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In loving memory of my dear mother

Let K be a field and G a finite group. Let G act on the rational function field $K(x(g) : g \in G)$ by K-automorphisms defined by $g \cdot x(h) = x(gh)$ for any g, $h \in G$. Denote by $K(G)$ the fixed field $K(x(g) : g \in G)^G$. Noether's problem then asks whether $K(G)$ is rational (i.e., purely transcendental) over K. The first main result of this article is that $K(G)$ is rational over K for a certain class of p-groups having an abelian subgroup of index p. The second main result is that $K(G)$ is rational over K for any group of order p^5 or p^6 (where p is an odd prime) having an abelian normal subgroup such that its quotient group is cyclic. (In both theorems we assume that if char $K \neq p$ then K contains a primitive p^e-th root of unity, where p^e is the exponent of G.)

1. Introduction

Let K be a field. A field extension L of K is called rational over K (or K-rational, for short) if $L \simeq K(x_1, \dots, x_n)$ for some integer n, with $x_1, \dots, x_n$ algebraically independent over K. Now let G be a finite group. Let G act on the rational function field $K(x(g) : g \in G)$ by K-automorphisms defined by $g \cdot x(h) = x(gh)$ for any $g, h \in G$. Denote by $K(G)$ the fixed field $K(x(g) : g \in G)^G$. *Noether's problem* then asks whether $K(G)$ is rational over K. This is related to the inverse Galois problem, to the existence of generic G-Galois extensions over K, and to the existence of versal G-torsors over K-rational field extensions [Swan 1983; Saltman 1982; Garibaldi et al. 2003, §33.1, p. 86]. Noether's problem for abelian groups was studied extensively by Swan, Voskresenskii, Endo, Miyata and Lenstra, etc. The reader is referred to [Swan 1983] for a survey of this problem. Fischer's theorem is a starting point of investigating Noether's problem for finite abelian groups in general.

Theorem 1.1 (Fischer [Swan 1983, Theorem 6.1]). *Let G be a finite abelian group of exponent e. Assume that* (i) *either* char $K = 0$ *or* char $K > 0$ *with* char $K \nmid e$, *and* (ii) *K contains a primitive e-th root of unity. Then $K(G)$ is rational over K.*

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On the other hand, just a handful of results about Noether's problem have been obtained when the groups are nonabelian. This is the case even when the group G is a p-group. The reader is referred to [Chu and Kang 2001; Hu and Kang 2010; Kang 2006; 2011; 2009] for previous results on Noether's problem for p-groups. The following theorem of Kang generalizes Fischer's theorem for the metacyclic p-groups.

Theorem 1.2 [Kang 2006, Theorem 1.5]. *Let G be a metacyclic p-group with exponent p^e, and let K be any field such that* (i) $\operatorname{char} K = p$, *or* (ii) $\operatorname{char} K \neq p$ *and K contains a primitive p^e-th root of unity. Then $K(G)$ is rational over K.*

The next job is to study Noether's problem for metabelian groups. Three results due to Haeuslein, Hajja and Kang, respectively, are known.

Theorem 1.3 [Haeuslein 1971]. *Let K be a field and G be a finite group. Assume that* (i) *G contains an abelian normal subgroup H such that G/H is cyclic of prime order p*, (ii) *$\mathbb{Z}[\zeta_p]$ is a unique factorization domain*, *and* (iii) *$\zeta_{p^e} \in K$, where e is the exponent of G. If $G \to \mathrm{GL}(V)$ is any finite-dimensional linear representation of G over K, then $K(V)^G$ is rational over K.*

Theorem 1.4 [Hajja 1983]. *Let K be a field and G be a finite group. Assume that* (i) *G contains an abelian normal subgroup H such that G/H is cyclic of order n*, (ii) *$\mathbb{Z}[\zeta_n]$ is a unique factorization domain*, *and* (iii) *K is algebraically closed with* $\operatorname{char} K = 0$. *If $G \to \mathrm{GL}(V)$ is any finite-dimensional linear representation of G over K, then $K(V)^G$ is rational over K.*

Theorem 1.5 [Kang 2009, Theorem 1.4]. *Let K be a field and G be a finite group. Assume that* (i) *G contains an abelian normal subgroup H such that G/H is cyclic of order n*, (ii) *$\mathbb{Z}[\zeta_n]$ is a unique factorization domain*, *and* (iii) *$\zeta_e \in K$, where e is the exponent of G. If $G \to \mathrm{GL}(V)$ is any finite-dimensional linear representation of G over K, then $K(V)^G$ is rational over K.*

Note that those integers n for which $\mathbb{Z}[\zeta_n]$ is a unique factorization domain are determined by Masley and Montgomery.

Theorem 1.6 [Masley and Montgomery 1976]. *$\mathbb{Z}[\zeta_n]$ is a unique factorization domain if and only if* $1 \leq n \leq 22$, *or* $n = 24, 25, 26, 27, 28, 30, 32, 33, 34, 35, 36, 38, 40, 42, 45, 48, 50, 54, 60, 66, 70, 84, 90$.

Therefore, Theorem 1.3 holds only for primes p such that $1 \leq p \leq 19$. One of the goals of our paper is to show that the this condition can be waived, under some additional assumptions regarding the structure of the abelian subgroup H.

Consider the following situation. Let G be a group of order p^n for $n \geq 2$ with an abelian subgroup H of order p^{n-1}. Bender [1927/28] determined some interesting properties of these groups. We study further the case when the p-th lower central subgroup $G_{(p)}$ is trivial. (Recall that $G_{(0)} = G$ and $G_{(i)} = [G, G_{(i-1)}]$ for $i \geq 1$ form the so-called lower central series.) For our purposes we need to classify with generators and relations these groups. We achieve this in the following lemma.

Lemma 1.7. *Let G be a group of order p^n for $n \geq 2$ with an abelian subgroup H of order p^{n-1}. Choose any $\alpha \in G$ such that α generates G/H, that is, $\alpha \notin H$, $\alpha^p \in H$. Define $H(p) = \{h \in H : h^p = 1, h \notin H^p\} \cup \{1\}$, and assume that $[H(p), \alpha] \subset H(p)$. Assume also that the p-th lower central subgroup $G_{(p)}$ is trivial. Then H is a direct product of normal subgroups of G belonging to four types*:

(1) $(C_p)^s$ *for some $s \geq 1$. There exist generators $\alpha_1, \ldots, \alpha_s$ of $(C_p)^s$ such that $[\alpha_j, \alpha] = \alpha_{j+1}$ for $1 \leq j \leq s-1$ and $\alpha_s \in Z(G)$.*

(2) C_{p^a} *for some $a \geq 1$. There exists a generator β of C_{p^a} such that $[\beta, \alpha] = \beta^{bp^{a-1}}$ for some $b : 0 \leq b \leq p-1$.*

(3) $C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_k}} \times (C_p)^s$ *for some $k \geq 1$, $a_i \geq 2$, $s \geq 1$. There exist generators $\alpha_{11}, \alpha_{21}, \ldots, \alpha_{k1}$ of $C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_k}}$ such that $[\alpha_{i,1}, \alpha] = \alpha_{i+1,1}^{p^{a_{i+1}-1}} \in Z(G)$ for $i = 1, \ldots, k-1$. There also exist generators $\alpha_{k,2}, \ldots, \alpha_{k,s+1}$ of $(C_p)^s$ such that $[\alpha_{k,j}, \alpha] = \alpha_{k,j+1}$ for $1 \leq j \leq s$ and $\alpha_{k,s+1} \in Z(G)$.*

(4) $C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_k}}$ *for some $k \geq 2$, $a_i \geq 2$. For any $i : 1 \leq i \leq k$ there exists a generator $\alpha_{i,1}$ of the factor $C_{p^{a_i}}$ such that $[\alpha_{i,1}, \alpha] = \alpha_{i+1,1}^{p^{a_i-1}} \in Z(G)$ and $[\alpha_{k,1}, \alpha] \in \langle \alpha_{1,1}^{p^{a_1-1}}, \ldots, \alpha_{k,1}^{p^{a_k-1}} \rangle$.*

The first main result of this paper is a generalization of Theorem 1.3:

Theorem 1.8. *Let G be a group of order p^n for $n \geq 2$ with an abelian subgroup H of order p^{n-1}, and let G be of exponent p^e. Choose any $\alpha \in G$ such that α generates G/H, that is, $\alpha \notin H$, $\alpha^p \in H$. Define $H(p) = \{h \in H : h^p = 1, h \notin H^p\} \cup \{1\}$, and assume that $[H(p), \alpha] \subset H(p)$. Let the p-th lower central subgroup $G_{(p)}$ be trivial. Assume that* (i) char $K = p > 0$, *or* (ii) char $K \neq p$ *and K contains a primitive p^e-th root of unity. Then $K(G)$ is rational over K.*

The key idea to prove Theorem 1.8 is to find a faithful G-subspace W of the regular representation space $\bigoplus_{g \in G} K \cdot x(g)$ and to show that W^G is rational over K. The subspace W is obtained as an induced representation from H by applying Lemma 1.7.

The next goal of our article is to study Noether's problem for some groups of orders p^5 and p^6 for any odd prime p. We use the list of generators and relations for these groups, given by James [1980]. It is known that $K(G)$ is always rational if G is a p-group of order at most p^4 and $\zeta_e \in K$, where e is the exponent of G

(see [Chu and Kang 2001]). However, in [Hoshi and Kang 2011] it is shown that there exists a group G of order p^5 such that $\mathbb{C}(G)$ is not rational over $\mathbb{C}$.

The second main result of this article is the following rationality criterion for the groups of orders p^5 and p^6 having an abelian normal subgroup such that its quotient group is cyclic.

Theorem 1.9. *Let G be a group of order p^n for $n \leq 6$ with an abelian normal subgroup H such that G/H is cyclic. Let G be of exponent p^e. Assume that* (i) $\operatorname{char} K = p > 0$, *or* (ii) $\operatorname{char} K \neq p$ *and K contains a primitive p^e-th root of unity. Then $K(G)$ is rational over K.*

We do not know whether Theorem 1.9 holds for any $n \geq 7$. However, we should not "overgeneralize" Theorem 1.9 to the case of any metabelian group because of the following theorem of Saltman.

Theorem 1.10 [Saltman 1984]. *For any prime number p and for any field K with* $\operatorname{char} K \neq p$ (*in particular, K may be an algebraically closed field*), *there is a metabelian p-group G of order p^9 such that $K(G)$ is not rational over K.*

We organize this paper as follows. We recall some preliminaries in Section 2 that will be used in the proofs of Theorems 1.8 and 1.9. There we also prove Lemma 2.5, which is a generalization of Kang's argument [2011, Case 5, Step II]. In Section 3 we prove Lemma 1.7, which is of independent interest, since it provides a list of generators and relations for any p-group G having an abelian subgroup H of index p, provided that $[H(p), \alpha] \subset H(p)$ and $G_{(p)} = 1$. Our main results — Theorems 1.8 and 1.9 — are proved in Sections 4 and 5, respectively.

2. Preliminaries

We list several results which will be used in the sequel.

Theorem 2.1 [Hajja and Kang 1995, Theorem 1]. *Let G be a finite group acting on $L(x_1, \dots, x_m)$, the rational function field of m variables over a field L, such that*

(1) *for any $\sigma \in G$, $\sigma(L) \subset L$,*

(2) *the restriction of the action of G to L is faithful,*

(3) *for any $\sigma \in G$,*

$$\begin{pmatrix} \sigma(x_1) \\ \vdots \\ \sigma(x_m) \end{pmatrix} = A(\sigma) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} + B(\sigma),$$

where $A(\sigma) \in \mathrm{GL}_m(L)$ and $B(\sigma)$ is an $m \times 1$ matrix over L. Then there exist $z_1, \dots, z_m \in L(x_1, \dots, x_m)$ such that $L(x_1, \dots, x_m)^G = L^G(z_1, \dots, z_m)$ and $\sigma(z_i) = z_i$ for any $\sigma \in G$ and $1 \leq i \leq m$.

