Let $G$ be a connected reductive algebraic group defined over a number field $k$. In this paper, we introduce the Ryshkov domain $R$ for the arithmetical minimum function $m_Q$ defined from a height function associated to a maximal $k$-parabolic subgroup $Q$ of $G$. The domain $R$ is a $Q(k)$-invariant subset of the adele group $G(\mathbb{A})$. We show that a fundamental domain $Q(k)\backslash R$ yields a fundamental domain for $G(k)\backslash G(\mathbb{A})$. We also see that any local maximum of $m_Q$ is attained on the boundary of $\Omega$.

Introduction

Let $P_n$ be the cone of positive definite $n$ by $n$ real symmetric matrices, and let $m(A)$ be the arithmetical minimum $\min_{0 \neq x \in \mathbb{Z}^n} \langle x, Ax \rangle$ of $A \in P_n$. The function $f : A \mapsto m(A)/(\det A)^{1/n}$ on $P_n$ is called the Hermite invariant. Since the maximum of $f$ gives the Hermite constant $\gamma_n$ for dimension $n$, the determination of local maxima of $f$ is a fundamental problem of lattice sphere packings in Euclidean spaces and the arithmetic theory of quadratic forms. Voronoi’s theorem [1908, Théorème 17] states that $f$ attains a local maximum at a point $A$ if and only if $A$ is perfect and eutactic. Moreover, perfect forms play an essential role in Voronoi’s reduction theory of $P_n$ with respect to the action of $GL_n(\mathbb{Z})$ (see, e.g., [Martinet 2003] and [Schürmann 2009]). Ryshkov [1970] introduced a locally finite polyhedron $R(m)$ in $P_n$ defined by the condition $m(A) \geq 1$. It is not difficult to show that $A$ is perfect with $m(A) = 1$ if and only if $A$ is a vertex of the boundary of $R(m)$. In particular, any local maximum of the Hermite invariant $f$ is attained on the boundary of $R(m)$. In this sense, we can say that the Ryshkov polyhedron $R(m)$ is well matched with $f$.

Let $G$ be a connected isotropic reductive algebraic group defined over a number field $k$, and let $Q$ be a maximal $k$-parabolic subgroup of $G$. In previous papers [Watanabe 2000; 2003], we investigated a constant $\gamma(G, Q, k)$ as a generalization of Hermite’s constant $\gamma_n$. Precisely, the constant $\gamma(G, Q, k)$ is defined to be

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the maximum of the function $m_Q(g) = \min_{x \in Q(k)\backslash G(k)} H_Q(xg)$ on $G(k)\backslash G(\mathbb{A})^1$, where $H_Q$ denotes the height function associated to $Q$. To prove the existence of the maximum of $m_Q$, we used Borel and Harish-Chandra’s reduction theory for the adele group $G(\mathbb{A})$ with respect to $G(k)$. However, a Siegel set in $G(\mathbb{A})$ is not well matched with $m_Q$ in a sense that one cannot obtain any information on locations of extreme points of $m_Q$ in a Siegel set.

The purpose of this paper is to construct a fundamental domain of $G(\mathbb{A})^1$ with respect to $G(k)$ which is well matched with $m_Q$. We first consider an analog of the Ryskoff polyhedron. We set $X_Q(g) = \{x \in Q(k)\backslash G(k) : m_Q(g) = H_Q(xg)\}$ for a given $g \in G(\mathbb{A})^1$. This is a finite subset of $Q(k)\backslash G(k)$ and is regarded as an analog of the set of minimal vectors of a positive definite real quadratic form. We define the domain $R(m_Q)$ as follows:

$$R(m_Q) = \{g \in G(\mathbb{A})^1 : \tilde{e} \in X_Q(g)\},$$

where $\tilde{e}$ denotes the trivial class $(k)$ in $Q(k)\backslash G(k)$. The set $R(m_Q)$ is a left $Q(k)$-invariant closed set with nonempty interior. The interior of $R(m_Q)$ is just a subset $R_1$ consisting of $g \in R(m_Q)$ such that $X_Q(g)$ is the one-point set $\{\tilde{e}\}$. We denote by $R_1^{-}$ the closure of $R_1$ in $G(\mathbb{A})^1$. Both $R_1$ and $R_1^{-}$ are also left $Q(k)$-invariant. By Baer and Levi’s theorem [1931, Satz 7], there exists an open fundamental domain $\Omega_Q$ of $R_1^{-}$ with respect to $Q(k)$, that is, $\Omega_Q$ is a relatively open subset of $R_1^{-}$ satisfying

- $Q(k)\Omega_Q^{-} = R_1^{-}$, where $\Omega_Q^{-}$ denotes the closure of $\Omega_Q$ in $R_1^{-}$, and
- $\gamma \Omega_Q \cap \Omega_Q^{-} = \emptyset$ for any $\gamma \in Q(k)\backslash \{e\}$.

Let $\Omega_Q^\circ$ denote the interior of $\Omega_Q$ in $G(\mathbb{A})^1$. Then our main theorem is stated as follows:

**Theorem.** The set $\Omega_Q^\circ$ is an open fundamental domain of $G(\mathbb{A})^1$ with respect to $G(k)$. Any local maximum of $m_Q$ is attained on the intersection of the boundary of $\Omega_Q^\circ$ and the boundary of $R_1^{-}$.

If we denote by $r_G$ the k-rank of the commutator subgroup of $G$, then $G$ has $r_G$ standard maximal $k$-parabolic subgroups. Since $\Omega_Q$ depends on $Q$, we obtain $r_G$ different kinds of fundamental domains of $G(\mathbb{A})^1$ with respect to $G(k)$. The method to construct $\Omega_Q$ may be viewed as a generalization of the highest point method (see [Grenier 1988] and [Terras 1988, §4,4]). For example, let $k = \mathbb{Q}$, $G = \text{GL}_n$ and $Q$ be a standard maximal $\mathbb{Q}$-parabolic subgroup such that $Q\backslash G$ is a projective space. Then our construction gives a fundamental domain $\Omega_Q$ whose Archimedean part is isomorphic with Grenier’s fundamental domain. If we choose another standard maximal $\mathbb{Q}$-parabolic subgroup of $\text{GL}_n$ as $Q$, then the
Archimedean part of \( \Omega_Q \) yields a new kind of fundamental domain of \( P_n \) with respect to \( \text{GL}_n(\mathbb{Z}) \) (see Example 3 in Section 7).

**Notation.** For a given ring \( \mathfrak{A} \), the set of all \( n \) by \( k \) matrices with entries in \( \mathfrak{A} \) is denoted by \( M_{n,k}(\mathfrak{A}) \). We write \( M_n(\mathfrak{A}) \) for \( M_{n,n}(\mathfrak{A}) \). The transpose of a given matrix \( a \in M_{n,k}(\mathfrak{A}) \) is denoted by \( a' \). In this paper, \( k \) denotes an algebraic number field of finite degree over \( \mathbb{Q} \) and \( \mathfrak{o} \) the ring of integers of \( k \). The sets of all infinite and finite places of \( k \) are denoted by \( \mathfrak{p}_\infty \) and \( \mathfrak{p}_f \), respectively. For \( \sigma \in \mathfrak{p}_\infty \cup \mathfrak{p}_f \), \( k_\sigma \) denotes the completion of \( k \) at \( \sigma \). For \( \sigma \in \mathfrak{p}_f \), \( \mathfrak{o}_\sigma \) denotes the closure of \( \mathfrak{o} \) in \( k_\sigma \). The étale \( \mathbb{R} \)-algebra \( k_\infty = k \otimes_{\mathbb{Q}} \mathbb{R} \) is identified with \( \prod_{\sigma \in \mathfrak{p}_\infty} k_\sigma \). Let \( \mathfrak{A} \) and \( \mathfrak{A}^\times \) denote the adele ring and the idèle group of \( k \), respectively. The idèle norm of \( \mathfrak{A}^\times \) is denoted by \( | \cdot |_{\mathfrak{A}} \).

1. **Height functions**

Let \( G \) be a connected affine algebraic group defined over \( k \). For any \( k \)-algebra \( \mathfrak{A} \), \( G(\mathfrak{A}) \) stands for the set of \( \mathfrak{A} \)-rational points of \( G \). Let \( X^*(G)_k \) be the free \( \mathbb{Z} \)-module consisting of all \( k \)-rational characters of \( G \). For each \( g \in G(\mathfrak{A}) \), we define the homomorphism \( \partial_G(g) : X^*(G)_k \to \mathbb{R}_{>0} \) by \( \partial_G(g)(\chi) = |\chi(g)|_{\mathfrak{A}} \) for \( \chi \in X^*(G)_k \). Then \( \partial_G \) is a homomorphism from \( G(\mathfrak{A}) \) into \( \text{Hom}_{\mathbb{R}}(X^*(G)_k, \mathbb{R}_{>0}) \). We write \( G(\mathfrak{A})^{1} \) for the kernel of \( \partial_G \).

