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RYSHKOV DOMAINS OF REDUCTIVE ALGEBRAIC GROUPS

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Dedicated to Professor Ichiro Satake on his 85th birthday

Let G be a connected reductive algebraic group defined over a number field k. In this paper, we introduce the Ryshkov domain R for the arithmetical minimum function m_Q defined from a height function associated to a maximal k-parabolic subgroup Q of G. The domain R is a $Q(\mathsf{k})$-invariant subset of the adele group $G(\mathbb{A})$. We show that a fundamental domain Ω for $Q(\mathsf{k})\backslash\mathsf{R}$ yields a fundamental domain for $G(\mathsf{k})\backslash G(\mathbb{A})$. We also see that any local maximum of m_Q is attained on the boundary of Ω.

Introduction

Let P_n be the cone of positive definite n by n real symmetric matrices, and let $\mathsf{m}(A)$ be the arithmetical minimum $\min_{0\neq x\in\mathbb{Z}^n} {}^t x A x$ of $A \in \mathsf{P}_n$. The function $f : A \mapsto \mathsf{m}(A)/(\det A)^{1/n}$ on P_n is called the Hermite invariant. Since the maximum of f gives the Hermite constant γ_n for dimension n, the determination of local maxima of f is a fundamental problem of lattice sphere packings in Euclidean spaces and the arithmetic theory of quadratic forms. Voronoi's theorem [1908, Théorème 17] states that f attains a local maximum at a point A if and only if A is perfect and eutactic. Moreover, perfect forms play an essential role in Voronoi's reduction theory of P_n with respect to the action of $\mathrm{GL}_n(\mathbb{Z})$ (see, e.g., [Martinet 2003] and [Schürmann 2009]). Ryshkov [1970] introduced a locally finite polyhedron $\mathsf{R}(\mathsf{m})$ in P_n defined by the condition $\mathsf{m}(A) \geq 1$. It is not difficult to show that A is perfect with $\mathsf{m}(A) = 1$ if and only if A is a vertex of the boundary of $\mathsf{R}(\mathsf{m})$. In particular, any local maximum of the Hermite invariant f is attained on the boundary of $\mathsf{R}(\mathsf{m})$. In this sense, we can say that the Ryshkov polyhedron $\mathsf{R}(\mathsf{m})$ is well matched with f.

Let G be a connected isotropic reductive algebraic group defined over a number field k, and let Q be a maximal k-parabolic subgroup of G. In previous papers [Watanabe 2000; 2003], we investigated a constant $\gamma(G, Q, \mathsf{k})$ as a generalization of Hermite's constant γ_n. Precisely, the constant $\gamma(G, Q, \mathsf{k})$ is defined to be

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the maximum of the function $\mathsf{m}_Q(g) = \min_{x\in Q(\mathsf{k})\backslash G(\mathsf{k})} H_Q(xg)$ on $G(\mathsf{k})\backslash G(\mathbb{A})^1$, where H_Q denotes the height function associated to Q. To prove the existence of the maximum of m_Q, we used Borel and Harish-Chandra's reduction theory for the adele group $G(\mathbb{A})$ with respect to $G(\mathsf{k})$. However, a Siegel set in $G(\mathbb{A})$ is not well matched with m_Q in a sense that one cannot obtain any information on locations of extreme points of m_Q in a Siegel set.

The purpose of this paper is to construct a fundamental domain of $G(\mathbb{A})^1$ with respect to $G(\mathsf{k})$ which is well matched with m_Q. We first consider an analog of the Ryshkov polyhedron. We set $X_Q(g) = \{x \in Q(\mathsf{k})\backslash G(\mathsf{k}) : m_Q(g) = H_Q(xg)\}$ for a given $g \in G(\mathbb{A})^1$. This is a finite subset of $Q(\mathsf{k})\backslash G(\mathsf{k})$ and is regarded as an analog of the set of minimal vectors of a positive definite real quadratic form. We define the domain $\mathsf{R}(\mathsf{m}_Q)$ as follows:

$$\mathsf{R}(\mathsf{m}_Q) = \{g \in G(\mathbb{A})^1 : \bar{e} \in X_Q(g)\},$$

where $\bar{e}$ denotes the trivial class $Q(\mathsf{k})$ in $Q(\mathsf{k})\backslash G(\mathsf{k})$. The set $\mathsf{R}(\mathsf{m}_Q)$ is a left $Q(\mathsf{k})$-invariant closed set with nonempty interior. The interior of $\mathsf{R}(\mathsf{m}_Q)$ is just a subset R_1 consisting of $g \in \mathsf{R}(\mathsf{m}_Q)$ such that $X_Q(g)$ is the one-point set $\{\bar{e}\}$. We denote by R_1^- the closure of R_1 in $G(\mathbb{A})^1$. Both R_1 and R_1^- are also left $Q(\mathsf{k})$-invariant. By Baer and Levi's theorem [1931, Satz 7], there exists an open fundamental domain Ω_Q of R_1^- with respect to $Q(\mathsf{k})$, that is, Ω_Q is a relatively open subset of R_1^- satisfying

- $Q(\mathsf{k})\Omega_Q^- = \mathsf{R}_1^-$, where Ω_Q^- denotes the closure of Ω_Q in R_1^-, and
- $\gamma\Omega_Q \cap \Omega_Q^- = \varnothing$ for any $\gamma \in Q(\mathsf{k}) \setminus \{e\}$.

Let Ω_Q° denote the interior of Ω_Q in $G(\mathbb{A})^1$. Then our main theorem is stated as follows:

Theorem. *The set Ω_Q° is an open fundamental domain of $G(\mathbb{A})^1$ with respect to $G(\mathsf{k})$. Any local maximum of m_Q is attained on the intersection of the boundary of Ω_Q° and the boundary of R_1^-.*

If we denote by r_G the k-rank of the commutator subgroup of G, then G has r_G standard maximal k-parabolic subgroups. Since Ω_Q depends on Q, we obtain r_G different kinds of fundamental domains of $G(\mathbb{A})^1$ with respect to $G(\mathsf{k})$. The method to construct Ω_Q may be viewed as a generalization of the highest point method (see [Grenier 1988] and [Terras 1988, §4,4]). For example, let $\mathsf{k} = \mathbb{Q}$, $G = \mathrm{GL}_n$ and Q be a standard maximal $\mathbb{Q}$-parabolic subgroup such that $Q\backslash G$ is a projective space. Then our construction gives a fundamental domain Ω_Q whose Archimedean part is isomorphic with Grenier's fundamental domain. If we choose another standard maximal $\mathbb{Q}$-parabolic subgroup of GL_n as Q, then the

Archimedean part of Ω_Q yields a new kind of fundamental domain of P_n with respect to $\mathrm{GL}_n(\mathbb{Z})$ (see Example 3 in Section 7).

Notation. For a given ring $\mathfrak{A}$, the set of all n by k matrices with entries in $\mathfrak{A}$ is denoted by $\mathrm{M}_{n,k}(\mathfrak{A})$. We write $\mathrm{M}_n(\mathfrak{A})$ for $\mathrm{M}_{n,n}(\mathfrak{A})$. The transpose of a given matrix $a \in \mathrm{M}_{n,k}(\mathfrak{A})$ is denoted by ${}^t a$. In this paper, k denotes an algebraic number field of finite degree over $\mathbb{Q}$ and o the ring of integers of k. The sets of all infinite and finite places of k are denoted by p_∞ and p_f, respectively. For $\sigma \in \mathsf{p}_\infty \cup \mathsf{p}_f$, k_σ denotes the completion of k at σ. For $\sigma \in \mathsf{p}_f$, o_σ denotes the closure of o in k_σ. The étale $\mathbb{R}$-algebra $\mathsf{k}_\infty = \mathsf{k} \otimes_{\mathbb{Q}} \mathbb{R}$ is identified with $\prod_{\sigma\in\mathsf{p}_\infty} \mathsf{k}_\sigma$. Let $\mathbb{A}$ and $\mathbb{A}^\times$ denote the adele ring and the idèle group of k, respectively. The idèle norm of $\mathbb{A}^\times$ is denoted by $|\cdot|_{\mathbb{A}}$.

1. Height functions

Let G be a connected affine algebraic group defined over k. For any k-algebra $\mathfrak{A}$, $G(\mathfrak{A})$ stands for the set of $\mathfrak{A}$-rational points of G. Let $X^*(G)_{\mathsf{k}}$ be the free $\mathbb{Z}$-module consisting of all k-rational characters of G. For each $g \in G(\mathbb{A})$, we define the homomorphism $\vartheta_G(g) : X^*(G)_{\mathsf{k}} \to \mathbb{R}_{>0}$ by $\vartheta_G(g)(\chi) = |\chi(g)|_{\mathbb{A}}$ for $\chi \in X^*(G)_{\mathsf{k}}$. Then ϑ_G is a homomorphism from $G(\mathbb{A})$ into $\mathrm{Hom}_{\mathbb{Z}}(X^*(G)_{\mathsf{k}}, \mathbb{R}_{>0})$. We write $G(\mathbb{A})^1$ for the kernel of ϑ_G.

In the following, let G be a connected isotropic reductive group defined over k. We fix a maximal k-split torus S of G and a minimal k-parabolic subgroup P_0 of G containing S. Denote by Φ_{k} and Δ_{k} the relative root system of G with respect to S and the set of simple roots of Φ_{k} corresponding to P_0, respectively. Let M_0 be the centralizer of S in G. Then P_0 has a Levi decomposition $P_0 = M_0 U_0$, where U_0 is the unipotent radical of P_0. A k-parabolic subgroup of G containing P_0 is called a standard k-parabolic subgroup of G. Every standard k-parabolic subgroup R of G has a unique Levi subgroup M_R containing M_0. We denote by U_R the unipotent radical of R and by Z_R the greatest central k-split torus in M_R. Throughout this paper, we fix a maximal compact subgroup $K = \prod_{\sigma\in\mathsf{p}_\infty} K_\sigma \times \prod_{\sigma\in\mathsf{p}_f} K_\sigma$ of $G(\mathbb{A})$ satisfying the following property: for every standard k-parabolic subgroup R of G, $K \cap M_R(\mathbb{A})$ is a maximal compact subgroup of $M_R(\mathbb{A})$, and $M_R(\mathbb{A})$ possesses an Iwasawa decomposition $(M_R(\mathbb{A}) \cap U_0(\mathbb{A}))M_0(\mathbb{A})(K \cap M_R(\mathbb{A}))$.