Theorem 2.2 [Ahmad et al. 2000, Theorem 3.1]. *Let G be a finite group acting on $L(x)$, the rational function field of one variable over a field L. Assume that, for any $\sigma \in G$, $\sigma(L) \subset L$ and $\sigma(x) = a_\sigma x + b_\sigma$ for any $a_\sigma, b_\sigma \in L$ with $a_\sigma \neq 0$. Then $L(x)^G = L^G(z)$ for some $z \in L[x]$.*

Theorem 2.3 [Chu and Kang 2001, Theorem 1.7]. *If* char $K = p > 0$ *and G is a finite p-group, then $K(G)$ is rational over K.*

The following lemma can be extracted from some proofs in [Kang 2011; Hu and Kang 2010].

Lemma 2.4. *Let $\langle\tau\rangle$ be a cyclic group of order $n > 1$, acting on $K(v_1, \dots, v_{n-1})$, the rational function field of $n-1$ variables over a field K, such that*

$$\tau : v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{n-1} \mapsto (v_1 \cdots v_{n-1})^{-1} \mapsto v_1.$$

Suppose that K contains a primitive n-th root of unity ξ. Then $K(v_1, \dots, v_{n-1}) = K(s_1, \dots, s_{n-1})$, where $\tau : s_i \mapsto \xi^i s_i$ for $1 \le i \le n-1$.

Proof. Define $w_0 = 1 + v_1 + v_1 v_2 + \cdots + v_1 v_2 \cdots v_{n-1}$, $w_1 = (1/w_0) - 1/n$, $w_{i+1} = (v_1 v_2 \cdots v_i / w_0) - 1/n$ for $1 \le i \le n-1$. Thus $K(v_1, \dots, v_{n-1}) = K(w_1, \dots, w_n)$ with $w_1 + w_2 + \cdots + w_n = 0$ and

$$\tau : w_1 \mapsto w_2 \mapsto \cdots \mapsto w_{n-1} \mapsto w_n \mapsto w_1.$$

Define $s_i = \sum_{1 \le j \le n} \xi^{-ij} w_j$ for $1 \le i \le n-1$. Then $\tau : s_i \mapsto \xi^i s_i$ for $1 \le i \le n-1$ and $K(w_1, \dots, w_n) = K(s_1, \dots, s_{n-1})$. □

Next, generalizing an argument used in [Kang 2011, Case 5, Step II], we obtain a result that will play an important role in our work.

Lemma 2.5. *Let $k > 1$, let p be any prime and let $\langle\alpha\rangle$ be a cyclic group of order p, acting on $K(y_{1i}, y_{2i}, \dots, y_{ki} : 1 \le i \le p-1)$, the rational function field of $k(p-1)$ variables over a field K, such that*

$$\alpha : y_{j1} \mapsto y_{j2} \mapsto \cdots \mapsto y_{jp-1} \mapsto (y_{j1} y_{j2} \cdots y_{jp-1})^{-1} \quad \text{for } 1 \le j \le k.$$

Assume that $K(v_{1i}, v_{2i}, \dots, v_{ki} : 1 \le i \le p-1) = K(y_{1i}, y_{2i}, \dots, y_{ki} : 1 \le i \le p-1)$ where for any $j : 1 \le j \le k$ and for any $i : 1 \le i \le p-1$ the variable v_{ji} is a monomial in the variables $y_{1i}, y_{2i}, \dots, y_{ki}$. Assume also that the action of α on $K(v_{1i}, v_{2i}, \dots, v_{ki} : 1 \le i \le p-1)$ is given by

$$\alpha : v_{j1} \mapsto v_{j1} v_{j2}^p,\ v_{j2} \mapsto v_{j3} \mapsto \cdots \mapsto v_{jp-1} \mapsto A_j (v_{j1} v_{j2}^{p-1} v_{j3}^{p-2} \cdots v_{jp-1}^2)^{-1}$$

for $1 \le j \le k$, where A_j is some monomial in $v_{1i}, \dots, v_{j-1i}$ for $2 \le j \le k$ and $A_1 = 1$. If K contains a primitive p-th root of unity ζ, then

$$K(v_{1i}, v_{2i}, \dots, v_{ki} : 1 \le i \le p-1) = K(s_{1i}, s_{2i}, \dots, s_{ki} : 1 \le i \le p-1),$$

where $\alpha : s_{ji} \mapsto \zeta^i s_{ji}$ *for* $1 \leq j \leq k$, $1 \leq i \leq p-1$.

Proof. We write the additive version of the multiplication action of α; that is, consider the $\mathbb{Z}[\pi]$-module $M = \bigoplus_{1\leq m\leq k}\big(\bigoplus_{1\leq i\leq p-1} \mathbb{Z}\cdot v_{mi}\big)$, where $\pi = \langle\alpha\rangle$. Define submodules $M_j = \bigoplus_{1\leq m\leq j}\big(\bigoplus_{1\leq i\leq p-1} \mathbb{Z}\cdot v_{mi}\big)$ for $1 \leq j \leq k$. Thus α has the following additive action

$$\alpha : v_{j1} \mapsto v_{j1} + pv_{j2},$$
$$v_{j2} \mapsto v_{j3} \mapsto \cdots \mapsto v_{jp-1} \mapsto A_j - v_{j1} - (p-1)v_{j2} - (p-2)v_{j3} - \cdots - 2v_{jp-1},$$

where $A_j \in M_{j-1}$.

By Lemma 2.4, M_1 is isomorphic to the $\mathbb{Z}[\pi]$-module $N = \bigoplus_{1\leq i\leq p-1} \mathbb{Z}\cdot u_i$, where $u_1 = v_{12}$, $u_i = \alpha^{i-1}\cdot v_{12}$ for $2 \leq i \leq p-1$, and

$$\alpha : u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p-1} \mapsto -u_1 - u_2 - \cdots - u_{p-1} \mapsto u_1.$$

Let $\Phi_p(T) \in \mathbb{Z}[T]$ be the p-th cyclotomic polynomial. Since $\mathbb{Z}[\pi]$ is isomorphic to $\mathbb{Z}[T]/(T^p - 1)$, we find that $\mathbb{Z}[\pi]/\Phi_p(\alpha) \simeq \mathbb{Z}[T]/\Phi_p(T) \simeq \mathbb{Z}[\omega]$, the ring of p-th cyclotomic integers. As $\Phi_p(\alpha)\cdot x = 0$ for any $x \in N$, the $\mathbb{Z}[\pi]$-module N can be regarded as a $\mathbb{Z}[\omega]$-module through the morphism $\mathbb{Z}[\pi] \to \mathbb{Z}[\pi]/\Phi_p(\alpha)$. When N is regarded as a $\mathbb{Z}[\omega]$-module, we have $N \simeq \mathbb{Z}[\omega]$, the rank-one free $\mathbb{Z}[\omega]$-module.

We claim that M itself can be regarded as a $\mathbb{Z}[\omega]$-module, that is, $\Phi_p(\alpha)\cdot M = 0$.

We return to multiplicative notation. Note that all v_{ji} are monomials in the y_{ji}. The action of α on y_{ji} given in the statement satisfies $\prod_{0\leq m\leq p-1} \alpha^m(y_{ji}) = 1$ for any $1 \leq j \leq k$, $1 \leq i \leq p-1$. Using the additive notations, we get $\Phi_p(\alpha)\cdot y_{ji} = 0$. Hence $\Phi_p(\alpha)\cdot M = 0$.

Define $M' = M/M_{k-1}$. We have a short exact sequence of $\mathbb{Z}[\pi]$-modules

$$0 \to M_{k-1} \to M \to M' \to 0. \tag{2-1}$$

Since M is a $\mathbb{Z}[\omega]$-module, (2-1) is a short exact sequence of $\mathbb{Z}[\omega]$-modules. Proceeding by induction, we obtain that M is a direct sum of free $\mathbb{Z}[\omega]$-modules isomorphic to N. Hence, $M \simeq \bigoplus_{1\leq j\leq k} N_j$, where $N_j \simeq N$ is a free $\mathbb{Z}[\omega]$-module and so a $\mathbb{Z}[\pi]$-module also (for $1 \leq j \leq k$).

Finally, we interpret the additive version of $M \simeq \bigoplus_{1\leq j\leq k} N_j \simeq N^k$ in terms of the multiplicative version as follows: There exist w_{ji} that are monomials in v_{ji} for $1 \leq j \leq k$, $1 \leq i \leq p-1$ such that $K(w_{ji}) = K(v_{ji})$ and α acts as

$$\alpha : w_{j1} \mapsto w_{j2} \mapsto \cdots \mapsto w_{jp-1} \mapsto (w_{j1}w_{j2}\cdots w_{jp-1})^{-1} \quad \text{for } 1 \leq j \leq k.$$

According to Lemma 2.4, the above action can be linearized as pointed out in the statement. □

Now, let G be any metacyclic p-group generated by two elements σ and τ with relations $\sigma^{p^a}=1$, $\tau^{p^b}=\sigma^{p^c}$ and $\tau^{-1}\sigma\tau=\sigma^{\varepsilon+\delta p^r}$ where $\varepsilon=1$ if p is odd, $\varepsilon=\pm 1$ if $p=2$, $\delta=0, 1$ and $a, b, c, r\geq 0$ are subject to some restrictions. For the description of these restrictions see, for example, [Kang 2006, p. 564].

Theorem 2.6 [Kang 2006, Theorem 4.1]. *Let p be a prime number, m, n and r positive integers, $k=1+p^r$ if $(p,r)\neq(2,1)$ or $k=-1+2^r$ if $p=2$ and $r\geq 2$. Let G be a split metacyclic p-group of order p^{m+n} and exponent p^e defined by $G=\langle\sigma,\tau:\sigma^{p^m}=\tau^{p^n}=1, \tau^{-1}\sigma\tau=\sigma^k\rangle$. Let K be any field such that* $\operatorname{char} K\neq p$ *and K contains a primitive p^e-th root of unity, and let ζ be a primitive p^m-th root of unity. Then $K(x_0,x_1,\ldots,x_{p^n-1})^G$ is rational over K, where G acts on $x_0,\ldots,x_{p^n-1}$ by*

$$\sigma: x_i\mapsto\zeta^{k^i}x_i,\quad \tau: x_0\mapsto x_1\mapsto\cdots\mapsto x_{p^n-1}\mapsto x_0.$$

3. Proof of Lemma 1.7

It is well known that H is a normal subgroup of G. We divide the proof into steps.

Step I. Let β_1 be any element of H that is not central. Since $G_{(p)}=\{1\}$, there exist $\beta_2,\ldots,\beta_k\in H$ for some $k:2\leq k\leq p$ such that $[\beta_j,\alpha]=\beta_{j+1}$, where $1\leq j\leq k-1$ and $\beta_k\neq 1$ is central. We are going to show now that the order of β_2 is not greater than p. In particular, from the multiplication rule $[a,\alpha][b,\alpha]=[ab,\alpha]$ (for any $a,b\in H$) it follows that all p-th powers are contained in the center of G.

