implies 3), since $\int_{\Omega^N} H(\boldsymbol{x}) \, d\pi(\boldsymbol{x}) = 0$ for any N-antisymmetric Hamiltonian H and any $\pi \in \mathcal{P}^\mu_{\text{sym}}(\Omega^N)$.

That 2) implies 1) is also straightforward since if π is the push-forward of μ by a map of the form $x \to (x, Sx, \ldots, S^{N-1}x)$, where S is a μ-measure-preserving S

with $S^N x = x$ μ-a.e. on Ω, then for all $h \in L^1(\Omega^N, d\pi)$, we have

$$\begin{aligned}\int_{\Omega^N} h(x_1, \dots, x_N)\, d\pi &= \int_\Omega h(x, Sx, \dots, S^{N-1}x)\, d\mu(x) \\ &= \int_\Omega h(Sx, S^2x, \dots, S^{N-1}x, S^N x)\, d\mu(x) \\ &= \int_\Omega h(Sx, S^2x, \dots, S^{N-1}x, x)\, d\mu(x) \\ &= \int_{\Omega^N} h(\sigma(x_1, \dots, x_N))\, d\pi.\end{aligned}$$

We now prove that 2) and 3) are equivalent. Assuming first that S is μ-measure-preserving such that $S^N = I$ μ-a.e., then for every Borel bounded N-antisymmetric H, we have

$$\begin{aligned}\int_\Omega H(x, Sx, S^2x, \dots, S^{N-1}x)\, d\mu &= \int_\Omega H(Sx, S^2x, \dots, S^{N-1}x, x)\, d\mu \\ &= \cdots = \int_\Omega H(S^{N-1}x, x, Sx, \dots, S^{N-2}x)\, d\mu.\end{aligned}$$

Since H is N-antisymmetric, we can see that

$$\begin{aligned}H(x, Sx, \dots, S^{N-1}x) + H(Sx, S^2x, \dots, S^{N-1}x, x)& \\ + \cdots + H(S^{N-1}x, x, Sx, \dots, S^{N-2}x) &= 0.\end{aligned}$$

It follows that $N \int_\Omega H(x, Sx, S^2x, \dots, S^{N-1}x)\, d\mu = 0$.

For the reverse implication, assume $\int_\Omega H(x, Sx, S^2x, \dots, S^{N-1}x)\, d\mu = 0$ for every N-antisymmetric Hamiltonian H. By testing this identity with the Hamiltonians

$$H(x_1, x_2, \dots, x_N) = f(x_1) - f(x_i),$$

where f is any continuous function on Ω, one gets that S is μ-measure-preserving. Now take the Hamiltonian

$$H(x_1, x_2, \dots, x_N) = |x_1 - Sx_N| - |Sx_1 - x_2| - |x_2 - Sx_1| + |Sx_2 - x_3|.$$

Note that $H \in \mathcal{H}_N(\Omega)$ since it is of the form

$$H(x_1, \dots, x_N) = f(x_1, x_2, x_N) - f(x_2, x_3, x_1).$$

Now test the above identity with such an H to obtain

$$0 = \int_\Omega H(x, Sx, S^2x, \dots, S^{N-1}x)\, d\mu = \int_\Omega |x - SS^{N-1}x|\, d\mu.$$

It follows that $S^N = I$ μ-a.e. on ω, and we are done. $\square$

Proof of Theorem 7. To show that 1) implies 2), it suffices to notice that if π is a σ-invariant probability measure on Ω^N such that $\mathrm{proj}_1\pi = \mu$, then

$$\begin{aligned}\int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_1 - x_{\ell+1}\rangle \, d\pi(x_1, \dots, x_N) \\ &= \frac{1}{N}\sum_{i=1}^{N}\int_{\Omega^N}\sum_{\ell=1}^{N-1}\langle u_\ell(x_i), x_i - x_{i+\ell}\rangle \, d\pi(x_1, \dots, x_N) \\ &= \frac{1}{N}\int_{\Omega^N}\left(\sum_{i=1}^{N}\sum_{\ell=1}^{N-1}\langle u_\ell(x_i), x_i - x_{i+\ell}\rangle\right) d\pi(x_1, \dots, x_N) \\ &\geq 0,\end{aligned}$$