Let Q be a standard proper maximal k-parabolic subgroup of G. There is only one simple root $\alpha_0 \in \Delta_{\mathsf{k}}$ such that the restriction of α_0 to Z_Q is nontrivial. Let n_Q be the positive integer such that $n_Q^{-1}\alpha_0|_{Z_Q}$ is a $\mathbb{Z}$-basis of $X^*(Z_Q/Z_G)_{\mathsf{k}}$. We write α_Q for $n_Q^{-1}\alpha_0|_{Z_Q}$ and $\hat{\alpha}_Q$ for $\hat{d}_Q n_Q^{-1}\alpha_0|_{Z_Q}$, where

$$\hat{d}_Q = [X^*(Z_Q/Z_G)_{\mathsf{k}} : X^*(M_Q/Z_G)_{\mathsf{k}}].$$

Then $\hat{\alpha}_Q$ is a $\mathbb{Z}$-basis of the submodule $X^*(M_Q/Z_G)_{\mathsf{k}}$ of $X^*(Z_Q/Z_G)_{\mathsf{k}}$. Define the map $z_Q : G(\mathbb{A}) \to Z_G(\mathbb{A})M_Q(\mathbb{A})^1\backslash M_Q(\mathbb{A})$ by $z_Q(g) = Z_G(\mathbb{A})M_Q(\mathbb{A})^1 m$ if $g = umh$ with $u \in U_Q(\mathbb{A})$, $m \in M_Q(\mathbb{A})$ and $h \in K$. This is well defined and left $Z_G(\mathbb{A})Q(\mathbb{A})^1$-invariant. Since $Z_G(\mathbb{A})^1 = Z_G(\mathbb{A}) \cap G(\mathbb{A})^1 \subset M_Q(\mathbb{A})^1$, z_Q gives rise to a map from $Y_Q = Q(\mathbb{A})^1\backslash G(\mathbb{A})^1$ to $M_Q(\mathbb{A})^1\backslash(M_Q(\mathbb{A}) \cap G(\mathbb{A})^1)$. Namely, we have the following commutative diagram, whose vertical arrows are natural maps:

$$\begin{array}{ccc} Y_Q & \xrightarrow{z_Q} & M_Q(\mathbb{A})^1\backslash(M_Q(\mathbb{A}) \cap G(\mathbb{A})^1) \\ \downarrow & & \downarrow \\ Z_G(\mathbb{A})Q(\mathbb{A})^1\backslash G(\mathbb{A}) & \xrightarrow{z_Q} & Z_G(\mathbb{A})M_Q(\mathbb{A})^1\backslash M_Q(\mathbb{A}). \end{array}$$

We define the height function $H_Q : G(\mathbb{A}) \to \mathbb{R}_{>0}$ by $H_Q(g) = |\widehat{\alpha}_Q(z_Q(g))|_{\mathbb{A}}^{-1}$ for $g \in G(\mathbb{A})$. We notice that the restriction of H_Q to $M_Q(\mathbb{A})$ is a homomorphism from $M_Q(\mathbb{A})$ onto $\mathbb{R}_{>0}$.

Example 1. Let G be a general linear group GL_n defined over the rational number field $\mathbb{Q}$, P_0 the group of upper triangular matrices in G and S the group of diagonal matrices in G. We fix an integer $k \in \{1, \dots, n-1\}$, and let

$$Q(\mathbb{Q}) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a \in \mathrm{GL}_k(\mathbb{Q}),\ b \in \mathrm{M}_{k,n-k}(\mathbb{Q}),\ d \in \mathrm{GL}_{n-k}(\mathbb{Q}) \right\}.$$

Then Q is a standard maximal $\mathbb{Q}$-parabolic subgroup of G. The rational character $\widehat{\alpha}_Q$ and the height H_Q are given by

$$\widehat{\alpha}_Q\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) = (\det a)^{(n-k)/r}(\det d)^{-k/r}$$

and

$$H_Q\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) = |\det a|_{\mathbb{A}}^{-(n-k)/r}|\det d|_{\mathbb{A}}^{k/r},$$

where r denotes the greatest common divisor of k and $n-k$. The height H_Q has another expression. To explain this, let $\mathbb{Q}^n$ be an n-dimensional column vector space over $\mathbb{Q}$ with standard basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$. The maximal parabolic subgroup $Q(\mathbb{Q})$ stabilizes the subspace spanned by $\boldsymbol{e}_1, \dots, \boldsymbol{e}_k$. Let $V_{n,k}(\mathbb{Q}) = \bigwedge^k \mathbb{Q}^n$ be the k-th exterior product of $\mathbb{Q}^n$. We set $V_{n,k}(\mathbb{A}) = V_{n,k}(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{A}$ and $V_{n,k}(\mathbb{Q}_\sigma) = V_{n,k}(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\sigma$ for $\sigma \in \mathfrak{p}_\infty \cup \mathfrak{p}_f$. A $\mathbb{Q}$-basis of $V_{n,k}(\mathbb{Q})$ is formed by the elements $\boldsymbol{e}_I = \boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_k}$ with $I = \{i_1 < i_2 < \cdots < i_k\} \subset \{1, \dots, n\}$. For a unique infinite place $\infty \in \mathfrak{p}_\infty$, we define the local height $H_\infty : V_{n,k}(\mathbb{Q}_\infty) \to \mathbb{R}_{>0}$ by

$$H_\infty\Big(\sum_I a_I \boldsymbol{e}_I\Big) = \Big(\sum_I |a_I|_\infty^2\Big)^{1/2},$$

where $|\cdot|_\infty$ denotes the usual absolute value of $\mathbb{Q}_\infty = \mathbb{R}$. For each finite prime $p \in \mathrm{p}_f$, we define the local height $H_p : V_{n,k}(\mathbb{Q}_p) \to \mathbb{R}_{>0}$ by

$$H_p\Big(\sum_I a_I \boldsymbol{e}_I\Big) = \sup_I |a_I|_p,$$

where $|\cdot|_p$ denotes the p-adic absolute value of $\mathbb{Q}_p$ normalized so that $|p|_p = p^{-1}$. Then the global height $H_{n,k} : V_{n,k}(\mathbb{Q}) \to \mathbb{R}_{>0}$ is defined to be a product of all local heights, that is, $H_{n,k}(x) = \prod_{\sigma\in\mathrm{p}_\infty\cup\mathrm{p}_f} H_\sigma(x)$ for $x \in V_{n,k}(\mathbb{Q})$. This $H_{n,k}$ is immediately extended to the subset $\mathrm{GL}(V_{n,k}(\mathbb{A}))V_{n,k}(\mathbb{Q})$ of the adele space $V_{n,k}(\mathbb{A})$ by

$$H_{n,k}(Ax) = \prod_{\sigma\in\mathrm{p}_\infty\cup\mathrm{p}_f} H_\sigma(A_\sigma x)$$

for $A = (A_\sigma) \in \mathrm{GL}(V_{n,k}(\mathbb{A}))$ and $x \in V_{n,k}(\mathbb{Q})$. In particular, for $g \in G(\mathbb{A}) = \mathrm{GL}_n(\mathbb{A})$, we can take the value $H_{n,k}(g\boldsymbol{e}_1 \wedge g\boldsymbol{e}_2 \wedge \cdots \wedge g\boldsymbol{e}_k)$. We choose a maximal compact subgroup K_∞ of $G(\mathbb{Q}_\infty)$ as $\{g \in G(\mathbb{Q}_\infty) : {}^t g^{-1} = g\}$. Let

$$K_f = \prod_{p\in\mathrm{p}_f} \mathrm{GL}_n(\mathbb{Z}_p) \quad \text{and} \quad K = K_\infty \times K_f.$$

Then, by elementary computations, we have

$$H_{n,k}(g\boldsymbol{e}_1 \wedge g\boldsymbol{e}_2 \wedge \cdots \wedge g\boldsymbol{e}_k) = |\det a|_{\mathbb{A}} \quad \text{if } g = h\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}$$

with $h \in K$, $a \in \mathrm{GL}_k(\mathbb{A})$, $b \in \mathrm{M}_{k,n-k}(\mathbb{A})$ and $d \in \mathrm{GL}_{n-k}(\mathbb{A})$. Therefore, if $g \in G(\mathbb{A})^1$, that is, $|\det g|_{\mathbb{A}} = 1$, then

$$H_Q(g) = H_{n,k}\big(g^{-1}\boldsymbol{e}_1 \wedge g^{-1}\boldsymbol{e}_2 \wedge \cdots \wedge g^{-1}\boldsymbol{e}_k\big)^{n/r}.$$

2. Twisted height functions restricted to one parameter subgroups

Let $N_G(S)$ be the normalizer of S in G and $W_G = N_G(S)(\mathsf{k})/M_0(\mathsf{k})$ the Weyl group of G with respect to S. For a simple root $\alpha \in \Delta_{\mathsf{k}}$, $s_\alpha \in W_G$ denotes the simple reflection corresponding to α. Then $\{s_\alpha\}_{\alpha\in\Delta_{\mathsf{k}}}$ generates W_G. We denote by W_G^Q the subgroup of W_G generated by $\{s_\alpha\}_{\alpha\in\Delta_{\mathsf{k}}\setminus\{\alpha_0\}}$. For each $w \in W_G$, we use the same notation w for a representative of w in $N_G(S)(\mathsf{k})$. The following cell decomposition of $G(\mathsf{k})$ holds via Bruhat decomposition [Borel and Tits 1965, Proposition 4.10, Corollaire 5.20]:

$$G(\mathsf{k}) = \bigsqcup_{[w]\in W_G^Q\backslash W_G/W_G^Q} Q(\mathsf{k})wQ(\mathsf{k}),$$

where $[w]$ stands for the class $W_G^Q w W_G^Q$ in $W_G^Q\backslash W_G/W_G^Q$.

The Weyl group W_G acts on $\boldsymbol{X}^*(S)_{\mathsf{k}}$ by $w\cdot\chi : t\mapsto \chi\big(w^{-1}tw\big)$ for $w\in W_G$ and $\chi\in \boldsymbol{X}^*(S)_{\mathsf{k}}$. We consider the restriction $\widehat{\alpha}_Q|_S$ of the rational character $\widehat{\alpha}_Q$ of M_Q to S.