From $[\beta_j,\alpha]=\beta_{j+1}$ there follows the well known formula

$$\alpha^{-p}\beta_1\alpha^p=\beta_1\beta_2^{\binom{p}{1}}\beta_3^{\binom{p}{2}}\cdots\beta_p^{\binom{p}{p-1}}\beta_{p+1}, \tag{3-1}$$

where we put $\beta_{k+1}=\cdots=\beta_{p+1}=1$. Since α^p is in H, we obtain the formula

$$\beta_2^{\binom{p}{1}}\beta_3^{\binom{p}{2}}\cdots\beta_k^{\binom{p}{k-1}}=1.$$

Hence $\bigl(\beta_2\cdot\prod_{j\neq 2}\beta_j^{a_j}\bigr)^p=1$ for some integers a_j. It is not hard to see that this identity is impossible if the order of β_2 exceeds p. Indeed, if $\ell=\max\{j:\beta_j^p\neq 1\}$, then β_ℓ^p is in the subgroup generated by $\beta_2^p,\ldots,\beta_{\ell-1}^p$. Thus $[\beta_\ell^p,\alpha]=[\beta_2^{b_2p}\cdots\beta_{\ell-1}^{b_{\ell-1}p},\alpha]=\beta_3^{b_2p}\cdots\beta_\ell^{b_{\ell-1}p}\neq 1$ for some $b_2,\ldots,b_{\ell-1}\in\mathbb{Z}_p$. On the other hand, $[\beta_\ell^p,\alpha]=\beta_{\ell+1}^p=1$, which is a contradiction.

Step II. Let us write the decomposition of H as a direct product of cyclic subgroups (not necessarily normal in G): $H\simeq(C_p)^t\times C_{p^{a_1}}\times C_{p^{a_2}}\times\cdots\times C_{p^{a_s}}$ for $0\leq t$, $2\leq a_1\leq a_2\leq\cdots\leq a_s$. Choose a generator $\alpha_{11}\in C_{p^{a_1}}$. Since $G_{(p)}=\{1\}$, there exist $\alpha_{12},\ldots,\alpha_{1k}\in H$ for some $k:2\leq k\leq p$ such that $[\alpha_{1j},\alpha]=\alpha_{1j+1}$, where $1\leq j\leq k-1$ and $\alpha_{1k}\neq 1$ is central. From Step I it follows that the order of α_{12} is not greater than p. We are going to define a normal subgroup of G which depends on the nature of the element α_{12}. We will denote it by $\langle\langle\alpha_{11}\rangle\rangle$, and call it *the commutator chain of* α_{11}. Simultaneously, we will define a complement in H denoted by $\overline{\langle\langle\alpha_{11}\rangle\rangle}$.

Case II.1. Let $\alpha_{12} = \alpha_{11}^{p^{a_1-1}} c_1$ for some $c_1 : 0 \le c_1 \le p-1$. Define $\langle\langle\alpha_{11}\rangle\rangle = \langle\alpha_{11}\rangle$ and $\overline{\langle\langle\alpha_{11}\rangle\rangle} = (C_p)^t \cdot \langle\alpha_{21}, \dots, \alpha_{s1}\rangle$. Clearly, $\langle\langle\alpha_{11}\rangle\rangle$ is a normal subgroup of type 2.

Case II.2. Let $\alpha_{12} \notin H^p$. According to the assumptions of our lemma, we have $[H(p), \alpha] \cap H^p = \{1\}$, so $\alpha_{1j} \notin H^p$ for all j. Define $\langle\langle\alpha_{11}\rangle\rangle = \langle\alpha_{11}, \dots, \alpha_{1k}\rangle$. Then $\langle\langle\alpha_{11}\rangle\rangle \simeq C_{p^{a_1}} \times (C_p)^{k-1}$ is a normal subgroup of type 3. Define $\overline{\langle\langle\alpha_{11}\rangle\rangle} = (C_p)^{t-k+1} \cdot \langle\alpha_{21}, \dots, \alpha_{s1}\rangle$, where $(C_p)^{t-k+1}$ is the complement of $(C_p)^{k-1}$ in $(C_p)^t$.

Case II.3. Let $\alpha_{12} \in H^p$. Then $\alpha_{12} = \prod_{i\in A} \alpha_{i1}^{p^{a_i-1}d_i}$, where $A \subset \{1, 2, \dots, s\}$, $1 \le d_i \le p-1$. Put $i_0 = \min\{i \in A\}$.

If $i_0 = 1$, then $\alpha_{12} = \big(\alpha_{11}^{d_1} \prod_{i\in A, i\ne 1} \alpha_{i1}^{p^{a_i-a_1}} d_i\big)^{p^{a_1-1}}$. We replace the generator α_{11} with $\alpha'_{11} = \alpha_{11}^{d_1} \prod_{i\in A, i\ne 1} \alpha_{i1}^{p^{a_i-a_1}d_i}$. Clearly, $\operatorname{ord}\alpha'_{11} = \operatorname{ord}\alpha_{11}$ and $[\alpha'_{11}, \alpha] \in \langle\alpha'_{11}\rangle$, so this case is reduced to Case I.

If $i_0 > 1$, then $\alpha_{12} = \big(\alpha_{i_0 1}^{d_{i_0}} \prod_{i\in A, i\ne i_0} \alpha_{i1}^{p^{a_i-a_{i_0}} d_i}\big)^{p^{a_{i_0}-1}}$. We replace the generator $\alpha_{i_0 1}$ with $\alpha'_{i_0 1} = \alpha_{i_0 1}^{d_{i_0}} \prod_{i\in A, i\ne i_0} \alpha_{i1}^{p^{a_i-a_{i_0}} d_i}$. Clearly, $\operatorname{ord}\alpha'_{i_0 1} = \operatorname{ord}\alpha_{i_0 1}$ and $\alpha'^{p^{a_{i_0}-1}}_{i_0 1} = \alpha_{12}$.

Abusing notation we will assume henceforth that $i_0 = 2$ and $\alpha_{21}^{p^{a_2-1}} = \alpha_{12}$. Consider $\alpha_{22} = [\alpha_{21}, \alpha]$. We have three possibilities now.

Subcase II.3.1. If $\alpha_{22} \in \langle\alpha_{11}^{p^{a_1-1}}, \alpha_{21}^{p^{a_1-1}}\rangle$, define $\langle\langle\alpha_{11}\rangle\rangle = \langle\alpha_{11}, \alpha_{21}\rangle$. Then $\langle\langle\alpha_{11}\rangle\rangle \simeq C_{p^{a_1}} \times C_{p^{a_2}}$ is a normal subgroup of type 4.

Subcase II.3.2. If $\alpha_{22} \notin H^p$, there exist $\alpha_{22}, \dots, \alpha_{2\ell} \in H$ for some $\ell : 2 \le \ell \le p$ such that $[\alpha_{2j}, \alpha] = \alpha_{2j+1}$, where $1 \le j \le \ell-1$ and $\alpha_{2\ell} \ne 1$ is central. Define $\langle\langle\alpha_{11}\rangle\rangle = \langle\alpha_{11}, \alpha_{21}, \alpha_{22}, \dots, \alpha_{2\ell}\rangle$. Then $\langle\langle\alpha_{11}\rangle\rangle \simeq C_{p^{a_1}} \times C_{p^{a_2}} \times (C_p)^{\ell-1}$ is a normal subgroup of type 3.

Subcase II.3.3. $\alpha_{22} \in H^p$. According to the observations we have just made, this subcase leads to the following two final possibilities.

- $\alpha_{22} = \alpha_{31}^{p^{a_3-1}}, \dots, \alpha_{r-12} = \alpha_{r1}^{p^{a_r-1}}, \alpha_{r2} \in \langle\alpha_{11}^{p^{a_1-1}}, \dots, \alpha_{r1}^{p^{a_r-1}}\rangle$. Define $\langle\langle\alpha_{11}\rangle\rangle = \langle\alpha_{11}, \alpha_{21}, \dots, \alpha_{r1}\rangle$. Then $\langle\langle\alpha_{11}\rangle\rangle \simeq C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_r}}$ is a normal subgroup of type 4. Define $\overline{\langle\langle\alpha_{11}\rangle\rangle} = (C_p)^t \cdot \langle\alpha_{r+11}, \dots, \alpha_{s1}\rangle$.
- $\alpha_{22} = \alpha_{31}^{p^{a_3-1}}, \dots, \alpha_{r-12} = \alpha_{r1}^{p^{a_r-1}}, \alpha_{r2} \notin H^p$. Then there exist $\alpha_{r2}, \dots, \alpha_{r\ell} \in H$ for some $\ell : 2 \le \ell \le p$ such that $[\alpha_{rj}, \alpha] = \alpha_{rj+1}$, where $1 \le j \le \ell-1$ and $\alpha_{r\ell} \ne 1$ is central. Define $\langle\langle\alpha_{11}\rangle\rangle = \langle\alpha_{11}, \alpha_{21}, \dots, \alpha_{r1}, \alpha_{r2}, \dots, \alpha_{r\ell}\rangle$. In this case $\langle\langle\alpha_{11}\rangle\rangle \simeq C_{p^{a_1}} \times C_{p^{a_1}} \times \cdots \times C_{p^{a_r}} \times (C_p)^{\ell-1}$ is a normal subgroup of type 3. Define $\overline{\langle\langle\alpha_{11}\rangle\rangle} = (C_p)^{t-\ell+1} \cdot \langle\alpha_{r+11}, \dots, \alpha_{s1}\rangle$, where $(C_p)^{t-\ell+1}$ is the complement of $(C_p)^{\ell-1}$ in $(C_p)^t$.

Step III. Put $H_1 = \langle\langle\alpha_{11}\rangle\rangle$ and $H_2 = \overline{\langle\langle\alpha_{11}\rangle\rangle}$. Note that $H_1 \cap H_2 = \{1\}$. However, H_2 may not be a normal subgroup of G. That is why we need to show that there exist a commutator chain $\mathcal{H}_1$ and a normal subgroup $\mathcal{H}_2$ of G such that $H = \mathcal{H}_1 \times \mathcal{H}_2$. In this step, we will describe a somewhat algorithmic approach which replaces the generators of H until the desired result is obtained.

Assume henceforth that H_2 is not normal in G. Then there exists a generator $\beta \in H_2$ such that $\alpha^{-1}\beta\alpha = hh_1$ for some $h \in H_2$, $h_1 \in H_1$, $h_1 \notin H_2$. Since $h = \beta h_2$ for some $h_2 \in H_2$, we get $[\beta, \alpha] = h_1 h_2$.

Let us assume first that $\operatorname{ord} \beta = p$. If $h_1 \in H^p$, then $h_2 \notin [H(p), \alpha]$; otherwise $[H(p), \alpha] \cap H^p \neq \{1\}$. In other words, h_2 does not appear in similar chains, so we can simply put $h_1 h_2$, instead of h_2, as a generator of H_2. In this way we obtain a group that is G-isomorphic to H_2. Thus we get that $[\beta, \alpha]$ is in this new copy of H_2. Similarly, if $h_1 \in H(p)$ and $h_2 \notin [H(p), \alpha]$, we can obtain a new copy of H_2 such that $[\beta, \alpha]$ is in H_2. If $h_2 \in [H(p), \alpha]$, we may assume that $[\beta, \alpha] \in H_1$. In this case $\langle\langle \alpha_{11} \rangle\rangle$ must be of type 3. Let $\langle\langle \alpha_{11} \rangle\rangle \simeq C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_k}} \times (C_p)^s$ be generated by elements $\alpha_{11}, \dots, \alpha_{k1}, \alpha_{k2}, \dots, \alpha_{ks+1}$ with relations given in the statement of the lemma. Assume that $\alpha_{k\ell} = [\beta, \alpha]$ for some $\ell : 2 \leq \ell \leq s + 1$. If $\ell > 2$, replace β with $\beta' = \beta\alpha_{k\ell-1}^{-1}$. Hence $[\beta', \alpha] = 1$. If $\ell = 2$, we can put $\alpha'_{k1} = \alpha_{k1}\beta^{-1}$, instead of α_{k1}, as a generator of H_1. In this way we obtain a group of type 4, since $[\alpha'_{k1}, \alpha] = 1$. Clearly, $[\beta, \alpha]$ is not in this new commutator chain $\mathcal{H}_1$. It is not hard to see that with similar replacements we can treat the general case $[\beta, \alpha] = \prod_i \alpha_{i1}^{p^{a_i-1}c_i} \cdot \prod_j \alpha_{kj}$. Thus we obtain the decomposition $H = \mathcal{H}_1 \times \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are normal subgroups of G.