In the following, let \( G \) be a connected isotropic reductive group defined over \( k \). We fix a maximal \( k \)-split torus \( S \) of \( G \) and a minimal \( k \)-parabolic subgroup \( P_0 \) of \( G \) containing \( S \). Denote by \( \Phi_k \) and \( \Delta_k \) the relative root system of \( G \) with respect to \( S \) and the set of simple roots of \( \Phi_k \) corresponding to \( P_0 \), respectively. Let \( M_0 \) be the centralizer of \( S \) in \( G \). Then \( P_0 \) has a Levi decomposition \( P_0 = M_0 U_0 \), where \( U_0 \) is the unipotent radical of \( P_0 \). A \( k \)-parabolic subgroup of \( G \) containing \( P_0 \) is called a standard \( k \)-parabolic subgroup of \( G \). Every standard \( k \)-parabolic subgroup \( R \) of \( G \) has a unique Levi subgroup \( M_R \) containing \( M_0 \). We denote by \( U_R \) the unipotent radical of \( R \) and by \( Z_R \) the greatest central \( k \)-split torus in \( M_R \). Throughout this paper, we fix a maximal compact subgroup \( K = \prod_{\sigma \in \mathfrak{p}_\infty} K_\sigma \times \prod_{\sigma \in \mathfrak{p}_f} K_\sigma \) of \( G(\mathfrak{A}) \) satisfying the following property: for every standard \( k \)-parabolic subgroup \( R \) of \( G \), \( K \cap M_R(\mathfrak{A}) \) is a maximal compact subgroup of \( M_R(\mathfrak{A}) \), and \( M_R(\mathfrak{A}) \) possesses an Iwasawa decomposition \((M_R(\mathfrak{A}) \cap U_0(\mathfrak{A})) M_0(\mathfrak{A}) (K \cap M_R(\mathfrak{A})) \).

Let \( Q \) be a standard proper maximal \( k \)-parabolic subgroup of \( G \). There is only one simple root \( \alpha_0 \in \Delta_k \) such that the restriction of \( \alpha_0 \) to \( Z_Q \) is nontrivial. Let \( n_Q \) be the positive integer such that \( n_Q^{-1} \alpha_0 \) is a \( \mathbb{Z} \)-basis of \( X^*(Z_Q/Z_G)_k \). We write \( \alpha_Q \) for \( n_Q^{-1} \alpha_0 |_{Z_Q} \) and \( \hat{\alpha}_Q \) for \( n_Q^{-1} \alpha_0 |_{Z_Q} \), where

\[
\hat{\alpha}_Q = [X^*(Z_Q/Z_G)_k : X^*(M_Q/Z_G)_k].
\]

Then \( \hat{\alpha}_Q \) is a \( \mathbb{Z} \)-basis of the submodule \( X^*(M_Q/Z_G)_k \) of \( X^*(Z_Q/Z_G)_k \). Define
the map \( z_Q : G(\mathbb{A}) \to Z_G(\mathbb{A})M_Q(\mathbb{A})^1 \setminus M_Q(\mathbb{A}) \) by \( z_Q(g) = Z_G(\mathbb{A})M_Q(\mathbb{A})^1m \) if \( g = umh \) with \( u \in U_Q(\mathbb{A}), \ m \in M_Q(\mathbb{A}) \) and \( h \in K \). This is well defined and left \( Z_G(\mathbb{A})Q(\mathbb{A})^1 \)-invariant. Since \( Z_G(\mathbb{A})^1 = Z_G(\mathbb{A}) \cap G(\mathbb{A})^1 \subset M_Q(\mathbb{A}) \), \( z_Q \) gives rise to a map from \( Y_Q = Q(\mathbb{A})^1 \setminus G(\mathbb{A})^1 \to M_Q(\mathbb{A})^1 \setminus (M_Q(\mathbb{A}) \cap G(\mathbb{A})^1) \). Namely, we have the following commutative diagram, whose vertical arrows are natural maps:

\[
\begin{array}{ccc}
Y_Q & \xrightarrow{z_Q} & M_Q(\mathbb{A})^1 \setminus (M_Q(\mathbb{A}) \cap G(\mathbb{A})^1) \\
\downarrow & & \downarrow \\
Z_G(\mathbb{A})Q(\mathbb{A})^1 \setminus G(\mathbb{A}) & \xrightarrow{z_Q} & Z_G(\mathbb{A})M_Q(\mathbb{A})^1 \setminus M_Q(\mathbb{A}).
\end{array}
\]

We define the height function \( H_Q : G(\mathbb{A}) \to \mathbb{R}_{>0} \) by \( H_Q(g) = |\hat{\alpha}_Q(z_Q(g))|_\alpha^{-1} \) for \( g \in G(\mathbb{A}) \). We notice that the restriction of \( H_Q \) to \( M_Q(\mathbb{A})^1 \) is a homomorphism from \( M_Q(\mathbb{A})^1 \) onto \( \mathbb{R}_{>0} \).

**Example 1.** Let \( G \) be a general linear group \( GL_n \) defined over the rational number field \( \mathbb{Q} \), \( P_0 \) the group of upper triangular matrices in \( G \) and \( S \) the group of diagonal matrices in \( G \). We fix an integer \( k \in \{1, \ldots, n-1\} \), and let

\[
Q(\mathbb{Q}) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a \in GL_k(\mathbb{Q}), \ b \in M_{k,n-k}(\mathbb{Q}), \ d \in GL_{n-k}(\mathbb{Q}) \right\}.
\]

Then \( Q \) is a standard maximal \( \mathbb{Q} \)-parabolic subgroup of \( G \). The rational character \( \hat{\alpha}_Q \) and the height \( H_Q \) are given by

\[
\hat{\alpha}_Q \left( \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right) = (\det a)^{(n-k)/r} (\det d)^{-k/r}
\]

and

\[
H_Q \left( \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right) = |\det a|_{\alpha}^{-(n-k)/r} |\det d|_{\alpha}^{k/r},
\]

where \( r \) denotes the greatest common divisor of \( k \) and \( n-k \). The height \( H_Q \) has another expression. To explain this, let \( \mathbb{Q}^n \) be an \( n \)-dimensional column vector space over \( \mathbb{Q} \) with standard basis \( e_1, \ldots, e_n \). The maximal parabolic subgroup \( Q(\mathbb{Q}) \) stabilizes the subspace spanned by \( e_1, \ldots, e_k \). Let \( V_{n,k}(\mathbb{Q}) = \wedge^k \mathbb{Q}^n \) be the \( k \)-th exterior product of \( \mathbb{Q}^n \). We set \( V_{n,k}(\mathbb{A}) = V_{n,k}(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{A} \) and \( V_{n,k}(\mathbb{Q}_\sigma) = V_{n,k}(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\sigma \) for \( \sigma \in p_\infty \cup p_f \). A \( \mathbb{Q} \)-basis of \( V_{n,k}(\mathbb{Q}) \) is formed by the elements \( e_I = e_{i_1} \wedge \cdots \wedge e_{i_k} \) with \( I = \{ i_1 < i_2 < \cdots < i_k \} \subset \{1, \ldots, n\} \). For a unique infinite place \( \infty \in p_\infty \), we define the local height \( H_\infty : V_{n,k}(\mathbb{Q}_\infty) \to \mathbb{R}_{>0} \) by

\[
H_\infty \left( \sum_I a_I e_I \right) = \left( \sum_I |a_I|_\infty^2 \right)^{1/2},
\]
where $| \cdot |_{\infty}$ denotes the usual absolute value of $\mathbb{Q}_{\infty} = \mathbb{R}$. For each finite prime $p \in P_f$, we define the local height $H_p : V_{n,k}(\mathbb{Q}_p) \to \mathbb{R}_{>0}$ by

$$H_p\left(\sum_I a_I e_I\right) = \sup_I |a_I|_p,$$

where $| \cdot |_p$ denotes the $p$-adic absolute value of $\mathbb{Q}_p$ normalized so that $|p|_p = p^{-1}$. Then the global height $H_{n,k} : V_{n,k}(\mathbb{Q}) \to \mathbb{R}_{>0}$ is defined to be a product of all local heights, that is, $H_{n,k}(x) = \prod_{\sigma \in \mathbb{P}_\infty \cup P_f} H_\sigma(x)$ for $x \in V_{n,k}(\mathbb{Q})$. This $H_{n,k}$ is immediately extended to the subset $GL(V_{n,k}(\mathbb{A}))V_{n,k}(\mathbb{Q})$ of the adele space $V_{n,k}(\mathbb{A})$ by

$$H_{n,k}(Ax) = \prod_{\sigma \in \mathbb{P}_\infty \cup P_f} H_\sigma(A\sigma x)$$

for $A = (A_\sigma) \in GL(V_{n,k}(\mathbb{A}))$ and $x \in V_{n,k}(\mathbb{Q})$. In particular, for $g \in G(\mathbb{A}) = GL_n(\mathbb{A})$, we can take the value $H_{n,k}(ge_1 \wedge ge_2 \wedge \cdots \vee ge_k)$. We choose a maximal compact subgroup $K_\infty$ of $G(\mathbb{Q}_\infty)$ as $\{g \in G(\mathbb{Q}_\infty) : |g|^{-1} = g\}$. Let

$$K_f = \prod_{p \in P_f} GL_n(\mathbb{Z}_p) \quad \text{and} \quad K = K_\infty \times K_f.$$