since $(u_1, \dots, u_{N-1})$ is jointly N-monotone. On the other hand, if π is the σ-invariant measure obtained by taking the image of $\mu := dx$ by $x \to (x, \dots, x)$, then

$$\int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_1 - x_{\ell+1}\rangle \, d\pi(x_1, \dots, x_N) = 0.$$

To show that 2) implies 3), let S be a μ-measure-preserving transformation on Ω such that $S^N = I$ μ-a.e. on Ω. Then the image π_S of μ by the map

$$x \to (x, Sx, S^2x, \dots, S^{N-1}x)$$

is σ-invariant, hence

$$\int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_1 - x_{\ell+1}\rangle \, d\pi_S(x_1, \dots, x_N) = \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_\ell(x), x - S^\ell x\rangle \, d\mu \geq 0.$$

By taking $S = I$, we get that the infimum is necessarily zero.

The equivalence of 3) and 4) follows immediately from developing the square.

We now show that 3) implies 1). Take N points $x_1, x_2, \dots, x_N$ in Ω, and let $R > 0$ be such that $B(x_i, R) \subset \Omega$. Consider the transformation

$$S_R(x) = \begin{cases} x - x_1 + x_2 & \text{for } x \in B(x_1, R), \\ x - x_2 + x_3 & \text{for } x \in B(x_2, R), \\ & \vdots \\ x - x_N + x_1 & \text{for } x \in B(x_N, R), \\ x & \text{otherwise.} \end{cases}$$

It is easy to see that S_R is a measure-preserving transformation and that $S_R^N = \mathrm{Id}$.

We then have

$$0 \le \int_\Omega \sum_{\ell=1}^{N-1} \langle u_\ell(x), x - S_R^\ell x\rangle \, d\mu \le \sum_{i=1}^{N} \int_{B(x_i,R)} \sum_{\ell=1}^{N-1} \langle u_\ell(x), x_i - x_{\ell+i}\rangle \, d\mu.$$

Letting $R \to 0$, we get from Lebesgue's density theorem that

$$\frac{1}{|B(x_i, R)|} \int_{B(x_i,R)} \langle u_\ell(x), x_i - x_{\ell+i}\rangle d\mu \to \langle u_\ell(x_i), x_i - x_{\ell+i}\rangle,$$

from which follows that $(u_1, \dots, u_{N-1})$ are jointly N-monotone a.e. on Ω. The fact that 1) is equivalent to 5) follows immediately from Theorem 2.

To prove that 5) implies 6), note that for all $p_i \in \mathbb{R}^d$, $x \in \Omega$, $y_i \in \Omega$, $i = 1, \dots, N-1$,

$$L_H(x, p_1, \dots, p_{N-1}) + H(x, y_1, \dots, y_{N-1}) \ge \sum_{i=1}^{N-1} \langle p_i, y_i\rangle,$$

which yields that for any $S \in \mathcal{S}_N(\Omega, \mu)$,

$$\int_\Omega [L_H(x, u_1(x), \dots, u_{N-1}(x))\, d\mu + H(x, Sx, \dots, S^{N-1}x)]\, d\mu \ge \int_\Omega \sum_{\ell=1}^{N-1} \langle u_\ell(x), S^\ell x\rangle \, d\mu.$$

If $H \in \mathcal{H}_N(\Omega)$ and $S \in \mathcal{S}_N(\Omega, \mu)$, we then have $\int_\Omega H(x, Sx, \dots, S^{N-1}x)\, d\mu = 0$, and therefore

$$\int_\Omega L_H(x, u_1(x), \dots, u_{N-1}(x))\, d\mu \ge \int_\Omega \sum_{\ell=1}^{N-1} \langle u_\ell(x), S^\ell x\rangle \, d\mu.$$