Next, we are going to assume that $\operatorname{ord} \beta > p$. According to the definition of the commutator chain of α_{11} we need to consider the three cases of Step II separately.

Case III.1. $\alpha_{12} = \alpha_{11}^{p^{a_1-1}c_1}$ for some $c_1 : 1 \leq c_1 \leq p - 1$. Here we must have $h_1 = \alpha_{11}^{p^{a_1-1}d_1}$ for some $d_1 : 1 \leq d_1 \leq p - 1$. We can replace β with $\beta' = \beta\alpha_{11}^{-d_1/c_1}$, so $[\beta', \alpha] = h_2$.

Case III.2. $\alpha_{12} \notin H^p$. If $h_1 = \prod_{j\geq 2} \alpha_{1j}^{d_j}$ for some $d_j : 0 \leq d_j \leq p - 1$, we can replace β with $\beta' = \beta \prod_{j\geq 2} \alpha_{1j-1}^{-d_j}$. Hence $[\beta', \alpha] = h_2$. This reduces the analysis to the case $h_1 = \alpha_{11}^{p^{a_1-1}d_1}$ for some $d_1 : 0 \leq d_1 \leq p - 1$. We now have three possibilities for h_2.

Subcase III.2.1. Let $h_2 \notin H^p$ and $h_2 \notin [H, \alpha]$. We can put $h_1 h_2$, instead of h_2, as a generator of H_2. In this way we obtain a group that is G-isomorphic to H_2. Thus we get that $[\beta, \alpha]$ is in this new copy of H_2.

Subcase III.2.2. Let $h_2 \notin H^p$ and $h_2 \in [H, \alpha]$, that is, there exists $\gamma \notin H^p$ such that $[\gamma, \alpha] = h_2$. Put $\beta' = \beta\gamma^{-1}$. Then $[\beta', \alpha] = h_1 = \alpha_{11}^{p^{a_1-1}d_1}$. Hence the commutator chain of α_{11} is contained in the commutator chain $\langle\langle \beta' \rangle\rangle$ which is a normal subgroup of G of type 3.

Subcase III.2.3. Let $h_2 \in H^p$; that is, $h_2 = \prod_{i\in B} \alpha_{i1}^{p^{a_i-1}d_i}$, where $B = \{i : \alpha_{i1} \in H_2\}$,

$0 \le d_i \le p-1$. We can replace α_{11} with $\alpha'_{11} = \alpha_{11}^{d_1} \prod_{i \in B} \alpha_{i1}^{p^{a_i - a_1} d_i}$. Now we have $[\beta, \alpha] = \alpha'^{p^{a_1-1}}_{11}$, so the commutator chain of α'_{11} is contained in the commutator chain $\langle\langle\beta\rangle\rangle$, which is a normal subgroup of G of type 3.

Case III.3. $\alpha_{12} \in H^p$. We have that either $\langle\langle\alpha_{11}\rangle\rangle \simeq C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_r}}$ is a normal subgroup of type 4, or $\langle\langle\alpha_{11}\rangle\rangle \simeq C_{p^{a_1}} \times C_{p^{a_1}} \times \cdots \times C_{p^{a_r}} \times (C_p)^{\ell-1}$ is a normal subgroup of type 3.

Similarly to Case III.2, if h_1 is a product of elements of order p that are not in $\langle\alpha_{11}^{p^{a_1-1}}\rangle$, by a suitable change of the generator β we will obtain $[\beta, \alpha] = h_2$. Thus we again reduce the considerations to the case $h_1 = \alpha_{11}^{p^{a_1-1} d_1}$ for some $d_1 : 0 \le d_1 \le p-1$. We have three possibilities for h_2, which are identical to the three subcases in Case III.2. The only slight difference is that the new commutator chain here can be of type 3 or type 4.

In this way, we have investigated all possibilities for the proper construction of the normal factors of H. The construction is algorithmic in nature. When we define a new commutator chain $\langle\langle\beta'\rangle\rangle$ or $\langle\langle\beta\rangle\rangle$ (as in Subcases III.2.2 and III.2.3), we have to start the same process all over again until we can not get a new commutator chain that contains the previous one. Denote by $\mathcal{H}_1$ the last commutator chain obtained by the described algorithm from H_1. We have that $\mathcal{H}_1$ is a normal subgroup of G of one of the types 1–4. Denote by $\mathcal{H}_2$ the subgroup obtained from H_2 by the replacements described above. Then H is a direct product of $\mathcal{H}_1$ and $\mathcal{H}_2$, where $\mathcal{H}_2$ is normal in G. Proceeding by induction we will obtain the decomposition given in the statement.

4. Proof of Theorem 1.8

If $\operatorname{char} K = p > 0$, we can apply Theorem 2.3. Therefore, we will assume that $\operatorname{char} K \neq p$.

According to Lemma 1.7, $H \simeq \mathcal{H}_1 \times \cdots \times \mathcal{H}_t$, where $\mathcal{H}_1, \ldots, \mathcal{H}_t$ are normal subgroups of G that are isomorphic to any of the four types described in Lemma 1.7.

Let V be a K-vector space whose dual space V^* is defined as $V^* = \bigoplus_{g \in G} K \cdot x(g)$, where G acts on V^* by $h \cdot x(g) = x(hg)$ for any $h, g \in G$. Therefore $K(V)^G = K(x(g) : g \in G)^G = K(G)$.

Now, for any subgroup $\mathcal{H}_i$ $(1 \le i \le t)$ we can define a faithful representation subspace $V_i = \bigoplus_{1 \le j \le k_i} K \cdot Y_j$, where k_i is the number of the generators of $\mathcal{H}_i$ as an abelian group. (For details see Cases I–IV.) Therefore, $\bigoplus_{1 \le i \le t} V_i$ is a faithful representation space of the subgroup H.

Next, for any subgroup $\mathcal{H}_i$ $(1 \le i \le t)$ we define $x_{jk} = \alpha^k \cdot Y_j$ for $1 \le j \le k_i$, $0 \le k \le p-1$. Define $W_i = \bigoplus_{j,k} K \cdot x_{jk} \subset V^*$. Then $W = \bigoplus_{1 \le i \le t} W_i$ is a faithful G-subspace of V^*. Thus, by Theorem 2.1 it suffices to show that W^G is rational over K. Note that $W^G = (W^H)^{\langle\alpha\rangle} = ((\cdots(W^{\mathcal{H}_1})^{\mathcal{H}_2}\cdots)^{\mathcal{H}_t})^{\langle\alpha\rangle} = ((\cdots(W_1^{\mathcal{H}_1} \bigoplus_{2 \le j \le t} W_j)^{\mathcal{H}_2}\cdots)^{\mathcal{H}_t})^{\langle\alpha\rangle} = \cdots = \bigoplus_{1 \le j \le t} (W_j^{\mathcal{H}_j})^{\langle\alpha\rangle}$. Therefore, we need to calculate $W_j^{\mathcal{H}_j}$ when $\mathcal{H}_j$ is isomorphic to any of the four types described in Lemma 1.7. Finally, we will show that the action of α on W^H can be linearized.

Case I. Assume that $\mathcal{H}_1$ is of type 3; that is, for some $k \geq 1$, $a_i \geq 2$, $s \geq 1$, $\mathcal{H}_1 \simeq C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_k}} \times (C_p)^s$. Denote by $\alpha_1, \ldots, \alpha_k$ the generators of $C_{p^{a_1}} \times \cdots \times C_{p^{a_k}}$, and by $\alpha_{k+1}, \ldots, \alpha_{k+s}$ the generators of $(C_p)^s$. According to Lemma 1.7, we have the relations $[\alpha_i, \alpha] = \alpha_{i+1}^{p^{a_{i+1}-1}} \in Z(G)$ for $1 \leq i \leq k-1$; $[\alpha_{k+j}, \alpha] = \alpha_{k+j+1}$ for $0 \leq j \leq s-1$; and $\alpha_{k+s} \in Z(G)$. Because of the frequent use of $k+s$ in this case, we put $r = k+s$.

We divide the proof into several steps.

<u>Step 1.</u> Define $X_1, X_2, \ldots, X_r \in V^*$ by

$$X_j = \sum_{\ell_1,\ldots,\ell_r} x\Big(\prod_{i \neq j} \alpha_i^{\ell_i}\Big) \quad \text{for } 1 \leq j \leq r.$$

Note that $\alpha_i \cdot X_j = X_j$ for $j \neq i$. Let $\zeta_{p^{a_i}} \in K$ be a primitive p^{a_i}-th root of unity for $1 \leq i \leq k$, and let ζ be a primitive p-th root of unity. Define $Y_1, Y_2, \ldots, Y_r \in V^*$ by

$$Y_i = \sum_{m=0}^{p^{a_i}-1} \zeta_{p^{a_i}}^{-m} \alpha_i^m \cdot X_i, \qquad Y_j = \sum_{m=0}^{p-1} \zeta^{-m} \alpha_j^m \cdot X_j,$$

for $1 \leq i \leq k$ and $k+1 \leq j \leq r$.

It follows that

$$\begin{aligned} \alpha_i &: Y_i \mapsto \zeta_{p^{a_i}} Y_i,\ Y_j \mapsto Y_j \quad \text{for } j \neq i \text{ and } 1 \leq i \leq k, \\ \alpha_j &: Y_j \mapsto \zeta Y_j,\ Y_i \mapsto Y_i \qquad \text{for } i \neq j \text{ and } k+1 \leq j \leq r. \end{aligned}$$

Thus $V_1 = \bigoplus_{1 \leq j \leq r} K \cdot Y_j$ is a faithful representation space of the subgroup $\mathcal{H}_1$.

Define $x_{ji} = \alpha^i \cdot Y_j$ for $1 \leq j \leq r$ and $0 \leq i \leq p-1$. Recall that $[\alpha_i, \alpha] = \alpha_{i+1}^{p^{a_{i+1}-1}} \in Z(G)$ for $1 \leq i \leq k-1$; $[\alpha_{k+j}, \alpha] = \alpha_{k+j+1}$ for $0 \leq j \leq s-1$; and $\alpha_r \in Z(G)$. Hence

$$\alpha^{-i} \alpha_j \alpha^i = \alpha_j \alpha_{j+1}^{i p^{a_{i+1}-1}} \quad \text{for } 1 \leq j \leq k-1,\ 1 \leq i \leq p-1$$

and

$$\alpha^{-i} \alpha_j \alpha^i = \alpha_j \alpha_{j+1}^{\binom{i}{1}} \alpha_{j+2}^{\binom{i}{2}} \cdots \alpha_r^{\binom{i}{r-j}} \quad \text{for } k \leq j \leq r-1,\ 1 \leq i \leq p-1.$$

It follows that

$$\alpha_\ell : x_{\ell i} \mapsto \zeta_{p^{a_\ell}} x_{\ell i},\ x_{\ell+1 i} \mapsto \zeta^i x_{\ell+1 i},\ x_{ji} \mapsto x_{ji} \quad \text{for } 1 \leq \ell \leq k-1,\ j \neq \ell, \ell+1,$$

$$\alpha_k : x_{ki} \mapsto \zeta_{p^{a_k}} x_{ki},\ x_{wi} \mapsto \zeta^{\binom{i}{w-k}} x_{wi},\ x_{vi} \mapsto x_{vi} \quad \text{for } 1 \leq v \leq k-1,\ k+1 \leq w \leq r,$$

$$\alpha_m : x_{ui} \mapsto \zeta^{\binom{i}{u-m}} x_{ui},\ x_{vi} \mapsto x_{vi} \qquad \text{for } k+1 \le m \le r,\ 1 \le v \le m-1, m \le u \le r,$$

$$\alpha : x_{j0} \mapsto x_{j1} \mapsto \cdots \mapsto x_{jp-1} \mapsto \zeta_{p^{c_j}}^{b_j} x_{j0} \qquad \text{for } 1 \le j \le r,$$

where $0 \le i \le p-1$, and c_j, b_j are some integers such that $0 \le b_j < p^{c_j} \le p^{a_j}$.