Then, by elementary computations, we have

$$H_{n,k}(ge_1 \wedge ge_2 \wedge \cdots \vee ge_k) = |\det a|_\mathbb{A} \quad \text{if} \ g = h \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

with $h \in K, a \in GL_k(\mathbb{A}), b \in M_{k,n-k}(\mathbb{A})$ and $d \in GL_{n-k}(\mathbb{A})$. Therefore, if $g \in G(\mathbb{A})^1$, that is, $|\det g|_\mathbb{A} = 1$, then

$$H_Q(g) = H_{n,k}(g^{-1}e_1 \wedge g^{-1}e_2 \wedge \cdots \vee g^{-1}e_k)^{n/r}.$$

2. Twisted height functions restricted to one parameter subgroups

Let $N_G(S)$ be the normalizer of $S$ in $G$ and $W_G = N_G(S)(k)/M_0(k)$ the Weyl group of $G$ with respect to $S$. For a simple root $\alpha \in \Delta_k, s_\alpha \in W_G$ denotes the simple reflection corresponding to $\alpha$. Then $\{s_\alpha\}_{\alpha \in \Delta_k}$ generates $W_G$. We denote by $W_G^Q$ the subgroup of $W_G$ generated by $\{s_\alpha\}_{\alpha \in \Delta_k \setminus \{a_0\}}$. For each $w \in W_G$, we use the same notation $w$ for a representative of $w$ in $N_G(S)(k)$. The following cell decomposition of $G(k)$ holds via Bruhat decomposition [Borel and Tits 1965, Proposition 4.10, Corollaire 5.20]:

$$G(k) = \bigsqcup_{[w] \in W_G^Q \setminus W_G/W_G^Q} Q(k)wQ(k),$$

where $[w]$ stands for the class $W_G^QwW_G^Q$ in $W_G^Q \setminus W_G/W_G^Q$. 
The Weyl group $W_G$ acts on $X^*(S)_k$ by $w \cdot \chi : t \mapsto \chi (w^{-1}tw)$ for $w \in W_G$ and $\chi \in X^*(S)_k$. We consider the restriction $\widehat{\alpha}_Q|_S$ of the rational character $\widehat{\alpha}_Q$ of $M_Q$ to $S$.

**Lemma 1.** The subgroup of $W_G$ fixing $\widehat{\alpha}_Q|_S$ is equal to $W^Q_G$.

**Proof.** Put $W' = \{ w \in W_G : w \cdot \widehat{\alpha}_Q|_S = \widehat{\alpha}_Q|_S \}$. Since a representative of $w \in W^Q_G$ is contained in $M_Q(k)$, we have $\widehat{\alpha}_Q(w^{-1}tw) = \widehat{\alpha}_Q(w)^{-1} \widehat{\alpha}_Q(t) \widehat{\alpha}_Q(w) = \widehat{\alpha}_Q(t)$ for all $t \in S$. Hence $W^Q_G$ is contained in $W'$. By [Humphreys 1990, §1.12 Theorem (a) and (c)], $W'$ is generated by a subset $W' \cap \{ s_{\alpha} \}_{\alpha \in \Delta_k}$ of simple reflections. From $W^Q_G \subset W'$, it follows $\{ s_{\alpha} \}_{\alpha \in \Delta_k \setminus \{ \alpha_0 \}} \subset W' \cap \{ s_{\alpha} \}_{\alpha \in \Delta_k} \subset \{ s_{\alpha} \}_{\alpha \in \Delta_k}$. Since $\widehat{\alpha}_Q$ is nontrivial on $S/Z_G$, $W' \cap \{ s_{\alpha} \}_{\alpha \in \Delta_k}$ must equal $\{ s_{\alpha} \}_{\alpha \in \Delta_k \setminus \{ \alpha_0 \}}$. Therefore $W'$ coincides with $W^Q_G$. \hfill $\Box$

Let $X_*(S)_k$ be the free $\mathbb{Z}$-module consisting of all $k$-rational cocharacters of $S$. A natural pairing

$$\langle \cdot , \cdot \rangle : X^*(S)_k \times X_*(S)_k \to \mathbb{Z}$$

defined as in [Borel 1991, §8.6] is a regular pairing over $\mathbb{Z}$.

**Lemma 2.** Let $w_1$ and $w_2$ be elements of $W_G$ such that $w_1^{-1}W^Q_G \neq w_2^{-1}W^Q_G$. Then there exist a cocharacter $\xi = \xi_{w_1,w_2} \in X_*(S)_k$ such that

$$H_Q(w_1 \xi(\lambda)w_1^{-1}) > H_Q(w_2 \xi(\lambda)w_2^{-1})$$

holds for all $\lambda \in \mathbb{A}^\times_{>1}$, where $\mathbb{A}^\times_{>1}$ denotes the set of $\lambda \in \mathbb{A}^\times$ satisfying $|\lambda|_\mathbb{A} > 1$.

**Proof.** Since $w_1^{-1} \cdot \widehat{\alpha}_Q|_S - w_2^{-1} \cdot \widehat{\alpha}_Q|_S \neq 0$ by Lemma 1, there is a $\xi \in X_*(S)_k$ such that $\langle w_1^{-1} \cdot \widehat{\alpha}_Q|_S - w_2^{-1} \cdot \widehat{\alpha}_Q|_S, \xi \rangle < 0$. The value $\ell = \langle w_1^{-1} \cdot \widehat{\alpha}_Q|_S - w_2^{-1} \cdot \widehat{\alpha}_Q|_S, \xi \rangle$ is a negative integer. We have

$$\widehat{\alpha}_Q(w_1 \xi(\lambda)w_1^{-1}) \cdot \widehat{\alpha}_Q(w_2 \xi(\lambda)w_2^{-1})^{-1} = \lambda^\ell$$

for all $\lambda \in G_m$. Therefore,

$$H_Q(w_1 \xi(\lambda)w_1^{-1}) H_Q(w_2 \xi(\lambda)w_2^{-1})^{-1} = |\lambda|_\mathbb{A}^{-\ell} > 1$$

holds for all $\lambda \in \mathbb{A}^\times_{>1}$. \hfill $\Box$

3. The Hermite function associated to $Q$ and minimal points

We set $X_Q = Q(k) \backslash G(k)$, which is regarded as a subset of $Y_Q = Q(\mathbb{A})^1 \backslash G(\mathbb{A})^1$. Let $\pi_X : G(k) \to X_Q$ be the natural quotient map. The symbol $\bar{e} = \pi_X(e) \in X_Q$ denotes the class of the unit element $e \in G(k)$. The Hermite function

$$m_Q : G(\mathbb{A})^1 \to \mathbb{R}_{>0}$$
is defined to be

$$m_Q(g) = \min_{x \in X_Q} H_Q(xg).$$

By definition, $m_Q$ is a positive valued continuous function on $G(k) \backslash G(\mathbb{A})^1 / K$.

For each $g \in G(\mathbb{A})^1$, we put

$$X_Q(g) = \{x \in X_Q : m_Q(g) = H_Q(xg)\},$$

which is a finite subset of $X_Q$. Thus we can define the counting function $n_Q(g) = \#X_Q(g)$.

**Lemma 3.** For any $g \in G(\mathbb{A})^1$, $\gamma \in G(k)$ and $h \in K$, one has $X_Q(\gamma gh) = X_Q(g)\gamma^{-1}$. Especially, the counting function $n_Q$ is left $G(k)$-invariant and right $K$-invariant.

The following lemma is proved by the same method as in [Watanabe 2012, Proof of Proposition 4.1].

**Lemma 4.** For $g \in G(\mathbb{A})^1$, there is a neighborhood $\mathfrak{U}$ of $g$ in $G(\mathbb{A})^1$ such that $X_Q(g') \subset X_Q(g)$ for all $g' \in \mathfrak{U}$.

