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We deal with two-sided complete hypersurfaces immersed in a Riemannian product space, whose base is assumed to have sectional curvature bounded from below. In this setting, we obtain sufficient conditions which assure that such a hypersurface is a slice of the ambient space, provided that its angle function has some suitable behavior. Furthermore, we establish a natural relation between our results and the classical problem of describing the geometry of a hypersurface immersed in the Euclidean space through the behavior of its Gauss map.

1. Introduction and statements of the main results

Let $\psi : \Sigma^n \to \mathbb{M}^{n+1}$ be an immersion of an orientable Riemannian manifold Σ^n in a Riemannian space form $\mathbb{M}^{n+1}$ and let N be the unit normal vector field along Σ^n. When $\mathbb{M}^{n+1}$ is the Euclidean space $\mathbb{R}^{n+1}$ and ψ is the complete graph of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, the image $N(\Sigma)$ of its Gauss map is contained in an open hemisphere of the unit Euclidean sphere $\mathbb{S}^n$. The behavior of the Gauss map has deeper consequences for the immersion. For instance, one of the most celebrated theorems of the theory of minimal surfaces in $\mathbb{R}^3$ is Bernstein's theorem [1910], which establishes that the only complete minimal graphs in $\mathbb{R}^3$ are planes. This result was extended under the weaker hypothesis that the image of the Gauss map of Σ^2 lies in an open hemisphere of $\mathbb{S}^2$, as we can see in [Barbosa and do Carmo 1974].

Meanwhile, Osserman [1959] answered a conjecture due to Nirenberg, showing that if a complete minimal surface Σ^2 in $\mathbb{R}^3$ is not a plane, then its normals must be everywhere dense on the unit sphere $\mathbb{S}^2$. More generally, Fujimoto [1988] proved that if the Gaussian image misses more than four points, then it is a plane. On the other hand, Hoffman, Osserman and Schoen [Hoffman et al. 1982] showed that if a complete oriented surface Σ^2 with constant mean curvature in $\mathbb{R}^3$ is such that the image of its Gauss map $N(\Sigma)$ lies in some open hemisphere of $\mathbb{S}^2$, then Σ^2 is a

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plane. Moreover, if $N(\Sigma)$ lies in a closed hemisphere, then Σ^2 is a plane or a right circular cylinder.

When the ambient space is a Riemannian product $\overline{M}^{n+1} = \mathbb{R} \times M^n$, the condition that the image of the Gauss map is contained in a closed hemisphere becomes that the angle function $\eta = \langle N, \partial_t \rangle$ does not change sign, as was observed in [Espinar and Rosenberg 2009]. Here, N denotes a unit normal vector field along a hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ and ∂_t stands for the unit vector field which determines on $\overline{M}^{n+1}$ a codimension-one foliation by totally geodesic slices $\{t\} \times M^n$. In this setting, our purpose in this work is to establish analogous results to those ones above described. In other words, we aim to give new satisfactory answers to the following question: *under what reasonable geometric restrictions on the angle function must a complete hypersurface immersed in a certain product space be a slice*?

We can truly say that one of the first remarkable results in this direction was the celebrated theorem of Bombieri, De Giorgi and Miranda [Bombieri et al. 1969], who proved that an entire minimal positive graph over $\mathbb{R}^n$ is a totally geodesic slice. Many other authors have approached problems in this branch. For instance, Rosenberg [2002] showed that when M^2 is a complete surface with nonnegative Gaussian curvature, an entire minimal graph in $\mathbb{R} \times M^2$ is totally geodesic. Hence, in this case, the graph is a horizontal slice or M^2 is a flat $\mathbb{R}^2$ and the graph is a tilted plane. Bérard and Sá Earp [2008] described all rotation hypersurfaces with constant mean curvature in $\mathbb{R} \times \mathbb{H}^n$, and used them as barriers to prove existence and characterization of certain vertical graphs with constant mean curvature and to give symmetry and uniqueness results for constant mean curvature compact hypersurfaces whose boundary is one or two parallel submanifolds in slices. Espinar and Rosenberg [2009] studied constant mean curvature surfaces in $\mathbb{R} \times M^2$, and classified them according to the infimum of the Gaussian curvature of their horizontal projection, under the assumption that the angle function does not change sign.