Let $W_1 = \bigoplus_{j,i} K \cdot x_{ji} \subset V^*$. As noted at the start of the proof, we must find $W_1^{\mathcal{H}_1}$.

Step 2. For $1 \le j \le r$ and for $1 \le i \le p-1$ define $y_{ji} = x_{ji}/x_{ji-1}$. Thus $W_1 = K(x_{j0}, y_{ji} : 1 \le j \le r, 1 \le i \le p-1)$ and for every $g \in G$,

$$g \cdot x_{j0} \in K(y_{ji} : 1 \le j \le r, 1 \le i \le p-1) \cdot x_{j0} \quad \text{for } 1 \le j \le r,$$

while the subfield $K(y_{ji} : 1 \le j \le r, 1 \le i \le p-1)$ is invariant by the action of G, that is,

$$\alpha_\ell : y_{\ell+1i} \mapsto \zeta y_{\ell+1i},\ y_{ji} \mapsto y_{ji} \qquad \text{for } 1 \le \ell \le k-1,\ j \ne \ell+1,$$

$$\alpha_m : y_{ui} \mapsto \zeta^{\binom{i-1}{u-m-1}} y_{ui},\ y_{vi} \mapsto y_{vi} \qquad \text{for } k \le m \le r-1,\ 1 \le v \le m,\ m+1 \le u \le r,$$

$$\alpha_r : y_{vi} \mapsto y_{vi} \qquad \text{for } 1 \le v \le r,$$

$$\alpha : y_{j1} \mapsto y_{j2} \mapsto \cdots \mapsto y_{jp-1} \mapsto \zeta_{p^{c_j}}^{b_j} (y_{j1} \cdots y_{jp-1})^{-1} \qquad \text{for } 1 \le j \le r.$$

From Theorem 2.2 it follows that if $K(y_{ji} : 1 \le j \le r, 1 \le i \le p-1)^G$ is rational over K, so is $K(x_{j0}, y_{ji} : 1 \le j \le r, 1 \le i \le p-1)^G$ over K.

Since K contains a primitive p^e-th root of unity ζ_{p^e}, where p^e is the exponent of G, K contains as well a primitive p^{c_j+1}-th root of unity, and we may replace the variables y_{ji} by $y_{ji}/\zeta_{p^{c_j+1}}^{b_j}$ so that we obtain a more convenient action of α without changing the actions of the α_j. Namely we may assume that

$$\alpha :\ y_{j1} \mapsto y_{j2} \mapsto \cdots \mapsto y_{jp-1} \mapsto (y_{j1} y_{j2} \cdots y_{jp-1})^{-1} \quad \text{for } 1 \le j \le r.$$

Define $u_{r1} = y_{r1}^p$, $u_{ri} = y_{ri}/y_{ri-1}$ for $2 \le i \le p-1$. Then

$$K(y_{ji}, u_{ri} : 1 \le j \le r-1, 1 \le i \le p-1) = K(y_{ji} : 1 \le j \le r, 1 \le i \le p-1)^{\langle \alpha_{r-1} \rangle}.$$

From Theorem 2.2 it follows that if $K(y_{ji}, u_{ri} : 1 \le j \le r-1, 2 \le i \le p-1)^G$ is rational over K, so is $K(y_{ji}, u_{ri} : 1 \le j \le r-1, 1 \le i \le p-1)^G$ over K. We have the actions

$$\alpha_\ell : u_{ri} \mapsto u_{ri} \qquad \text{for } 1 \le \ell \le k-1,$$

$$\alpha_m : u_{ri} \mapsto \zeta^{\binom{i-2}{r-m-2}} u_{ri} \qquad \text{for } 2 \le i \le p-1, k \le m \le r-2,$$

$$\alpha : u_{r2} \mapsto u_{r3} \mapsto \cdots \mapsto u_{rp-1} \mapsto (u_{r1}u_{r2}^{p-1}u_{r3}^{p-2}\cdots u_{rp-1}^{2})^{-1} \mapsto u_{r1}u_{r2}^{p-2}u_{r3}^{p-3}\cdots u_{rp-2}^{2}u_{rp-1}.$$

For $2 \le i \le p-1$ define

$$v_{ri} = u_{ri}\, y_{r-1i}^{-1}\, y_{r-2i}\, y_{r-3i}^{-1} \cdots y_{k+2i}^{(-1)^{r-k}}\, y_{k+1i}^{(-1)^{r-k+1}},$$

and put $v_{r1} = u_{r1}$.

With the aid of the well known property $\binom{n}{m} - \binom{n-1}{m} = \binom{n-1}{m-1}$, it is not hard to verify the identity

$$\binom{i-2}{r-m-2} - \binom{i-1}{r-m-2} + \binom{i-1}{r-m-3} - \binom{i-1}{r-m-4} + \cdots$$
$$\cdots + (-1)^{r-m-1}\binom{i-1}{2} + (-1)^{r-m}\binom{i-1}{1} + (-1)^{r-m+1}\binom{i-1}{0} = 0.$$

It follows that

$$\alpha_m :\ v_{ri} \mapsto v_{ri} \qquad \text{for } 1 \le i \le p-1 \text{ and } 1 \le m \le r-2,$$
$$\alpha :\ v_{r2} \mapsto v_{r3} \mapsto \cdots \mapsto v_{rp-1} \mapsto A_r \cdot (v_{r1}v_{r2}^{p-1}v_{r3}^{p-2}\cdots v_{rp-1}^{2})^{-1}.$$

where A_r is some monomial in y_{ji} for $2 \le j \le r-1$, $1 \le i \le p-1$.

Define $u_{r-11} = y_{r-11}^{p}$, $u_{r-1i} = y_{r-1i}/y_{r-1i-1}$ for $2 \le i \le p-1$. Then

$$K(y_{ji}, u_{r-1i} : 1 \le j \le r-2,\ 1 \le i \le p-1) = K(y_{ji} : 1 \le j \le r-1,\ 1 \le i \le p-1)^{\langle \alpha_{r-2} \rangle}.$$

From Theorem 2.2 it follows that if $K(y_{ji}, u_{r-1i} : 1 \le j \le r-2, 2 \le i \le p-1)^G$ is rational over K, so is $K(y_{ji}, u_{r-1i} : 1 \le j \le r-2, 1 \le i \le p-1)^G$ over K. Similarly to the definition of v_{ri}, we can define v_{r-1i} so that $\alpha_m(v_{r-1i}) = v_{r-1i}$ for $2 \le i \le p-1$ and $1 \le m \le r-3$. It is obvious that we can proceed in the same way, defining elements $v_{r-2i}, v_{r-3i}, \ldots, v_{k+1i}$ such that α_m acts trivially on all the v_{ji} for $k \le m \le r-3$.

Recall that the actions of α_ℓ on the y_{ji} for $1 \le \ell \le k-1$ are

$$\alpha_\ell :\ y_{\ell+1i} \mapsto \zeta y_{\ell+1i},\ \ y_{ji} \mapsto y_{ji}, \quad \text{for } 1 \le i \le p-1, 1 \le \ell \le k-1,\ j \ne \ell+1.$$

For any $1 \le \ell \le k-1$ define $v_{\ell+11} = y_{\ell+11}^{p}$, $v_{\ell+1i} = y_{\ell+1i}/y_{\ell+1i-1}$, where $2 \le i \le p-1$. Put also $v_{1i} = y_{1i}$ for $1 \le i \le p-1$. Then

$$K(v_{ji} : 1 \le j \le r, 1 \le i \le p-1) = K(y_{ji} : 1 \le j \le r, 1 \le i \le p-1)^{\mathcal{H}_1}.$$

The action of α is given by

$$\alpha : v_{11} \mapsto v_{12} \mapsto \cdots \mapsto v_{1p-1} \mapsto (v_{11}v_{12}\cdots v_{1p-1})^{-1},\ \ v_{m1} \mapsto v_{m1}v_{m2}^{p},$$
$$v_{m2} \mapsto v_{m3} \mapsto \cdots \mapsto v_{mp-1} \mapsto A_m \cdot (v_{m1}v_{m2}^{p-1}v_{m3}^{p-2}\cdots v_{mp-1}^{2})^{-1},$$

for $2 \le m \le r$, where A_m is some monomial in $v_{k+1i}, \dots, v_{m-1i}$ for $k+2 \le m \le r$ and $A_2 = A_3 = \cdots = A_{k+1} = 1$. From Lemmas 2.4 and 2.5 it follows that the action of α on $K(v_{ji} : 1 \le j \le r, 1 \le i \le p-1)$ can be linearized.

Case II. Assume that $\mathcal{H}_1$ is of type 1; that is, $\mathcal{H}_1 \simeq (C_p)^{s+1}$ for some $s \ge 0$. Denote by $\beta_1, \dots, \beta_{s+1}$ the generators of $(C_p)^{s+1}$. According to Lemma 1.7, we have the relations $[\beta_j, \alpha] = \beta_{j+1}$ for $1 \le j \le s$ and $\beta_{s+1} \in Z(G)$.

Define $X_1, X_2, \dots, X_{s+1} \in V^*$ by

$$X_j = \sum_{\ell_1,\dots,\ell_{s+1}} x\Big(\prod_{m \neq j} \beta_m^{\ell_m}\Big)$$

for $1 \le j \le s+1$. Note that $\beta_j \cdot X_i = X_i$ for $j \neq i$. Let ζ be a primitive p-th root of unity. Define $Y_1, Y_2, \dots, Y_{s+1} \in V^*$ by

$$Y_j = \sum_{r=0}^{p-1} \zeta^{-r} \beta_j^r \cdot X_j$$

for $1 \le j \le s+1$.

It follows that

$$\beta_j : Y_j \mapsto \zeta Y_j,\ Y_i \mapsto Y_i \quad \text{for } i \neq j \text{ and } 1 \le j \le s+1.$$

Thus $V_1 = \bigoplus_{1 \le j \le s+1} K \cdot Y_j$ is a representation space of the subgroup $\mathcal{H}_1$.