Lemma 1. *The subgroup of W_G fixing $\widehat{\alpha}_Q|_S$ is equal to W_G^Q.*

Proof. Put $W'=\{w\in W_G\ :\ w\cdot\widehat{\alpha}_Q|_S=\widehat{\alpha}_Q|_S\}$. Since a representative of $w\in W_G^Q$ is contained in $M_Q(\mathsf{k})$, we have $\widehat{\alpha}_Q(w^{-1}tw)=\widehat{\alpha}_Q(w)^{-1}\widehat{\alpha}_Q(t)\widehat{\alpha}_Q(w)=\widehat{\alpha}_Q(t)$ for all $t\in S$. Hence W_G^Q is contained in W'. By [Humphreys 1990, §1.12 Theorem (a) and (c)] , W' is generated by a subset $W'\cap\{s_\alpha\}_{\alpha\in\Delta_{\mathsf{k}}}$ of simple reflections. From $W_G^Q\subset W'$, it follows $\{s_\alpha\}_{\alpha\in\Delta_{\mathsf{k}}\setminus\{\alpha_0\}}\subset W'\cap\{s_\alpha\}_{\alpha\in\Delta_{\mathsf{k}}}\subset\{s_\alpha\}_{\alpha\in\Delta_{\mathsf{k}}}$. Since $\widehat{\alpha}_Q$ is nontrivial on S/Z_G, $W'\cap\{s_\alpha\}_{\alpha\in\Delta_{\mathsf{k}}}$ must equal $\{s_\alpha\}_{\alpha\in\Delta_{\mathsf{k}}\setminus\{\alpha_0\}}$. Therefore W' coincides with W_G^Q. ◻

Let $\boldsymbol{X}_*(S)_{\mathsf{k}}$ be the free $\mathbb{Z}$-module consisting of all k-rational cocharacters of S. A natural pairing

$$\langle\cdot,\cdot\rangle : \boldsymbol{X}^*(S)_{\mathsf{k}}\times\boldsymbol{X}_*(S)_{\mathsf{k}}\to\mathbb{Z}$$

defined as in [Borel 1991, §8.6] is a regular pairing over $\mathbb{Z}$.

Lemma 2. *Let w_1 and w_2 be elements of W_G such that $w_1^{-1}W_G^Q\neq w_2^{-1}W_G^Q$. Then there exist a cocharacter $\xi=\xi_{w_1,w_2}\in\boldsymbol{X}_*(S)_{\mathsf{k}}$ such that*

$$H_Q\big(w_1\xi(\lambda)w_1^{-1}\big)>H_Q\big(w_2\xi(\lambda)w_2^{-1}\big)$$

holds for all $\lambda\in\mathbb{A}^\times_{>1}$, where $\mathbb{A}^\times_{>1}$ denotes the set of $\lambda\in\mathbb{A}^\times$ satisfying $|\lambda|_{\mathbb{A}}>1$.

Proof. Since $w_1^{-1}\cdot\widehat{\alpha}_Q|_S-w_2^{-1}\cdot\widehat{\alpha}_Q|_S\neq 0$ by Lemma 1, there is a $\xi\in\boldsymbol{X}_*(S)_{\mathsf{k}}$ such that $\langle w_1^{-1}\cdot\widehat{\alpha}_Q|_S-w_2^{-1}\cdot\widehat{\alpha}_Q|_S,\xi\rangle<0$. The value $\ell=\langle w_1^{-1}\cdot\widehat{\alpha}_Q|_S-w_2^{-1}\cdot\widehat{\alpha}_Q|_S,\xi\rangle$ is a negative integer. We have

$$\widehat{\alpha}_Q(w_1\xi(\lambda)w_1^{-1})\cdot\widehat{\alpha}_Q(w_2\xi(\lambda)w_2^{-1})^{-1}=\lambda^\ell$$

for all $\lambda\in\boldsymbol{G}_m$. Therefore,

$$H_Q(w_1\xi(\lambda)w_1^{-1})H_Q(w_2\xi(\lambda)w_2^{-1})^{-1}=|\lambda|_{\mathbb{A}}^{-\ell}>1$$

holds for all $\lambda\in\mathbb{A}^\times_{>1}$. ◻

3. The Hermite function associated to Q and minimal points

We set $X_Q=Q(\mathsf{k})\backslash G(\mathsf{k})$, which is regarded as a subset of $Y_Q=Q(\mathbb{A})^1\backslash G(\mathbb{A})^1$. Let $\pi_X : G(\mathsf{k})\to X_Q$ be the natural quotient map. The symbol $\bar{e}=\pi_X(e)\in X_Q$ denotes the class of the unit element $e\in G(\mathsf{k})$. The Hermite function

$$\mathsf{m}_Q : G(\mathbb{A})^1\to\mathbb{R}_{>0}$$

is defined to be

$$\mathsf{m}_Q(g) = \min_{x \in X_Q} H_Q(xg).$$

By definition, m_Q is a positive valued continuous function on $G(\mathsf{k})\backslash G(\mathbb{A})^1/K$.

For each $g \in G(\mathbb{A})^1$, we put

$$X_Q(g) = \{x \in X_Q : \mathsf{m}_Q(g) = H_Q(xg)\},$$

which is a finite subset of X_Q. Thus we can define the counting function $\mathsf{n}_Q(g) = \# X_Q(g)$.

Lemma 3. *For any $g \in G(\mathbb{A})^1$, $\gamma \in G(\mathsf{k})$ and $h \in K$, one has $X_Q(\gamma g h) = X_Q(g)\gamma^{-1}$. Especially, the counting function n_Q is left $G(\mathsf{k})$-invariant and right K-invariant.*

The following lemma is proved by the same method as in [Watanabe 2012, Proof of Proposition 4.1].

Lemma 4. *For $g \in G(\mathbb{A})^1$, there is a neighborhood $\mathcal{U}$ of g in $G(\mathbb{A})^1$ such that $X_Q(g') \subset X_Q(g)$ for all $g' \in \mathcal{U}$.*

Example 2. Let G be a general linear group GL_n defined over $\mathbb{Q}$. We keep notations used in Example 1. In this case, we can express m_Q in terms of some minimum of positive definite symmetric matrices. Since $\mathrm{GL}_n/\mathbb{Q}$ is of class number one, $G(\mathbb{A})^1 = \{g \in \mathrm{GL}_n(\mathbb{A}) : |\det g|_{\mathbb{A}} = 1\}$ has the following decomposition:

$$G(\mathbb{A})^1 = G(\mathbb{Q})(G(\mathbb{Q}_\infty)^1 \times K_f),$$

where $G(\mathbb{Q}_\infty)^1 = \{g \in \mathrm{GL}_n(\mathbb{Q}_\infty) : \det g = \pm 1\}$ and $K_f = \prod_{p \in \mathsf{p}_f} \mathrm{GL}_n(\mathbb{Z}_p)$. We fix $g = \delta(g_\infty \times g_f) \in G(\mathbb{A})^1$ with $\delta \in G(\mathbb{Q})$, $g_\infty \in G(\mathbb{Q}_\infty)^1$ and $g_f \in K_f$. From the left $G(\mathbb{Q})$-invariance and the right K-invariance of m_Q, it follows that

$$m_Q(g) = m_Q(g_\infty) = \min_{x \in X_Q} H_Q(x g_\infty) = \min_{\gamma \in G(\mathbb{Q})} H_Q(\gamma g_\infty).$$

Furthermore, since $G(\mathbb{Q}) = Q(\mathbb{Q})\,\mathrm{GL}_n(\mathbb{Z})$ and H_Q is left $Q(\mathbb{Q})$-invariant, we have

$$m_Q(g) = \min_{\gamma \in \mathrm{GL}_n(\mathbb{Z})} H_Q(\gamma g_\infty).$$

An elementary proof of the decomposition $G(\mathbb{Q}) = Q(\mathbb{Q})\,\mathrm{GL}_n(\mathbb{Z})$ is found in [Shimura 1994, Theorem 3]. By Example 1,

$$\begin{aligned} H_Q(\gamma g_\infty) &= H_{n,k}\big(g_\infty^{-1}\gamma^{-1}\boldsymbol{e}_1 \wedge \cdots \wedge g_\infty^{-1}\gamma^{-1}\boldsymbol{e}_k\big)^{n/r} \\ &= H_\infty\big(g_\infty^{-1}\gamma^{-1}\boldsymbol{e}_1 \wedge \cdots \wedge g_\infty^{-1}\gamma^{-1}\boldsymbol{e}_k\big)^{n/r} \prod_{p \in \mathsf{p}_f} H_p\big(\gamma^{-1}\boldsymbol{e}_1 \wedge \cdots \wedge \gamma^{-1}\boldsymbol{e}_k\big)^{n/r} \\ &= H_\infty\big(g_\infty^{-1}\gamma^{-1}\boldsymbol{e}_1 \wedge \cdots \wedge g_\infty^{-1}\gamma^{-1}\boldsymbol{e}_k\big)^{n/r}. \end{aligned}$$

Here we notice that $H_p(\gamma^{-1}\boldsymbol{e}_1 \wedge \cdots \wedge \gamma^{-1}\boldsymbol{e}_k) = 1$ for all $p \in \mathsf{p}_f$ and $\gamma \in \mathrm{GL}_n(\mathbb{Z})$. For a given $\gamma \in \mathrm{GL}_n(\mathbb{Z})$, X_γ stands for the n by k matrix consisting of the first k columns of γ. Binet's formula (see [Bombieri and Gubler 2006, Proposition 2.8.8]) yields

$$H_\infty\big(g_\infty^{-1}\gamma^{-1}\boldsymbol{e}_1 \wedge \cdots \wedge g_\infty^{-1}\gamma^{-1}\boldsymbol{e}_k\big) = \det\big({}^tX_{\gamma^{-1}}\,{}^tg_\infty^{-1}g_\infty^{-1}X_{\gamma^{-1}}\big)^{1/2}.$$

As a consequence, we obtain

$$m_Q(g) = \min_{X \in \mathrm{M}_{n,k}(\mathbb{Z})^*} \det\big({}^tX\,{}^tg_\infty^{-1}g_\infty^{-1}X\big)^{n/2r},$$

where $\mathrm{M}_{n,k}(\mathbb{Z})^*$ denotes the set of X_γ for all $\gamma \in \mathrm{GL}_n(\mathbb{Z})$. In the case of $k = 1$, $\mathrm{M}_{n,1}(\mathbb{Z})^*$ is just the set of primitive vectors of the lattice $\mathbb{Z}^n$, and hence $m_Q(g)$ coincides with the $n/2$ power of the arithmetical minimum of the positive definite symmetric matrix ${}^tg_\infty^{-1}g_\infty^{-1}$.

