Pacific Journal of Mathematics

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Volume 270 No. 2 August 2014

PACIFIC JOURNAL OF MATHEMATICS
Vol. 270, No. 2, 2014
dx.doi.org/10.2140/pjm.2014.270.393

LOCALLY LIPSCHITZ CONTRACTIBILITY OF ALEXANDROV SPACES AND ITS APPLICATIONS

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We prove that any finite-dimensional Alexandrov space with a lower curvature bound is locally Lipschitz contractible. As an application, we obtain a sufficient condition for solving the Plateau problem in an Alexandrov space, as considered by Mese and Zulkowski.

1. Introduction

Alexandrov spaces appear naturally in the collapsing and convergence theory of Riemannian manifolds and play an important role in Riemannian geometry. In this paper, when we say simply an Alexandrov space, we mean an Alexandrov space *of curvature bounded from below locally* and *of finite dimension*. The fundamental properties of such spaces were well studied in [Burago et al. 1992]. Perelman [1991] carried out a remarkable study of topological structures for Alexandrov spaces, proving a topological stability theorem: if two compact Alexandrov spaces of the same dimension are very close in the Gromov–Hausdorff topology, then they are homeomorphic. See also [Kapovitch 2007]. This further implies that, for any point in an Alexandrov space, its small open ball is homeomorphic to its tangent cone. In particular, an open ball of small radius with respect to its center is contractible. It is expected by geometers that corresponding statements obtained by replacing homeomorphic by bi-Lipschitz homeomorphic could be proved. Until now, we did not know any Lipschitz structure of an Alexandrov space around singular points. The main purpose of this paper is to prove that any finite-dimensional Alexandrov space with a lower curvature bound is *strongly locally Lipschitz contractible* in the sense defined later. For short, SLLC denotes this property. The SLLC-condition is a strong version of the LLC-condition introduced in [Yamaguchi 1997] (see Remark 4.5).

We define strongly locally Lipschitz contractibility. We denote by $U(p, r)$ an open ball centered at p of radius r in a metric space.

MSC2010: 53C20, 53C21, 53C23.
Keywords: Alexandrov space, Lipschitz contractibility.

Definition 1.1. A metric space X is *strongly locally Lipschitz contractible*, or SLLC, if for every point $p \in X$, there exists $r > 0$ and a map

$$h : U(p, r) \times [0, 1] \to U(p, r)$$

such that h is a homotopy from $h(\,\cdot\,, 0) = \mathrm{id}_{U(p,r)}$ to $h(\,\cdot\,, 1) = p$ and h is Lipschitz (i.e., there exists $C, C' > 0$ such that

$$d(h(x, s), h(y, t)) \le Cd(x, y) + C'|s - t|$$

for every $x, y \in U(p, r)$ and $s, t \in [0, 1]$) and such that for every $r' < r$, the image of h restricted to $U(p, r') \times [0, 1]$ is $U(p, r')$.

We call such a ball $U(p, r)$ a Lipschitz contractible ball and h a Lipschitz contraction on $U(p, r)$.

A main result in the present paper is the following.

Theorem 1.2. *Any finite-dimensional Alexandrov space is strongly locally Lipschitz contractible.*

In [Yamaguchi 1997], a weaker form of Theorem 1.2 was conjectured.

For metric spaces P and X and possibly empty subsets $Q \subset P$ and $A \subset X$, we denote by $f : (P, Q) \to (X, A)$ a map from P to X with $f(Q) \subset A$. Two maps f and g from (P, Q) to (X, A) are *homotopic* (resp. *Lipschitz homotopic*) if there exists a continuous (resp. Lipschitz) map

$$h : (P \times [0, 1], Q \times [0, 1]) \to (X, A)$$

such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x \in P$. Then, we write $f \sim g$ (resp. $f \sim_{\mathrm{Lip}} g$). Let us denote by

$$[(P, Q), (X, A)] \quad \text{and} \quad [(P, Q), (X, A)]_{\mathrm{Lip}}$$

respectively the set of all homotopy classes of continuous maps from (P, Q) to (X, A) and the set of all Lipschitz homotopy classes of Lipschitz maps from (P, Q) to (X, A).

Let us consider a *Lipschitz simplicial complex*: a metric space which admits a triangulation such that each simplex is a bi-Lipschitz image of a simplex in a Euclidean space. For a precise definition, see Section 4.

Corollary 1.3. *Let P be a finite Lipschitz simplicial complex and Q a possibly empty subcomplex of P. Let X be an Alexandrov space and A an open subset of X. Then, the natural map from $[(P, Q), (X, A)]_{\mathrm{Lip}}$ to $[(P, Q), (X, A)]$ is bijective.*

For a metric space X, a point $x_0 \in X$, and $k \in \mathbb{N}$, we define *the k-th Lipschitz homotopy group $\pi_k^{\mathrm{Lip}}(X, x_0)$* by setting $\pi_k^{\mathrm{Lip}}(X, x_0) = [(S^k, *), (X, x_0)]_{\mathrm{Lip}}$ as sets, where $* \in S^k$ is an arbitrary point; it is equipped with the group operation of the usual homotopy groups.

Corollary 1.4. *For an Alexandrov space X, a point $x_0 \in X$, and $k \in \mathbb{N}$, the natural map*

$$\pi_k^{\mathrm{Lip}}(X, x_0) \to \pi_k(X, x_0).$$

is an isomorphism of groups.

Application: the Plateau problem. The Plateau problem in an Alexandrov space was considered by Mese and Zulkowski [2010] as follows. Let $W^{1,2}(D^2, X)$ denote the $(1, 2)$-Sobolev space from D^2 to an Alexandrov space X, in the sense of the Sobolev space of a metric space target defined by Korevaar and Schoen [1993]. Giving a closed Jordan curve Γ in X, we set

$$\mathcal{F}_\Gamma := \bigl\{u \in W^{1,2}(D^2, X) \cap C(D^2, X) \bigm| u|_{\partial D^2} \text{ parametrizes } \Gamma \text{ monotonically}\bigr\}.$$

Mese and Zulkowski defined the area $A(u)$ of a Sobolev map $u \in W^{1,2}(D^2, X)$. Under these settings, the Plateau problem is stated as follows:

Find a map $u \in W^{1,2}(D^2, X)$ such that $A(u) = \inf\{A(v) \mid v \in \mathcal{F}_\Gamma\}$.

Theorem 1.5 [Mese and Zulkowski 2010]. *Let X be a finite-dimensional compact Alexandrov space and Γ a closed Jordan curve in X. If $\mathcal{F}_\Gamma \neq \varnothing$, then there exists a solution to the Plateau problem.*

For Alexandrov spaces, no condition on Γ implying $\mathcal{F}_\Gamma \neq \varnothing$ was known. As an application of Theorem 1.2, we can obtain such a condition of Γ.

Corollary 1.6. *Let Γ be a rectifiable closed Jordan curve in an Alexandrov space X. If Γ is topologically contractible in X, then $\mathcal{F}_\Gamma \neq \varnothing$.*

Application: simplicial volume. Yamaguchi [1997, Theorem 0.5] proved, assuming an LLC-condition on an Alexandrov space, an inequality between Gromov's simplicial volume and the Hausdorff measure of the Alexandrov space. As an immediate consequence of Theorem 1.2, we obtain:

Corollary 1.7 [Gromov 1982; Yamaguchi 1997]. *Let X be a compact orientable n-dimensional Alexandrov space without boundary, of curvature $\geq \kappa$ for $\kappa < 0$. Then $\|X\| \leq n!\,(n-1)^n \sqrt{-\kappa}^{\,n}\, \mathcal{H}^n(X)$.*

Here, $\|X\|$ is Gromov's simplicial volume, which is the ℓ_1-norm of the fundamental class of X, and $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure. For precise terminology, we refer to [Gromov 1982; Yamaguchi 1997].

Further, if we assume "a lower Ricci curvature bound" for X in the sense of [Bacher and Sturm 2010], then we obtain the following:

Theorem 1.8. *Let X be a compact orientable n-dimensional Alexandrov space without boundary. Let m be a locally finite Borel measure on X with full support that is absolutely continuous with respect to $\mathcal{H}^n$. If the metric measure space*

(X, m) satisfies the reduced curvature-dimension condition $\mathrm{CD}^*(K, N)$ *locally for* $K, N \in \mathbb{R}$ *with* $N \geq 1$ *and* $K < 0$, *then*

$$\|X\| \leq n! \sqrt{-(N-1)K}^{\,n} \mathcal{H}^n(X).$$

Theorem 1.8 is new even if X is a manifold, because a reference measure m can be freely chosen.

Organization. We review fundamental properties of Alexandrov spaces in Section 2. In particular, we recall the theory of the gradient flow of distance functions on an Alexandrov space, established in [Petrunin 1995] and [Perelman and Petrunin 1994]. In Section 3, we prove that the distance function from a metric sphere at each point in an Alexandrov space is regular on a much smaller concentric punctured ball. Then, using the gradient flow of the distance function, we prove Theorem 1.2. In Section 4, we recall precise terminology of the applications in the introduction, and prove Corollaries 1.3, 1.4 and 1.6. In Section 5, we note that our proof given in Section 3 also works for infinite-dimensional Alexandrov spaces whenever the space of directions is compact. In Section 6, we recall several notions of a lower Ricci curvature bound on a metric space together with a Borel measure and their relation. By using the Bishop–Gromov-type volume growth inequality, we prove Theorem 1.8.

2. Preliminaries

This section consists of a review of the definition of Alexandrov spaces and a somewhat detailed review of the gradient flow theory of semiconcave functions on Alexandrov spaces. For further details, we refer to [Burago et al. 1992; 2001] or [Petrunin 2007].