Define $x_{ji} = \alpha^i \cdot Y_j$ for $1 \le j \le s+1, 0 \le i \le p-1$. Recall that $[\beta_j, \alpha] = \beta_{j-1}$. Hence

$$\alpha^{-i} \beta_j \alpha^i = \beta_j \beta_{j+1}^{\binom{i}{1}} \beta_{j+2}^{\binom{i}{2}} \cdots \beta_{s+1}^{\binom{i}{s+1-j}}.$$

It follows that

$$\begin{aligned}
&\beta_1 : x_{1i} \mapsto \zeta x_{1i},\ x_{ji} \mapsto \zeta^{\binom{i}{j-1}} x_{ji} && \text{for } 2 \le j \le s+1, 0 \le i \le p-1,\\
&\beta_j : x_{\ell i} \mapsto x_{\ell i},\ x_{mi} \mapsto \zeta^{\binom{i}{m-j}} x_{mi} && \text{for } 1 \le \ell \le j-1,\\
& && \quad j \le m \le s+1, 0 \le i \le p-1,\\
&\alpha : x_{j0} \mapsto x_{j1} \mapsto \cdots \mapsto x_{jp-1} \mapsto \zeta^{b_j} x_{j0} && \text{for } 1 \le j \le s+1, 0 \le b_j \le p-1.
\end{aligned}$$

Compare the actions of $\alpha, \beta_1, \dots, \beta_{s+1}$ with the actions of $\alpha, \alpha_k, \dots, \alpha_{k+s}$ from Case I, Step 1. They are almost the same. Apply the proof of Case I.

Case III. Assume that $\mathcal{H}_1$ is of type 2; that is, $\mathcal{H}_1 \simeq C_{p^a}$ for some $a \ge 1$. Denote by β the generator of C_{p^a}. Then $[\beta, \alpha] = \beta^{bp^{a-1}}$ for some $b : 0 \le b \le p-1$. Let $\zeta_{p^a} \in K$ be a primitive p^a-th root of unity, and let ζ be a primitive p-th root of unity. Define $X = \sum_i \zeta_{p^a}^{-i} x(\beta^i)$. Then $\beta(X) = \zeta_{p^a} X$, and define $x_i = \alpha^i \cdot X$ for

$0 \le i \le p-1$. It follows that

$$\begin{aligned} \beta &: x_i \mapsto \zeta_{p^a}\zeta^{ib}x_i && \text{for } 0 \le i \le p-1, \\ \alpha &: x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p-1} \mapsto \zeta_{p^a}^c x_0 && \text{for } 0 \le c \le p^a-1. \end{aligned}$$

Define $W_1 = \bigoplus_i K \cdot x_i \subset V^*$. For $1 \le i \le p-1$ define $y_i = x_i/x_{i-1}$. Thus $W_1 = K(x_0, y_i : 1 \le i \le p-1)$ and for every $g \in G$

$$g \cdot x_0 \in K(y_i : 1 \le i \le p-1) \cdot x_0,$$

while the subfield $K(y_i : 1 \le i \le p-1)$ is invariant by the action of G, that is,

$$\begin{aligned} \beta &: y_i \mapsto \zeta^b y_i && \text{for } 1 \le i \le p-1, \\ \alpha &: y_1 \mapsto y_2 \mapsto \cdots \mapsto \zeta_{p^a}^c (y_1 \cdots y_{p-1})^{-1} && \text{for } 0 \le c \le p^a-1. \end{aligned}$$

From Theorem 2.2 it follows that if $K(y_i : 1 \le i \le p-1)^G$ is rational over K, so is $K(x_0, y_i : 1 \le i \le p-1)^G$ over K.

Since K contains a primitive p^e-th root of unity ζ_{p^e}, where p^e is the exponent of G, K contains $\zeta_{p^{a+1}}^c$ as well. We may replace the variables y_i by $y_i/\zeta_{p^{a+1}}^c$ so that we obtain

$$\alpha : y_1 \mapsto y_2 \mapsto \cdots \mapsto y_{p-1} \mapsto (y_1 y_2 \cdots y_{p-1})^{-1}.$$

Define $u_1 = y_1^p$, $u_i = y_i/y_{i-1}$ for $2 \le i \le p-1$. Then $K(u_i : 1 \le i \le p-1) = K(y_i : 1 \le i \le p-1)^{\langle\beta\rangle}$. The action of α is given by

$$\alpha : u_1 \mapsto u_1 u_2^p,\ u_2 \mapsto u_3 \mapsto \cdots \mapsto u_{p-1} \mapsto (u_1 u_2^{p-1} u_3^{p-2} \cdots u_{p-1}^2)^{-1}.$$

From Lemma 2.4 (or 2.5) it follows that the action of α can be linearized.

Case IV. Assume that $\mathcal{H}_1$ is of type 4, that is, $\mathcal{H}_1 \simeq C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_k}}$ for some $k \ge 2$. Denote by $\alpha_1, \dots, \alpha_k$ the generators of $\mathcal{H}_1$. According to Lemma 1.7, we have the relations $[\alpha_i, \alpha] = \alpha_{i+1}^{p^{a_{i+1}-1}} \in Z(G)$ for $1 \le i \le k-1$ and $[\alpha_k, \alpha] = \prod_{j=1}^k \alpha_j^{p^{a_j-1}c_j} \in Z(G)$ for some $0 \le c_j \le p-1$.

Similarly to the previous cases, define $Y_1, Y_2, \dots, Y_k \in V^*$ so that

$$\alpha_i :\ Y_i \mapsto \zeta_{p^{a_i}} Y_i,\ Y_j \mapsto Y_j \quad \text{for } j \ne i \text{ and } 1 \le i \le k.$$

Thus $V_1 = \bigoplus_{1 \le j \le k} K \cdot Y_j$ is a faithful representation space of the subgroup $\mathcal{H}_1$.

Next, define $x_{ji} = \alpha^i \cdot Y_j$ for $1 \le j \le k$, $0 \le i \le p-1$. Note that

$$\alpha^{-i}\alpha_j\alpha^i = \alpha_j \alpha_{j+1}^{ip^{a_{j+1}-1}} \quad \text{for } 1 \le j \le k-1,\ 1 \le i \le p-1$$

and

$$\alpha^{-i}\alpha_k\alpha^i = \alpha_k \prod_{j=1}^k \alpha_j^{ip^{a_j-1}c_j} \quad \text{for } 1 \le i \le p-1.$$

It follows that

$$\begin{aligned}
&\alpha_\ell : x_{\ell i} \mapsto \zeta_{p^{a_\ell}} x_{\ell i},\ x_{\ell+1i} \mapsto \zeta^i x_{\ell+1i},\ x_{ji} \mapsto x_{ji} && \text{for } 1 \le \ell \le k-1,\ j \ne \ell, \ell+1,\\
&\alpha_k : x_{ki} \mapsto \zeta_{p^{a_k}} \zeta^{ic_k} x_{ki},\ x_{ji} \mapsto \zeta^{ic_j} x_{ji} && \text{for } 1 \le j \le k-1,\\
&\alpha : x_{j0} \mapsto x_{j1} \mapsto \cdots \mapsto x_{jp-1} \mapsto \zeta_{p^{a_j}}^{b_j} x_{j0} && \text{for } 1 \le j \le k,
\end{aligned}$$

where $0 \le i \le p-1$, $0 \le c_j \le p-1$ and $0 \le b_j \le p^{a_j} - 1$.

Define $W_1 = \bigoplus_{j,i} K \cdot x_{ji} \subset V^*$, and for $1 \le i \le p-1$ define $y_i = x_i / x_{i-1}$. Thus $W_1 = K(x_{j0}, y_{ji} : 1 \le j \le k, 1 \le i \le p-1)$ and for every $g \in G$,

$$g \cdot x_{j0} \in K(y_{ji} : 1 \le j \le k, 1 \le i \le p-1) \cdot x_{j0} \quad \text{for } 1 \le j \le k,$$

while the subfield $K(y_{ji} : 1 \le j \le k, 1 \le i \le p-1)$ is invariant by the action of G, that is,

$$\begin{aligned}
&\alpha_\ell : y_{\ell+1i} \mapsto \zeta y_{\ell+1i},\ y_{ji} \mapsto y_{ji} && \text{for } 1 \le i \le p-1,\\
& && \quad 1 \le \ell \le k-1,\ j \ne \ell+1,\\
&\alpha_k : y_{ji} \mapsto \zeta^{c_j} y_{ji} && \text{for } 1 \le i \le p-1,\ 1 \le j \le k,\\
&\alpha : y_{j1} \mapsto y_{j2} \mapsto \cdots \mapsto y_{jp-1} \mapsto \zeta_{p^{a_j}}^{b_j} (y_{j1} \cdots y_{jp-1})^{-1}.
\end{aligned}$$

From Theorem 2.2 it follows that if $K(y_{ji} : 1 \le j \le k, 1 \le i \le p-1)^G$ is rational over K, so is $K(x_{j0}, y_{ji} : 1 \le j \le k, 1 \le i \le p-1)^G$ over K. As before, we can again assume that α acts in this way:

$$\alpha :\ y_{j1} \mapsto y_{j2} \mapsto \cdots \mapsto y_{jp-1} \mapsto (y_{j1} y_{j2} \cdots y_{jp-1})^{-1}.$$

Now, assume that $0 < c_1 \le p-1$. For $2 \le j \le k$ choose e_j such that $c_1 e_j + c_j \equiv 0 \pmod p$, and define $u_{1i} = y_{1i}$, $u_{ji} = y_{1i}^{e_j} y_{ji}$. It follows that

$$\begin{aligned}
&\alpha_\ell : u_{\ell+1i} \mapsto \zeta u_{\ell+1i},\ u_{ji} \mapsto u_{ji} && \text{for } 1 \le i \le p-1,\\
& && \quad 1 \le \ell \le k-1,\ j \ne \ell+1,\\
&\alpha_k : u_{1i} \mapsto \zeta^{c_1} u_{1i},\ u_{ji} \mapsto u_{ji} && \text{for } 1 \le i \le p-1,\ 2 \le j \le k.
\end{aligned}$$

Define $v_{j1} = u_{j1}^p$, $v_{ji} = u_{ji}/u_{ji-1}$ for $2 \le i \le p-1$, $1 \le j \le k$. Then

$$K(v_{ji} : 1 \le j \le k, 1 \le i \le p-1) = K(u_{ji} : 1 \le j \le k, 1 \le i \le p-1)^{\mathcal{H}_1}.$$

The action of α is given by

$$\alpha : v_{j1} \mapsto v_{j1} v_{j2}^p,\ v_{j2} \mapsto v_{j3} \mapsto \cdots \mapsto v_{jp-1} \mapsto (v_{j1} v_{j2}^{p-1} v_{j3}^{p-2} \cdots v_{jp-1}^2)^{-1}$$

for $2 \le j \le k$. Lemma 2.5 implies the action of α on $K(v_{ji} : 1 \le j \le k, 1 \le i \le p-1)$ can be linearized.

Finally, let $c_1 = 0$. Define $v_{j1} = u_{j1}^p$, $v_{ji} = u_{ji}/u_{ji-1}$ for $2 \le i \le p-1$, $2 \le j \le k$. Then $K(u_{1i}, v_{ji} : 2 \le j \le k, 1 \le i \le p-1) = K(u_{ji} : 1 \le j \le k, 1 \le i \le p-1)^{\mathcal{H}_1}$. The action of α again can be linearized as before. We are done.

5. Proof of Theorem 1.9

By studying the classification of all groups of order p^5 made by James [1980], we see that the nonabelian groups with an abelian subgroup of index p and that are not direct products of smaller groups are precisely the groups from the isoclinic families with numbers 2, 3, 4 and 9. Notice that all these groups satisfy the conditions of Theorem 1.8. The isoclinic family 8 contains only the group $\Phi_8(32)$ which is metacyclic, so we can apply Theorem 1.2. It is not hard to see that there are no other groups of order p^5 containing a normal abelian subgroup H such that G/H is cyclic.