4. The Ryshkov domain of G associated to Q

We define the Ryshkov domain $\mathsf{R} = \mathsf{R}(\mathsf{m}_Q)$ of m_Q by

$$\mathsf{R} = \mathsf{R}(\mathsf{m}_Q) = \big\{g \in G(\mathbb{A})^1 : \mathsf{m}_Q(g)/H_Q(g) \geq 1\big\}.$$

Since $\mathsf{m}_Q(g) \leq H_Q(g)$ holds for all $g \in G(\mathbb{A})^1$, we have

$$\begin{aligned}\mathsf{R} &= \big\{g \in G(\mathbb{A})^1 : \mathsf{m}_Q(g) = H_Q(g)\big\} \\ &= \big\{g \in G(\mathbb{A})^1 : \bar{e} \in X_Q(g)\big\}.\end{aligned}$$

Since both H_Q and m_Q are continuous, R is a closed subset in $G(\mathbb{A})^1$.

Lemma 5. *One has $Q(\mathsf{k})\mathsf{R}K = \mathsf{R}$ and $G(\mathbb{A})^1 = G(\mathsf{k})\mathsf{R}$.*

Proof. The first assertion is obvious by the definition of H_Q. To prove the second assertion, we choose a minimal point $x \in X_Q(g)$ for a given $g \in G(\mathbb{A})^1$. There is a $\gamma \in G(\mathsf{k})$ such that $x = \pi_X(\gamma)$. Then $H_Q(xg) = H_Q(\gamma g) = \mathsf{m}_Q(g) = \mathsf{m}_Q(\gamma g)$ since m_Q is left $G(\mathsf{k})$-invariant. Therefore, $\gamma g \in \mathsf{R}$. □

Lemma 6. *Let C be an arbitrary subset of $G(\mathbb{A})^1$, and let $g \in G(\mathbb{A})^1$ and $\gamma \in G(\mathsf{k})$.*

(1) $\gamma g \in \mathsf{R}$ *if and only if* $\pi_X(\gamma) \in X_Q(g)$.

(2) $X_Q(g) = \pi_X(\{\gamma \in G(\mathsf{k}) : \gamma g \in \mathsf{R}\})$.

(3) $\gamma C \subset \mathsf{R}$ *if and only if* $\pi_X(\gamma) \in \bigcap_{g \in C} X_Q(g)$.

(4) $\bigcap_{g \in \mathsf{R}} X_Q(g) = \{\bar{e}\}$.

(5) $\gamma \mathsf{R} \subset \mathsf{R}$ *if and only if* $\gamma \in Q(\mathsf{k})$.

Proof. By definition, $\gamma g \in \mathsf{R}$ if and only if $\mathsf{m}_Q(\gamma g) = H_Q(\gamma g)$. This is equivalent to $\pi_X(\gamma) \in X_Q(g)$ because $\mathsf{m}_Q(\gamma g) = \mathsf{m}_Q(g)$. Both (2) and (3) follow from (1). For a point $x = \pi_X(\gamma) \in \bigcap_{g \in \mathsf{R}} X_Q(g)$, we have $\gamma Q(\mathsf{k})\mathsf{R} \subset \mathsf{R}$; in other words, $xQ(\mathsf{k}) \subset \bigcap_{g \in \mathsf{R}} X_Q(g)$. Since $xQ(\mathsf{k})$ is an infinite set for $x \neq \bar{e}$ by Bruhat decomposition, we must have $x = \bar{e}$. This shows (4). Item (5) follows from (3) and (4). □

Lemma 7. *Let $g_0 \in \mathsf{R}$ be an element such that $\mathsf{n}_Q(g_0) > 1$ and x_0 an arbitrary element in $X_Q(g_0)$. Then, any neighborhood $\mathcal{U}$ of g_0 in $G(\mathbb{A})^1$ contains a point g such that $X_Q(g) \subset X_Q(g_0)$ and $x_0 \notin X_Q(g)$.*

Proof. We may assume $\mathcal{U}$ satisfies $X_Q(g) \subset X_Q(g_0)$ for all $g \in \mathcal{U}$ by Lemma 4. Since $\mathsf{n}_Q(g_0) > 1$, there is an $x \in X_Q(g_0)$ such that $x \neq \bar{e}$. This x is of the form $\pi_X(w\gamma)$ with $w \in W_G \setminus W_G^Q$ and $\gamma \in Q(\mathsf{k})$. By Lemma 2, there is a cocharacter $\xi = \xi_{w,e} \in X_*(S)_{\mathsf{k}}$ such that $H_Q\big(w\xi(\lambda)w^{-1}\big) > H_Q(\xi(\lambda))$ holds for all $\lambda \in \mathbb{A}^\times_{>1}$. Let $\lambda \in \mathbb{A}^\times$ be an element sufficiently close to 1 so that $g_\lambda = \gamma^{-1}\xi(\lambda)\gamma g_0$ is contained in $\mathcal{U}$. We have

$$\begin{aligned} H_Q(g_\lambda) &= H_Q(\xi(\lambda)\gamma g_0) = H_Q(\xi(\lambda))H_Q(\gamma g_0) \\ &= H_Q(\xi(\lambda))H_Q(g_0) = H_Q(\xi(\lambda))\mathsf{m}_Q(g_0) \end{aligned}$$

and

$$\begin{aligned} H_Q(xg_\lambda) &= H_Q(w\xi(\lambda)\gamma g_0) = H_Q(w\xi(\lambda)w^{-1})H_Q(w\gamma g_0) \\ &= H_Q(w\xi(\lambda)w^{-1})\mathsf{m}_Q(g_0). \end{aligned}$$

If $x_0 = \bar{e}$, then we choose λ sufficiently close to 1 satisfying $\lambda^{-1} \in \mathbb{A}^\times_{>1}$. Since $X_Q(g_\lambda) \subset X_Q(g_0)$ and $\mathsf{m}_Q(g_\lambda) \le H_Q(xg_\lambda) < H_Q(g_\lambda)$, $X_Q(g_\lambda)$ does not contain $\bar{e}$. If $x_0 \neq \bar{e}$, then we choose x as x_0 and $\lambda \in \mathbb{A}^\times_{>1}$ sufficiently close to 1. Since $\mathsf{m}_Q(g_\lambda) \le H_Q(g_\lambda) < H_Q(x_0 g_\lambda)$, $X_Q(g_\lambda)$ does not contain x_0. □

Lemma 8. $\min_{g \in G(\mathbb{A})^1} \mathsf{n}_Q(g) = \min_{g \in \mathsf{R}} \mathsf{n}_Q(g) = 1$.

Proof. From Lemma 5 and the $G(\mathsf{k})$-invariance of n_Q, it follows that

$$\min_{g \in G(\mathbb{A})^1} \mathsf{n}_Q(g) = \min_{g \in \mathsf{R}} \mathsf{n}_Q(g).$$

If $g_0 \in \mathsf{R}$ satisfies $\min_{g \in \mathsf{R}} \mathsf{n}_Q(g) = \mathsf{n}_Q(g_0) > 1$, then by Lemmas 5 and 7, there exist a point $g_1 \in G(\mathbb{A})^1$ and $\gamma_1 \in G(\mathsf{k})$ such that $\mathsf{n}_Q(\gamma_1 g_1) = \mathsf{n}_Q(g_1) < \mathsf{n}_Q(g_0)$ and $\gamma_1 g_1 \in \mathsf{R}$. This is a contradiction. □

We define the subset R_1 of R by

$$\mathsf{R}_1 = \{g \in \mathsf{R} : \mathsf{n}_Q(g) = 1\} = \big\{g \in G(\mathbb{A})^1 : X_Q(g) = \{\bar{e}\}\big\}.$$

Lemma 9. R_1 *coincides with the interior* R° *of* R *in* $G(\mathbb{A})^1$.

Proof. For $g \in \mathsf{R}_1$, we choose a neighborhood $\mathcal{U}$ of g in $G(\mathbb{A})^1$ as in Lemma 4. Then $\mathcal{U} \subset \mathsf{R}_1$. Therefore, R_1 is open and is contained in R°. If there exists an element $g_0 \in \mathsf{R}^\circ$ such that $\mathsf{n}_Q(g_0) > 1$, then, by Lemma 7, R° contains an element g satisfying $\bar{e} \notin X_Q(g)$. This contradicts $g \in \mathsf{R}$. □

It is obvious that $G(\mathsf{k})\mathsf{R}_1 = \{g \in G(\mathbb{A})^1 : \mathsf{n}_Q(g) = 1\}$.

Lemma 10. *$G(\mathsf{k})\mathsf{R}_1$ is open and dense in $G(\mathbb{A})^1$.*

Proof. Since R_1 is open in $G(\mathbb{A})^1$, so is $G(\mathsf{k})\mathsf{R}_1$. We assume $G(\mathbb{A})^1 \setminus G(\mathsf{k})\mathsf{R}_1$ has an interior point g_0. Let $\mathcal{U}$ be a neighborhood of g_0 in $G(\mathbb{A})^1$ so that $\mathcal{U} \cap G(\mathsf{k})\mathsf{R}_1 = \varnothing$. By Lemma 5, we can take $\gamma_0 \in G(\mathsf{k})$ such that $\gamma_0 g_0 \in \mathsf{R}$. Since $\mathsf{n}_Q(\gamma_0 g_0) = \mathsf{n}_Q(g_0) > 1$, by Lemmas 5 and 7, there exist $g_1 \in \gamma_0\mathcal{U}$ and $\gamma_1 \in G(\mathsf{k})$ such that $\mathsf{n}_Q(g_1) < \mathsf{n}_Q(g_0)$ and $\gamma_1 g_1 \in \mathsf{R}$. If $\mathsf{n}_Q(g_1) > 1$, then there exist $g_2 \in \gamma_1\gamma_0\mathcal{U}$ and $\gamma_2 \in G(\mathsf{k})$ such that $\mathsf{n}_Q(g_2) < \mathsf{n}_Q(g_1)$ and $\gamma_2 g_2 \in R$. This process terminates after finitely many iterations. At the last step, we obtain an element $g_\ell \in \gamma_{\ell-1} \cdots \gamma_0\mathcal{U}$ such that $\mathsf{n}_Q(g_\ell) = 1$. Then $(\gamma_{\ell-1} \cdots \gamma_0)^{-1} g_\ell$ is contained in $\mathcal{U} \cap G(\mathsf{k})\mathsf{R}_1$. This contradicts $\mathcal{U} \cap G(\mathsf{k})\mathsf{R}_1 = \varnothing$. Therefore, $G(\mathbb{A})^1 \setminus G(\mathsf{k})\mathsf{R}_1$ is nowhere dense in $G(\mathbb{A})^1$. □