If now H is the N-sub-antisymmetric Hamiltonian obtained by 5), which is concave in the first variable and convex in the last $N-1$ variables, then

$$L_H(x, u_1(x), \dots, u_{N-1}(x)) + H(x, x, \dots, x) = \sum_{\ell=1}^{N-1} \langle u_\ell(x), x\rangle \quad \text{for all } x \in \Omega\setminus\Omega_0,$$

and therefore $\int_\Omega L_H(x, u_1(x), \dots, u_{N-1}(x))\, d\mu = \sum_{\ell=1}^{N-1} \int_\Omega \langle u_\ell(x), x\rangle \, d\mu$.

Now consider

$$\bar{H}(\boldsymbol{x}) = \frac{1}{N}\Big((N-1)H(\boldsymbol{x}) - \sum_{i=1}^{N-1} H(\sigma^i(\boldsymbol{x}))\Big).$$

As before, we have $\overline{H} \in \mathcal{H}_N(\Omega)$ and $\overline{H} \geq H$. Since $L_{\overline{H}} \leq L_H$, we have

$$\int_\Omega L_{\overline{H}}(x, u_1(x), \ldots, u_{N-1}(x))\, d\mu = \sum_{\ell=1}^{N-1} \int_\Omega \langle u_\ell(x), x\rangle\, d\mu$$

and 6) is proved.

Finally, note that 6) readily implies 3), which means that $(u_1, \ldots, u_{N-1})$ is then jointly N-monotone. □

We now consider again the case of a single N-cyclically monotone vector field.

Corollary 9. *Let $u : \Omega \to \mathbb{R}^d$ be a bounded measurable vector field. The following properties are then equivalent*:

1) *The vector field u is N-cyclically monotone a.e.*, *that is*, *there exists a measure-zero set Ω_0 such that u is N-cyclically monotone on $\Omega \setminus \Omega_0$.*
2) *The infimum of the Monge–Kantorovich problem*

$$\inf\left\{\int_{\Omega^N} \langle u(x_1), x_1 - x_2\rangle\, d\pi(\boldsymbol{x}) : \pi \in \mathcal{P}^\mu_{\mathrm{sym}}(\Omega^N)\right\} \tag{38}$$

is equal to zero, *and is therefore attained by the push-forward of μ by the map $x \to (x, x, \ldots, x)$.*

3) *The vector field u is in the polar of $\mathcal{S}_N(\Omega, \mu)$*, *that is*,

$$\inf\left\{\int_\Omega \langle u(x), x - Sx\rangle\, d\mu : S \in \mathcal{S}_N(\Omega, \mu)\right\} = 0. \tag{39}$$

4) *The projection of u on $\mathcal{S}_N(\Omega, \mu)$ is the identity map*, *that is*,

$$\inf\left\{\int_\Omega |u(x) - Sx|^2\, d\mu : S \in \mathcal{S}_N(\Omega, \mu)\right\} = \int_\Omega |u(x) - x|^2\, d\mu. \tag{40}$$

5) *There exists an N-cyclically sub-antisymmetric function H of two variables*, *which is concave in the first variable*, *convex in the second variable*, *vanishing on the diagonal and such that*

$$u(x) = \nabla_2 H(x, x) \quad \textit{for a.e. } x \in \Omega. \tag{41}$$

6) *The following duality holds*:

$$\inf\left\{\int_\Omega L_H(x, u(x), 0, \ldots, 0)\, d\mu : H \in \mathcal{H}_N(\Omega)\right\}$$
$$= \sup\left\{\int_\Omega \langle u(x), Sx\rangle\, d\mu : S \in \mathcal{S}_N(\Omega, \mu)\right\}$$

and the latter is attained at the identity map.