The groups of order p^6 with an abelian subgroup of index p and that are not direct products of smaller groups are precisely the groups from the isoclinic families with numbers 2, 3, 4 and 9. Again, all these groups satisfy the conditions of Theorem 1.8. The groups of order p^6, containing a normal abelian subgroup H such that G/H is cyclic of order $> p$ are precisely the groups from the isoclinic families with numbers 8 and 14. Note that the groups $\Phi_8(42)$, $\Phi_8(33)$, $\Phi_{14}(42)$ are metacyclic, and the group $\Phi_8(321)a$ is a direct product of the metacyclic group $\Phi_8(32)$ and the cyclic group C_p. Therefore, we need to consider the remaining groups, whose presentations we write down for convenience of the reader.

$$\Phi_8(321)b = \langle \alpha_1, \alpha_2, \beta, \gamma : [\alpha_1, \alpha_2] = \beta = \alpha_1^p,\ [\beta, \alpha_2] = \beta^p = \gamma^p,\ \alpha_2^{p^2} = \beta^{p^2} = 1 \rangle,$$
$$\Phi_8(321)c_r = \langle \alpha_1, \alpha_2, \beta : [\alpha_1, \alpha_2] = \beta,\ [\beta, \alpha_2]^{r+1} = \beta^{p(r+1)} = \alpha_1^{p^2},\ \alpha_2^{p^2} = \beta^{p^2} = 1 \rangle,$$
$$\Phi_8(321)c_{p-1} = \langle \alpha_1, \alpha_2, \beta : [\alpha_1, \alpha_2] = \beta,\ [\beta, \alpha_2] = \beta^p = \alpha_2^{p^2},\ \alpha_1^{p^2} = \beta^{p^2} = 1 \rangle,$$
$$\Phi_8(222) = \langle \alpha_1, \alpha_2, \beta : [\alpha_1, \alpha_2] = \beta,\ [\beta, \alpha_2] = \beta^p,\ \alpha_1^{p^2} = \alpha_2^{p^2} = \beta^{p^2} = 1 \rangle,$$
$$\Phi_{14}(321) = \langle \alpha_1, \alpha_2, \beta : [\alpha_1, \alpha_2] = \beta,\ \alpha_1^{p^2} = \beta^p,\ \alpha_2^{p^2} = \beta^{p^2} = 1 \rangle,$$
$$\Phi_{14}(222) = \langle \alpha_1, \alpha_2, \beta : [\alpha_1, \alpha_2] = \beta,\ \alpha_1^{p^2} = \alpha_2^{p^2} = \beta^{p^2} = 1 \rangle.$$

Case I. $G = \Phi_8(321)b$. Denote by H the abelian normal subgroup of G generated by α_1 and γ. Then $H = \langle \alpha_1, \gamma\beta^{-1} \rangle \simeq C_{p^3} \times C_p$ and $G/H = \langle \alpha_2 \rangle \simeq C_{p^2}$.

Let V be a K-vector space whose dual space V^* is defined as $V^* = \bigoplus_{g \in G} K \cdot x(g)$, where G acts on V^* by $h \cdot x(g) = x(hg)$ for any $h, g \in G$. Thus we have $K(V)^G = K(x(g) : g \in G)^G = K(G)$.

Define $X_1, X_2 \in V^*$ by

$$X_1 = \sum_{i=0}^{p-1} x((\gamma\beta^{-1})^i), \quad X_2 = \sum_{i=0}^{p^3-1} x(\alpha_1^i).$$

Note that $\gamma\beta^{-1} \cdot X_1 = X_1$ and $\alpha_1 \cdot X_2 = X_2$.

Let $\zeta_{p^3} \in K$ be a primitive p^3-th root of unity and put $\zeta = \zeta_{p^3}^{p^2}$, a primitive p-th root of unity. Define $Y_1, Y_2 \in V^*$ by

$$Y_1 = \sum_{i=0}^{p^3-1} \zeta_{p^3}^{-i} \alpha_1^i \cdot X_1, \quad Y_2 = \sum_{i=0}^{p-1} \zeta^{-i} (\gamma\beta^{-1})^i \cdot X_2.$$

It follows that

$$\begin{aligned} \alpha_1 &: Y_1 \mapsto \zeta_{p^3} Y_1, \;\; Y_2 \mapsto Y_2, \\ \gamma\beta^{-1} &: Y_1 \mapsto Y_1, \qquad Y_2 \mapsto \zeta Y_2, \\ \gamma &: Y_1 \mapsto \zeta_{p^2} Y_1, \;\; Y_2 \mapsto \zeta Y_2. \end{aligned}$$

Thus $K \cdot Y_1 + K \cdot Y_2$ is a representation space of the subgroup H.

Define $x_i = \alpha_2^i \cdot Y_1$, $y_i = \alpha_2^i \cdot Y_2$ for $0 \le i \le p^2 - 1$. From the relations $\alpha_1 \alpha_2^i = \alpha_2^i \alpha_1 \beta^i \beta^{\binom{i}{2} p}$ it follows that

$$\begin{aligned} \alpha_1 &: x_i \mapsto \zeta_{p^3} \zeta_{p^2}^i \zeta^{\binom{i}{2}} x_i, \;\; y_i \mapsto y_i, \\ \gamma &: x_i \mapsto \zeta_{p^2} x_i, \qquad\quad y_i \mapsto \zeta y_i, \\ \alpha_2 &: x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^2-1} \mapsto x_0, \\ &\quad\; y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p^2-1} \mapsto y_0, \end{aligned}$$

for $0 \le i \le p^2 - 1$.

We find that $Y = \left(\bigoplus_{0 \le i \le p^2-1} K \cdot x_i\right) \oplus \left(\bigoplus_{0 \le i \le p^2-1} K \cdot y_i\right)$ is a faithful G-subspace of V^*. Thus, by Theorem 2.1, it suffices to show that $K(x_i, y_i : 0 \le i \le p^2-1)^G$ is rational over K.

For $1 \le i \le p^2 - 1$, define $u_i = x_i / x_{i-1}$ and $v_i = y_i / y_{i-1}$. Thus

$$K(x_i, y_i : 0 \le i \le p^2 - 1) = K(x_0, y_0, u_i, v_i : 1 \le i \le p^2 - 1)$$

and for every $g \in G$

$$g \cdot x_0 \in K(u_i, v_i : 1 \le i \le p^2 - 1) \cdot x_0, \;\; g \cdot y_0 \in K(u_i, v_i : 1 \le i \le p^2 - 1) \cdot y_0,$$

while the subfield $K(u_i, v_i : 1 \le i \le p^2 - 1)$ is invariant by the action of G. Thus $K(x_i, y_i : 0 \le i \le p^2 - 1)^G = K(u_i, v_i : 1 \le i \le p^2 - 1)^G(u, v)$ for some u, v such that $\alpha_1(v) = \gamma(v) = \alpha_2(v) = v$ and $\alpha_1(u) = \gamma(u) = \alpha_2(u) = u$. We now have

$$\begin{aligned} \alpha_1 &: u_i \mapsto \zeta_{p^2} \zeta^{i-1} u_i, \;\; v_i \mapsto v_i, \\ \gamma &: u_i \mapsto u_i, \qquad\qquad v_i \mapsto v_i, \\ \alpha_2 &: u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p^2-1} \mapsto (u_1 u_2 \cdots u_{p^2-1})^{-1}, \\ &\quad\; v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p^2-1} \mapsto (v_1 v_2 \cdots v_{p^2-1})^{-1}, \end{aligned} \tag{5-1}$$

for $1 \le i \le p^2-1$. If $K(u_i, v_i : 1 \le i \le p^2-1)^G(u, v)$ is rational over K, it follows from Theorem 2.2 that $K(x_i, y_i : 0 \le i \le p^2-1)^G$ is rational over K.

Since γ acts trivially on $K(u_i, v_i : 1 \le i \le p^2-1)$, we find that

$$K(u_i, v_i : 1 \le i \le p^2-1)^G = K(u_i, v_i : 1 \le i \le p^2-1)^{\langle \alpha_1, \alpha_2 \rangle}.$$

Now, consider the metacyclic p-group

$$\widetilde{G} = \langle \sigma, \tau : \sigma^{p^3} = \tau^{p^2} = 1, \tau^{-1}\sigma\tau = \sigma^k, k = 1+p \rangle.$$

Define $X = \sum_{0 \le j \le p^3-1} \zeta_{p^3}^{-j} x(\sigma^j)$, $V_i = \tau^i X$ for $0 \le i \le p^2-1$. It follows that

$$\begin{aligned} \sigma &: V_i \mapsto \zeta_{p^3}^{k^i} V_i, \\ \tau &: V_0 \mapsto V_1 \mapsto \cdots \mapsto V_{p^2-1} \mapsto V_0. \end{aligned}$$

Note that $K(V_0, V_1, \ldots, V_{p^2-1})^{\widetilde{G}}$ is rational by Theorem 2.6.

Define $U_i = V_i/V_{i-1}$ for $1 \le i \le p^2-1$. Then $K(V_0, V_1, \ldots, V_{p^2-1})^{\widetilde{G}} = K(U_1, U_2, \ldots, U_{p^2-1})^{\widetilde{G}}(U)$, where

$$\begin{aligned} &\sigma : U \mapsto U, \quad U_i \mapsto \zeta_{p^3}^{k^i-k^{i-1}} U_i, \\ &\tau : U \mapsto U, \quad U_1 \mapsto U_2 \mapsto \cdots \mapsto U_{p^2-1} \mapsto (U_1 U_2 \cdots U_{p^2-1})^{-1}. \end{aligned}$$

Notice that $k^i - k^{i-1} = (1+p)^{i-1}p \equiv (1+(i-1)p)p \pmod{p^3}$, so $\zeta_{p^3}^{k^i-k^{i-1}} = \zeta_{p^2}^{1+(i-1)p}$. Compare the first and third entries of (5-1) (i.e., the actions of α_1, α_2 on $K(u_i : 1 \le i \le p^2-1)$) with the actions of $\widetilde{G}$ on $K(U_i : 1 \le i \le p^2-1)$. They are the same. Hence, according to Theorem 2.6, we get that $K(u_1, \ldots, u_{p^2-1})^G(u) \cong K(U_1, \ldots, U_{p^2-1})^{\widetilde{G}}(U) = K(V_0, V_1, \ldots, V_{p^2-1})^{\widetilde{G}}$ is rational over K. Since by Lemma 2.4 we can linearize the action of α_2 on $K(v_i : 1 \le i \le p^2-1)$, we finally obtain that $K(u_i, v_i : 1 \le i \le p^2-1)^{\langle \alpha_1, \alpha_2 \rangle}$ is rational over K.

Case II. $G = \Phi_8(321)c_r$. Denote by H the abelian normal subgroup of G generated by α_1 and β. Then $H = \langle \alpha_1, \alpha_1^{-p}\beta^{r+1} \rangle \simeq C_{p^3} \times C_p$ and $G/H = \langle \alpha_2 \rangle \simeq C_{p^2}$. Let $a = (r+1)^{-1} \in \mathbb{Z}_{p^2}$, hence $\beta = \alpha_1^{ap}(\alpha_1^{-p}\beta^{r+1})^a$. Similarly to Case I, we can define $Y_1, Y_2 \in V^*$ such that

$$\begin{aligned} \alpha_1 &: Y_1 \mapsto \zeta_{p^3} Y_1, \quad Y_2 \mapsto Y_2, \\ \alpha_1^{-p}\beta^{r+1} &: Y_1 \mapsto Y_1, \qquad Y_2 \mapsto \zeta Y_2, \\ \beta &: Y_1 \mapsto \zeta_{p^2}^a Y_1, \quad Y_2 \mapsto \zeta^a Y_2. \end{aligned}$$

Thus $K \cdot Y_1 + K \cdot Y_2$ is a representation space of the subgroup H.