In [Aquino and Lima 2011] and [Lima and Parente 2012], we applied the well-known generalized maximum principle of Omori [1967] and Yau [1975], and an extension of it due to Akutagawa [1987], in order to obtain rigidity theorems concerning complete vertical graphs with constant mean curvature in $\mathbb{R} \times \mathbb{M}^n$. In [Lima 2014], the first author extended the technique developed in [Yau 1976] in order to investigate the rigidity of entire vertical graphs in a Riemannian product space $\mathbb{R} \times M^n$, whose base M^n is assumed to have Ricci curvature with strict sign. Under a suitable restriction on the norm of the gradient of the function u which determines such a graph $\Sigma^n(u)$, he proved that $\Sigma^n(u)$ must be a slice $\{t\} \times M^n$.

Now, motivated by the previous discussion, we will state our results. In what follows, $H_2 = 2/(n(n-1))S_2$ stands for the mean value of the second elementary symmetric function S_2 on the eigenvalues of the Weingarten operator A of the hypersurface Σ^n. Moreover, we recall that a hypersurface is said to be *two-sided* if its normal bundle is trivial, that is, if there is a globally defined unit normal vector field on it.

Theorem 1. *Let* $\overline{M}^{n+1} = \mathbb{R} \times M^n$ *be a Riemannian product space whose base* M^n *has sectional curvature* K_M *such that* $K_M \geq -\kappa$ *for some* $\kappa > 0$, *and let* $\psi : \Sigma^n \to \overline{M}^{n+1}$ *be a two-sided complete hypersurface with constant mean curvature* H *and* H_2 *bounded from below. Suppose that the angle function* η *of* Σ^n *is bounded away from zero and that its height function* h *satisfies one of the following conditions*:

$$|\nabla h|^2 \leq \frac{\alpha}{(n-1)\kappa}|A|^2 \tag{1-1}$$

for some constant $0 < \alpha < 1$; *or*

$$|\nabla h|^2 \leq \frac{n}{(n-1)\kappa}H^2. \tag{1-2}$$

Then, Σ^n *is a slice of* $\overline{M}^{n+1}$.

As a consequence of Example 10 given in Section 4, we cannot extend estimate (1-1) to the limit case $\alpha = 1$. On the other hand, taking into account estimate (1-2), when $M^n = \mathbb{R}^n$, we note that Theorem 1 reads as follows:

Corollary 2. *Let* Σ^n *be a two-sided complete hypersurface of* $\mathbb{R}^{n+1}$ *with constant mean curvature and scalar curvature bounded from below. If the closure of the image of the Gauss map of* Σ^n *is contained in an open hemisphere of* $\mathbb{S}^n$, *then* Σ^n *is minimal.*

Proceeding, we treat the case where the mean curvature H is not assumed to be constant, but is just assumed to not change sign along the hypersurface:

Theorem 3. *Let* $\overline{M}^{n+1} = \mathbb{R} \times M^n$ *be a Riemannian product space whose base* M^n *has sectional curvature bounded from below*, *and let* $\psi : \Sigma^n \to \overline{M}^{n+1}$ *be a two-sided complete hypersurface that lies between two slices of* $\overline{M}^{n+1}$. *Suppose the angle function* η *of* Σ^n *is bounded away from* 1 *or from* -1. *If* H_2 *is bounded from below and* H *is bounded and does not change sign on* Σ^n, *then* $\inf_\Sigma H = 0$. *In particular*, *if* H *is constant*, *then* Σ^n *is minimal.*

Thanks to the result of Osserman already cited, Theorem 3 yields:

Corollary 4. *The only two-sided complete constant mean curvature surfaces of* $\mathbb{R}^3$ *with Gaussian curvature bounded from below*, *lying between two planes and such that both poles of* $\mathbb{S}^2$ *are not in the closure of the image of the Gauss map*, *that are orthogonal to such planes*, *are planes of* $\mathbb{R}^3$.

On the other hand, Example 10 will show that the assumption that Σ^n lies between two slices of $\mathbb{R} \times M^n$ is necessary in Theorem 3 in order to conclude that the mean curvature of Σ^n cannot be globally bounded away from zero. Moreover, we observe that the horizontal circular cylinder $\mathscr{C} \subset \mathbb{R}^3$ satisfies almost all the hypothesis of Corollary 4, except the one which requires that neither pole of $\mathbb{S}^2$ orthogonal to $\mathscr{C}$ is in the closure of the image of the Gauss map N of $\mathscr{C}$. Actually, $\mathscr{C}$ is unbounded in all directions where N is isolated.