Lemma 11. *For $\gamma \in G(\mathsf{k})$, $\mathsf{R}_1 \cap \gamma\mathsf{R} \neq \varnothing$ if and only if $\gamma \in Q(\mathsf{k})$.*

Proof. If $\mathsf{R}_1 \cap \gamma\mathsf{R}$ has an element g, then $\pi_X(\gamma^{-1}) \in X_Q(g) = \{\bar{e}\}$ by Lemma 6. □

Lemma 12. *Let R_1^- be the closure of R_1. Then we have the following subdivision of $G(\mathbb{A})^1$:*

$$G(\mathbb{A})^1 = \bigcup_{\gamma Q(\mathsf{k}) \in G(\mathsf{k})/Q(\mathsf{k})} \gamma\mathsf{R}_1^-.$$

Proof. We fix an arbitrary $g \in G(\mathbb{A})^1$. By Lemma 10, there exists a sequence $\{g_n\} \subset G(\mathsf{k})\mathsf{R}_1$ such that $\lim_{n\to\infty} g_n = g$. We take a neighborhood $\mathcal{U}$ of g as in Lemma 4 and may assume that $\{g_n\} \subset \mathcal{U}$. Since $g_n \in G(\mathsf{k})\mathsf{R}_1$, $X_Q(g_n)$ consists of a single element $\pi_X(\gamma_n)$, where $\gamma_n \in G(\mathsf{k})$. From $g_n \in \mathcal{U}$, it follows that $\pi_X(\gamma_n) \in X_Q(g)$ for all n. Since $X_Q(g)$ is a finite set, we can take a subsequence $\{g_{n_j}\}$ such that $\pi_X(\gamma_{n_j}) = \pi_X(\gamma) \in X_Q(g)$ for all n_j. Then $\{g_{n_j}\} \subset \gamma^{-1}\mathsf{R}_1$, and g is contained in the closure of $\gamma^{-1}\mathsf{R}_1$. □

For $g \in G(\mathbb{A})^1$, we put

$$S_Q(g) = \pi_X(\{\gamma \in G(\mathsf{k}) : \gamma g \in \mathsf{R}_1^-\}).$$

By Lemmas 6 and 12, $S_Q(g)$ is a nonempty subset of $X_Q(g)$.

Lemma 13. *For $g_0 \in G(\mathbb{A})^1$, there is a neighborhood $\mathcal{U}$ of g_0 in $G(\mathbb{A})^1$ such that $S_Q(g) \subset S_Q(g_0)$ for all $g \in \mathcal{U}$.*

Proof. Let $\mathcal{U}$ be a neighborhood of g_0 such that $X_Q(g) \subset X_Q(g_0)$ for all $g \in \mathcal{U}$. Since $g_0 \notin \gamma^{-1}\mathsf{R}_1^-$ for any $\pi_X(\gamma) \in X_Q(g_0) \setminus S_Q(g_0)$, we can take a sufficiently small $\mathcal{U}$ so that $\mathcal{U} \cap \gamma^{-1}\mathsf{R}_1^- = \varnothing$ for all $\pi_X(\gamma) \in X_Q(g_0) \setminus S_Q(g_0)$. Then, for any $g \in \mathcal{U}$, $S_Q(g) \cap X_Q(g_0) \setminus S_Q(g_0)$ is empty; that is, $S_Q(g) \subset S_Q(g_0)$. □

Remark. We do not know whether $\mathsf{R}_1^- = \mathsf{R}$ holds or not in general. If it does, then $S_Q(g) = X_Q(g)$ holds for all g.

5. A fundamental domain of $G(\mathbb{A})^1$ with respect to $G(\mathsf{k})$

Definition. Let T be a locally compact Hausdorff space and Γ be a discrete group acting on T from the left. Assume that the action of Γ on T is properly discontinuous. An open subset Ω of T is called an open fundamental domain of T with respect to Γ if Ω satisfies the following conditions:

(1) $T = \Gamma\Omega^-$, where Ω^- stands for the closure of Ω in T, and

(2) $\Omega \cap \gamma\Omega^- = \varnothing$ if $\gamma \in \Gamma \setminus \{e\}$.

A subset F of T is called a fundamental domain of T with respect to Γ if there is an open fundamental domain Ω as above such that $\Omega \subset \mathsf{F} \subset \Omega^-$.

By Baer and Levi's theorem [1931] (see also [van der Waerden 1935, §10]), an open fundamental domain of T with respect to Γ exists if the set of points stabilized by some nontrivial element of Γ is discrete in T. Thus there exists an open fundamental domain Ω_Q of R_1^- with respect to $Q(\mathsf{k})$. For a given subset A of R_1^-, A° and A^- denote the interior and the closure of A in $G(\mathbb{A})^1$, respectively. Since R_1^- is closed in $G(\mathbb{A})^1$, the closure of A in R_1^- coincides with A^-.

Lemma 14. *Let Ω_Q be an open fundamental domain of R_1^- with respect to $Q(\mathsf{k})$. Then one has $\Omega_Q^\circ = \Omega_Q \cap \mathsf{R}_1$ and $\Omega_Q^- = (\Omega_Q \cap \mathsf{R}_1)^-$.*

Proof. Since Ω_Q is an open set in R_1^- with respect to the relative topology, there is an open set $\mathcal{U}$ in $G(\mathbb{A})^1$ such that $\Omega_Q = \mathsf{R}_1^- \cap \mathcal{U}$. Therefore, $\Omega_Q \cap \mathsf{R}_1 = \mathcal{U} \cap \mathsf{R}_1$ is open in $G(\mathbb{A})^1$, and hence $\Omega_Q^\circ = \Omega_Q \cap \mathsf{R}_1$. Since R_1 is dense in R_1^- and Ω_Q is relatively open in R_1^-, the closure of $\Omega_Q \cap \mathsf{R}_1$ in R_1^- contains Ω_Q, that is, $\Omega_Q \subset (\Omega_Q \cap \mathsf{R}_1)^-$. Hence $\Omega_Q^- = (\Omega_Q \cap \mathsf{R}_1)^-$. □

Theorem 15. *Let Ω_Q be an open fundamental domain of R_1^- with respect to $Q(\mathsf{k})$. Then Ω_Q° is an open fundamental domain of $G(\mathbb{A})^1$ with respect to $G(\mathsf{k})$.*

Proof. From $\mathsf{R}_1^- = Q(\mathsf{k})\Omega_Q^-$ and Lemma 12, it follows $G(\mathbb{A})^1 = G(\mathsf{k})\Omega_Q^-$. For $\gamma \in G(\mathsf{k})$, we assume $\Omega_Q^\circ \cap \gamma\Omega_Q^- \neq \varnothing$. By Lemma 11, γ is contained in $Q(\mathsf{k})$. Since Ω_Q is an open fundamental domain of R_1^- with respect to $Q(\mathsf{k})$, γ must be equal to e. □

For a given subset A of $G(\mathbb{A})^1$, we denote by ∂A the boundary of A.

Lemma 16. *If $g_0 \in \mathsf{R}_1^-$ attains a local maximum of m_Q, then g_0 is in $\partial \mathsf{R}_1^-$.*

Proof. Suppose $g_0 \in \mathsf{R}_1$. Since R_1 is open, zg_0 is contained in R_1 if $z \in Z_Q(\mathbb{A})$ is sufficiently close to e. Then

$$\mathsf{m}_Q(zg_0) = H_Q(zg_0) = H_Q(z)H_Q(g_0) = H_Q(z)\mathsf{m}_Q(g_0).$$

Since $H_Q(z)$ can vary on the interval $(1-\epsilon, 1+\epsilon)$ for a sufficiently small $\epsilon > 0$, $\mathsf{m}_Q(g_0)$ is not a local maximum of m_Q. □

Since $(\Omega_Q^-)^\circ = \Omega_Q^\circ \subset \mathsf{R}_1$, the following theorem immediately follows from Lemma 16.

Theorem 17. *Let Ω_Q be the same as in Theorem 15. If $g_0 \in \Omega_Q^-$ attains a local maximum of m_Q, then g_0 is in $\partial\Omega_Q^- \cap \partial\mathsf{R}_1^-$.*

Remark. A point $g_0 \in G(\mathbb{A})^1$ is said to be extreme if g_0 attains a local maximum of m_Q. By Theorem 17, any extreme point is contained in $G(\mathsf{k})(\partial\Omega_Q^- \cap \partial\mathsf{R}_1^-)$. A candidate of the notion analogous to perfect quadratic forms is the following: a point $g \in G(\mathbb{A})^1$ is said to be Q-perfect if there is a neighborhood $\mathcal{U}$ of g such that

$$\mathcal{U} \cap \bigcap_{\pi_X(\delta)\in S_Q(g)} \delta^{-1}\mathsf{R}_1^- = \{g\}.$$

6. The case when G is of class number one

We put $K_f = \prod_{\sigma\in\mathsf{p}_f} K_\sigma$, $G_{\mathbb{A},\infty} = G(\mathsf{k}_\infty) \times K_f$, $G_{\mathbb{A},\infty}^1 = G_{\mathbb{A},\infty} \cap G(\mathbb{A})^1$ and $G_\circ = G(\mathsf{k}) \cap G_{\mathbb{A},\infty}$. By identifying $G(\mathsf{k}_\infty)$ with the subgroup

$$\{(g_\sigma) \in G(\mathbb{A}) : g_\sigma = e \text{ for all } \sigma \in \mathsf{p}_f\}$$

of $G(\mathbb{A})$, we put $G(\mathsf{k}_\infty)^1 = G(\mathsf{k}_\infty) \cap G(\mathbb{A})^1$. The number $n_\mathsf{k}(G)$ of double cosets in $G(\mathbb{A})$ modulo $G(\mathsf{k})$ and $G_{\mathbb{A},\infty}$ is called the class number of G. For example, $n_\mathsf{k}(\mathrm{GL}_n)$ is equal to the class number of k. If G is almost k-simple, k-isotropic and simply connected, then $n_\mathsf{k}(G) = 1$ by the strong approximation theorem. In this section, we assume that $n_\mathsf{k}(G) = 1$. Then $G(\mathbb{A})^1 = G(\mathsf{k})G_{\mathbb{A},\infty}^1$. Let h_Q be the number of double cosets of $G(\mathsf{k})$ modulo $Q(\mathsf{k})$ and $G_\circ$. By [Borel 1963, Proposition 7.5], h_Q is equal to the class number of M_Q. Let $\{\xi_1 = e, \xi_2, \dots, \xi_{h_Q}\}$ be a complete system of representatives of $Q(\mathsf{k})\backslash G(\mathsf{k})/G_\circ$. For each ξ_i, we define

$$\mathsf{R}_{\xi_i,\infty} = \bigl\{g_\infty \in G(\mathsf{k}_\infty)^1 : \mathsf{m}_Q(g_\infty) = H_Q(\xi_i g_\infty)\bigr\}.$$