We recall the definition of Alexandrov spaces:

Definition 2.1 [Burago et al. 1992; 2001]. Let $\kappa \in \mathbb{R}$. We call a complete metric space X an *Alexandrov space of curvature* $\geq \kappa$ if it satisfies the following:

(1) X is a geodesic space; i.e., for every x and y in X, there is a curve $\gamma : [0, |x, y|] \to X$ such that $\gamma(0) = x$, $\gamma(|x, y|) = y$, and the length of γ equals $|x, y|$. Here, $|x, y|$ denotes the distance between x and y, written also as $|xy|$ or $d(x, y)$. We call such a curve γ a geodesic between x and y, and denote it by xy.

(2) X has curvature $\geq \kappa$; i.e., for every $p, q, r \in X$ (with $|p, q| + |q, r| + |r, p| < 2\pi/\sqrt{\kappa}$ if $\kappa > 0$) and every x in a geodesic qr between q and r, taking a comparison triangle $\triangle \tilde{p}\tilde{q}\tilde{r} = \tilde{\triangle} pqr$ in a simply connected complete surface $\mathbb{M}_\kappa$ of constant curvature κ and a corresponding point $\tilde{x}$ in $\tilde{q}\tilde{r}$, we have

$$|p, x| \geq |\tilde{p}, \tilde{x}|.$$

We simply say that a complete metric space X is an *Alexandrov space* if it is a geodesic space, and for any $p \in X$, there exist a neighborhood U of p and $\kappa \in \mathbb{R}$ such that U has curvature $\geq \kappa$ in the sense that it satisfies condition (2); i.e., any triangle in U (whose sides are contained in U) is not thinner than its comparison triangle in $\mathbb{M}_\kappa$.

If X is compact, then it has a uniform lower curvature bound. Throughout the paper, we do not need a uniform lower curvature bound, since we are mainly interested in local properties. It is known that if X has a uniform lower curvature bound, say κ, then X has curvature $\geq \kappa$ [Burago et al. 1992].

Semiconcave functions. In this subsection, we refer to [Petrunin 2007; 1995].

Definition 2.2. Let I be an interval and $\lambda \in \mathbb{R}$. We say a function $f : I \to \mathbb{R}$ is λ-*concave* if the function

$$\bar{f}(t) = f(t) - \frac{\lambda}{2} t^2$$

is concave on I. That is, for any $t < t' < t''$ in I, we have

$$\frac{\bar{f}(t') - \bar{f}(t)}{t' - t} \geq \frac{\bar{f}(t'') - \bar{f}(t')}{t'' - t'}.$$

We say a function $f : I \to \mathbb{R}$ is λ-*concave in the barrier sense* if for any $t_0 \in \operatorname{int} I$, there exist a neighborhood I_0 of t_0 in I and a twice-differentiable function $g : I_0 \to \mathbb{R}$ such that

$$g(t_0) = f(t_0),\ g \geq f \text{ and } g'' \leq \lambda \text{ on } \operatorname{int} I.$$

Lemma 2.3 [Petrunin 1995]. *Let $f : I \to \mathbb{R}$ be a continuous function on an interval I and $\lambda \in \mathbb{R}$. Then the following are equivalent*:

(1) *f is λ-concave in the sense of Definition 2.2.*

(2) *For any $t_0 \in I$, there is $A \in \mathbb{R}$ such that*

$$f(t) \leq f(t_0) + A(t - t_0) + \frac{\lambda}{2}(t - t_0)^2$$

for any $t \in I$.

(3) *f is λ-concave in the barrier sense.*

Proof. By considering $f(t) - (\lambda/2)\, t^2$, we may assume that $\lambda = 0$.

Let us prove the implication (1) $\Rightarrow$ (2). Let us take $t_0 \in I$, not equal to the supremum of I. By the concavity of f, the value

$$A = \lim_{\varepsilon \to 0+} \frac{f(t_0 + \varepsilon) - f(t_0)}{\varepsilon}$$

is well-defined. And, the concavity of f implies

$$f(t) \leq f(t_0) + A(t - t_0).$$

When $t_0 \in I$ is the supremum of I, we obtain the same inequality as above by replacing A with $\lim_{\varepsilon \to 0+}(f(t_0 - \varepsilon) - f(t_0))/\varepsilon$.

The implication (2) $\Rightarrow$ (3) is trivial.

Let us assume that f satisfies (3), and take t_0 in the interior of I. Then there exists a twice-differentiable function $g : I \to \mathbb{R}$ such that

$$g(t_0) = f(t_0), \; g \geq f \text{ and } g'' \leq 0.$$

Hence, for any $t' < t_0 < t$, we have

$$\frac{f(t) - f(t_0)}{t - t_0} \leq \frac{g(t) - g(t_0)}{t - t_0} \leq \frac{g(t_0) - g(t')}{t_0 - t'} \leq \frac{f(t_0) - f(t')}{t_0 - t'}.$$

Therefore, f is concave. □

Let X be a geodesic space and U be an open subset of X. Let $f : U \to \mathbb{R}$ be a function. We say that f is *λ-concave* on U if for every geodesic $\gamma : I \to U$, the function $f \circ \gamma : I \to \mathbb{R}$ is λ-concave on I. For a function $g : U \to \mathbb{R}$, we say that f is *g-concave* if for any $p \in U$ and $\varepsilon > 0$, there is an open neighborhood V of p in U, such that f is $(g(p) + \varepsilon)$-concave on V. We say that $f : U \to \mathbb{R}$ is *g-concave in the barrier sense* if for any $p \in U$ and $\varepsilon > 0$, there exists an open neighborhood V of p in U such that for every geodesic γ contained in V, $f \circ \gamma$ is $(g(p) + \varepsilon)$-concave in the barrier sense. By an argument similar to the proof of Lemma 2.3, f is g-concave if and only if f is g-concave in the barrier sense.

From now on, we fix an Alexandrov space X. We use results and notions on Alexandrov spaces obtained in [Burago et al. 1992], and we refer to [Burago et al. 2001] and [Petrunin 2007]. T_pX denotes the tangent cone of X at p and Σ_pX denotes the space of directions of X at p.

For any λ-concave function $f : U \to \mathbb{R}$ on an open subset U of X, $p \in U$, and $\delta > 0$, a function $f_\delta : \delta^{-1}U \to \mathbb{R}$ is defined as the same function $f_\delta = f$ on the same domain $\delta^{-1}U = U$ as sets. Since the metric of $\delta^{-1}U$ is the metric of U multiplied by δ^{-1}, f_δ is $\delta^2\lambda$-concave on $\delta^{-1}U$. In addition, if f is Lipschitz near p, then the blow-up $d_pf : T_pX \to \mathbb{R}$, that is, the limit with respect to some sequence $\delta_i \to 0$,

$$\lim_{i \to \infty} f_{\delta_i} : \lim_{i \to \infty} (\delta_i^{-1}U, p) \to \mathbb{R}$$

is 0-concave on T_pX. We call d_pf the *differential* of f at p. Note that the differential of a locally Lipschitz semiconcave function always exists and does not depend on the choice of sequence (δ_i). Actually, $d_p f(\xi)$ is calculated by

$$d_p f(\xi) = \lim_{t \to 0+} \frac{f(\exp_p(t\xi)) - f(p)}{t}$$

if $\xi \in \Sigma'_p$ is a geodesic direction, where $\exp_p(t\xi)$ denotes the geodesic starting from p with the direction ξ.

Distance functions as semiconcave functions. For any real number κ, let us define "trigonometric functions" sn_κ and cs_κ by the following ODE:

$$\begin{cases} \mathrm{sn}''_\kappa(t) + \kappa\,\mathrm{sn}_\kappa(t) = 0, & \mathrm{sn}_\kappa(0) = 0, \quad \mathrm{sn}'_\kappa(0) = 1; \\ \mathrm{cs}''_\kappa(t) + \kappa\,\mathrm{cs}_\kappa(t) = 0, & \mathrm{cs}_\kappa(0) = 1, \quad \mathrm{cs}'_\kappa(0) = 0. \end{cases}$$

They are explicitly represented as follows.

$$\mathrm{sn}_\kappa(t) = \sum_{n=0}^{\infty} \frac{(-\kappa)^n}{(2n+1)!} t^{2n+1} = \begin{cases} \dfrac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}\, t) & \text{if } \kappa > 0, \\ t & \text{if } \kappa = 0, \\ \dfrac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}\, t) & \text{if } \kappa < 0, \end{cases}$$

$$\mathrm{cs}_\kappa(t) = \mathrm{sn}'_\kappa(t) = \sum_{n=0}^{\infty} \frac{(-\kappa)^n}{(2n)!} t^{2n} = \begin{cases} \cos(\sqrt{\kappa}\, t) & \text{if } \kappa > 0, \\ 1 & \text{if } \kappa = 0, \\ \cosh(\sqrt{-\kappa}\, t) & \text{if } \kappa < 0. \end{cases}$$

These functions are elementary for the space form $\mathbb{M}_\kappa$ in the sense that they satisfy the following: Let us take any points $p, q, r \in \mathbb{M}_\kappa$ with $|pq| + |qr| + |rp| < 2\,\mathrm{diam}\,\mathbb{M}_\kappa$, and set $\theta := \angle qpr$. Let γ be the geodesic pr with $\gamma(0) = p$ and $\gamma(|p, r|) = r$. We set $\ell(t) = |q, \gamma(t)|$. When $\kappa \neq 0$, the cosine formula states

$$\mathrm{cs}_\kappa(\ell(t)) = \mathrm{cs}_\kappa |pq|\, \mathrm{cs}_\kappa\, t + \kappa\, \mathrm{sn}_\kappa |pq|\, \mathrm{sn}_\kappa\, t \cos\theta.$$

Also, we have

$$(\mathrm{cs}_\kappa(\ell(t)))'' + \kappa\ \mathrm{cs}_\kappa(\ell(t)) = 0. \tag{2-1}$$