Rosenberg, Schulze and Spruck [Rosenberg et al. 2013] showed that an entire minimal graph with nonnegative height function in a product space $\mathbb{R} \times M^n$, whose base M^n is a complete Riemannian manifold having nonnegative Ricci curvature and with sectional curvature bounded from below, must be a slice. Consequently, Theorem 3 yields:

Corollary 5. *Let M^n be a complete Riemannian manifold with nonnegative Ricci curvature and whose sectional curvature is bounded from below. Let $\Sigma^n(u) = \{(u(x), x) : x \in M^n\} \subset \mathbb{R} \times M^n$ be the entire graph of a nonnegative smooth function $u : M^n \to \mathbb{R}$, with H constant and H_2 bounded from below. If u is bounded, then $u \equiv t_0$ for some $t_0 \in \mathbb{R}$.*

Again from Theorem 3, this time combined with Theorem 1.2 of [Rosenberg et al. 2013], we obtain:

Corollary 6. *Let M^n be a parabolic complete Riemannian manifold with bounded sectional curvature. Let $\Sigma^n(u) = \{(u(x), x) : x \in M^n\} \subset \mathbb{R} \times M^n$ be the entire graph of a smooth function $u : M^n \to \mathbb{R}$, with H constant and H_2 bounded from below. If u is bounded, then $u \equiv t_0$ for some $t_0 \in \mathbb{R}$.*

In the situation of Theorem 3, we saw that a constant mean curvature hypersurface satisfying the hypotheses of the theorem must be minimal. Theorem 1 suggests an interesting, related question: If a constant mean curvature hypersurface trapped between two planes is a graph, and the closure of the image of the Gauss map does not contain either pole, must the hypersurface be trivial? Osserman's theorem asserts that the hypersurface is indeed a plane when the ambient space is $\mathbb{R}^3$. When the ambient space is a product whose base has nonnegative Ricci curvature and sectional curvature bounded from below, Corollary 5 also gives a positive answer for this question, provided that the hypersurface is already a graph of a bounded and nonnegative function, while Corollary 6 deals with the parabolic case using the conformal invariance of parabolicity.

The proofs of Theorems 1 and 3 are given in Section 3.

2. Preliminaries

We consider an $(n+1)$-dimensional product space $\overline{M}^{n+1}$ of the form $\mathbb{R} \times M^n$, where M^n is an n-dimensional connected Riemannian manifold and $\overline{M}^{n+1}$ is endowed with the standard product metric

$$\langle\ ,\ \rangle = \pi_{\mathbb{R}}^*(dt^2) + \pi_M^*(\langle\ ,\ \rangle_M),$$

where $\pi_{\mathbb{R}}$ and π_M denote the canonical projections from $\mathbb{R} \times M^n$ onto each factor and $\langle\ ,\ \rangle_M$ is the Riemannian metric on M^n. For simplicity, we will just write $\overline{M}^{n+1} = \mathbb{R} \times M^n$ and $\langle\ ,\ \rangle = dt^2 + \langle\ ,\ \rangle_M$. For a fixed $t_0 \in \mathbb{R}$, we say that $M^n_{t_0} = \{t_0\} \times M^n$ is a *slice* of $\overline{M}^{n+1}$. It is not difficult to prove that such a slice of $\overline{M}^{n+1}$ is a totally geodesic hypersurface (see, for instance, [O'Neill 1983]).

Throughout this paper, we will deal with two-sided complete hypersurfaces $\psi : \Sigma^n \to \mathbb{R} \times M^n$. Let $\overline{\nabla}$ and ∇ denote the Levi-Civita connections in $\mathbb{R} \times M^n$ and Σ^n, respectively. The Gauss and Weingarten formulas for ψ are respectively

$$\overline{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle N \tag{2-1}$$

and

$$AX = -\overline{\nabla}_X N \tag{2-2}$$

where $X, Y \in \mathfrak{X}(\Sigma)$ are tangent vector fields, and $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is the Weingarten operator of Σ^n with respect to its orientation (unit normal vector field) N.

We will consider two particular functions naturally attached to such a hypersurface Σ^n: the (vertical) height function $h = (\pi_{\mathbb{R}})|_{\Sigma}$ and the angle function $\eta = \langle N, \partial_t\rangle$. Since Σ^n is assumed to be two-sided its angle function η is globally defined.