**Example 2.** Let $G$ be a general linear group $GL_n$ defined over $\mathbb{Q}$. We keep notations used in Example 1. In this case, we can express $m_Q$ in terms of some minimum of positive definite symmetric matrices. Since $GL_n(\mathbb{Q})$ is of class number one, $G(\mathbb{A})^1 = \{g \in GL_n(\mathbb{A}) : |\det g|_\mathbb{A} = 1\}$ has the following decomposition:

$$G(\mathbb{A})^1 = G(\mathfrak{Q})(G(\mathbb{Q}_\infty)^1 \times K_f),$$

where $G(\mathbb{Q}_\infty)^1 = \{g \in GL_n(\mathbb{Q}_\infty) : \det g = \pm 1\}$ and $K_f = \prod_{p \in \mathfrak{p}_f} GL_n(\mathbb{Z}_p)$. We fix $g = \delta(g_\infty \times g_f) \in G(\mathbb{A})^1$ with $\delta \in G(\mathfrak{Q})$, $g_\infty \in G(\mathbb{Q}_\infty)^1$ and $g_f \in K_f$. From the left $G(\mathfrak{Q})$-invariance and the right $K$-invariance of $m_Q$, it follows that

$$m_Q(g) = m_Q(g_\infty) = \min_{x \in X_Q} H_Q(xg_\infty) = \min_{\gamma \in G(\mathfrak{Q})} H_Q(\gamma g_\infty).$$

Furthermore, since $G(\mathfrak{Q}) = Q(\mathfrak{Q}) GL_n(\mathfrak{Z})$ and $H_Q$ is left $Q(\mathfrak{Q})$-invariant, we have

$$m_Q(g) = \min_{\gamma \in GL_n(\mathfrak{Z})} H_Q(\gamma g_\infty).$$

An elementary proof of the decomposition $G(\mathfrak{Q}) = Q(\mathfrak{Q}) GL_n(\mathfrak{Z})$ is found in [Shimura 1994, Theorem 3]. By Example 1,

$$H_Q(\gamma g_\infty) = H_{n,k}(g_\infty^{-1} \gamma^{-1} e_1 \wedge \cdots \wedge g_\infty^{-1} \gamma^{-1} e_k)^{n/r}$$

$$= H_\infty(g_\infty^{-1} \gamma^{-1} e_1 \wedge \cdots \wedge g_\infty^{-1} \gamma^{-1} e_k)^{n/r} \prod_{p \in \mathfrak{p}_f} H_p(\gamma^{-1} e_1 \wedge \cdots \wedge \gamma^{-1} e_k)^{n/r}$$

$$= H_\infty(g_\infty^{-1} \gamma^{-1} e_1 \wedge \cdots \wedge g_\infty^{-1} \gamma^{-1} e_k)^{n/r}.$$
Here we notice that $H_p(\gamma^{-1}e_1 \wedge \cdots \wedge \gamma^{-1}e_k) = 1$ for all $p \in p_f$ and $\gamma \in \text{GL}_n(\mathbb{Z})$. For a given $\gamma \in \text{GL}_n(\mathbb{Z})$, $X_\gamma$ stands for the $n$ by $k$ matrix consisting of the first $k$ columns of $\gamma$. Binet’s formula (see [Bombieri and Gubler 2006, Proposition 2.8.8]) yields

$$H_\infty(\gamma^{-1}e_1 \wedge \cdots \wedge \gamma^{-1}e_k) = \det(tX_\gamma^{-1} t\gamma^{-1} tX_\gamma^{-1})^{1/2}.$$  

As a consequence, we obtain

$$m_Q(g) = \min_{X \in \mathcal{M}_{n,k}(\mathbb{Z})^*} \det(tX_\infty^{-1} X)^{n/2r},$$

where $\mathcal{M}_{n,k}(\mathbb{Z})^*$ denotes the set of $X_\gamma$ for all $\gamma \in \text{GL}_n(\mathbb{Z})$. In the case of $k = 1$, $\mathcal{M}_{n,1}(\mathbb{Z})^*$ is just the set of primitive vectors of the lattice $\mathbb{Z}^n$, and hence $m_Q(g)$ coincides with the $n/2$ power of the arithmetical minimum of the positive definite symmetric matrix $t\gamma^{-1} \gamma^{-1}$.

4. The Ryshkov domain of $G$ associated to $Q$

We define the Ryshkov domain $R = R(m_Q)$ of $m_Q$ by

$$R = R(m_Q) = \{g \in G(\mathbb{A})^1 : m_Q(g) / H_Q(g) \geq 1\}.$$  

Since $m_Q(g) \leq H_Q(g)$ holds for all $g \in G(\mathbb{A})^1$, we have

$$R = \{g \in G(\mathbb{A})^1 : m_Q(g) = H_Q(g)\} = \{g \in G(\mathbb{A})^1 : \tilde{e} \in X_Q(g)\}.$$  

Since both $H_Q$ and $m_Q$ are continuous, $R$ is a closed subset in $G(\mathbb{A})^1$.

**Lemma 5.** One has $Q(k)RK = R$ and $G(\mathbb{A})^1 = G(k)R$.

**Proof.** The first assertion is obvious by the definition of $H_Q$. To prove the second assertion, we choose a minimal point $x \in X_Q(g)$ for a given $g \in G(\mathbb{A})^1$. There is a $\gamma \in G(k)$ such that $x = \pi_X(\gamma)$. Then $H_Q(xg) = H_Q(\gamma g) = m_Q(g) = m_Q(\gamma g)$ since $m_Q$ is left $G(k)$-invariant. Therefore, $\gamma g \in R$. \hfill $\square$

**Lemma 6.** Let $C$ be an arbitrary subset of $G(\mathbb{A})^1$, and let $g \in G(\mathbb{A})^1$ and $\gamma \in G(k)$.

1. $\gamma g \in R$ if and only if $\pi_X(\gamma) \in X_Q(g)$.
2. $X_Q(g) = \pi_X(\{\gamma \in G(k) : \gamma g \in R\})$.
3. $\gamma C \subset R$ if and only if $\pi_X(\gamma) \in \bigcap_{g \in C} X_Q(g)$.
4. $\bigcap_{g \in R} X_Q(g) = \{\tilde{e}\}$.
5. $\gamma R \subset R$ if and only if $\gamma \in Q(k)$. 

Proof. By definition, \(\gamma g \in \mathbb{R}\) if and only if \(m_Q(\gamma g) = H_Q(\gamma g)\). This is equivalent to \(\pi_X(\gamma) \in X_Q(g)\) because \(m_Q(\gamma g) = m_Q(g)\). Both (2) and (3) follow from (1). For a point \(x = \pi_X(\gamma) \in \bigcap_{g \in \mathbb{R}} X_Q(g)\), we have \(\gamma Q(k)R \subset \mathbb{R}\); in other words, \(x Q(k) \subset \bigcap_{g \in \mathbb{R}} X_Q(g)\). Since \(x Q(k)\) is an infinite set for \(x \neq \hat{e}\) by Bruhat decomposition, we must have \(x = \hat{e}\). This shows (4). Item (5) follows from (3) and (4). \(\square\)

Lemma 7. Let \(g_0 \in \mathbb{R}\) be an element such that \(n_Q(g_0) > 1\) and \(x_0\) an arbitrary element in \(X_Q(g_0)\). Then, any neighborhood \(\mathcal{U}\) of \(g_0\) in \(G(\mathbb{A})^1\) contains a point \(g\) such that \(X_Q(g) \subset X_Q(g_0)\) and \(x_0 \notin X_Q(g)\).

Proof. We may assume \(\mathcal{U}\) satisfies \(X_Q(g) \subset X_Q(g_0)\) for all \(g \in \mathcal{U}\) by Lemma 4. Since \(n_Q(g_0) > 1\), there is an \(x \in X_Q(g_0)\) such that \(x \neq \hat{e}\). This \(x\) is of the form \(\pi_X(w \gamma)\) with \(w \in W_G \setminus W_Q\) and \(\gamma \in Q(k)\). By Lemma 2, there is a cocharacter \(\xi = \xi_{w,e} \in X_n(S_k)\) such that \(H_Q(w \xi(\lambda w^{-1}) > H_Q(\xi(\lambda))\) holds for all \(\lambda \in \mathbb{A}_x^1\). Let \(\lambda \in \mathbb{A}_x^1\) be an element sufficiently close to 1 so that \(g_\lambda = \gamma^{-1} \xi(\lambda) \gamma g_0\) is contained in \(\mathcal{U}\). We have

\[
H_Q(g_\lambda) = H_Q(\xi(\lambda) \gamma g_0) = H_Q(\xi(\lambda)) H_Q(\gamma g_0) = H_Q(\xi(\lambda)) m_Q(g_0)
\]

and

\[
H_Q(x g_\lambda) = H_Q(w \xi(\lambda) \gamma g_0) = H_Q(w \xi(\lambda) w^{-1} \gamma g_0) = H_Q(w \xi(\lambda) w^{-1}) m_Q(g_0).
\]

If \(x_0 = \hat{e}\), then we choose \(\lambda\) sufficiently close to 1 satisfying \(\lambda^{-1} \in \mathbb{A}_x^1\). Since \(X_Q(g_\lambda) \subset X_Q(g_0)\) and \(m_Q(g_\lambda) \leq H_Q(x g_\lambda) < H_Q(g_\lambda)\), \(X_Q(g_\lambda)\) does not contain \(\hat{e}\). If \(x_0 \neq \hat{e}\), then we choose \(x\) as \(x_0\) and \(\lambda \in \mathbb{A}_x^1\) sufficiently close to 1. Since \(m_Q(g_\lambda) \leq H_Q(g_\lambda) < H_Q(x_0 g_\lambda)\), \(X_Q(g_\lambda)\) does not contain \(x_0\). \(\square\)

Lemma 8. \(\min_{g \in G(\mathbb{A})^1} n_Q(g) = \min_{g \in \mathbb{R}} n_Q(g) = 1\).