Lemma 2.4 [Perelman and Petrunin 1994]. *The distance function d_A from a closed subset A in an Alexandrov space X of curvature $\geq \kappa$ is $(\mathrm{cs}_\kappa(d_A)/\mathrm{sn}_\kappa(d_A))$-concave on $(X - A) \cap \{d_A < \pi/(2\sqrt{\kappa})\}$. Here, if $\kappa \leq 0$, then we consider $\pi/(2\sqrt{\kappa})$ as $+\infty$, and if $\kappa = 0$, then we consider $\mathrm{cs}_\kappa(d_A)/\mathrm{sn}_\kappa(d_A)$ as $1/d_A$.*

Proof. We consider the case that $\kappa \neq 0$. Let us take any geodesic γ contained in $(X - A) \cap \{d_A < \pi/(2\sqrt{\kappa})\}$. We take x on γ and reparametrize γ as $x = \gamma(0)$. We choose $w \in A$ such that $|Ax| = |wx|$. We set $\ell(t) := |A, \gamma(t)|$. Let us take a geodesic $\tilde{\gamma}$ and a point $\tilde{w}$ in the κ-plane $\mathbb{M}_\kappa$ such that $|\tilde{w}\tilde{\gamma}(0)| = |wx|$ and $\angle(\uparrow_{\tilde{x}}^{\tilde{w}}$

$, \tilde{\gamma}^+(0)) = \angle(\uparrow_x^w, \gamma^+(0))$. Let us set $\tilde{\ell}(t) := |\tilde{w}, \tilde{\gamma}(t)|$. By Alexandrov convexity, $\ell(t) \le \tilde{\ell}(t)$.

From (2-1), a standard calculation implies

$$\tilde{\ell}'' = \frac{\mathrm{cs}_\kappa(\tilde{\ell})}{\mathrm{sn}_\kappa(\tilde{\ell})}(1 - (\tilde{\ell}')^2) \le \frac{\mathrm{cs}_\kappa(\tilde{\ell})}{\mathrm{sn}_\kappa(\tilde{\ell})}.$$

Therefore, ℓ is $(\mathrm{cs}_\kappa(\ell)/\,\mathrm{sn}_\kappa(\ell))$-concave. The proof is complete if $\kappa \ne 0$. When X has nonnegative curvature, taking a negative number κ as a lower curvature bound of X and letting κ tend to 0, we obtain $\mathrm{cs}_\kappa(d_A)/\,\mathrm{sn}_\kappa(d_A) \to 1/d_A$. □

Gradient flows. For vectors v, w in the tangent cone T_pX, setting $o = o_p$, the origin of T_pX, we define $|v| = |o, v|$ and

$$\langle v, w\rangle = \begin{cases} |v||w| \cos \angle vow & \text{if } |v|, |w| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.5 [Perelman and Petrunin 1994; Petrunin 1995]. Let f be a λ-concave function on an open subset U of X. We say that a vector $g \in T_pX$ at $p \in U$ is a *gradient* of f at p if it satisfies

(1) $df_p(v) \le \langle v, g\rangle$ for all $v \in T_pX$;

(2) $df_p(g) = \langle g, g\rangle$.

We recall that a unique such g exists, which is denoted by $\nabla_p f = \nabla f(p)$.

We say that f is *regular at* p if $d_p f(v) > 0$ for some $v \in T_pX$, or equivalently, $|\nabla_p f| > 0$. Otherwise, f is said to be *critical at* p.

Definition 2.6 [Perelman and Petrunin 1994; Petrunin 1995]. Let $f : U \to \mathbb{R}$ be a semiconcave function on an open subset U of an Alexandrov space. A Lipschitz curve $\gamma : [0, a) \to X$ on an interval $[0, a)$ is said to be a *gradient curve on U for f* if for any $t \in [0, a)$ with $\gamma(t) \in U$,

$$\lim_{\varepsilon \to 0+} \frac{f \circ \gamma(t + \varepsilon) - f \circ \gamma(t)}{\varepsilon}$$

exists and is equal to $|\nabla f|^2(\gamma(t))$.

Note that if f is critical at $\gamma(t)$, the gradient curve γ for f satisfies $\gamma(t') = \gamma(t)$ for any $t' \ge t$.

The (multivalued) logarithm map $\log_p : X \to T_pX$ is defined for $x \ne p$ as $\log_p(x) = |px| \cdot \uparrow_p^x$, where $\uparrow_p^x$ is a direction of a geodesic px, and for $x = p$ as $\log_p(x) = o_p$. If γ is a gradient curve on U, then for t with $\gamma(t) \in U$, the forward direction

$$\gamma^+(t) := \lim_{\varepsilon \to 0+} \frac{\log_{\gamma(t)}(\gamma(t + \varepsilon))}{\varepsilon} \in T_{\gamma(t)}X$$

exists and is equal to the gradient $\nabla f(\gamma(t))$.

Proposition 2.7 [Kapovitch et al. 2010; Petrunin 2007; Petrunin 1995; Perelman and Petrunin 1994]. *Letting γ and η be gradient curves starting from $x = \gamma(0)$ and $y = \eta(0)$ in an open subset U for a λ-concave function $f : U \to \mathbb{R}$, we obtain*

$$|\gamma(s)\eta(s)| \leq e^{\lambda s}|xy|$$

for every $s \geq 0$.

This proposition implies a gradient curve starting at $x \in U$ is unique on its domain.

Theorem 2.8 [Petrunin 1995; 2007; Perelman and Petrunin 1994]. *For any open subset U of an Alexandrov space, a semiconcave function f on U, and $x \in U$, there exists a unique maximal gradient curve*

$$\gamma : [0, a) \to U$$

with $\gamma(0) = x$ for f, where γ is maximal, if for every gradient curve $\eta : [0, b) \to U$ for f with $\eta(0) = x$, we have $b \leq a$.

Definition 2.9 [Perelman and Petrunin 1994; Petrunin 1995]. Let U be an open subset of an Alexandrov space X and $f : U \to \mathbb{R}$ a semiconcave function. Let $\{[0, a_x)\}_{x \in U}$ be a family of intervals for $a_x > 0$. A map

$$\Phi : \bigcup_{x \in U} \{x\} \times [0, a_x) \to U$$

is a *gradient flow* of f on U (with respect to $\{[0, a_x)\}_{x \in U}$) if for every $x \in U$, $\Phi(x, 0) = x$ and the restriction

$$\Phi(x, \cdot\,) : [0, a_x) \to U$$

is a gradient curve of f on U.

A gradient flow Φ is *maximal* if each domain $[0, a_x)$ of the gradient curve is maximal.

By Theorem 2.8 and Proposition 2.7, a maximal gradient flow on U always exists and is unique.

Let Φ be the gradient flow of a semiconcave function on an open subset U. By a standard argument, we obtain

$$\Phi(x, s + t) = \Phi(\Phi(x, s), t)$$

for every $x \in U$ and $s, t \geq 0$, wherever the formula is defined.

3. Proof of Theorem 1.2

The purpose of this section is to prove Theorem 1.2. Let us fix a finite-dimensional Alexandrov space X. As we show in Section 5, the proof works for an infinite-dimensional Alexandrov space with an additional assumption.

We first prove the following. Consider the distance function $f = d(S(p, R), \cdot)$ from a metric sphere $S(p, R) = \{q \in X \mid |pq| = R\}$. We may assume that a neighborhood of p has curvature ≥ -1 by rescaling the metric of X if necessary. We denote by $B(p, R)$ the closed ball centered at p of radius R.

Proposition 3.1. *For any $p \in X$ and $\varepsilon > 0$, there exists $R > 0$ and $\delta_0 = \delta_0(\varepsilon, R) > 0$ such that the distance function*

$$f = d(S(p, R), \cdot)$$

from the metric sphere $S(p, R)$ satisfies

$$d_x f(\uparrow_x^p) > \cos\varepsilon \tag{3-1}$$

for every $x \in B(p, \delta_0 R) - \{p\}$. In particular, f is regular on $B(p, \delta_0 R) - \{p\}$.

Proposition 3.1 is key in our paper, which implies the important Lemma 3.3 later.

For a subset A of an Alexandrov space and $x \notin A$, we denote by A'_x the set of all directions of geodesics from x to A of length $|x, A|$.

Proof of Proposition 3.1. Since the tangent cone $T_p X$ is isometric to the metric cone $K(\Sigma_p)$ over the space of directions Σ_p, there exists a positive constant R satisfying the following:

(3-2) For any $v \in \Sigma_p$, there is $q \in S(p, R)$ such that $\angle(v, \uparrow_p^q) \leq \varepsilon$.