A simple computation shows that the gradient of $\pi_{\mathbb{R}}$ on $\mathbb{R} \times M^n$ is given by

$$\overline{\nabla}\pi_{\mathbb{R}} = \langle \overline{\nabla}\pi_{\mathbb{R}}, \partial_t\rangle \partial_t = \partial_t. \tag{2-3}$$

Consequently, from (2-3) we have that the gradient of h on Σ^n is

$$\nabla h = (\overline{\nabla}\pi_{\mathbb{R}})^{\top} = \partial_t^{\top} = \partial_t - \eta N, \tag{2-4}$$

where $(\ )^{\top}$ denotes the tangential component of a vector field in $\mathfrak{X}(\overline{M}^{n+1})$ along Σ^n. Hence, from (2-4) we get the relation

$$|\nabla h|^2 = 1 - \eta^2, \tag{2-5}$$

where $|\ |$ denotes the norm of a vector field on Σ^n. From Proposition 7.35 of [O'Neill 1983] we have

$$\overline{\nabla}_X \partial_t = 0 \tag{2-6}$$

for every $X \in \mathfrak{X}(\Sigma)$. Thus, from (2-4) and (2-6) we get

$$\nabla_X(\nabla h) = \nabla_X(\partial_t^{\top}) = \eta AX \tag{2-7}$$

for every tangent vector field $X \in \mathfrak{X}(\Sigma)$. Therefore, the Laplacian on Σ^n of the height function is given by

$$\Delta h = nH\eta, \tag{2-8}$$

where $H = (1/n)\,\mathrm{tr}(A)$ is the mean curvature of Σ^n relative to N. Moreover, as a particular case of Proposition 3.1 of [Caminha and Lima 2009], we obtain a useful formula for the Laplacian on Σ^n of the angle function η:

Lemma 7. *Let $\psi : \Sigma^n \to \mathbb{R} \times M^n$ be a hypersurface with orientation N, and let $\eta = \langle N, \partial_t \rangle$ be its angle function. If Σ^n has constant mean curvature H, then*

$$\Delta\eta = -(\mathrm{Ric}_M(N^*, N^*) + |A|^2)\eta,$$

where Ric_M *denotes the Ricci curvature of the base M^n, N^* is the projection of the unit normal vector field N onto the base M^n and $|A|$ is the Hilbert–Schmidt norm of the shape operator A.*

On the other hand, as in [O'Neill 1983], the curvature tensor R of a hypersurface $\psi : \Sigma^n \to \mathbb{R} \times M^n$ is given by

$$R(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where [,] denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(\Sigma)$. A well known fact is that, using (2-1) and (2-2), we can describe the curvature tensor R of the hypersurface Σ^n in terms of the shape operator A and the curvature tensor $\bar{R}$ of $\mathbb{R} \times M^n$ by the so-called Gauss equation given by

$$R(X, Y)Z = (\bar{R}(X, Y)Z)^{\top} + \langle AX, Z\rangle AY - \langle AY, Z\rangle AX \tag{2-9}$$

for tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma)$.

To close this section, we recall the generalized maximum principle of Omori [1967] and Yau [1975], which will be the main analytical tool used in proving to prove our Bernstein-type results:

Lemma 8. *Let Σ^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below, and $f : \Sigma^n \to \mathbb{R}$ a smooth function which is bounded from below on Σ^n. Then there is a sequence of points (p_k) in Σ^n such that*

$$\lim_{k\to\infty} f(p_k) = \inf_{\Sigma} f, \quad \lim_{k\to\infty} |\nabla f(p_k)| = 0 \quad \textit{and} \quad \liminf_{k\to\infty} \Delta f(p_k) \geq 0.$$

3. Proofs of Theorems 1 and 3

Proof of Theorem 1. Since we are assuming that η is bounded away from zero, we can suppose that $\eta > 0$ and, consequently, $\inf \eta > 0$. From Lemma 7, we have

$$\Delta\eta = -(\mathrm{Ric}_M(N^*, N^*) + |A|^2)\eta. \tag{3-1}$$

Since we are also assuming that the sectional curvature K_M of the base M^n is such that $K_M \geq -\kappa$ for some $\kappa > 0$, with a straightforward computation we get

$$\mathrm{Ric}_M(N^*, N^*) \geq -(n-1)\kappa|N^*|^2 = -(n-1)\kappa(1-\eta^2),$$

where N^* stands for the component of N tangent to M^n. Then, from (2-5) and (3-1) we obtain

$$\Delta\eta \leq -(|A|^2 - (n-1)\kappa|\nabla h|^2)\eta. \tag{3-2}$$

Thus, if we assume that the height function of Σ^n satisfies the hypothesis (1-1), from (1-1) and (3-2) we have