Proof. From Lemma 5 and the \(G(k)\)-invariance of \(n_Q\), it follows that

\[
\min_{g \in G(\mathbb{A})^1} n_Q(g) = \min_{g \in \mathbb{R}} n_Q(g).
\]

If \(g_0 \in \mathbb{R}\) satisfies \(\min_{g \in \mathbb{R}} n_Q(g) = n_Q(g_0) > 1\), then by Lemmas 5 and 7, there exist a point \(g_1 \in G(\mathbb{A})^1\) and \(\gamma_1 \in G(k)\) such that \(n_Q(\gamma_1 g_1) = n_Q(g_1) < n_Q(g_0)\) and \(\gamma_1 g_1 \in \mathbb{R}\). This is a contradiction. \(\square\)

We define the subset \(R_1\) of \(\mathbb{R}\) by

\[
R_1 = \{ g \in \mathbb{R} : n_Q(g) = 1 \} = \{ g \in G(\mathbb{A})^1 : X_Q(g) = \{ \hat{e} \} \}.
\]

Lemma 9. \(R_1\) coincides with the interior \(R^o\) of \(\mathbb{R}\) in \(G(\mathbb{A})^1\).
Lemma 11. \( S_g \) of a single element \( \sigma \) and may assume that \( f \) \( G \).

Proof. If \( G \) \( H5121 \) such that \( n \) \( g \) \( f \) finitely many iterations. At the last step, we obtain an element \( n \) \( k \) \( 1 \).

By Lemma 5, we can take \( g \) \( Q \) \( 2 \).

Proof. If \( G \) \( H5121 \) and \( g \) \( Q \) \( 2 \). \( \gamma \) \( 0 \) \( g \) \( f \).

By Lemmas 6 and 12, \( \pi_X(\gamma) \in X_Q(g) = \{ e \} \) by Lemma 6. \( \square \)

Lemma 12. Let \( R_1 \) be the closure of \( R_1 \). Then we have the following subdivision of \( G(\mathbb{A})^1 \):

\[
G(\mathbb{A})^1 = \bigcup_{\gamma \in Q(1) \cap G(k)} \gamma R_1^-.
\]

Proof. We fix an arbitrary \( g \in G(\mathbb{A})^1 \). By Lemma 10, there exists a sequence \( \{ g_n \} \subset G(k)R_1 \) such that \( \lim_{n \to \infty} g_n = g \). We take a neighborhood \( \mathcal{U} \) of \( g \) as in Lemma 4 and may assume that \( \{ g_n \} \subset \mathcal{U} \). Since \( g_n \in G(k)R_1 \), \( X_Q(g_n) \) consists of a single element \( \pi_X(\gamma_n) \), where \( \gamma_n \in G(k) \). From \( g_n \in \mathcal{U} \), it follows that \( \pi_X(\gamma_n) \in X_Q(g) \) for all \( n \). Since \( X_Q(g) \) is a finite set, we can take a subsequence \( \{ g_{n_j} \} \) such that \( \pi_X(\gamma_{n_j}) = \pi_X(\gamma) \in X_Q(g) \) for all \( n_j \). Then \( \{ g_{n_j} \} \subset \gamma^{-1}R_1 \), and \( g \) is contained in the closure of \( \gamma^{-1}R_1 \). \( \square \)

For \( g \in G(\mathbb{A})^1 \), we put

\[
S_Q(g) = \pi_X(\{ \gamma \in G(k) : \gamma g \in R_1^- \}).
\]

By Lemmas 6 and 12, \( S_Q(g) \) is a nonempty subset of \( X_Q(g) \).

Lemma 13. For \( g_0 \in G(\mathbb{A})^1 \), there is a neighborhood \( \mathcal{U} \) of \( g_0 \) in \( G(\mathbb{A})^1 \) such that \( S_Q(g) \subset S_Q(g_0) \) for all \( g \in \mathcal{U} \).
Proof. Let \( \mathcal{U} \) be a neighborhood of \( g_0 \) such that \( X_Q(g) \subseteq X_Q(g_0) \) for all \( g \in \mathcal{U} \). Since \( g_0 \not\in \gamma^{-1}R_1^- \) for any \( \pi_X(\gamma) \in X_Q(g_0) \setminus S_Q(g_0) \), we can take a sufficiently small \( \mathcal{U} \) so that \( \mathcal{U} \cap \gamma^{-1}R_1^- = \emptyset \) for all \( \pi_X(\gamma) \in X_Q(g_0) \setminus S_Q(g_0) \). Then, for any \( g \in \mathcal{U} \), \( S_Q(g) \cap X_Q(g_0) \setminus S_Q(g_0) \) is empty; that is, \( S_Q(g) \subseteq S_Q(g_0) \). \( \square \)

Remark. We do not know whether \( R_1^- = R \) holds or not in general. If it does, then \( S_Q(g) = X_Q(g) \) holds for all \( g \).

5. A fundamental domain of \( G(\mathbb{A})^1 \) with respect to \( G(k) \)

Definition. Let \( T \) be a locally compact Hausdorff space and \( \Gamma \) be a discrete group acting on \( T \) from the left. Assume that the action of \( \Gamma \) on \( T \) is properly discontinuous. An open subset \( \Omega \) of \( T \) is called an open fundamental domain of \( T \) with respect to \( \Gamma \) if \( \Omega \) satisfies the following conditions:

(1) \( T = \Gamma \Omega^- \), where \( \Omega^- \) stands for the closure of \( \Omega \) in \( T \), and

(2) \( \Omega \cap \gamma \Omega^- = \emptyset \) if \( \gamma \in \Gamma \setminus \{ e \} \).

A subset \( F \) of \( T \) is called a fundamental domain of \( T \) with respect to \( \Gamma \) if there is an open fundamental domain \( \Omega \) as above such that \( \Omega \subseteq F \subseteq \Omega^- \).

By Baer and Levi’s theorem [1931] (see also [van der Waerden 1935, §10]), an open fundamental domain of \( T \) with respect to \( \Gamma \) exists if the set of points stabilized by some nontrivial element of \( \Gamma \) is discrete in \( T \). Thus there exists an open fundamental domain \( \Omega_Q \) of \( R_1^- \) with respect to \( Q(k) \). For a given subset \( A \) of \( R_1^- \), \( A^\circ \) and \( A^- \) denote the interior and the closure of \( A \) in \( G(\mathbb{A})^1 \), respectively. Since \( R_1^- \) is closed in \( G(\mathbb{A})^1 \), the closure of \( A \) in \( R_1^- \) coincides with \( A^- \).

Lemma 14. Let \( \Omega_Q \) be an open fundamental domain of \( R_1^- \) with respect to \( Q(k) \). Then one has \( \Omega_Q^\circ = \Omega_Q \cap R_1 \) and \( \Omega_Q^- = (\Omega_Q \cap R_1)^- \).

Proof. Since \( \Omega_Q \) is an open set in \( R_1^- \) with respect to the relative topology, there is an open set \( \mathcal{U} \) in \( G(\mathbb{A})^1 \) such that \( \Omega_Q = R_1^- \cap \mathcal{U} \). Therefore, \( \Omega_Q \cap R_1 = \mathcal{U} \cap R_1 \) is open in \( G(\mathbb{A})^1 \), and hence \( \Omega_Q^\circ = \Omega_Q \cap R_1 \). Since \( R_1 \) is dense in \( R_1^- \) and \( \Omega_Q \) is relatively open in \( R_1^- \), the closure of \( \Omega_Q \cap R_1 \) in \( R_1^- \) contains \( \Omega_Q \), that is, \( \Omega_Q \subseteq (\Omega_Q \cap R_1)^- \). Hence \( \Omega_Q^- = (\Omega_Q \cap R_1)^- \). \( \square \)

Theorem 15. Let \( \Omega_Q \) be an open fundamental domain of \( R_1^- \) with respect to \( Q(k) \). Then \( \Omega_Q^\circ \) is an open fundamental domain of \( G(\mathbb{A})^1 \) with respect to \( G(k) \).

Proof. From \( R_1^- = Q(k)\Omega_Q^- \) and Lemma 12, it follows \( G(\mathbb{A})^1 = G(k)\Omega_Q^- \). For \( \gamma \in G(k) \), we assume \( \Omega_Q^\circ \cap \gamma \Omega_Q^- \neq \emptyset \). By Lemma 11, \( \gamma \) is contained in \( Q(k) \). Since \( \Omega_Q \) is an open fundamental domain of \( R_1^- \) with respect to \( Q(k) \), \( \gamma \) must be equal to \( e \). \( \square \)

For a given subset \( A \) of \( G(\mathbb{A})^1 \), we denote by \( \partial A \) the boundary of \( A \).
Lemma 16. If \( g_0 \in R_1^- \) attains a local maximum of \( m_Q \), then \( g_0 \) is in \( \partial R_1^- \).

Proof. Suppose \( g_0 \in R_1 \). Since \( R_1 \) is open, \( zg_0 \) is contained in \( R_1 \) if \( z \in Z_Q(\mathbb{A}) \) is sufficiently close to \( e \). Then

\[
m_Q(zg_0) = H_Q(zg_0) = H_Q(z)H_Q(g_0) = H_Q(z)m_Q(g_0).
\]

Since \( H_Q(z) \) can vary on the interval \((1 - \epsilon, 1 + \epsilon)\) for a sufficiently small \( \epsilon > 0 \), \( m_Q(g_0) \) is not a local maximum of \( m_Q \). \( \square \)

Since \( (\Omega_Q^-)^o = \Omega_Q^o \subset R_1 \), the following theorem immediately follows from Lemma 16.