From now on, we set $S := S(p, R)$. For any $x \in S(p, \delta R)$, fixing a direction $\uparrow_p^x \in x'_p$, let us take $q_1, q_2 \in S$ such that

$$|x, q_1| = |x, S| := \min_{q \in S} |x, q| \tag{3-3}$$

and

$$\angle xpq_2 = \angle(\uparrow_p^x, \uparrow_p^{q_2}) = \angle(\uparrow_p^x, S'_p) := \min_{v \in S'_p} \angle(\uparrow_p^x, v). \tag{3-4}$$

By the condition (3-2), we have

$$\tilde{\angle} xpq_2 \leq \angle xpq_2 \leq \varepsilon.$$

Then, by the law of sines, we obtain

$$\sin \tilde{\angle} pxq_2 = \frac{\sinh R}{\sinh |xq_2|} \sin \tilde{\angle} xpq_2 \leq \frac{\sinh R}{\sinh R(1-\delta)} \sin\varepsilon. \tag{3-5}$$

On the other hand, by the law of cosines, we obtain

$$\begin{aligned}\cosh |xq_2| &= \cosh \delta R \cosh R - \sinh \delta R \sinh R \cos \tilde{\angle} xpq_2 \\ &\le \cosh \delta R \cosh R - \sinh \delta R \sinh R \cos \varepsilon\end{aligned}$$

and

$$\begin{aligned}-\sinh \delta R \sinh |xq_2| \cos \tilde{\angle} pxq_2 &= \cosh R - \cosh \delta R \cosh |xq_2| \\ &\ge \cosh R\{1 - \cosh^2 \delta R\} + \sinh R \sinh \delta R \cos \varepsilon.\end{aligned}$$

Therefore, if δ is smaller than some constant, then

$$-\cos \tilde{\angle} pxq_2 > 0. \tag{3-6}$$

By (3-5) and (3-6), we obtain

$$\tilde{\angle} pxq_2 \ge \pi - (1 + \tau(\delta))\varepsilon. \tag{3-7}$$

Next, let us consider the point q_1 taken as in (3-3). Then, it satisfies

$$\tilde{\angle} xpq_1 = \min_{q \in S} \tilde{\angle} xpq \le \min_{q \in S} \angle xpq \le \varepsilon.$$

By a similar argument with q_1 instead of q_2, we obtain

$$\tilde{\angle} pxq_1 \ge \pi - (1 + \tau(\delta))\varepsilon. \tag{3-8}$$

By the quadruple condition, with (3-7) and (3-8), we obtain

$$\tilde{\angle} q_1xq_2 \le 2\pi - \tilde{\angle} pxq_1 - \tilde{\angle} pxq_2 \le (2 + \tau(\delta))\varepsilon.$$

If δ is small with respect to ε, then we obtain

$$|q_1q_2| \le 3R\varepsilon.$$

Therefore, we obtain

$$\tilde{\angle} q_1pq_2 \le 4\varepsilon. \tag{3-9}$$

For any $y \in px - \{p, x\}$, we set $q_3 = q_3(y) \in S$ to be such that

$$|y, q_3| = |y, S|.$$

By an argument similar to above, we obtain

$$\tilde{\angle} pyq_3 \ge \pi - (1 + \tau(|py|/R))\varepsilon > \pi - 2\varepsilon. \tag{3-10}$$

Then, we have

$$\tilde{\angle} xyq_3 < 2\varepsilon.$$

By the Gauss–Bonnet theorem, if y is near x, then

$$\tilde{\angle} yxq_3 > \pi - 3\varepsilon.$$

By the first variation formula, we obtain

$$df_x(\uparrow_x^p) = \lim_{xp \ni y \to x} \frac{|Sy| - |Sx|}{|xy|} \geq \liminf_{xp \ni y \to x} \frac{|q_3 y| - |q_3 x|}{|xy|} \geq \cos 3\varepsilon.$$

This completes the proof. □

We fix δ_0 as in the conclusion of Proposition 3.1, and fix $\delta \leq \delta_0$.

Lemma 3.2. *For any $x \in B(p, \delta R) - \{p\}$, we have*

$$\angle(\nabla_x f, \uparrow_x^p) < \varepsilon \text{ and } |\nabla_x f, \uparrow_x^p| < \sqrt{2}\varepsilon.$$

Proof. By Proposition 3.1, we have

$$df_x(\uparrow_x^p) > \cos \varepsilon.$$

By the definition of the gradient, we obtain

$$df_x(\uparrow_x^p) \leq |\nabla_x f| \cos \angle(\nabla_x f, \uparrow_x^p) \leq \cos \angle(\nabla f, \uparrow_x^p).$$

Therefore, we have $\angle(\nabla_x f, \uparrow_x^p) < \varepsilon$.

Since f is 1-Lipschitz, $|\nabla f| \leq 1$. And, by the above inequality,

$$|\nabla f|_x = \max_{\xi \in \Sigma_x} df_x(\xi) \geq df(\uparrow_x^p) > \cos \varepsilon.$$

Then, we obtain

$$|\nabla_x f, \uparrow_x^p|^2 < |\nabla f|^2 + 1 - 2|\nabla f| \cos \varepsilon \leq 2 \sin^2 \varepsilon.$$

Therefore, $|\nabla f, \uparrow_x^p| < \sqrt{2}\varepsilon$. □

Let us consider the gradient flow Φ_t of $f = d(S, \cdot)$.

Lemma 3.3. *For every $x \in B(p, \delta R)$,*

$$|\Phi_t(x), p| \leq |x, p| - \cos \varepsilon \cdot t,$$

whenever this formula is defined. In particular, for any $t \geq \delta R / \cos \varepsilon$, we have $\Phi_t(x) = p$.

Proof. Let us set $\gamma(t) = \Phi_t(x)$, the gradient curve for f starting from $\gamma(0) = x$. If $\gamma(t_0) \neq p$, then

$$\left.\frac{d}{dt}\right|_{t=t_0+} |\Phi_t(x), p| = -\langle \nabla_{\gamma(t_0)} f, \uparrow_{\gamma(t_0)}^p \rangle < -\cos \varepsilon.$$

Integrating this, we have

$$|\Phi_{t_0}(x), p| - |x, p| \leq -\cos\varepsilon \cdot t_0.$$

This completes the proof. □

Finally, we estimate the Lipschitz constant of the flow Φ on $B(p, \delta R)$. Let us recall that f is λ-concave on $B(p, \delta R)$ for some λ. By Lemma 2.4, λ can be given as follows:

$$\frac{\cosh(f)}{\sinh(f)} \leq \frac{\cosh R}{\sinh(R(1-\delta))} = \lambda.$$

By Proposition 2.7, for any $x, y \in B(p, \delta R)$,

$$|\Phi(x,t), \Phi(y,t)| \leq e^{\lambda t}|xy|.$$

Since f is 1-Lipschitz, for $x \in B(p, \delta R)$ and $t' < t$, we have

$$|\Phi(x,t), \Phi(x,t')| \leq \int_{t'}^{t} \left| \frac{d}{ds}^{+} \Phi(x,s) \right| ds = \int_{t'}^{t} |\nabla f|(\Phi(x,s))\, ds \leq t - t'.$$

Therefore, we obtain the following:

Lemma 3.4. *For any $x, y \in B(p, \delta R)$ and $t \geq s \geq 0$,*

$$|\Phi(x,s), \Phi(y,t)| \leq e^{\lambda s}|x, y| + t - s.$$

Note that, by Lemma 3.3, setting $\ell = \delta_0 R / \cos\varepsilon$, the term $e^{\lambda\ell}$ can be bounded from above by a constant arbitrary close to 1 if we choose δ_0 and R small enough.

By Lemma 3.4, we obtain a Lipschitz homotopy

$$\varphi : B(p, \delta_0 R) \times [0,1] \to B(p, \delta_0 R)$$

with $\varphi(\,\cdot\,, 1) = p$, defined by $\varphi(x,t) = \Phi(x, \ell t)$ for $(x,t) \in B(p, \delta_0 R) \times [0,1]$. This completes the proof of Theorem 1.2.

Remark 3.5. In the above argument, we employ the distance function from $S(p, R)$ to prove Theorem 1.2. Similarly, one can use the averaged distance function constructed in [Perelman 1993] and [Kapovitch 2005] to prove Theorem 1.2.

4. Proof of applications

Proof of Corollaries 1.3 and 1.4. Let V be a metric space, U a subset of V, and $p \in V$. We say that U is *Lipschitz contractible to p in V* if there exists a Lipschitz map

$$h : U \times [0,1] \to V$$

such that

$$h(x,0) = x \quad \text{and} \quad h(x,1) = p$$

for any $x \in U$. We call such an h a *Lipschitz contraction* from U to p in V. We say that U is *Lipschitz contractible in V* if U is Lipschitz contractible to some point in V.

Lemma 4.1. *Let U be Lipschitz contractible in a metric space V. For any Lipschitz map $\varphi : S^{n-1} \to U$, there exists a Lipschitz map $\tilde{\varphi} : D^n \to V$ such that $\tilde{\varphi}|_{S^{n-1}} = \varphi$.*

Proof. By definition, there exist $p \in V$ and a Lipschitz map

$$h : U \times [0, 1] \to V$$

such that

$$h(x, 0) = x \text{ and } h(x, 1) = p$$

for any $x \in U$. We define a map

$$\varphi_1 : S^{n-1} \times [0, 1] \to V$$

by $\varphi_1 = h \circ (\varphi \times \mathrm{id})$. Then, φ_1 is Lipschitz with Lipschitz constant at most $\mathrm{Lip}\,(h) \cdot \max\{1, \mathrm{Lip}\,(\varphi)\}$. We define a map

$$\varphi_2 : D^n \times \{1\} \to V$$

by $\varphi_2(v, 1) = p$ for all $v \in D^n$. And we consider a space

$$Y = S^{n-1} \times [0, 1] \cup D^n \times \{1\}$$

equipped with a length metric with respect to a gluing $S^{n-1} \times \{1\} \ni (v, 1) \mapsto (v, 1) \in \partial D^n \times \{1\}$. Now we define a map $\varphi_3 : Y \to V$ by

$$\varphi_3 = \begin{cases} \varphi_1 & \text{on } S^{n-1} \times [0, 1], \\ \varphi_2 & \text{on } D^n \times \{1\}. \end{cases}$$

This is well-defined. Then, φ_3 is $\mathrm{Lip}\,(\varphi_1)$-Lipschitz. Indeed, for $x \in S^{n-1} \times [0, 1]$ and $y \in D^n \times \{1\}$, we have

$$|\varphi_3(x), \varphi_3(y)| = |\varphi_3(x), p|.$$

Let $\bar{x} \in S^{n-1} \times \{1\}$ be the foot of a perpendicular segment from x to $S^{n-1} \times \{1\}$. We note that $|x, \bar{x}| \le |x, y|$ and $\varphi_3(\bar{x}) = p$. Then, we obtain

$$|\varphi_3(x), p| = |\varphi_3(x), \varphi_3(\bar{x})| = |\varphi_1(x), \varphi_1(\bar{x})| \le \mathrm{Lip}\,(\varphi_1)|x, \bar{x}| \le \mathrm{Lip}\,(\varphi_1)|x, y|.$$

Obviously, there exists a bi-Lipschitz homeomorphism

$$f : D^n \to Y$$

with $f(0) = (0, 1) \in D^n \times \{1\}$ preserving the boundaries, in the sense that $f(v) = (v, 0) \in S^{n-1} \times \{0\}$ for any $v \in S^{n-1}$. Then, we obtain a Lipschitz map $\tilde{\varphi} := \varphi_3 \circ f$ satisfying the desired condition. $\square$

Definition 4.2. We say that a metric space Y is a *Lipschitz simplicial complex* if there exists a triangulation T of Y satisfying the following: For each simplex $S \in T$, there exists a bi-Lipschitz homeomorphism $\varphi_S : \triangle^{\dim S} \to S$. Here, the simplex $\triangle^{\dim S}$ is a standard simplex equipped with the Euclidean metric and S is given the restricted metric of Y. We say that such a triangulation T is a *Lipschitz triangulation* of Y. The dimension of Y is given by $\dim Y = \sup_{S \in T} \dim S$. We only deal with Y such that $\dim Y < \infty$.