$$\Delta\eta \leq -(1-\alpha)|A|^2\eta. \tag{3-3}$$

On the other hand, we claim that the Ricci curvature of Σ^n is bounded from below. Therefore we can apply Lemma 8 to the function η, obtaining a sequence of points $p_k \in \Sigma^n$ such that $\liminf_{k\to\infty} \Delta\eta(p_k) \geq 0$ and $\lim_{k\to\infty} \eta(p_k) = \inf_{p\in\Sigma} \eta(p)$. Consequently, since we are assuming that the Weingarten operator A is bounded on Σ^n, from (3-3), up to a subsequence, we get

$$0 \leq \liminf_{k\to\infty} \Delta\eta(p_k) \leq -(1-\alpha) \lim_{k\to\infty} |A|^2(p_k) \inf_{p\in\Sigma} \eta(p) \leq 0.$$

Thus, we obtain that $\lim_{k\to\infty} |A|(p_k) = 0$ and, from (1-1), $\lim_{k\to\infty} |\nabla h|(p_k) = 0$. Hence, from (2-5) we conclude that $\inf_{p\in\Sigma} \eta(p) = 1$ and, consequently, $\eta \equiv 1$. Therefore, Σ is a slice.

It just remains to prove our claim that the Ricci curvature of Σ^n is bounded from below. For this, let us consider $X \in \mathfrak{X}(\Sigma)$ and a local orthonormal frame $\{E_1, \dots, E_n\}$ of $\mathfrak{X}(\Sigma)$. Then, it follows from the Gauss equation (2-9) that

$$\mathrm{Ric}_\Sigma(X, X) = \sum_i \langle \bar{R}(X, E_i)X, E_i\rangle + nH\langle AX, X\rangle - \langle AX, AX\rangle. \tag{3-4}$$

Thus, taking into account once more the lower bound of the sectional curvature of the base M^n, we have

$$\langle \bar{R}(X, E_i)X, E_i\rangle \geq -\kappa\big(\langle X^*, X^*\rangle_{M^n}\langle E_i^*, E_i^*\rangle_{M^n} - \langle X^*, E_i^*\rangle_{M^n}^2\big), \tag{3-5}$$

where $X^* = X - \langle X, \partial_t\rangle\partial_t$ and $E_i^* = E_i - \langle E_i, \partial_t\rangle\partial_t$ are the projections of the tangent vector fields X and E_i onto M^n, respectively. Then, adding up the relation (3-5) we get

$$\begin{aligned}\sum_i \langle \bar{R}(X, E_i)X, E_i\rangle &\geq -\kappa\big((n-1)|X|^2 - |\nabla h|^2|X|^2 - (n-2)\langle X, \nabla h\rangle^2\big)\\ &\geq -\kappa(n-1)|X|^2.\end{aligned}$$

Therefore, from (3-4), and using the Cauchy–Schwarz inequality, we have that the Ricci curvature of Σ^n satisfies the lower estimate

$$\mathrm{Ric}_\Sigma(X, X) \geq -\big((n-1)\kappa - |A||A - nHI|\big)|X|^2 \tag{3-6}$$

for all $X \in \mathfrak{X}(\Sigma)$. Therefore, taking into account that

$$|A|^2 = n^2 H^2 - n(n-1)H_2, \tag{3-7}$$

our restrictions on H and H_2 guarantee that the Ricci curvature tensor of Σ^n is bounded from below and, hence, we conclude the first part of the proof of Theorem 1.

Now, let us suppose that the height function of Σ^n satisfies the hypothesis (1-2). In this case, from (3-2) and (3-7) we obtain

$$\Delta\eta \leq -n(n-1)(H^2 - H_2)\eta. \tag{3-8}$$

Consequently, in a similar way as in the previous case, we can apply Lemma 8 in order to obtain a sequence of points $p_k \in \Sigma^n$ such that

$$0 \leq \liminf_{k\to\infty} \Delta\eta(p_k) \leq -n(n-1) \liminf_{k\to\infty}(H^2 - H_2)(p_k) \inf_{p\in\Sigma} \eta(p) \leq 0.$$

Hence, up to a subsequence, $\lim_{k\to\infty}(H^2 - H_2)(p_k) = 0$. Moreover, since H is assumed to be constant, we get from (3-7) that

$$\lim_{k\to\infty} |A|^2(p_k) = nH^2.$$

Now we recall that $|A|^2 = \sum_i \kappa_i^2$, where the κ_i are the eigenvalues of A. Thus, up to taking a subsequence, for all $1 \leq i \leq n$ we have that $\lim_k \kappa_i(p_k) = \kappa_i^*$ for some $\kappa_i^* \in \mathbb{R}$. Motivated by this fact, we set

$$\frac{n(n-1)}{2}\overline{H}_2 = \sum_{i<j} \kappa_i^* \kappa_j^*,$$

and we note that $H = \frac{1}{n}\sum_i \kappa_i^*$. Thus $H^2 = \overline{H}_2$ and $\kappa_i^* = H$ for all $1 \leq i \leq n$. So, let $\{e_i\}$ be a local orthonormal frame of eigenvectors associated to the eigenvalues $\{\kappa_i\}$ of A. We can write $\nabla h = \sum_i \lambda_i e_i$, where the λ_i are continuous functions on Σ^n.