Theorem 17. Let \( \Omega_Q \) be the same as in Theorem 15. If \( g_0 \in \Omega_Q^- \) attains a local maximum of \( m_Q \), then \( g_0 \) is in \( \partial \Omega_Q^- \cap \partial R_1^- \).

Remark. A point \( g_0 \in G(\mathbb{A})^1 \) is said to be extreme if \( g_0 \) attains a local maximum of \( m_Q \). By Theorem 17, any extreme point is contained in \( G(k)/\partial \Omega_Q^- \cap \partial R_1^- \). A candidate of the notion analogous to perfect quadratic forms is the following: a point \( g \in G(\mathbb{A})^1 \) is said to be \( Q \)-perfect if there is a neighborhood \( \mathcal{U} \) of \( g \) such that

\[
\mathcal{U} \cap \bigcap_{\pi(x) \in S_Q(g)} \delta^{-1}R_1^- = \{g\}.
\]

6. The case when \( G \) is of class number one

We put \( K_f = \prod_{\sigma \in p_f} K_{\sigma}, \ G_{\mathbb{A}, \infty} = G(k_{\infty}) \times K_f, \ G_{\mathbb{A}, \infty}^1 = G_{\mathbb{A}, \infty} \cap G(\mathbb{A})^1 \) and \( G_o = G(k) \cap G_{\mathbb{A}, \infty} \). By identifying \( G(k_{\infty}) \) with the subgroup

\[
\{(g_\sigma) \in G(\mathbb{A}) : g_\sigma = e \text{ for all } \sigma \in p_f\}
\]

of \( G(\mathbb{A}) \), we put \( G(k_{\infty})^1 = G(k_{\infty}) \cap G(\mathbb{A})^1 \). The number \( n_k(G) \) of double cosets in \( G(\mathbb{A}) \) modulo \( G(k) \) and \( G_{\mathbb{A}, \infty} \) is called the class number of \( G \). For example, \( n_k(GL_n) \) is equal to the class number of \( k \). If \( G \) is almost \( k \)-simple, \( k \)-isotropic and simply connected, then \( n_k(G) = 1 \) by the strong approximation theorem. In this section, we assume that \( n_k(G) = 1 \). Then \( G(\mathbb{A})^1 = G(k)G_{\mathbb{A}, \infty}^1 \). Let \( h_Q \) be the number of double cosets of \( G(k) \) modulo \( Q(k) \) and \( G_o \). By [Borel 1963, Proposition 7.5], \( h_Q \) is equal to the class number of \( M_Q \). Let \( \{\xi_1 = e, \xi_2, \ldots, \xi_{h_Q}\} \) be a complete system of representatives of \( Q(k) \backslash G(k)/G_o \). For each \( \xi_i \), we define

\[
R_{\xi_i, \infty} = \{g_{\infty} \in G(k_{\infty})^1 : m_Q(g_{\infty}) = H_Q(\xi_i g_{\infty})\}.
\]

Since \( G(k) \) is a disjoint union of \( Q(k)\xi_i G_o \) for \( i = 1, \ldots, h_Q \), \( m_Q(g_{\infty}) \) equals

\[
\min_{1 \leq i \leq h_Q} \min_{\delta \in G_o} H_Q(\xi_i \delta g_{\infty}).
\]
Lemma 18. 
\[ R = \bigcup_{i=1}^{h_Q} Q(k)\xi_i (R_{\xi_i,\infty} \times K_f). \]

Proof. For each \( i \), \( Q(k)\xi_i (R_{\xi_i,\infty} \times K_f) \subset R \) is trivial. Since

\[ G(\mathbb{A})^1 = \bigcup_{i=1}^{h_Q} Q(k)\xi_i G_{\mathbb{F},\infty}^1 \]

by [Borel 1963, §7], a given \( g \in R \) is represented as \( g = \gamma \xi_i (g_{\infty} \times g_f) \) for some \( i, \gamma \in Q(k) \) and \( g_{\infty} \times g_f \in G_{\mathbb{F},\infty}^1 \). Then \( m_{Q}(g) = H_{Q}(g) \) implies \( m_{Q}(g_{\infty}) = H_{Q}(\xi_i g_{\infty}) \). Therefore, \( g_{\infty} \in R_{\xi_i,\infty} \). \( \square \)

We write \( Q_i \) for the conjugate \( \xi_i^{-1} Q \xi_i \) of \( Q \). This \( Q_i \) is a maximal \( k \)-parabolic subgroup of \( G \). We put \( Q_{i,0} = Q_i(k) \cap G_{\mathbb{F},\infty} \).

Lemma 19. If \( g(R_{\xi_i,\infty} \times K_f) \cap (R_{\xi_i,\infty} \times K_f) \) is nonempty for \( g \in Q_i(k) \), then \( g \in Q_{i,0} \).

Proof. If there is an \( h \in R_{\xi_i,\infty} \times K_f \) such that \( gh \in R_{\xi_i,\infty} \times K_f \), then

\[ g \in (R_{\xi_i,\infty} \times K_f)h^{-1} \subset G_{\mathbb{F},\infty}. \] \( \square \)

It is easy to prove that the group \( Q_{i,0} \) stabilizes \( R_{\xi_i,\infty} \times K_f \) by left multiplication.

We fix a complete system \( \{\gamma_{ij}\}_j \) of representatives of \( Q_i(k)/Q_{i,0} \). It follows from Lemma 19 that \( \gamma_{ij}(R_{\xi_i,\infty} \times K_f) \cap \gamma_{ik}(R_{\xi_i,\infty} \times K_f) = \emptyset \) if \( j \neq k \). Therefore, we obtain the following subdivision of \( R \):

(1) \[ R = \bigcup_{i=1}^{h_Q} \bigcup_{j} \xi_i \gamma_{ij} (R_{\xi_i,\infty} \times K_f). \]

Let \( R_{\xi_i,\infty}^0 \) be the interior of \( R_{\xi_i,\infty} \) and \( R_{\xi_i,\infty}^* \) the closure of \( R_{\xi_i,\infty}^0 \) in \( G(k_\infty)^1 \). Since the union of (1) is disjoint, it is obvious that

(2) \[ R_1 = \bigcup_{i=1}^{h_Q} \bigcup_{j} \xi_i \gamma_{ij} (R_{\xi_i,\infty}^* \times K_f). \]

Proposition 20. Let \( \Omega_{i,\infty} \) be an open fundamental domain of \( R_{\xi_i,\infty}^* \) with respect to \( Q_{i,0} \) for \( i = 1, \ldots, h_Q \). Then the set

\[ \Omega = \bigcup_{i=1}^{h_Q} \xi_i (\Omega_{i,\infty} \times K_f) \]

gives an open fundamental domain of \( R_1 \) with respect to \( Q(k) \).
Proof. Let \( \Omega_{i,\infty}^- \) denote the closure of \( \Omega_{i,\infty}^- \) in \( G(k_{\infty})^1 \). For \( g \in Q(k) \), we assume \( \Omega \cap g\Omega^- \neq \emptyset \). Then, for some \( i, j \),

\[
(3) \quad \xi_i(\Omega_{i,\infty}^- \times K_f) \cap g_\xi_j(\Omega_{j,\infty}^- \times K_f) \neq \emptyset.
\]

There exist \( \gamma_{jk} \) and \( \delta \in Q_{i,o} \) such that \( \xi_j^{-1}g_\xi_j = \gamma_{jk}\delta \). Then (3) is the same as

\[
\xi_i(\Omega_{i,\infty}^- \times K_f) \cap \xi_j \gamma_{jk}(\delta \Omega_{j,\infty}^- \times K_f) \neq \emptyset.
\]

By (1), we have \( i = j \), \( \gamma_{jk} = e \) and \( \Omega_{i,\infty}^- \cap \delta \Omega_{j,\infty}^- \neq \emptyset \). Since \( \Omega_{j,\infty}^- \) is an open fundamental domain of \( \mathbb{R}^*_{\xi_j,\infty} \) with respect to \( Q_{j,o} \), \( \delta \) must be equal to \( e \). Therefore, \( \Omega \cap g\Omega^- \neq \emptyset \) implies \( g = e \). Finally, \( Q(k)\Omega^- = \mathbb{R}_1^- \) follows from (2) and \( Q_{i,o}\Omega_{i,\infty}^- = \mathbb{R}^*_{\xi_i,\infty} \).

By Theorem 17, we obtain the following.