A Lipschitz simplicial complex Y is called *finite* if it has a Lipschitz triangulation consisting of finitely many elements.

Note that a subdivision (for instance, the barycentric one) of a Lipschitz triangulation is also a Lipschitz triangulation.

Proposition 4.3. *Let X be an SLLC space, Y a Lipschitz simplicial complex, and $f : Y \to X$ a continuous map. Then, there exists a homotopy*

$$h : Y \times [0, 1] \to X$$

from $h_0 = f$ such that h_1 is Lipschitz on each simplex of Y.

Further, if f is Lipschitz on a subcomplex A of Y, then a homotopy h can be chosen that is relative to A, that is, satisfying $h(a, t) = a$ for any $a \in A$ and $t \in [0, 1]$.

Proof. If $\dim Y = 0$, then we set $h(x, t) = f(x)$ for $x \in Y$ and $t \in [0, 1]$. Then, h is the desired homotopy.

We assume that the assertion holds for $\dim Y \leq k - 1$. First, we prove that for any $f : \triangle^k \to X$, there exists a homotopy

$$h : \triangle^k \times [0, 1] \to X$$

from $h_0 = f$ to a Lipschitz map h_1. Taking a subdivision if necessary, let us take a finite Lipschitz triangulation T of $\triangle^k$ satisfying the following: For any k-simplex $E \in T$, there exists an open subset U_E of X which is a Lipschitz contractible ball such that $f(E) \subset U_E$. For any simplex $F \in T$ of $\dim F \leq k - 1$, we set

$$U_F = \bigcap_{F \subset E \in T} U_E.$$

This is an open subset of X. Let us denote by Z a $(k - 1)$-skeleton of $\triangle^k$ with respect to T. By the inductive assumption, there exists a homotopy

$$h : Z \times [0, 1] \to X$$

from $h_0 = f|_Z$ such that for every simplex F of Z, the following hold:

- $h_1|_F$ is Lipschitz.
- $h(F \times [0, 1]) \subset U_F$.

- If $f|_F$ is Lipschitz, then $h_t|_F = f|_F$ for any t.

Let E be a k-simplex of $\triangle^k$ with respect to T. We denote by $h^{\partial E}$ the restriction of h to $\partial E \times [0, 1]$. Then, the image of $h^{\partial E}$ is contained in $\bigcup_{T \ni F \subset \partial E} U_F \subset U_E$. Since the pair $(E, \partial E)$ has the homotopy extension property, there exists a homotopy

$$h^E : E \times [0, 1] \to U_E$$

from $f|_E$ which is an extension of $h^{\partial E}$. Then, h_1^E is Lipschitz on ∂E. For another k-simplex E' of $\triangle^k$ with common face $E \cap E'$,

$$h_t^E = h_t^{E'}$$

on $E \cap E'$ for all t. Since U_E is a Lipschitz contractible ball, by Lemma 4.1 there is a homotopy

$$\bar{h}^E : E \times [0, 1] \to X$$

relative to ∂E from $\bar{h}_0^E = h_1^E$ to a Lipschitz map $\bar{h}_1^E : E \to X$. Let us define a homotopy $\hat{h}^E : E \to X$ by

$$\hat{h}^E(x, t) = \begin{cases} h^E(x, t) & \text{if } t \in [0, 1/2], \\ \bar{h}^E(x, t) & \text{if } t \in [1/2, 1]. \end{cases}$$

We define $\hat{h} : \triangle^k \times [0, 1] \to X$ by

$$\hat{h}(x, t) = \hat{h}^E(x, t)$$

for $x \in E \in T$. Then, $\hat{h}_0 = f$ and $\hat{h}_1$ is Lipschitz.

Next, we consider a continuous map $f : Y \to X$ from a Lipschitz simplicial complex Y with $\dim Y = k$. Let Z be a $(k-1)$-simplex of Y. By the inductive assumption, there exists a homotopy

$$h : Z \times [0, 1] \to X$$

from $h_0 = f|_Z$, and h_1 is Lipschitz on every simplex of Z. From now on, let us denote by E a k-skeleton of Y. By using the homotopy extension property for $(E, \partial E)$ and Lemma 4.1, we obtain a homotopy

$$h^E : E \times [0, 1] \to X$$

which is an extension of $h|_{\partial E \times [0,1]}$, with $h_0^E = f|_E$. Since $h_1^E|_{\partial E} = h_1|_{\partial E}$ is Lipschitz, there exists a homotopy

$$\bar{h}^E : E \times [0, 1] \to X$$

relative to ∂E from $\bar{h}_0^E = h_1^E$ to a Lipschitz map $\bar{h}_1^E$. We set $\bar{h}(x,t) = h(x,1)$ for $x \in Z$ and $t \in [0,1]$. And, we define a homotopy $\hat{h} : Y \times [0,1] \to X$ by

$$\hat{h}(x,t) = \begin{cases} h(x,2t) & \text{if } x \in Z \text{ and } t \in [0,1/2], \\ \bar{h}(x,2t-1) & \text{if } x \in Z \text{ and } t \in [1/2,1], \\ h^E(x,2t) & \text{if } x \in E \subset Y \text{ and } t \in [0,1/2], \\ \bar{h}^E(x,2t-1) & \text{if } x \in E \subset Y \text{ and } t \in [1/2,1]. \end{cases}$$

Then, $\hat{h}_0 = f$ and $\hat{h}_1$ is Lipschitz on every simplex. □

Corollary 4.4. *Let Y be a Lipschitz simplicial complex, X an SLLC space, and $f : Y \to X$ a continuous map. Let T be a Lipschitz triangulation of Y and $\{U_F \mid F \in T\}$ a family of open subsets of X satisfying the following properties*:

- *$f(F) \subset U_F$ for $F \in T$.*
- *$U_F \subset U_E$ for $F, E \in T$ with $F \subset E$.*

Then, there exists a homotopy $h : Y \times [0,1] \to X$ from $h_0 = f$ such that for every $F \in T$:

- *h_1 is Lipschitz on F.*
- *$h(F \times [0,1]) \subset U_F$.*
- *If f is Lipschitz on F, then $h_t = f$ on F for all t.*

For instance, fixing $\varepsilon > 0$ and setting U_F an ε-neighborhood of $f(F)$ for every $F \in T$, the family $\{U_F \mid F \in T\}$ satisfies the assumption of Corollary 4.4.

Proof of Corollary 4.4. If $\dim Y = 0$, the assertion is trivial. We assume that Corollary 4.4 holds when $\dim Y \le k-1$ for some $k \ge 1$. Let Y be a Lipschitz simplicial complex with $\dim Y = k$ and T a Lipschitz triangulation of Y. Let us take a family $\{U_F \mid F \in T\}$ of open subsets satisfying the assumption of Corollary 4.4. By inductive assumption, there exists a homotopy

$$h : Y^{(k-1)} \times [0,1] \to X$$

from $h_0 = f|_{Y^{(k-1)}}$, and h_1 is Lipschitz on each $F \in T$ of $\dim \le k-1$ and $h_t(F) \subset U_F$ for all t. Let us denote by E a k-simplex in T. By Proposition 4.3, there exists a homotopy

$$h^E : E \times [0,1] \to U_E$$

from $h_0^E = f|_E$ to a Lipschitz map h_1^E such that $h_t^E = h_t$ on ∂E for all t. Then, the concatenation map

$$\hat{h}(x,t) = \begin{cases} h(x,t) & \text{if } x \in Y^{(k-1)}, \\ h^E(x,t) & \text{if } x \in E, \end{cases}$$

is the desired homotopy. □

Remark 4.5. We note that Proposition 4.3 and Corollary 4.4 above can be also proved assuming X is just LLC instead of SLLC. Here, we say that a metric space X is locally Lipschitz contractible, for short LLC, if for any $p \in X$ and $\varepsilon > 0$, there exist $r \in (0, \varepsilon]$ and a Lipschitz contraction φ from $U(p, r)$ to p in $U(p, \epsilon)$. We also remark that Corollaries 1.3 and 1.4 are true if X is just LLC.

Let us start to prove Corollaries 1.3 and 1.4.

Proof of Corollaries 1.3 and 1.4. Let us take a finite Lipschitz simplicial complex pair (P, Q), with Q possibly empty. We prove Corollaries 1.3 and 1.4 assuming X to be SLLC. Let A be an open subset in X. Let us consider a continuous map $f : (P, Q) \to (X, A)$. By Corollary 4.4 and Theorem 1.2, we obtain a homotopy

$$\varphi : (P, Q) \times [0, 1] \to (X, A)$$

from $\varphi_0 = f$ to a Lipschitz map $\varphi_1 : (P, Q) \to (X, A)$. Here, we note that since A is open in X, the homotopy φ_t can be chosen so that $\varphi_t(Q) \subset A$. Then, we obtain a correspondence

$$C((P, Q), (X, A)) \ni f \mapsto \varphi_1 \in \mathrm{Lip}((P, Q), (X, A)), \tag{4-1}$$

where $C(*, **)$ (resp. $\mathrm{Lip}(*, **)$) denotes the set of all continuous (resp. Lipschitz) maps from $*$ to $**$.