On the other hand, from (2-4) and (2-6) we have

$$X(\eta) = -\langle A(X), \partial_t\rangle = -\langle X, A(\partial_t^\top)\rangle = -\langle X, A(\nabla h)\rangle$$

for all $X \in \mathfrak{X}(\Sigma)$. Thus,

$$\nabla\eta = -A(\nabla h). \tag{3-9}$$

By applying Lemma 8 once more to the function η, from (3-9) we then get

$$\begin{aligned} 0 = \lim_k |A(\nabla h)|^2(p_k) &= \sum_i \lim_k(\kappa_i^2\lambda_i^2)(p_k) \\ &= \sum_i (\kappa_i^*)^2 \lim_k \lambda_i^2(p_k) = H^2 \sum_i \lim_k \lambda_i^2(p_k), \end{aligned}$$

up to taking a subsequence. If $H = 0$, from hypothesis (1-1) we have immediately that Σ^n is a slice. If $H^2 > 0$, then for all $1 \le i \le n$ we have $\lim_k \lambda_i(p_k) = 0$. Thus, $\lim_k |\nabla h|(p_k) = 0$ and, from (2-5),

$$\inf_{p\in\Sigma} \eta(p) = \lim_{k\to\infty} \eta(p_k) = 1.$$

Therefore, $\eta = 1$ on Σ^n, and hence Σ^n is a slice. □

Proof of Theorem 3. Note that, as in the proof of Theorem 1, our restrictions on the sectional curvature of the base M^n and the hypothesis on the mean curvatures H and H_2 guarantee that the Ricci curvature of Σ^n is bounded from below.

Now, suppose for instance that $H \ge 0$ on Σ^n. Thus, since Σ^n lies between two slices of $\mathbb{R} \times M^n$, from (2-8) and Lemma 8 we obtain a sequence of points $p_k \in \Sigma^n$ such that

$$0 \ge \limsup_{k\to\infty} \Delta h(p_k) = n \limsup_{k\to\infty} (H\eta)(p_k).$$

From (2-5) we also have

$$0 = \lim_{k\to\infty} |\nabla h|(p_k) = 1 - \lim_{k\to\infty} \eta^2(p_k).$$

Thus, if we suppose, for instance, that -1 is not in the closure of the image of η, we get $\lim\limits_{k\to\infty} \eta(p_k) = 1$. Consequently,

$$0 \ge \limsup_{k\to\infty} \Delta h(p_k) = n \limsup_{k\to\infty} H(p_k) \ge 0,$$

and, hence, we conclude that

$$\limsup_{k\to\infty} H(p_k) = 0.$$

If $H \le 0$, from (2-8) and (2-5) we can once more apply Lemma 8 in order to obtain a sequence $q_k \in \Sigma^n$ such that $0 \le \liminf_{k\to\infty} \Delta h(q_k) = n \liminf_{k\to\infty} (H\eta)(q_k)$, and, supposing once more that -1 is not in the closure of the image of η, we get

$$0 \le \liminf_{k\to\infty} \Delta h(p_k) = n \liminf_{k\to\infty} H(p_k) \le 0.$$

Consequently, we have

$$\liminf_{k\to\infty} H(p_k) = 0.$$

Therefore, in this case we also conclude that $\inf_\Sigma H = 0$. □

4. Entire vertical graphs in $\mathbb{R} \times M^n$

We recall that a *vertical graph* over a connected domain Ω of a complete Riemannian manifold M^n is determined by a smooth function $u \in C^\infty(\Omega)$, and is given by

$$\Sigma^n(u) = \{(u(x), x) : x \in \Omega\} \subset \mathbb{R} \times M^n.$$

From the product metric on the ambient space, $\Sigma^n(u)$ induces on Ω the metric