**Corollary 21.** If \( g_0 \in \Omega^- \) attains a local maximum of \( m_Q \), then \( g_0 \) is contained in the set

\[
\bigcup_{i=1}^{h_Q} \xi_i((\partial \Omega_{i,\infty}^- \cap \partial \mathbb{R}^*_{\xi_i,\infty}) \times K_f).
\]

We consider the infinite part \( \Omega_{\infty} \) of \( \Omega \) given in Proposition 20, that is,

\[
\Omega_{\infty} = \bigcup_{i=1}^{h_Q} \xi_i \Omega_{i,\infty}.
\]

Let \( \Omega_{\infty} \) and \( \Omega^- \) be the interior and the closure of \( \Omega_{\infty} \) in \( G(k_{\infty})^1 \), respectively. The projection from \( G(\mathbb{A})^1 = G(k)G_{h,\infty}^1 \) to the infinite component \( G(k_{\infty})^1 \) gives an isomorphism \( G(k)\setminus G(\mathbb{A})^1 / K_f \cong G_0 \setminus G(k_{\infty})^1 \). Since \( \Omega \) is a fundamental domain of \( G(\mathbb{A})^1 \) with respect to \( G(k) \) by Theorem 15, we have \( G_0\Omega_{\infty}^- = G(k_{\infty})^1 \).

**Corollary 22.** If \( h_Q = 1 \), then \( \Omega_{\infty} \) is a fundamental domain of \( G(k_{\infty})^1 \) with respect to \( G_0 \).

**Proof.** Since \( \Omega_{\infty} = \Omega_{1,\infty} \) is a relatively open set in \( \mathbb{R}^*_{\xi,\infty} \), we have \( \Omega_{\infty} = \Omega_{\infty} \cap R_{\xi,\infty}^0 \). Thus the closure of \( \Omega_{\infty}^0 \) coincides with \( \Omega_{\infty}^- \). If \( \Omega_{\infty}^0 \cap g\Omega_{\infty}^- \neq \emptyset \) for \( g \in G_0 \), then \( (\Omega_{\infty}^0 \times K_f) \cap g(\Omega_{\infty}^- \times K_f) \neq \emptyset \) because \( gK_f = K_f \). This implies \( g = e \) since \( \Omega_{\infty}^0 \times K_f \) is an open fundamental domain of \( G(\mathbb{A})^1 \) with respect to \( G(k) \).

7. Examples

**Example 3.** Let \( G \) be a general linear group \( \text{GL}_n \) defined over \( \mathbb{Q} \). We continue an illustration given in Examples 1 and 2. We fix an integer \( k \in \{1, \ldots, n-1\} \), and
We define the closed subset \( Q(\mathbb{Q}) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a \in \text{GL}_k(\mathbb{Q}), \ b \in \text{M}_{k,n-k}(\mathbb{Q}), \ d \in \text{GL}_{n-k}(\mathbb{Q}) \right\} \).

Since \( h_Q = 1 \), we have \( \xi_1 = e \) and \( Q_1 = Q \).

Let \( P_n \) be the cone of positive definite \( n \) by \( n \) real symmetric matrices, and let \( P_1 \) be the intersection of \( P_n \) and \( \text{SL}_n(\mathbb{R}) \). The group \( G(\mathbb{Q}_\infty) = \text{GL}_n(\mathbb{R}) \) acts on \( P_n \) from the right by \( (A, g) \mapsto A[g] = {^t}gAg \) for \( (A, g) \in P_n \times G(\mathbb{Q}_\infty) \). The maximal compact subgroup \( K_\infty \) of \( G(\mathbb{Q}_\infty) \), defined as in Example 2, stabilizes the identity matrix \( I_n \in P_n \). The map \( \pi : g \mapsto {^t}g^{-1}g^{-1} \) from \( G(\mathbb{Q}_\infty) \) onto \( P_n \) gives an isomorphism between \( G(\mathbb{Q}_\infty)/K_\infty \) and \( P_n \). Since

\[
G(\mathbb{Q}_\infty)^1 = \{ g \in G(\mathbb{Q}_\infty) : \det g = \pm 1 \},
\]

we have \( G(\mathbb{Q}_\infty)^1/K_\infty \cong \pi(G(\mathbb{Q}_\infty)^1) = P_1 \). An element \( A \in P_n \) is written as

\[
A = \begin{pmatrix} I_k & 0 \\ {^t}u & I_{n-k} \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} I_k & u \\ 0 & I_{n-k} \end{pmatrix},
\]

where \( v \in P_k, w \in P_{n-k} \) and \( u \in \text{M}_{k,n-k}(\mathbb{R}) \). We write \( u_A, A[k] \) and \( A[n-k] \) for \( u, v \) and \( w \), respectively.

By definition, \( G_Z = G(\mathbb{Q}) \cap G_{\mathbb{R},\infty} \) and \( Q_Z = Q(\mathbb{Q}) \cap G_{\mathbb{R},\infty} \) are just the groups \( \text{GL}_n(\mathbb{Z}) \) and \( Q(\mathbb{Q}) \cap \text{GL}_n(\mathbb{Z}) \) of unimodular integral matrices in \( G(\mathbb{Q}) \) and \( Q(\mathbb{Q}) \), respectively. As in Example 2, \( X_\gamma \) stands for the \( n \) by \( k \) matrix consisting of the first \( k \)-columns of \( \gamma \in G_Z \), and \( \text{M}_{n,k}(\mathbb{Z})^* \) stands for the set of \( X_\gamma \) for all \( \gamma \in G_Z \).

We define the closed subset \( F_{n,k} \) of \( P_n \) as follows:

\[
F_{n,k} = \{ A \in P_n : \det A[k] \leq \det({^t}XAX) \text{ for all } X \in \text{M}_{n,k}(\mathbb{Z})^* \}.
\]

In Example 2, we showed

\[
H_Q(\gamma g) = \det({^t}X_{\gamma-1} \pi(\gamma)X_{\gamma-1})^{n/2r}
\]

for any \( \gamma \in G_Z \) and \( g \in G(\mathbb{Q}_\infty)^1 \). Since \( H_Q(g) = (\det \pi(g)[k])^{n/2r} \), we obtain

\[
R_{e,\infty}/K_\infty \cong \pi(R_{e,\infty}) = F_{n,k} \cap \text{SL}_n(\mathbb{R}).
\]

Therefore, \( Q_Z \setminus R_{e,\infty}/K_\infty \) is isomorphic to \( (F_{n,k} \cap \text{SL}_n(\mathbb{R}))/Q_Z \). If \( \gamma \in Q_Z \) is of the form

\[
\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}
\]

with \( a \in \text{GL}_k(\mathbb{Z}), d \in \text{GL}_{n-k}(\mathbb{Z}) \) and \( b \in \text{M}_{k,n-k}(\mathbb{Z}) \), then components of \( {^t}\gamma A \gamma \) for \( A \in P_n \) are given by

\[
u {^t}\gamma A \gamma = a^{-1}(u_A d + b), \quad \gamma A[k] = t a A[k] a, \quad \gamma A[k] = t A[n-k] a.
\]
Let $\mathcal{D}$ and $\mathcal{E}$ be arbitrary fundamental domains for the quotients $P_{k}/GL_{k}(\mathbb{Z})$ and $P_{n-k}/GL_{n-k}(\mathbb{Z})$, respectively. We define the subset $F_{n,k}(\mathcal{D}, \mathcal{E})$ of $F_{n,k}$ as

$$F_{n,k}(\mathcal{D}, \mathcal{E}) = \{ A \in F_{n,k} : A^{[k]} \in \mathcal{D}, A_{[n-k]} \in \mathcal{E}, u_{A} = (u_{ij}), -\frac{1}{2} \leq u_{ij} \leq \frac{1}{2} \text{ for all } i, j, \text{ and } 0 \leq u_{11} \}.$$ 

Since $F_{n,k}(\mathcal{D}, \mathcal{E})$ is a fundamental domain of $F_{n,k}$ with respect to $Q_{\mathbb{Z}}$, the inverse image $\pi^{-1}(F_{n,k}(\mathcal{D}, \mathcal{E}) \cap SL_{n}(\mathbb{R}))$ of $F_{n,k}(\mathcal{D}, \mathcal{E}) \cap SL_{n}(\mathbb{R})$ gives a fundamental domain of $R_{e,\infty}$ with respect to $Q_{\mathbb{Z}}$. As a consequence of Theorem 15 and Proposition 20, the set

$$\pi^{-1}(F_{n,k}(\mathcal{D}, \mathcal{E}) \cap SL_{n}(\mathbb{R})) \times K_{f}$$

gives a fundamental domain of $G(\mathbb{A})^{1}$ with respect to $G(\mathbb{Q})$. Moreover, from Corollary 22, it follows that $F_{n,k}(\mathcal{D}, \mathcal{E})$ is a fundamental domain of $P_{n}$ with respect to $GL_{n}(\mathbb{Z})$.

In the case of $k = 1$, this gives an inductive construction of a fundamental domain $\Omega_{n}$ of $P_{n}$ with respect to $GL_{n}(\mathbb{Z})$ as follows. First, put $\Omega_{2} = F_{2,1}(P_{1}, P_{1})$. By definition, $\Omega_{2}$ is Minkowski’s fundamental domain of $P_{2}$. Then we define inductively $\Omega_{3} = F_{3,1}(P_{1}, \Omega_{2}), \ldots, \Omega_{n} = F_{n,1}(P_{1}, \Omega_{n-1})$. The domain $\Omega_{n}$ coincides with Grenier’s fundamental domain [1988].