Proof. This is an immediate application of Theorem 7 applied to the $(N-1)$-tuplet vector fields $(u, 0, \dots, 0)$, which is clearly jointly N-monotone on $\Omega \setminus \Omega_0$, whenever u is N-monotone on $\Omega \setminus \Omega_0$. □

Remark 10. The sets of μ-measure-preserving N-involutions $(\mathcal{S}_N(\Omega, \mu))_N$ do not form a nested family, that is, $\mathcal{S}_N(\Omega, \mu)$ is not necessarily included in $\mathcal{S}_M(\Omega, \mu)$, whenever $N \leq M$, unless of course M is a multiple of N. On the other hand, the above theorem shows that their polar sets, i.e.,

$$\mathcal{S}_N(\Omega, \mu)^0 = \left\{ u \in L^2(\Omega, \mathbb{R}^d) : \int_\Omega \langle u(x), x - Sx \rangle \, d\mu \geq 0 \text{ for all } S \in \mathcal{S}_N(\Omega, \mu) \right\},$$

which coincide with the N-cyclically monotone maps, satisfy

$$\mathcal{S}_{N+1}(\Omega, \mu)^0 \subset \mathcal{S}_N(\Omega, \mu)^0,$$

for every $N \geq 1$. This can also be seen directly. Indeed, it is clear that a 2-involution is a 4-involution but not necessarily a 3-involution. On the other hand, assume that u is a 3-cyclically monotone operator. Then for any transformation $S : \Omega \to \Omega$, we have

$$\int_\Omega \langle u(x), x - Sx \rangle \, d\mu + \int_\Omega \langle u(Sx), Sx - S^2 x \rangle \, d\mu + \int_\Omega \langle u(S^2 x), S^2 x - x \rangle \, d\mu \geq 0.$$

Now if S is measure-preserving, we have

$$\int_\Omega \langle u(x), x - Sx \rangle \, d\mu + \int_\Omega \langle u(x), x - Sx \rangle \, d\mu + \int_\Omega \langle u(S^2 x), S^2 x - x \rangle \, d\mu \geq 0,$$

and if $S^2 = I$, then $\int_\Omega \langle u(x), x - Sx \rangle \, d\mu \geq 0$, which means that $u \in \mathcal{S}_2(\Omega, \mu)^0$. Similarly, one can show that any $(N+1)$-cyclically monotone operator belongs to $\mathcal{S}_N(\Omega, \mu)^0$. In other words, $\mathcal{S}_{N+1}(\Omega, \mu)^0 \subset \mathcal{S}_N(\Omega, \mu)^0$ for all $N \geq 2$. Note that $\mathcal{S}_1(\Omega, \mu)^0 = \{I\}^0 = L^2(\Omega, \mathbb{R}^d)$, while

$$\begin{aligned} \mathcal{S}(\Omega, \mu)^0 &= \textstyle\bigcap_N \mathcal{S}_N(\Omega, \mu)^0 \\ &= \{u \in L^2(\Omega, \mathbb{R}^d), u = \nabla\phi \text{ for some convex function } \phi \text{ in } W^{1,2}(\mathbb{R}^d)\}, \end{aligned}$$

in view of classical results of Rockafellar [1970] and Brenier [1991].

Remark 11. In [Ghoussoub and Moameni 2013b], the preceding result is extended to give a similar decomposition for any family of bounded measurable vector fields $u_1, u_2, \dots, u_{N-1}$ on Ω. It is shown there that there exists a measure-preserving N-involution S on Ω and an N-antisymmetric Hamiltonian H on Ω^N such that for $i = 1, \dots, N-1$, we have

$$u_i(x) = \nabla_{i+1} H(x, Sx, S^2 x, \dots, S^{N-1} x) \quad \text{for a.e. } x \in \Omega.$$

Acknowledgement

We are grateful to the anonymous referee for a careful reading of this paper, which led to several improvements.

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Received January 7, 2013. Revised October 2, 2013.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 269 No. 2 June 2014