Let us consider two homotopic continuous maps f and g from (P, Q) to (X, A). From the correspondence (4-1), we obtain Lipschitz maps f' and g' from (P, Q) to (X, A) which are homotopic to f and g, respectively. Connecting these homotopies, we obtain a homotopy

$$H : (P, Q) \times [0, 1] \to (X, A)$$

between $H(\cdot, 0) = f'$ and $H(\cdot, 1) = g'$. Now, we consider a Lipschitz simplicial complex $\tilde{P} = P \times [0, 1]$ and a subcomplex $\tilde{R} = P \times \{0, 1\}$. Then, the map H is Lipschitz on $\tilde{R}$. Hence, by Proposition 4.3, we obtain a homotopy

$$\tilde{H} : \tilde{P} \times [0, 1] \to X$$

relative to $\tilde{R}$ from $\tilde{H}(\cdot, 0) = H$ to a Lipschitz map $\tilde{H}(\cdot, 1)$. Then, $\tilde{H}(\cdot, 1)$ is a Lipschitz homotopy between f' and g'. Therefore, we conclude that the correspondence (4-1) sends a homotopy to a Lipschitz homotopy. This completes the proof of Corollary 1.3.

Let us consider a pointed n-sphere (S^n, p_0) and an Alexandrov space X with point $x_0 \in X$. Then, for any map $f : (S^n, p_0) \to (X, x_0)$, the restriction $f|_{\{p_0\}}$ is always Lipschitz. Hence, by an argument as above and Proposition 4.3, we obtain the conclusion of Corollary 1.4. □

Plateau problem. We first recall the definition of the Sobolev space of a metric space target in order to state the setting of Plateau problem in an Alexandrov space as in the introduction, referring to [Korevaar and Schoen 1993] and [Mese and Zulkowski 2010]. For a complete metric space X and a domain Ω in a Riemannian manifold having compact closure, a function $u : \Omega \to X$ is said to be an *L^2-map* if u is Borel measurable and, for some (equivalently, any) point $p_0 \in X$, the integral

$$\int_\Omega |u(x),\, p_0|^2 \, d\mu$$

is finite, where μ is the Riemannian volume measure. The set of all L^2-maps from Ω to X is denoted by $L^2(\Omega, X)$. We recall the definition of the energy of $u \in L^2(\Omega, X)$: For any $\varepsilon > 0$, we set $\Omega_\varepsilon = \{x \in \Omega \mid d(\partial\Omega, x) > \varepsilon\}$, and define an approximate energy density $e^u_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ by

$$e^u_\varepsilon(x) = \frac{1}{\omega_n} \int_{S(x,\varepsilon)} \frac{d(u(x), u(y))^2}{\varepsilon^2} \frac{d\sigma}{\varepsilon^{n-1}}.$$

Here, $n = \dim \Omega$, $S(x, \varepsilon)$ is the metric sphere around x with radius ε and σ is the surface measure on it. By [Korevaar and Schoen 1993, 1.2(iii)], we obtain

$$\int_{\Omega_\varepsilon} e^u_\varepsilon(x)\, d\mu \le C\varepsilon^{-2}.$$

Let us take a Borel measure ν on the interval $(0, 2)$ satisfying

$$\nu \ge 0, \quad \nu((0, 2)) = 1, \quad \text{and} \quad \int_0^2 \lambda^{-2}\, d\nu(\lambda) < \infty.$$

An averaged approximate energy density ${}_\nu e^u_\varepsilon(x)$ is defined by

$${}_\nu e^u_\varepsilon(x) = \begin{cases} \displaystyle\int_0^2 e^u_{\lambda\varepsilon}(x)\, d\nu(\lambda) & \text{if } x \in \Omega_{2\varepsilon}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $C_c(\Omega)$ be the set of all continuous function on Ω with compact support. We define a functional $E^u_\varepsilon : C_c(\Omega) \to \mathbb{R}$ by

$$E^u_\varepsilon(f) := \int_\Omega f(x)\, {}_\nu e^u_\varepsilon \, d\mu(x).$$

Then, the *energy* of u is defined by

$$E^u = \sup_{\substack{f \in C_c(\Omega) \\ 0 \le f \le 1}} \limsup_{\varepsilon \to 0} E^u_\varepsilon(f).$$

The $(1, 2)$-Sobolev space is defined as

$$W^{1,2}(\Omega, X) = \{u \in L^2(\Omega, X) \mid E^u < \infty\}.$$

We start to prove Corollary 1.6.

Proof of Corollary 1.6. Let Γ be a rectifiable closed Jordan curve in an Alexandrov space X which is topologically contractible. Since Γ is rectifiable, we can take a Lipschitz monotonic parametrization

$$\gamma : S^1 \to \Gamma.$$

By the contractibility of Γ, there exists a continuous map

$$h : \Gamma \times [0, 1] \to X$$

such that $h(\,\cdot\,, 0) = \mathrm{id}_\Gamma$ and $h(\,\cdot\,, 1) = p$ for some $p \in X$. We define a map $f : S^1 \times [0, 1] \to X$ by $f(x, t) = h(\gamma(x), t)$. Further, we set $f(y, 1) = p$ for $y \in D^2$. By taking a reparametrization of $f : S^1 \times [0, 1] \cup D^2 \times \{1\} \to X$, we obtain a continuous map

$$g : D^2 \to X$$

such that $g|_{\partial D^2} = \gamma$.

By Proposition 4.3, there exists a homotopy

$$\tilde{h} : D^2 \times [0, 1] \to X$$

relative to ∂D^2 such that $\tilde{h}(\,\cdot\,, 0) = g$ and $\tilde{h}(\,\cdot\,, 1)$ is Lipschitz. Thus, we obtain a Lipschitz map $\tilde{g} = \tilde{h}(\,\cdot\,, 1)$ such that $\tilde{g}|_{\partial D^2} = \gamma$. By the definition of the energy, we obtain

$$E(\tilde{g}) \le \mathrm{Lip}(\tilde{g})^2 < \infty.$$

Here, $\mathrm{Lip}(\tilde{g})$ is the Lipschitz constant of $\tilde{g}$. Therefore, we conclude $\tilde{g} \in \mathscr{F}_\Gamma$. □

5. A note on the infinite-dimensional case

It is known that the (Hausdorff) dimension of an Alexandrov space is a nonnegative integer or is infinite. There are only a few works on infinite-dimensional Alexandrov spaces. It is not known whether an infinite-dimensional Alexandrov space is locally contractible.

When we consider an Alexandrov space of *possibly infinite dimension*, we somewhat generalize Definition 2.1 as follows: A complete metric space X is called an *Alexandrov space* if it is a length metric space and satisfies the quadruple condition locally. Here, a complete metric space X is a *length* metric space if for every two points $p, q \in X$ and any $\varepsilon > 0$, there exists a point $r \in X$ satisfying $\max\{|pr|, |rq|\} < |pq|/2 + \varepsilon$. Since a length metric space has no geodesics in general, to define a notion of a lower curvature bound, we change the triangle comparison condition to the quadruple condition. Here, an open subset U of a length space X satisfies the *quadruple condition* modeled on the κ-plane $\mathbb{M}_\kappa$ if for any four distinct points p_0, p_1, p_2 and p_3 in U, we have

$$\tilde{\angle} p_1 p_0 p_2 + \tilde{\angle} p_2 p_0 p_3 + \tilde{\angle} p_3 p_0 p_1 \leq 2\pi,$$

where $\tilde{\angle} = \tilde{\angle}_\kappa$ denotes the comparison angle modeled on $\mathbb{M}_\kappa$.

By a standard argument, any geodesic triangle (if one exists) in an Alexandrov space of possibly infinite dimension satisfies the triangle comparison condition. It is known that finite-dimensional Alexandrov spaces are proper metric space; in particular, by Hopf–Rinow theorem, they are geodesic spaces.

Plaut proved that an Alexandrov space of infinite dimension is an “almost” geodesic space. Precisely:

Theorem 5.1 [Plaut 1996]. *Let X be an Alexandrov space of infinite dimension. For any $p \in X$, the subset $J_p \subset X$ defined by*

$$J_p = \bigcap_{\delta > 0} \{q \in X - \{p\} \mid \text{there exists } x \in X - \{p, q\} \text{ with } \tilde{\angle} pqx > \pi - \delta\}$$

is a dense G_δ-subset in X, and, for every $q \in J_p$, there exists a unique geodesic connecting p and q.

We now show that the compactness of the space of directions at some point implies Lipschitz contractibility around the point.