Since $G(\mathsf{k})$ is a disjoint union of $Q(\mathsf{k})\xi_i G_\circ$ for $i = 1, \dots, h_Q$, $\mathsf{m}_Q(g_\infty)$ equals

$$\min_{1\le i\le h_Q} \min_{\delta\in G_\circ} H_Q(\xi_i\delta g_\infty).$$

Lemma 18.
$$\mathsf{R} = \bigsqcup_{i=1}^{h_Q} Q(\mathsf{k})\xi_i(\mathsf{R}_{\xi_i,\infty} \times K_f).$$

Proof. For each i, $Q(\mathsf{k})\xi_i(\mathsf{R}_{\xi_i,\infty} \times K_f) \subset \mathsf{R}$ is trivial. Since

$$G(\mathbb{A})^1 = \bigsqcup_{i=1}^{h_Q} Q(\mathsf{k})\xi_i G^1_{\mathbb{A},\infty}$$

by [Borel 1963, §7], a given $g \in \mathsf{R}$ is represented as $g = \gamma\xi_i(g_\infty \times g_f)$ for some i, $\gamma \in Q(\mathsf{k})$ and $g_\infty \times g_f \in G^1_{\mathbb{A},\infty}$. Then $\mathsf{m}_Q(g) = H_Q(g)$ implies $\mathsf{m}_Q(g_\infty) = H_Q(\xi_i g_\infty)$. Therefore, $g_\infty \in \mathsf{R}_{\xi_i,\infty}$. □

We write Q_i for the conjugate $\xi_i^{-1} Q \xi_i$ of Q. This Q_i is a maximal k-parabolic subgroup of G. We put $Q_{i,\circ} = Q_i(\mathsf{k}) \cap G_{\mathbb{A},\infty}$.

Lemma 19. *If* $g(\mathsf{R}_{\xi_i,\infty} \times K_f) \cap (\mathsf{R}_{\xi_i,\infty} \times K_f)$ *is nonempty for* $g \in Q_i(\mathsf{k})$, *then* $g \in Q_{i,\circ}$.

Proof. If there is an $h \in \mathsf{R}_{\xi_i,\infty} \times K_f$ such that $gh \in \mathsf{R}_{\xi_i,\infty} \times K_f$, then

$$g \in (\mathsf{R}_{\xi_i,\infty} \times K_f)h^{-1} \subset G_{\mathbb{A},\infty}. \qquad \square$$

It is easy to prove that the group $Q_{i,\circ}$ stabilizes $\mathsf{R}_{\xi_i,\infty} \times K_f$ by left multiplication. We fix a complete system $\{\gamma_{ij}\}_j$ of representatives of $Q_i(\mathsf{k})/Q_{i,\circ}$. It follows from Lemma 19 that $\gamma_{ij}(\mathsf{R}_{\xi_i,\infty} \times K_f) \cap \gamma_{ik}(\mathsf{R}_{\xi_i,\infty} \times K_f) = \varnothing$ if $j \neq k$. Therefore, we obtain the following subdivision of R:

$$\mathsf{R} = \bigsqcup_{i=1}^{h_Q} \bigsqcup_{j} \xi_i\gamma_{ij}(\mathsf{R}_{\xi_i,\infty} \times K_f). \tag{1}$$

Let $\mathsf{R}^\circ_{\xi_i,\infty}$ be the interior of $\mathsf{R}_{\xi_i,\infty}$ and $\mathsf{R}^*_{\xi_i,\infty}$ the closure of $\mathsf{R}^\circ_{\xi_i,\infty}$ in $G(\mathsf{k}_\infty)^1$. Since the union of (1) is disjoint, it is obvious that

$$\mathsf{R}_1^- = \bigsqcup_{i=1}^{h_Q} \bigsqcup_{j} \xi_i\gamma_{ij}(\mathsf{R}^*_{\xi_i,\infty} \times K_f). \tag{2}$$

Proposition 20. *Let* $\Omega_{i,\infty}$ *be an open fundamental domain of* $\mathsf{R}^*_{\xi_i,\infty}$ *with respect to* $Q_{i,\circ}$ *for* $i = 1, \ldots, h_Q$. *Then the set*

$$\Omega = \bigsqcup_{i=1}^{h_Q} \xi_i(\Omega_{i,\infty} \times K_f)$$

gives an open fundamental domain of R_1^- *with respect to* $Q(\mathsf{k})$.

Proof. Let $\Omega^-_{i,\infty}$ denote the closure of $\Omega_{i,\infty}$ in $G(\mathsf{k}_\infty)^1$. For $g \in Q(\mathsf{k})$, we assume $\Omega \cap g\Omega^- \neq \varnothing$. Then, for some i, j,

$$\xi_i(\Omega_{i,\infty} \times K_f) \cap g\xi_j(\Omega^-_{j,\infty} \times K_f) \neq \varnothing. \tag{3}$$

There exist γ_{jk} and $\delta \in Q_{j,\circ}$ such that $\xi_j^{-1} g \xi_j = \gamma_{jk}\delta$. Then (3) is the same as

$$\xi_i(\Omega_{i,\infty} \times K_f) \cap \xi_j \gamma_{jk}(\delta\Omega^-_{j,\infty} \times K_f) \neq \varnothing.$$

By (1), we have $i = j$, $\gamma_{jk} = e$ and $\Omega_{j,\infty} \cap \delta\Omega^-_{j,\infty} \neq \varnothing$. Since $\Omega_{j,\infty}$ is an open fundamental domain of $\mathsf{R}^*_{\xi_j,\infty}$ with respect to $Q_{j,\circ}$, δ must be equal to e. Therefore, $\Omega \cap g\Omega^- \neq \varnothing$ implies $g = e$. Finally, $Q(\mathsf{k})\Omega^- = \mathsf{R}^-_1$ follows from (2) and $Q_{i,\circ}\Omega^-_{i,\infty} = \mathsf{R}^*_{\xi_i,\infty}$. □

By Theorem 17, we obtain the following.

Corollary 21. *If $g_0 \in \Omega^-$ attains a local maximum of m_Q, then g_0 is contained in the set*

$$\bigsqcup_{i=1}^{h_Q} \xi_i\big((\partial\Omega^-_{i,\infty} \cap \partial\mathsf{R}^*_{\xi_i,\infty}) \times K_f\big).$$

We consider the infinite part Ω_∞ of Ω given in Proposition 20, that is,

$$\Omega_\infty = \bigcup_{i=1}^{h_Q} \xi_i \Omega_{i,\infty}.$$

Let Ω°_∞ and Ω^-_∞ be the interior and the closure of Ω_∞ in $G(\mathsf{k}_\infty)^1$, respectively. The projection from $G(\mathbb{A})^1 = G(\mathsf{k})G^1_{\mathbb{A},\infty}$ to the infinite component $G(\mathsf{k}_\infty)^1$ gives an isomorphism $G(\mathsf{k})\backslash G(\mathbb{A})^1/K_f \cong G_\circ\backslash G(\mathsf{k}_\infty)^1$. Since Ω is a fundamental domain of $G(\mathbb{A})^1$ with respect to $G(\mathsf{k})$ by Theorem 15, we have $G_\circ\Omega^-_\infty = G(\mathsf{k}_\infty)^1$.

Corollary 22. *If $h_Q = 1$, then Ω_∞ is a fundamental domain of $G(\mathsf{k}_\infty)^1$ with respect to $G_\circ$.*

Proof. Since $\Omega_\infty = \Omega_{1,\infty}$ is a relatively open set in $\mathsf{R}^*_{e,\infty}$, we have $\Omega^\circ_\infty = \Omega_\infty \cap \mathsf{R}^\circ_{e,\infty}$. Thus the closure of Ω°_∞ coincides with Ω^-_∞. If $\Omega^\circ_\infty \cap g\Omega^-_\infty \neq \varnothing$ for $g \in G_\circ$, then $(\Omega^\circ_\infty \times K_f) \cap g(\Omega^-_\infty \times K_f) \neq \varnothing$ because $gK_f = K_f$. This implies $g = e$ since $\Omega^\circ_\infty \times K_f$ is an open fundamental domain of $G(\mathbb{A})^1$ with respect to $G(\mathsf{k})$. □

7. Examples

Example 3. Let G be a general linear group GL_n defined over $\mathbb{Q}$. We continue an illustration given in Examples 1 and 2. We fix an integer $k \in \{1, \ldots, n-1\}$, and let

$$Q(\mathbb{Q}) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a \in \mathrm{GL}_k(\mathbb{Q}),\ b \in \mathrm{M}_{k,n-k}(\mathbb{Q}),\ d \in \mathrm{GL}_{n-k}(\mathbb{Q}) \right\}.$$

Since $h_Q = 1$, we have $\xi_1 = e$ and $Q_1 = Q$.