$$\langle\ ,\ \rangle = du^2 + \langle\ ,\ \rangle_{M^n}. \tag{4-1}$$

A vertical graph $\Sigma^n(u)$ is said to be *entire* if $\Omega = M^n$. Now, when the base M^n is complete, any entire vertical graph $\Sigma^n(u)$ in the product space $\mathbb{R} \times M^n$ is complete, because such a graph is properly immersed in $\mathbb{R} \times M^n$, which is obviously complete if M^n is. (Alternatively one can argue as follows: the Cauchy–Schwarz inequality and (4-1) give

$$\langle X, X\rangle = \langle X, X\rangle_{M^n} + \langle Du, X\rangle^2_{M^n} \geq (1 + |Du|^2)\langle X, X\rangle_{M^n}$$

for every tangent vector field X on Σ^n. Hence, $\langle X, X\rangle \geq \langle X, X\rangle_{M^n}$. This implies that $L \geq L_{M^n}$, where L and L_{M^n} denote the length of a curve on $\Sigma^n(u)$ with respect to the Riemannian metrics $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_{M^n}$; the completeness of $\Sigma^n(u)$ follows.)

Let $\Sigma^n(u) = \{(u(x), x) : x \in M^n\} \subset \mathbb{R} \times M^n$ be an entire vertical graph. The function $g : \mathbb{R} \times M^n \to \mathbb{R}$ given by $g(t, x) = t - u(x)$ is such that $\Sigma^n(u) = g^{-1}(0)$. Moreover, for all tangent vector fields X on $\mathbb{R} \times M^n$,

$$X(g) = \langle X, \partial_t\rangle\partial_t(g) + X^*(g) = \langle \partial_t - Du, X\rangle,$$

where $X^* = X - \langle X, \partial_t\rangle\partial_t$ is the projection of X onto the base M^n and Du is the gradient of u in M^n. Thus,

$$\overline{\nabla} g(u(x), x) = \partial_t|_{(u(x),x)} - Du(x) \quad \text{for all } x \in M^n.$$

Hence, the unit vector field

$$N(x) = \frac{1}{\sqrt{1 + |Du|^2}}(\partial_t|_{(u(x),x)} - Du(x)), \quad x \in M^n \tag{4-2}$$

gives an orientation for $\Sigma^n(u)$ such that $\eta > 0$ on it. Consequently, taking into account (2-5), from (4-2) we get

$$|\nabla h|^2 = \frac{|Du|^2}{1 + |Du|^2}. \tag{4-3}$$

Let us study the shape operator A of $\Sigma^n(u)$ with respect to the orientation given by (4-2). For any $X \in \mathfrak{X}(\Sigma(u))$, since $X = \langle Du, X\rangle_{M^n}\partial_t + X^*$, we have

$$AX = -\overline{\nabla}_X N = -\langle Du, X\rangle\overline{\nabla}_{\partial_t} N - \overline{\nabla}_X N. \tag{4-4}$$

Consequently, from (4-2) and (4-4), and with the aid of Proposition 7.35 of [O'Neill 1983], we verify that

$$AX = \frac{1}{\sqrt{1 + |Du|^2}} D_X Du + \frac{\langle D_X Du, Du\rangle}{(1 + |Du|^2)^{3/2}} Du, \tag{4-5}$$

where D denotes the Levi-Civita connection in M^n with respect to its metric $\langle\ ,\ \rangle_{M^n}$. Consequently, the mean curvature of $\Sigma^n(u)$ is given by

$$nH = \operatorname{Div} \frac{Du}{\sqrt{1+|Du|^2}}, \tag{4-6}$$

where Div stands for the divergence on the base M^n.

Remark 9. Salavessa [1989] showed that when the base M^n is complete noncompact, an entire graph $\Sigma^n(u)$ in $\mathbb{R}\times M^n$ with constant mean curvature H is minimal provided that the Cheeger constant $\mathfrak{h}(M)$ of the base M^n vanishes. We recall that

$$\mathfrak{h}(M) = \inf_D \frac{A(\partial D)}{V(D)},$$

where D ranges over all open submanifolds of M^n with compact closure in M^n and smooth boundary, and where $V(D)$, $A(\partial D)$ are the volume of D and the area of ∂D, respectively, relative to the metric of M^n.