Define $x_i = \alpha_2^i \cdot Y_1$, $y_i = \alpha_2^i \cdot Y_2$ for $0 \le i \le p^2-1$. From the relations $\alpha_1\alpha_2^i = \alpha_2^i\alpha_1\beta^i\beta^{\binom{i}{2}p}$ and $\beta\alpha_2^i = \alpha_2^i\beta^{1+ip}$ it follows that, for $0 \le i \le p^2-1$,

$$\begin{aligned}
\alpha_1 : x_i &\mapsto \zeta_{p^3}\zeta_{p^2}^{ai}\zeta^{a\binom{i}{2}}x_i, \quad y_i \mapsto \zeta^{ai}y_i,\\
\beta : x_i &\mapsto \zeta_{p^2}^{a}\zeta^{ai}x_i, \quad y_i \mapsto \zeta^{a}y_i,\\
\alpha_2 : x_0 &\mapsto x_1 \mapsto \cdots \mapsto x_{p^2-1} \mapsto x_0,\\
y_0 &\mapsto y_1 \mapsto \cdots \mapsto y_{p^2-1} \mapsto y_0.
\end{aligned}$$

We find that $Y = \left(\bigoplus_{0\le i\le p^2-1} K\cdot x_i\right)\oplus\left(\bigoplus_{0\le i\le p^2-1} K\cdot y_i\right)$ is a faithful G-subspace of V^*. Thus, by Theorem 2.1, it suffices to show that $K(x_i, y_i : 0\le i\le p^2-1)^G$ is rational over K.

For $1 \le i \le p^2-1$, define $u_i = x_i/x_{i-1}$ and $v_i = y_i/y_{i-1}$. We now have

$$\begin{aligned}
\alpha_1 : u_i &\mapsto \zeta_{p^2}^{a}\zeta^{a(i-1)}u_i, \quad v_i \mapsto \zeta^{a}v_i,\\
\beta : u_i &\mapsto \zeta^{a}u_i, \quad v_i \mapsto v_i,\\
\alpha_2 : u_1 &\mapsto u_2 \mapsto \cdots \mapsto u_{p^2-1} \mapsto (u_1u_2\cdots u_{p^2-1})^{-1},\\
v_1 &\mapsto v_2 \mapsto \cdots \mapsto v_{p^2-1} \mapsto (v_1v_2\cdots v_{p^2-1})^{-1},
\end{aligned}$$

for $1\le i\le p^2-1$. Theorem 2.2 implies that if $K(u_i, v_i : 1\le i\le p^2-1)^G(u, v)$ is rational over K, so is $K(x_i, y_i : 0\le i\le p^2-1)^G$ over K.

Since β acts in the same way as α_1^p on $K(u_i, v_i : 1\le i\le p^2-1)$, we find that $K(u_i, v_i : 1\le i\le p^2-1)^G = K(u_i, v_i : 1\le i\le p^2-1)^{\langle\alpha_1,\alpha_2\rangle}$.

For $1\le i\le p^2-1$ define $V_i = v_i/u_i^p$. It follows that, for $1\le i\le p^2-1$,

$$\begin{aligned}
\alpha_1 :\ & u_i \mapsto \zeta_{p^2}^{a}\zeta^{a(i-1)}u_i, \quad V_i \mapsto V_i,\\
\alpha_2 :\ & u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p^2-1} \mapsto (u_1u_2\cdots u_{p^2-1})^{-1},\\
& V_1 \mapsto V_2 \mapsto \cdots \mapsto V_{p^2-1} \mapsto (V_1V_2\cdots V_{p^2-1})^{-1}.
\end{aligned}\tag{5-2}$$

Compare (5-2) with (5-1). They look almost the same. Apply the proof of Case I.

Case III. $G = \Phi_8(321)c_{p-1}$. Denote by H the abelian normal subgroup of G generated by α_1 and β. Then $H \simeq C_{p^2}\times C_{p^2}$ and $G/H \simeq C_{p^2}$. Similarly to Case I, we can define $Y_1, Y_2 \in V^*$ such that

$$\begin{aligned}
\alpha_1 : Y_1 &\mapsto \zeta_{p^2}Y_1, \quad Y_2 \mapsto Y_2,\\
\beta : Y_1 &\mapsto Y_1, \quad Y_2 \mapsto \zeta_{p^2}Y_2.
\end{aligned}$$

Thus $K\cdot Y_1 + K\cdot Y_2$ is a representation space of the subgroup H.

Define $x_i = \alpha_2^i\cdot Y_1$, $y_i = \alpha_2^i\cdot Y_2$ for $0\le i\le p^2-1$. From the relations $\alpha_1\alpha_2^i = \alpha_2^i\alpha_1\beta^i\beta^{\binom{i}{2}p}$ and $\beta\alpha_2^i = \alpha_2^i\beta^{1+ip}$ it follows that, for $0\le i\le p^2-1$,

$$\begin{aligned}
\alpha_1 : x_i &\mapsto \zeta_{p^2}x_i, \quad y_i \mapsto \zeta_{p^2}^{i}\zeta^{\binom{i}{2}}y_i,\\
\beta : x_i &\mapsto x_i, \qquad y_i \mapsto \zeta_{p^2}\zeta^{i}y_i,
\end{aligned}$$

$$\alpha_2 : x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^2-1} \mapsto x_0,$$
$$y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p^2-1} \mapsto \zeta y_0.$$

For $1 \le i \le p^2-1$, define $u_i = x_i/x_{i-1}$ and $v_i = y_i/y_{i-1}$. We now have

$$\begin{aligned}
\alpha_1 &: u_i \mapsto u_i, \quad v_i \mapsto \zeta_{p^2}\zeta^{i-1} v_i,\\
\beta &: u_i \mapsto u_i, \quad v_i \mapsto \zeta v_i,\\
\alpha_2 &: u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p^2-1} \mapsto (u_1u_2\cdots u_{p^2-1})^{-1},\\
&\quad v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p^2-1} \mapsto \zeta(v_1v_2\cdots v_{p^2-1})^{-1},
\end{aligned}$$

for $1 \le i \le p^2-1$. Since β acts in the same way as α_1^p on $K(u_i, v_i : 1 \le i \le p^2-1)$, we find that $K(u_i, v_i : 1 \le i \le p^2-1)^G = K(u_i, v_i : 1 \le i \le p^2-1)^{\langle \alpha_1, \alpha_2 \rangle}$.

Let $\zeta_{p^3} \in K$ be a primitive p^3-th root of unity such that $\zeta_{p^3}^{p^2} = \zeta$. For $1 \le i \le p^2-1$ define $w_i = v_i/\zeta_{p^3}$. It follows that

$$\begin{aligned}
\alpha_1 &: \ u_i \mapsto u_i, \quad w_i \mapsto \zeta_{p^2}\zeta^{i-1} w_i,\\
\alpha_2 &: \ u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p^2-1} \mapsto (u_1u_2\cdots u_{p^2-1})^{-1},\\
&\quad w_1 \mapsto w_2 \mapsto \cdots \mapsto w_{p^2-1} \mapsto (w_1w_2\cdots w_{p^2-1})^{-1},
\end{aligned} \tag{5-3}$$

for $1 \le i \le p^2-1$. Compare (5-3) with (5-1) or (5-2). They look almost the same. Apply the proof of Case I.

Case IV. $G = \Phi_8(222)$. Denote by H the abelian normal subgroup of G generated by α_1 and β. Then $H \simeq C_{p^2} \times C_{p^2}$ and $G/H \simeq C_{p^2}$. The proof henceforth is almost the same as Case III.

Case V. $G = \Phi_{14}(321)$. Denote by H the abelian normal subgroup of G generated by α_2 and β. Then $H \simeq C_{p^2} \times C_{p^2}$ and $G/H \simeq C_{p^2}$.

As before, we can define $Y_1, Y_2 \in V^*$ such that

$$\begin{aligned}
\alpha_2 &: \ Y_1 \mapsto \zeta_{p^2} Y_1, \quad Y_2 \mapsto Y_2,\\
\beta &: \ Y_1 \mapsto Y_1, \quad Y_2 \mapsto \zeta_{p^2} Y_2.
\end{aligned}$$

Thus $K \cdot Y_1 + K \cdot Y_2$ is a representation space of the subgroup H.

Define $x_i = \alpha_1^i \cdot Y_1$, $y_i = \alpha_1^i \cdot Y_2$ for $0 \le i \le p^2-1$. From the relations $\alpha_2\alpha_1^i = \alpha_1^i\alpha_2\beta^{-i}$ it follows that, for $0 \le i \le p^2-1$,

$$\begin{aligned}
\alpha_2 &: x_i \mapsto \zeta_{p^2} x_i, \quad y_i \mapsto \zeta_{p^2}^{-i} y_i,\\
\beta &: x_i \mapsto x_i, \qquad y_i \mapsto \zeta_{p^2} y_i,\\
\alpha_1 &: x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^2-1} \mapsto x_0,\\
&\quad y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p^2-1} \mapsto \zeta y_0.
\end{aligned}$$

For $1 \leq i \leq p^2-1$, define $u_i = x_i/x_{i-1}$ and $v_i = y_i/y_{i-1}$. We now have

$$\begin{aligned}
\alpha_2 &: u_i \mapsto u_i, \quad v_i \mapsto \zeta_{p^2}^{-1} v_i, \\
\beta &: u_i \mapsto u_i, \quad v_i \mapsto v_i, \\
\alpha_1 &: u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p^2-1} \mapsto (u_1 u_2 \cdots u_{p^2-1})^{-1}, \\
&\quad v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p^2-1} \mapsto \zeta (v_1 v_2 \cdots v_{p^2-1})^{-1},
\end{aligned}$$

for $1 \leq i \leq p^2-1$. Since β acts trivially on $K(u_i, v_i : 1 \leq i \leq p^2-1)$, we find that $K(u_i, v_i : 1 \leq i \leq p^2-1)^G = K(u_i, v_i : 1 \leq i \leq p^2-1)^{\langle \alpha_1, \alpha_2 \rangle}$.

Define $w_1 = v_1^{p^2} \zeta^{-1}$, $w_i = v_i/v_{i-1}$ for $2 \leq i \leq p^2-1$. We now have

$$K(v_1, \ldots, v_{p^2-1})^{\langle \alpha_2 \rangle} = K(w_1, \ldots, w_{p^2-1})$$

and

$$\begin{aligned}
\alpha_1 &: w_1 \mapsto w_2^{p^2} w_1, \\
&\quad w_2 \mapsto w_3 \mapsto \cdots \mapsto w_{p^2-1} \mapsto 1/(w_1 w_2^{p^2-1} w_3^{p^2-2} \cdots w_{p^2-1}^2).
\end{aligned}$$

Define $z_1 = w_2$, $z_i = \alpha_1^{i-1} \cdot w_2$ for $2 \leq i \leq p^2-1$. Then $K(w_i : 1 \leq i \leq p^2-1) = K(z_i : 1 \leq i \leq p^2-1)$ and

$$\alpha_1 : \; z_1 \mapsto z_2 \mapsto \cdots \mapsto z_{p^t-1} \mapsto (z_1 z_2 \cdots z_{p^2-1})^{-1}.$$

The action of α_1 can be linearized by Lemma 2.4. Thus $K(u_i, z_i : 1 \leq i \leq p^2-1)^{\langle \alpha_1 \rangle}$ is rational over K by Theorem 2.1. We are done.

Case VI. $G = \Phi_{14}(222)$. Denote by H the abelian normal subgroup of G generated by α_2 and β. Then $H \simeq C_{p^2} \times C_{p^2}$ and $G/H \simeq C_{p^2}$. The proof henceforth is almost the same as Case V.

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