Proposition 5.2. *Let X be an Alexandrov space of infinite dimension. Suppose that there exists a point $p \in X$ such that the space of directions Σ_p at p is compact. Then, the following are true:*

(i) *The pointed Gromov–Hausdorff limit as $r \to \infty$ of the scaling space (rX, p) exists and is isometric to the cone over Σ_p.*

(ii) *Σ_p is a geodesic space.*

(iii) *X is proper.*

(iv) *There exists $R_0 > 0$, depending on p, such that for every $R \leq R_0$, $U(p, R)$ is Lipschitz contractible to p in itself.*

Proof. (i) Let $K = K(\Sigma_p)$ be the Euclidean cone over Σ_p and B be the unit ball around the origin o. Let J_p be the set defined in Theorem 5.1. For any $\varepsilon > 0$, we take a finite ε-net $\{v_\alpha\}_\alpha \subset B$. We may assume that every v_α is contained in $K(\Sigma'_p) - \{o\}$. That is, there exists $r > 0$ such that for every α, there is a geodesic γ_α starting from p having direction $v_\alpha/|v_\alpha|$, with length at least r. Let $x_\alpha \in B(p, r)$ be defined by $x_\alpha = \gamma_\alpha(r|v_\alpha|)$. Then, $\{x_\alpha\}_\alpha$ is an ε-net in $(1/r)B(p, r)$. Indeed, for any $x \in B(p, r) \cap J_p$, setting $v = \log_p(x) \in K(\Sigma_p)$, we have $(1/r)v \in B$. Then, there exists α such that $|v_\alpha, (1/r)v| \le \varepsilon$. Therefore, $|rv_\alpha, v| \le r\varepsilon$. We may assume that a lower curvature bound of X is less than or equal to 0. Then

$$\exp_p : B(o, r) \cap \mathrm{dom}(\exp_p) \to B(p, r)$$

is 1-Lipschitz, where $\mathrm{dom}(\exp_p)$ is the domain of $\exp_p$. Therefore, $|x_\alpha, x|_X \le r\varepsilon$.

Let us retake r to be small enough that

$$\left| \frac{|x_\alpha, x_\beta|}{r} - |v_\alpha, v_\beta| \right| \le \varepsilon.$$

Then, the map $v_\alpha \mapsto x_\alpha$ implies a $C\varepsilon$-approximation between B and $(1/r)B(p, r)$ for any small r. Here, C is a constant not depending on any other term. Therefore, the pointed spaces $((1/r)X, p)$ are Gromov–Hausdorff convergent to $(K(\Sigma_p), o)$ as $r \to 0$.

Clearly (ii) holds by (i) and (iii). We prove (iii). Let us consider any closed ball $B(p, r)$ centered at p. Let us take any sequence $\{x_i\} \subset B(p, r)$. We take $y_i \in B(p, r) \cap J_p$ such that $|x_i, y_i| \le 1/i$. Then, $v_i = \log_p(y_i) \in B(o, r) \subset T_pX$ is well-defined. By (i), T_pX is proper. Hence, there exists a convergent subsequence $\{v_{n(i)}\}_i$ of $\{v_i\}_i$. Since $\exp_p$ is Lipschitz, $\{x_{n(i)}\}$ is convergent.

We recall that the proof of Theorem 1.2 started from the assertion (3-2) in the proof of Proposition 3.1. The assertion (i) guarantees (3-2). Therefore, one can prove (iv) in the same way as the proof of Theorem 1.2. □

6. An estimation of simplicial volume of Alexandrov spaces

In this section, we consider an Alexandrov space having a lower *Ricci* curvature bound, and we prove an estimation of the simplicial volume of such a space as stated in Theorem 1.8. The original form of Theorem 1.8 was proved by Gromov [1982] when X is a Riemannian manifold with a lower Ricci curvature bound.

Gromov's original proof was depending on the well-known Bishop–Gromov volume inequality. For an Alexandrov space of curvature $\ge \kappa$ for some $\kappa \in \mathbb{R}$, its Hausdorff measure is known to satisfy the Bishop–Gromov-type volume growth estimate. The second author's proof of Corollary 1.7 was depending on this volume growth estimate [Yamaguchi 1997]. It is known that several natural generalized notions of a lower Ricci curvature bound induce a volume growth estimate. Among them, the local reduced curvature-dimension condition introduced by Bacher and Sturm [2010] can be used as a general condition implying the inequality in Theorem 1.8. For completeness, we recall the definitions of several generalized notions of lower Ricci curvature bound, and prove Theorem 1.8.

Several notions of lower Ricci curvature bound. We recall several generalized notions of a lower bound of Ricci curvature, defined on a pair consisting of a metric space and a Borel measure on it. For the theory, history and undefined terms of the following, we refer to [Sturm 2006a; 2006b; Bacher and Sturm 2010; Cavalletti and Sturm 2012; Ohta 2007] and their references.

In this section, we denote by M a complete separable metric space. By $\mathcal{P}_2(M)$ we denote the set of all Borel probability measures μ on M with finite second moment. A metric called the L_2-Wasserstein distance W_2 is defined on $\mathcal{P}_2(M)$. Let us fix a locally finite Borel measure m on M. Such a pair (M, m) is called a metric measure space. Let us denote by $\mathcal{P}_\infty(M, m)$ the subset of $\mathcal{P}_2(M)$ consisting of all measures which are absolutely continuous in m and have bounded support.

From now on, K and N denote real numbers with $N \geq 1$. For $\nu \in \mathcal{P}_\infty(M, m)$ with density $\rho = d\nu/dm$, its Rényi entropy with respect to m is given by

$$S_N(\nu|m) := -\int_M \rho^{1-1/N}\, dm = -\int_M \rho^{-1/N}\, d\nu.$$

For $t \in [0, 1]$, a function $\sigma^{(t)}_{K,N} : (0, \infty) \to [0, \infty)$ is defined as

$$\sigma^{(t)}_{K,N}(\theta) = \begin{cases} +\infty & \text{if } K\theta^2 \geq N\pi^2, \\ \dfrac{\mathrm{sn}_{K/N}(t\theta)}{\mathrm{sn}_{K/N}(\theta)} & \text{otherwise.} \end{cases}$$

And, we set $\tau^{(t)}_{K,N}(\theta) = t^{1/N}\sigma^{(t)}_{K,N-1}(\theta)^{(N-1)/N}$.

Definition 6.1 [Bacher and Sturm 2010; Cavalletti and Sturm 2012; Sturm 2006b]. Let K and N be real numbers with $N \geq 1$. Let (M, m) be a metric measure space. We say that (M, m) satisfies the *reduced curvature-dimension condition* $\mathrm{CD}^*(K, N)$ *locally* — denoted by $\mathrm{CD}^*_{\mathrm{loc}}(K, N)$ — if for any $p \in M$ there exists a neighborhood $M(p)$ such that for all ν_0, $\nu_1 \in \mathcal{P}_\infty(M, m)$ supported on $M(p)$, denoting those densities by ρ_0, ρ_1 with respect to m, there exist an optimal coupling q of ν_0 and ν_1 and a geodesic $\Gamma : [0, 1] \to \mathcal{P}_\infty(M, m)$, parametrized proportionally to arclength, connecting $\nu_0 = \Gamma(0)$ and $\nu_1 = \Gamma(1)$, such that

$$S_{N'}(\Gamma(t)|m) \leq -\int_{M\times M} \big[\sigma^{(1-t)}_{K,N'}(d(x_0, x_1))\rho_0^{-1/N'}(x_0) + \sigma^{(t)}_{K,N'}(d(x_0, x_1))\rho_1^{-1/N'}(x_1)\big]\, dq(x_0, x_1)$$

holds for all $t \in [0, 1]$ and all $N' \geq N$.

We say that (M, m) satisfies the *curvature-dimension condition* $\mathrm{CD}(K, N)$ *locally* — denoted by $\mathrm{CD}_{\mathrm{loc}}(K, N)$ — if it satisfies $\mathrm{CD}^*_{\mathrm{loc}}(K, N)$ with $\sigma^{(s)}_{K,N'}$ replaced by $\tau^{(s)}_{K,N'}$ for each $s \in [0, 1]$ and $N' \geq N$.

The (global) conditions $\mathrm{CD}^*(K, N)$ and $\mathrm{CD}(K, N)$ are defined similarly, and imply corresponding local conditions.

From the inequality $\tau^{(t)}_{K,N}(\theta) \geq \sigma^{(t)}_{K,N}(\theta)$, $\mathrm{CD}(K, N)$ implies $\mathrm{CD}^*(K, N)$ (and $\mathrm{CD}_{\mathrm{loc}}(K, N)$ implies $\mathrm{CD}^*_{\mathrm{loc}}(K, N)$). Further, it is known that the local CD-conditions are equivalent in the following sense:

When a mathematical condition $\varphi(K)$ is given for each $K \in \mathbb{R}$, we say that an mathematical object P satisfies $\varphi(K-)$ if P satisfies $\varphi(K')$ for all $K' < K$.

Theorem 6.2 [Bacher and Sturm 2010, Proposition 5.5]. *Let $K, N \in \mathbb{R}$ with $N \geq 1$ and let (M, m) be a metric measure space. Then, (M, m) satisfies* $\mathrm{CD}^*_{\mathrm{loc}}(K-, N)$ *if and only if it satisfies* $\mathrm{CD}_{\mathrm{loc}}(K-, N)$.

There is another notion of a lower Ricci curvature bound in metric measure spaces which is called the *measure contraction property*, denoted by $\mathrm{MCP}(K, N)$. Since we do not use its theory to prove Theorem 1.8 in this paper, we omit its definition. For the definition and theory, we refer to [Ohta 2007] and [Sturm 2006b].

A metric measure space (M, m) is called nonbranching if M is a geodesic space and is nonbranching in the sense that for any four points x, y, z_1, z_2 in M, if y is a common midpoint of x and z_1 and of x and z_2, then $z_1 = z_2$. It is known that a nonbranching metric measure space satisfying $\mathrm{CD}(K, N)$ satisfies $\mathrm{MCP}(K, N)$. Recently, Cavalletti and Sturm proved:

Theorem 6.3 [2012, Theorem 1.1]. *Let (M, m) be a nonbranching metric measure space. Let $K, N \in \mathbb{R}$ with $N \geq 1$. If (M, m) satisfies* $\mathrm{CD}_{\mathrm{loc}}(K, N)$, *then it satisfies* $\mathrm{MCP}(K, N)$.