Returning to the context of Theorem 1, we observe the condition that the angle function η of the hypersurface Σ^n is bounded away from zero assures that Σ^n is, in fact, an entire vertical graph $\Sigma^n(u)$ for some smooth function $u : M^n \to \mathbb{R}$. Consequently, considering the case that there exists a hypersurface with positive constant mean curvature H, and supposing that (1-2) holds, from (4-3) and (4-6) we see that Salavessa's argument allows us to get

$$\begin{aligned} nHV(D) &\le \int_D nH\,dV = \int_D \operatorname{Div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\,dV \\ &= \oint_{\partial D} \left\langle \frac{Du}{\sqrt{1+|Du|^2}}, \nu \right\rangle dA \le \sqrt{\frac{n}{(n-1)\kappa}}\,HA(\partial D), \end{aligned}$$

where ν is the outward unit normal of ∂D. This yields the following lower estimate for the Cheeger constant of the base M^n:

$$\sqrt{n(n-1)\kappa} \le \mathfrak{h}(M).$$

Furthermore, recalling the stability operator $\mathscr{L} = -\Delta - \operatorname{Ric}(N, N) - |A|^2$, a constant mean curvature hypersurface Σ^n is said to be stable if

$$\int_\Sigma (\mathscr{L}f)\,f \ge 0 \quad \text{for all } f \in C_0^2(\Sigma). \tag{4-7}$$

We also note that under the stated hypothesis of Theorem 1, the hypersurface is a slice and therefore $\operatorname{Ric}(\partial_t, \partial_t) = 0$ and $|A|^2 \equiv 0$. Hence, in this case from (4-7) we see that the minimal hypersurface is stable.

We close our paper by presenting a suitable example of a nontrivial complete vertical graph $\Sigma^2(u)$ with constant mean curvature in the product space $\mathbb{R}\times\mathbb{H}^2$, which is directly related to the hypothesis of Theorems 1 and 3 (see the comments in Section 1).

Example 10. We consider the upper half-plane model for the two-dimensional hyperbolic space $\mathbb{H}^2$; that is, $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, endowed with the complete metric $\langle\ ,\ \rangle_{\mathbb{H}^2} = (1/y^2)(dx^2 + dy^2)$.

In this setting, let us define the smooth function $u : \mathbb{H}^2 \to \mathbb{R}$ by $u(x, y) = a \ln y$, $a \in \mathbb{R}$, and consider the entire vertical graph

$$\Sigma^2(u) = \{(a \ln y, x, y) : y > 0\} \subset \mathbb{R} \times \mathbb{H}^2.$$

We have $Du(x, y) = (0, ay)$ and hence $|Du(x, y)|^2 = a^2$. Moreover, the height function h of $\Sigma^2(u)$ satisfies

$$|\nabla h|^2 = \frac{|Du|^2}{1 + |Du|^2} = \frac{a^2}{1 + a^2}.$$

Thus, from (2-5) we have that the angle function η of $\Sigma^2(u)$ with respect to the orientation (4-2) is given by

$$\eta = \frac{1}{\sqrt{1 + |a|^2}}.$$

Consequently, by using that $\mathrm{Div} = \mathrm{Div}_0 - (2/y)dy$, where Div_0 denotes the divergent on $\mathbb{R}^2$, with a straightforward computation we verify that

$$2Hr^3 = r^2 y^2 \Delta_0 u - y^3 (y Q(u) + u_y |D_0 u|_0^2), \tag{4-8}$$

where Δ_0, D_0 and $|\ |_0$ stand for the Laplacian, the gradient and the norm in the canonical Euclidean metric, $r = \sqrt{1 + |Du|^2} = \sqrt{1 + a^2}$ and

$$Q(u) = u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy}.$$

Thus, replacing $u(x, y) = a \ln y$ in (4-8), we obtain

$$H = \frac{a}{2\sqrt{1 + a^2}}$$

and, since η is a positive constant, from Lemma 7 we get

$$0 = \Delta\eta = -(|A|^2 - |\nabla h|^2)\eta, \tag{4-9}$$

and, hence,

$$|\nabla h|^2 = |A|^2.$$

Furthermore, from (3-7) we easily see that $H_2 = 0$ on $\Sigma^2(u)$. But, $H_2 = \kappa_1\kappa_2$, where κ_1, κ_2 denote the eigenvalues of A. Therefore, considering $\kappa_2 = 0$ and using that $H = (\kappa_1 + \kappa_2)/2 = \kappa_1/2$, we obtain that $\kappa_1 = a/\sqrt{1 + a^2}$.

Finally, according to the stability criteria given in (4-7), from (4-9) we also conclude that $\Sigma^2(u)$ constitutes a nontrivial example of a stable surface in $\mathbb{R} \times \mathbb{H}^2$. Consequently, concerning the context of Theorem 1, we see that the stability of the hypersurface cannot alone guarantee the uniqueness result.

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