Let P_n be the cone of positive definite n by n real symmetric matrices, and let P_n^1 be the intersection of P_n and $\mathrm{SL}_n(\mathbb{R})$. The group $G(\mathbb{Q}_\infty) = \mathrm{GL}_n(\mathbb{R})$ acts on P_n from the right by $(A, g) \mapsto A[g] = {}^tgAg$ for $(A, g) \in \mathsf{P}_n \times G(\mathbb{Q}_\infty)$. The maximal compact subgroup K_∞ of $G(\mathbb{Q}_\infty)$, defined as in Example 2, stabilizes the identity matrix $I_n \in \mathsf{P}_n$. The map $\pi : g \mapsto {}^tg^{-1}g^{-1}$ from $G(\mathbb{Q}_\infty)$ onto P_n gives an isomorphism between $G(\mathbb{Q}_\infty)/K_\infty$ and P_n. Since

$$G(\mathbb{Q}_\infty)^1 = \{g \in G(\mathbb{Q}_\infty) : \det g = \pm 1\},$$

we have $G(\mathbb{Q}_\infty)^1/K_\infty \cong \pi(G(\mathbb{Q}_\infty)^1) = \mathsf{P}_n^1$. An element $A \in \mathsf{P}_n$ is written as

$$A = \begin{pmatrix} I_k & 0 \\ {}^tu & I_{n-k} \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} I_k & u \\ 0 & I_{n-k} \end{pmatrix},$$

where $v \in \mathsf{P}_k$, $w \in \mathsf{P}_{n-k}$ and $u \in \mathrm{M}_{k,n-k}(\mathbb{R})$. We write u_A, $A^{[k]}$ and $A_{[n-k]}$ for u, v and w, respectively.

By definition, $G_\mathbb{Z} = G(\mathbb{Q}) \cap G_{\mathbb{A},\infty}$ and $Q_\mathbb{Z} = Q(\mathbb{Q}) \cap G_{\mathbb{A},\infty}$ are just the groups $\mathrm{GL}_n(\mathbb{Z})$ and $Q(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$ of unimodular integral matrices in $G(\mathbb{Q})$ and $Q(\mathbb{Q})$, respectively. As in Example 2, X_γ stands for the n by k matrix consisting of the first k-columns of $\gamma \in G_\mathbb{Z}$, and $\mathrm{M}_{n,k}(\mathbb{Z})^*$ stands for the set of X_γ for all $\gamma \in G_\mathbb{Z}$. We define the closed subset $\mathsf{F}_{n,k}$ of P_n as follows:

$$\mathsf{F}_{n,k} = \left\{ A \in \mathsf{P}_n : \det A^{[k]} \le \det({}^tXAX) \text{ for all } X \in \mathrm{M}_{n,k}(\mathbb{Z})^* \right\}.$$

In Example 2, we showed

$$H_Q(\gamma g) = \det\left({}^tX_{\gamma^{-1}}\pi(g)X_{\gamma^{-1}}\right)^{n/2r}$$

for any $\gamma \in G_\mathbb{Z}$ and $g \in G(\mathbb{Q}_\infty)^1$. Since $H_Q(g) = \left(\det \pi(g)^{[k]}\right)^{n/2r}$, we obtain

$$\mathsf{R}_{e,\infty}/K_\infty \cong \pi(\mathsf{R}_{e,\infty}) = \mathsf{F}_{n,k} \cap \mathrm{SL}_n(\mathbb{R}).$$

Therefore, $Q_\mathbb{Z}\backslash\mathsf{R}_{e,\infty}/K_\infty$ is isomorphic to $(\mathsf{F}_{n,k} \cap \mathrm{SL}_n(\mathbb{R}))/Q_\mathbb{Z}$. If $\gamma \in Q_\mathbb{Z}$ is of the form

$$\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

with $a \in \mathrm{GL}_k(\mathbb{Z})$, $d \in \mathrm{GL}_{n-k}(\mathbb{Z})$ and $b \in \mathrm{M}_{k,n-k}(\mathbb{Z})$, then components of ${}^t\gamma A\gamma$ for $A \in \mathsf{P}_n$ are given by

$$u_{{}^t\gamma A\gamma} = a^{-1}(u_A d + b), \quad \left({}^t\gamma A\gamma\right)^{[k]} = {}^taA^{[k]}a, \quad \left({}^t\gamma A\gamma\right)_{[n-k]} = {}^tdA_{[n-k]}d.$$

Let $\mathfrak{D}$ and $\mathfrak{E}$ be arbitrary fundamental domains for the quotients $\mathsf{P}_k/\mathrm{GL}_k(\mathbb{Z})$ and $\mathsf{P}_{n-k}/\mathrm{GL}_{n-k}(\mathbb{Z})$, respectively. We define the subset $\mathsf{F}_{n,k}(\mathfrak{D},\mathfrak{E})$ of $\mathsf{F}_{n,k}$ as

$$\mathsf{F}_{n,k}(\mathfrak{D},\mathfrak{E}) = \{A \in \mathsf{F}_{n,k} : A^{[k]} \in \mathfrak{D},\ A_{[n-k]} \in \mathfrak{E},\\ u_A = (u_{ij}),\ -\tfrac{1}{2} \le u_{ij} \le \tfrac{1}{2} \text{ for all } i, j, \text{ and } 0 \le u_{11}\}.$$

Since $\mathsf{F}_{n,k}(\mathfrak{D},\mathfrak{E})$ is a fundamental domain of $\mathsf{F}_{n,k}$ with respect to $Q_{\mathbb{Z}}$, the inverse image $\pi^{-1}(\mathsf{F}_{n,k}(\mathfrak{D},\mathfrak{E}) \cap \mathrm{SL}_n(\mathbb{R}))$ of $\mathsf{F}_{n,k}(\mathfrak{D},\mathfrak{E}) \cap \mathrm{SL}_n(\mathbb{R})$ gives a fundamental domain of $\mathsf{R}_{e,\infty}$ with respect to $Q_{\mathbb{Z}}$. As a consequence of Theorem 15 and Proposition 20, the set

$$\pi^{-1}(\mathsf{F}_{n,k}(\mathfrak{D},\mathfrak{E}) \cap \mathrm{SL}_n(\mathbb{R})) \times K_f$$

gives a fundamental domain of $G(\mathbb{A})^1$ with respect to $G(\mathbb{Q})$. Moreover, from Corollary 22, it follows that $\mathsf{F}_{n,k}(\mathfrak{D},\mathfrak{E})$ is a fundamental domain of P_n with respect to $\mathrm{GL}_n(\mathbb{Z})$.

In the case of $k = 1$, this gives an inductive construction of a fundamental domain Ω_n of P_n with respect to $\mathrm{GL}_n(\mathbb{Z})$ as follows. First, put $\Omega_2 = \mathsf{F}_{2,1}(\mathsf{P}_1,\mathsf{P}_1)$. By definition, Ω_2 is Minkowski's fundamental domain of P_2. Then we define inductively $\Omega_3 = \mathsf{F}_{3,1}(\mathsf{P}_1,\Omega_2), \dots, \Omega_n = \mathsf{F}_{n,1}(\mathsf{P}_1,\Omega_{n-1})$. The domain Ω_n coincides with Grenier's fundamental domain [1988].

Finally, we show that, in the case of $k = 1$, $\mathsf{R}_{e,\infty}/K_\infty$ corresponds to a face of the Ryshkov polyhedron $\mathsf{R}(\mathsf{m}) = \left\{A \in \mathsf{P}_n : \mathsf{m}(A) = \min_{0 \ne x \in \mathbb{Z}^n} {}^t x A x \ge 1\right\}$. For $A \in \mathsf{P}_n$, let $S(A)$ denote the set of minimal integral vectors of A:

$$S(A) = \{x \in \mathbb{Z}^n : \mathsf{m}(A) = {}^t x A x\}.$$

We take $\boldsymbol{e}_1 = {}^t(1, 0, \dots, 0) \in \mathbb{Z}^n$. It is obvious that the subset $\{A \in \mathsf{P}_n : \boldsymbol{e}_1 \in S(A)\}$ of P_n coincides with $\mathsf{F}_{n,1}$. As was shown in [Watanabe 2012, Lemma 1.5], $\mathscr{F}_{\{\boldsymbol{e}_1\}} = \mathsf{F}_{n,1} \cap \partial\mathsf{R}(\mathsf{m}) = \{A \in \mathsf{F}_{n,1} : \mathsf{m}(A) = 1\}$ is a face of $\mathsf{R}(\mathsf{m})$. It is easy to see that the map $A \mapsto \mathsf{m}(A)^{-1}A$ gives a bijection from $\mathsf{F}_{n,1} \cap \mathrm{SL}_n(\mathbb{R})$ onto $\mathscr{F}_{\{\boldsymbol{e}_1\}}$. Therefore, $\mathsf{R}_{e,\infty}/K_\infty \cong \pi(\mathsf{R}_{e,\infty})$ corresponds to $\mathscr{F}_{\{\boldsymbol{e}_1\}}$.

Example 4. Let k be a totally real number field of degree r and $n = 2m$ be an even integer. We consider a symplectic group

$$G(\mathsf{k}) = \mathrm{Sp}_n(\mathsf{k}) = \left\{ g \in \mathrm{GL}_{2m}(\mathsf{k}) \ :\ {}^t g \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} \right\}.$$

For a fixed $k \in \{1, 2, \dots, m\}$, let Q denote the maximal parabolic subgroup of G given by

$$Q(\mathsf{k}) = U(\mathsf{k})M(\mathsf{k}),$$

where

$$M(\mathsf{k}) = \left\{ \delta(a,b) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b_{11} & 0 & b_{12} \\ 0 & 0 & {}^t a^{-1} & 0 \\ 0 & b_{21} & 0 & b_{22} \end{pmatrix} \ : \ \begin{array}{c} a \in \mathrm{GL}_k(\mathsf{k}), \\ b = (b_{ij}) \in \mathrm{Sp}_{2(m-k)}(\mathsf{k}) \end{array} \right\},$$

$$U(\mathsf{k}) = \left\{ \begin{pmatrix} I_k & * & * & * \\ 0 & I_{m-k} & * & 0 \\ 0 & 0 & I_k & 0 \\ 0 & 0 & * & I_{m-k} \end{pmatrix} \in G(\mathsf{k}) \right\}.$$

The module of k-rational characters $X^*(M)_{\mathsf{k}}$ of M is a free $\mathbb{Z}$-module of rank 1 generated by the character

$$\widehat{\alpha}_Q(\delta(a,b)) = \det a.$$

The height $H_Q : G(\mathbb{A}) \to \mathbb{R}_{>0}$ is given by $H_Q(g) = |\det a|_{\mathbb{A}}^{-1}$ if $g = u\delta(a,b)h$ with $u \in U(\mathbb{A})$, $\delta(a,b) \in M(\mathbb{A})$ and $h \in K$.