Finally, we show that, in the case of $k = 1$, $R_{e,\infty}/K_{\infty}$ corresponds to a face of the Ryshkov polyhedron $R(m) = \{ A \in P_{n} : m(A) = \min_{0 \neq x \in \mathbb{Z}^{n}} \langle x, Ax \rangle \geq 1 \}$. For $A \in P_{n}$, let $S(A)$ denote the set of minimal integral vectors of $A$:

$$S(A) = \{ x \in \mathbb{Z}^{n} : m(A) = \langle x, Ax \rangle \}.$$ 

We take $e_{1} = \langle 1, 0, \ldots, 0 \rangle \in \mathbb{Z}^{n}$. It is obvious that the subset $\{ A \in P_{n} : e_{1} \in S(A) \}$ of $P_{n}$ coincides with $F_{n,1}$. As was shown in [Watanabe 2012, Lemma 1.5], $\mathcal{F}_{\{e_{1}\}} = F_{n,1} \cap \partial R(m) = \{ A \in F_{n,1} : m(A) = 1 \}$ is a face of $R(m)$. It is easy to see that the map $A \mapsto m(A)^{-1}A$ gives a bijection from $F_{n,1} \cap SL_{n}(\mathbb{R})$ onto $\mathcal{F}_{\{e_{1}\}}$. Therefore, $R_{e,\infty}/K_{\infty} \cong \pi(R_{e,\infty})$ corresponds to $\mathcal{F}_{\{e_{1}\}}$.

**Example 4.** Let $k$ be a totally real number field of degree $r$ and $n = 2m$ be an even integer. We consider a symplectic group

$$G(k) = Sp_{n}(k) = \left\{ g \in GL_{2m}(k) : \langle g \begin{pmatrix} 0 & -I_{m} \\ I_{m} & 0 \end{pmatrix} g, \begin{pmatrix} 0 & -I_{m} \\ I_{m} & 0 \end{pmatrix} \rangle \right\}.$$ 

For a fixed $k \in \{1, 2, \ldots, m\}$, let $Q$ denote the maximal parabolic subgroup of $G$ given by

$$Q(k) = U(k)M(k),$$
where

\[
M(k) = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b_{11} & 0 & b_{12} \\ 0 & 0 & t_a^{-1} & 0 \\ 0 & b_{21} & 0 & b_{22} \end{pmatrix} : a \in \text{GL}_k(k), \quad b = (b_{ij}) \in \text{Sp}_{2(m-k)}(k) \right\},
\]

\[
U(k) = \left\{ \begin{pmatrix} I_k & * & * & * \\ 0 & I_{m-k} & * & 0 \\ 0 & 0 & I_k & 0 \\ 0 & 0 & * & I_{m-k} \end{pmatrix} \in G(k) \right\}.
\]

The module of $k$-rational characters $X^*(M)_k$ of $M$ is a free $\mathbb{Z}$-module of rank 1 generated by the character

\[
\hat{\alpha}_Q(\delta(a, b)) = \det a.
\]

The height $H_Q : G(\mathbb{A}) \to \mathbb{R}_{>0}$ is given by $H_Q(g) = |\det a|_{\mathbb{A}}^{-1}$ if $g = u\delta(a, b)h$ with $u \in U(\mathbb{A})$, $\delta(a, b) \in M(\mathbb{A})$ and $h \in K$.

We restrict ourselves to the case $k = m$. An element of $M(\mathbb{A})$ is denoted by

\[
\delta(a) = \begin{pmatrix} a & 0 \\ 0 & t_a^{-1} \end{pmatrix}, \quad a \in \text{GL}_m(\mathbb{A}).
\]

Let

\[
H_m = \{ Z \in M_m(\mathbb{C}) : ^tZ = Z, \ \text{Im}Z \in P_m \}
\]

be the Siegel upper half space and $H'_m$ the direct product of $r$ copies of $H_m$. For $Z = (Z_\sigma)_{\sigma \in \rho_{\infty}} \in H'_m$, $\text{Re}Z$, $\text{Im}Z$ and $\text{det} Z$ stand for $(\text{Re}Z_\sigma)_{\sigma \in \rho_{\infty}}$, $(\text{Im}Z_\sigma)_{\sigma \in \rho_{\infty}}$ and $(\text{det} Z_\sigma)_{\sigma \in \rho_{\infty}}$, respectively. The group $G(k_{\infty})$ acts transitively on $H'_m$ by

\[
g\{Z\} = ((a_\sigma Z_\sigma + b_\sigma)(c_\sigma Z_\sigma + d_\sigma)^{-1})_{\sigma \in \rho_{\infty}}
\]

for $Z = (Z_\sigma) \in H'_m$ and

\[
g = (g_\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}_{\sigma \in \rho_{\infty}} \in G(k_{\infty}).
\]

The stabilizer $K_{\infty}$ of $Z_0 = (\sqrt{-1}I_m, \ldots, \sqrt{-1}I_m) \in H'_m$ in $G(k_{\infty})$ is a maximal compact subgroup of $G(k_{\infty})$. We choose $K$ as $K_{\infty} \times \prod_{\sigma \in \rho_f} \text{Sp}_n(\mathbb{O}_\sigma)$. The map $\pi : g_{\infty} \mapsto g\{Z_0\}$ from $G(k_{\infty})$ onto $H'_m$ gives an isomorphism $G(k_{\infty})/K_{\infty} \cong H'_m$, and hence $G(k) \backslash G(\mathbb{A})/K \cong G_{\infty} \backslash H'_m$. Since $\text{Im}\{u\delta(a)h\{Z_0\}\} = a t_a$ holds for $u \in U(k_{\infty})$, $a \in \text{GL}_m(k_{\infty})$ and $h \in K_{\infty}$, we have

\[
H_Q(g_{\infty}) = \text{Nr}_{k_{\infty}/\mathbb{R}}(\text{det} \text{Im}\{g_{\infty}(Z_0)\})^{-1/2} = \left( \prod_{\sigma \in \rho_{\infty}} \text{det} \text{Im}\{g_\sigma(\sqrt{-1}I_m)\} \right)^{-1/2}
\]

for any $g_{\infty} = (g_\sigma) \in G(k_{\infty})$, where $\text{Nr}_{k_{\infty}/\mathbb{R}}$ denotes the norm of $k_{\infty}$ over $\mathbb{R}$. 
The class number $h_Q$ of $M \cong \text{GL}_m$ defined over $k$ is equal to the class number $h_k$ of $k$. We assume $h_k = 1$ for simplicity. Then $G(k) = Q(k)G_0$ and $G(\mathbb{A}) = Q(k)G_{\mathbb{A},\infty}$, and hence

$$m_Q(g_\infty) = \min_{\gamma \in G_0} H_Q(\gamma g_\infty).$$

Since

$$\text{Nr}_{k_\infty/\mathbb{R}}(\det \text{Im}\{\gamma (Z)\}) = \prod_{\sigma \in \mathbb{P}_\infty} |\det(\sigma(c)Z_\sigma + \sigma(d))|^{-2} \text{Nr}_{k_\infty/\mathbb{R}}(\det \text{Im}Z)$$

for $Z = (Z_\sigma) \in H_m^r$ and

$$\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in G_0 = \text{Sp}_n(o),$$

the condition $m_Q(g_\infty) = H_Q(g_\infty)$ of $g_\infty$ is equivalent with the following condition of $Z = g_\infty(Z_0)$:

$$\prod_{\sigma \in \mathbb{P}_\infty} |\det(\sigma(c)Z_\sigma + \sigma(d))| \geq 1 \quad \text{for all } \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in G_0.$$

Therefore, the domain $R_{e,\infty}$ modulo $K_\infty$ is isomorphic to

$$F = \left\{ (Z_\sigma) \in H_m^r : \prod_{\sigma \in \mathbb{P}_\infty} |\det(\sigma(c)Z_\sigma + \sigma(d))| \geq 1 \text{ for all } \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in G_0 \right\}.$$

Let $C$ be an arbitrary fundamental domain of the additive group $M_m(k_\infty)$ with respect to $M_m(o)$, and let $D$ be an arbitrary fundamental domain of $P_m^r$ with respect to $\text{GL}_m(o)$. It is easy to see that

$$F(C, D) = \{Z \in F : \text{Re}Z \in C, \text{Im}Z \in D\}$$

is a fundamental domain of $F$ with respect to $Q_o$. By Corollary 22, $F(C, D)$ is a fundamental domain of $H_m^r$ with respect to $G_0$.

If $k = \mathbb{Q}$ and $D$ is Minkowski’s fundamental domain, then $F(C, D)$ coincides with Siegel’s fundamental domain [1939].

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References


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TAKAO WATANABE
GRADUATE SCHOOL OF SCIENCE
OSAKA UNIVERSITY
MACHIKANEYAMA 1-1
TOYONAKA 560-0043
JAPAN
 twatanabe@math.sci.osaka-u.ac.jp
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