We restrict ourselves to the case $k = m$. An element of $M(\mathbb{A})$ is denoted by

$$\delta(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}, \quad a \in \mathrm{GL}_m(\mathbb{A}).$$

Let

$$\mathsf{H}_m = \left\{ Z \in \mathrm{M}_m(\mathbb{C}) \ : \ {}^t Z = Z, \ \ \mathrm{Im} Z \in \mathsf{P}_m \right\}$$

be the Siegel upper half space and H_m^r the direct product of r copies of H_m. For $Z = (Z_\sigma)_{\sigma \in \mathsf{p}_\infty} \in \mathsf{H}_m^r$, $\mathrm{Re} Z$, $\mathrm{Im} Z$ and $\det Z$ stand for $(\mathrm{Re} Z_\sigma)_{\sigma\in\mathsf{p}_\infty}$, $(\mathrm{Im} Z_\sigma)_{\sigma\in\mathsf{p}_\infty}$ and $(\det Z_\sigma)_{\sigma\in\mathsf{p}_\infty}$, respectively. The group $G(\mathsf{k}_\infty)$ acts transitively on H_m^r by

$$g\langle Z\rangle = \left((a_\sigma Z_\sigma + b_\sigma)(c_\sigma Z_\sigma + d_\sigma)^{-1}\right)_{\sigma\in\mathsf{p}_\infty}$$

for $Z = (Z_\sigma) \in \mathsf{H}_m^r$ and

$$g = (g_\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}_{\sigma\in\mathsf{p}_\infty} \in G(\mathsf{k}_\infty).$$

The stabilizer K_∞ of $Z_0 = (\sqrt{-1}I_m, \ldots, \sqrt{-1}I_m) \in \mathsf{H}_m^r$ in $G(\mathsf{k}_\infty)$ is a maximal compact subgroup of $G(\mathsf{k}_\infty)$. We choose K as $K_\infty \times \prod_{\sigma\in\mathsf{p}_f} \mathrm{Sp}_n(\mathsf{o}_\sigma)$. The map $\pi : g_\infty \mapsto g\langle Z_0\rangle$ from $G(\mathsf{k}_\infty)$ onto H_m^r gives an isomorphism $G(\mathsf{k}_\infty)/K_\infty \cong \mathsf{H}_m^r$, and hence $G(\mathsf{k})\backslash G(\mathbb{A})/K \cong G_{\mathsf{o}}\backslash \mathsf{H}_m^r$. Since $\mathrm{Im}\left\{(u\delta(a)h)\langle Z_0\rangle\right\} = a\,{}^t a$ holds for $u \in U(\mathsf{k}_\infty)$, $a \in \mathrm{GL}_m(\mathsf{k}_\infty)$ and $h \in K_\infty$, we have

$$H_Q(g_\infty) = \mathrm{Nr}_{\mathsf{k}_\infty/\mathbb{R}}(\det \mathrm{Im}\{g_\infty\langle Z_0\rangle\})^{-1/2} = \left(\prod_{\sigma\in\mathsf{p}_\infty} \det \mathrm{Im}\left\{g_\sigma\left\langle\sqrt{-1}I_m\right\rangle\right\}\right)^{-1/2}$$

for any $g_\infty = (g_\sigma) \in G(\mathsf{k}_\infty)$, where $\mathrm{Nr}_{\mathsf{k}_\infty/\mathbb{R}}$ denotes the norm of k_∞ over $\mathbb{R}$.

The class number h_Q of $M \cong \mathrm{GL}_m$ defined over k is equal to the class number h_{k} of k. We assume $h_{\mathsf{k}} = 1$ for simplicity. Then $G(\mathsf{k}) = Q(\mathsf{k})G_{\mathsf{o}}$ and $G(\mathbb{A}) = Q(\mathsf{k})G_{\mathbb{A},\infty}$, and hence

$$\mathsf{m}_Q(g_\infty) = \min_{\gamma \in G_{\mathsf{o}}} H_Q(\gamma g_\infty).$$

Since

$$\mathrm{Nr}_{\mathsf{k}_\infty/\mathbb{R}}(\det \mathrm{Im}\{\gamma\langle Z\rangle\}) = \prod_{\sigma \in \mathsf{p}_\infty} |\det(\sigma(c)Z_\sigma + \sigma(d))|^{-2} \mathrm{Nr}_{\mathsf{k}_\infty/\mathbb{R}}(\det \mathrm{Im} Z)$$

for $Z = (Z_\sigma) \in \mathsf{H}_m^r$ and

$$\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in G_{\mathsf{o}} = \mathrm{Sp}_n(\mathsf{o}),$$

the condition $\mathsf{m}_Q(g_\infty) = H_Q(g_\infty)$ of g_∞ is equivalent with the following condition of $Z = g_\infty\langle Z_0\rangle$:

$$\prod_{\sigma \in \mathsf{p}_\infty} |\det(\sigma(c)Z_\sigma + \sigma(d))| \geq 1 \qquad \text{for all} \quad \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in G_{\mathsf{o}}.$$

Therefore, the domain $R_{e,\infty}$ modulo K_∞ is isomorphic to

$$\mathsf{F} = \left\{ (Z_\sigma) \in \mathsf{H}_m^r \ : \ \prod_{\sigma \in \mathsf{p}_\infty} |\det(\sigma(c)Z_\sigma + \sigma(d))| \geq 1 \text{ for all } \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in G_{\mathsf{o}} \right\}.$$

Let $\mathfrak{C}$ be an arbitrary fundamental domain of the additive group $\mathrm{M}_m(\mathsf{k}_\infty)$ with respect to $\mathrm{M}_m(\mathsf{o})$, and let $\mathfrak{D}$ be an arbitrary fundamental domain of P_m^r with respect to $\mathrm{GL}_m(\mathsf{o})$. It is easy to see that

$$\mathsf{F}(\mathfrak{C}, \mathfrak{D}) = \{Z \in \mathsf{F} : \mathrm{Re} Z \in \mathfrak{C},\ \mathrm{Im} Z \in \mathfrak{D}\}$$

is a fundamental domain of F with respect to Q_{o}. By Corollary 22, $\mathsf{F}(\mathfrak{C}, \mathfrak{D})$ is a fundamental domain of H_m^r with respect to G_{o}.

If $\mathsf{k} = \mathbb{Q}$ and $\mathfrak{D}$ is Minkowski's fundamental domain, then $\mathsf{F}(\mathfrak{C}, \mathfrak{D})$ coincides with Siegel's fundamental domain [1939].

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References

[Baer and Levi 1931] R. Baer and F. Levi, "Stetige Funktionen in topologischen Räumen", *Math. Z.* **34**:1 (1931), 110–130. MR 1545244 Zbl 0002.16002 JFM 57.0731.02

[Bombieri and Gubler 2006] E. Bombieri and W. Gubler, *Heights in Diophantine geometry*, New Mathematical Monographs **4**, Cambridge University Press, 2006. MR 2007a:11092 Zbl 1115.11034

[Borel 1963] A. Borel, "Some finiteness properties of adele groups over number fields", *Inst. Hautes Études Sci. Publ. Math.* **16** (1963), 5–30. MR 34 #2578 Zbl 0135.08902

[Borel 1991] A. Borel, *Linear algebraic groups*, 2nd ed., Graduate Texts in Mathematics **126**, Springer, New York, 1991. MR 92d:20001 Zbl 0726.20030

[Borel and Tits 1965] A. Borel and J. Tits, "Groupes réductifs", *Inst. Hautes Études Sci. Publ. Math.* **27** (1965), 55–150. MR 34 #7527 Zbl 0145.17402

[Grenier 1988] D. Grenier, "Fundamental domains for the general linear group", *Pacific J. Math.* **132**:2 (1988), 293–317. MR 89d:11055 Zbl 0699.10045

[Humphreys 1990] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics **29**, Cambridge University Press, 1990. MR 92h:20002 Zbl 0725.20028

[Martinet 2003] J. Martinet, *Perfect lattices in Euclidean spaces*, Grundlehren der Mathematischen Wissenschaften **327**, Springer, Berlin, 2003. MR 2003m:11099 Zbl 1017.11031

[Ryshkov 1970] S. S. Ryshkov, "The polyhedron $\mu(m)$ and some extremal problems in the geometry of numbers", *Dokl. Akad. Nauk SSSR* **194** (1970), 514–517. In Russian; translated in *Sov. Math. Dokl.* **11** (1970), 1240–1244. MR 43 #2613 Zbl 0229.10013

[Schürmann 2009] A. Schürmann, *Computational geometry of positive definite quadratic forms: polyhedral reduction theories, algorithms, and applications*, University Lecture Series **48**, Amer. Math. Soc., Providence, RI, 2009. MR 2010a:11130 Zbl 1185.52016

[Shimura 1994] G. Shimura, "Fractional and trigonometric expressions for matrices", *Amer. Math. Monthly* **101**:8 (1994), 744–758. MR 96e:15053 Zbl 0836.15004

[Siegel 1939] C. L. Siegel, "Einführung in die Theorie der Modulfunktionen n-ten Grades", *Math. Ann.* **116** (1939), 617–657. MR 1,203f Zbl 0021.20302

[Terras 1988] A. Terras, *Harmonic analysis on symmetric spaces and applications, II*, Springer, Berlin, 1988. MR 89k:22017 Zbl 0668.10033

[Voronoï 1908] G. Voronoï, "Nouvelles applications des paramètres continus à la théorie des formes quadratiques, premier mémoire: sur quelques propriétés des formes quadratiques positives parfaites", *J. Reine Angew. Math.* **133** (1908), 97–178. JFM 38.0261.01

[van der Waerden 1935] B. L. van der Waerden, *Gruppen von linearen Transformationen*, Ergebnisse der Mathematik und ihrer Grenzgebiete **4**:2, Springer, Berlin, 1935. Zbl 0011.10102 JFM 61.0105.03

[Watanabe 2000] T. Watanabe, "On an analog of Hermite's constant", *J. Lie Theory* **10**:1 (2000), 33–52. MR 2001a:11111 Zbl 1029.11031

[Watanabe 2003] T. Watanabe, "Fundamental Hermite constants of linear algebraic groups", *J. Math. Soc. Japan* **55**:4 (2003), 1061–1080. MR 2005f:11260 Zbl 1103.11033

[Watanabe 2012] T. Watanabe, "A survey on Voronoï's theorem", pp. 334–377 in *Geometry and analysis of automorphic forms of several variables* (Tokyo, 2009), edited by Y. Hamahata et al., Ser. Number Theory Appl. **7**, World Scientific, Hackensack, NJ, 2012. MR 2908043 Zbl 1268.11090

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