Bishop–Gromov volume growth estimate. Let (M, m) be a metric measure space and $x \in \mathrm{supp}(m)$. We set

$$v_x(r) := m(B(x, r)).$$

For $K, N \in \mathbb{R}$ with $N > 1$, we define

$$\bar{v}_{K,N}(r) = \int_0^r \mathrm{sn}^{N-1}_{K/(N-1)}(t)\,dt.$$

A metric measure space (M, m) satisfies the *Bishop–Gromov volume growth estimate* $\mathrm{BG}(K, N)$ if for any $x \in \mathrm{supp}(m)$, the function

$$v_x(r)/\bar{v}_{K,N}(r)$$

is nonincreasing in $r \in (0, \infty)$ (with $r \leq \pi\sqrt{(N-1)/K}$ if $K > 0$).

Since $\bar{v}_{K,N}(r)$ is continuous in K, $\mathrm{BG}(K-, N)$ implies $\mathrm{BG}(K, N)$. The Bishop–Gromov volume growth estimate is implied by several lower Ricci curvature bounds, for instance the measure contraction property.

Theorem 6.4 [Ohta 2007, Theorem 5.1; Sturm 2006b, Remark 5.3]. *If (M, m) satisfies* $\mathrm{MCP}(K, N)$, *then it satisfies* $\mathrm{BG}(K, N)$.

Summarizing the above facts, we can state the following implications: Let $K, N \in \mathbb{R}$ with $N \geq 1$. For a nonbranching metric measure space (M, m),

$$
\begin{aligned}
\mathrm{CD}^*_{\mathrm{loc}}(K, N) \Longrightarrow \mathrm{CD}^*_{\mathrm{loc}}(K-, N) &\Longleftrightarrow \mathrm{CD}_{\mathrm{loc}}(K-, N) \\
&\Longrightarrow \mathrm{MCP}(K-, N) \Longrightarrow \mathrm{BG}(K-, N) \Longrightarrow \mathrm{BG}(K, N).
\end{aligned}
\tag{6-1}
$$

Universal covering space with lifted measure. Let X be a semilocally simply connected space. Then, there is a universal covering $\pi : Y \to X$. In addition, if X is a length space, then Y can also be considered as a length space. The map π becomes a local isometry.

In addition, we assume that (X, m) is a proper metric measure space. Let $\mathcal{V}$ be the family of all open sheets of the universal covering $\pi : Y \to X$. We define a set function $m_Y : \mathcal{V} \to [0, \infty]$ by

$$m_Y(V) = m(\pi(V)).$$

One can naturally extend m_Y to a Borel measure on Y. We also write this measure as m_Y, and call it the *lift* of m. Since m is locally finite, so is m_Y.

In general, for a geodesic $\Gamma : [0, 1] \to \mathcal{P}_2(M)$, if $\Gamma(0)$ and $\Gamma(1)$ are supported on $U(x, r)$ for some $x \in X$ and $r > 0$, then $\Gamma(t)$ is supported on $U(x, 2r)$ for every $t \in (0, 1)$ [Sturm 2006a, Lemma 2.11]. Therefore, we obtain:

Proposition 6.5 [Bacher and Sturm 2010, Theorem 7.10]. *The local* (*reduced*) *curvature-dimension condition is inherited by the lift. Namely*, *let* $K, N \in \mathbb{R}$ *with* $N \geq 1$ *and let* (X, m) *and* (Y, m_Y) *be as above. If* (X, m) *satisfies* $\mathrm{CD}_{\mathrm{loc}}(K, N)$ (*resp.* $\mathrm{CD}^*_{\mathrm{loc}}(K, N)$), *then* (Y, m_Y) *also satisfies* $\mathrm{CD}_{\mathrm{loc}}(K, N)$ (*resp.* $\mathrm{CD}^*_{\mathrm{loc}}(K, N)$).

Proof of Theorem 1.8. Let X be an n-dimensional compact orientable Alexandrov space without boundary. Let m be a locally finite Borel measure on X with full support. We assume that (X, m) satisfies $\mathrm{CD}^*_{\mathrm{loc}}(K, N)$ for $K < 0$ and $N \geq 1$. By Proposition 6.5, the universal covering Y of X with lift m_Y of m also satisfies $\mathrm{CD}^*_{\mathrm{loc}}(K, N)$. And, Y is an n-dimensional Alexandrov space. Since m has full support, so does m_Y. By the implication (6-1), (Y, m_Y) satisfies $\mathrm{BG}(K, N)$. Therefore, as mentioned in the preface of this section, the original proof of Gromov's theorem relying on the Bishop–Gromov volume comparison works in our setting (see [Gromov 1982, §2; Yamaguchi 1997, Appendix]). Hence, we can prove Theorem 1.8 with a similar such an argument. For undefined terms appearing and for facts used in the following argument, we refer to [Gromov 1982; Yamaguchi 1997].

Let $\mathcal{M}$ (resp. $\mathcal{M}_+$) be the Banach space (resp. the set) of all finite signed (resp. positive) Borel measures on Y, where $\mathcal{M}$ is equipped with the norm $\|\mu\| = \int_Y d|\mu| \in [0, \infty)$. Due to the general theory established in [Gromov 1982, §2] and [Yamaguchi 1997, Appendix], if a differentiable averaging operator $S : Y \to \mathcal{M}_+$ exists, then for any $\alpha \in H_n(X)$,

$$\|\alpha\|_1 \le n!\,(\mathscr{L}[S])^n \operatorname{mass}(\alpha) \tag{6-2}$$

holds. Here, the value $\mathscr{L}[S]$ is defined as follows: For $y \in Y$,

$$\mathscr{L}S_y = \limsup_{z \to y} \frac{\|S(z) - S(y)\|}{d(z, y)} \quad \text{and} \quad \mathscr{L}[S] = \sup_{y \in Y} \frac{\mathscr{L}S_y}{\|S(y)\|}.$$

We recall a concrete construction of a differentiable averaging operator. For $R > 0$ and $y \in Y$, we set $S_R(y) \in \mathscr{M}_+$ to be

$$S_R(y) = 1_{B(y,R)} \cdot m_Y.$$

Here, 1_A is the characteristic function of $A \subset Y$. For $\epsilon > 0$, we define $S_{R,\epsilon} : Y \to \mathscr{M}_+$ by

$$S_{R,\epsilon}(y) = \frac{1}{\epsilon} \int_{R-\epsilon}^{R} S_{R'}(y)\, dR'.$$

Its norm is $\|S_{R,\epsilon}(y)\| = (1/\epsilon) \int_{R-\epsilon}^{R} v_y(R')\, dR'$ and is not less than $v_y(R - \epsilon)$. Here, $v_z(r) = m_Y(B(z, r))$ for $z \in Y$ and $r > 0$. Given the Lipschitz function $\psi = \psi_{R,\epsilon} : [0, \infty) \to [0, 1]$ defined by

$$\psi(t) = \begin{cases} 1 & \text{if } t \le R - \epsilon, \\ (R - t)/\epsilon & \text{if } t \in [R - \epsilon, R], \\ 0 & \text{if } t \ge R, \end{cases}$$

we can write $S_{R,\epsilon}(y) = \psi(d(y, \cdot))\, m_Y$ for any $y \in Y$.

We can check $S_{R,\epsilon}$ is a differentiable averaging operator as follows: Since m_Y is $\pi_1(X)$-invariant, the maps S_R and $S_{R,\epsilon}$ are $\pi_1(X)$-equivariant. Since m is absolutely continuous in $\mathscr{H}^n_X$, so is m_Y in $\mathscr{H}^n_Y$. One can check that $S_{R,\epsilon}$ is differentiable m_Y-almost everywhere with respect to the differentiable structure of Y, where the differentiable structure on Alexandrov spaces are defined by Otsu and Shioya [1994]. Indeed, the differential $D_y S_{R,\epsilon}(\gamma^+(0))$ of $S_{R,\epsilon}$ at y along a geodesic γ starting from $y = \gamma(0)$ is calculated by

$$\big(D_y S_{R,\epsilon}(\gamma^+(0))\big)(A) = \frac{1}{\epsilon} \int_{A \cap A(y;\, R-\epsilon, R)} \cos \angle(z'_y, \gamma^+(0))\, dm_Y(z)$$

for any Borel set $A \subset Y$, where $A(z; r, r')$ is the annulus around $z \in Y$ of inner radius r and outer radius r', for $r \le r'$.

To estimate $\mathscr{L}[S_{R,\epsilon}]$, we use the Bishop–Gromov volume growth estimate as follows. We obtain

$$\mathscr{L}(S_{R,\epsilon})_y = \sup_{\xi \in \Sigma_y} \|D_y S_{R,\epsilon}(\xi)\| \le \frac{m_Y(A(y;\, R-\epsilon, R))}{\epsilon}.$$

It follows from $\mathrm{BG}(K, N)$ that

$$\frac{\mathcal{L}(S_{R,\epsilon})_y}{\|S_{R,\epsilon}(y)\|} \le \frac{v_y(R) - v_y(R-\epsilon)}{\epsilon \cdot v_y(R-\epsilon)} \le C_{K,N}(R,\epsilon).$$

Here, setting

$$\bar{v}(R') = \bar{v}_{K,N}(R') = \int_0^{R'} \mathrm{sn}_{K/(N-1)}^{N-1}(t)\,dt,$$

we have

$$C_{K,N}(R,\epsilon) := \frac{\bar{v}(R) - \bar{v}(R-\epsilon)}{\epsilon \cdot \bar{v}(R-\epsilon)}.$$

Since $\mathrm{mass}([X]) = \mathcal{H}^n(X)$ [Yamaguchi 1997, Theorem 0.1], by using (6-2) and by letting $\epsilon \to 0$ and $R \to \infty$, we obtain

$$\|X\| \le n!\sqrt{-K(N-1)}^{\,n}\,\mathcal{H}^n(X).$$

This completes the proof of Theorem 1.8. □

Remark 6.6. By [Petrunin 2011] and [Zhang and Zhu 2010], it is known that for an n-dimensional Alexandrov space X of curvature $\ge \kappa$, the metric measure space $(X, \mathcal{H}^n)$ satisfies the curvature-dimension condition $\mathrm{CD}((n-1)\kappa, n)$. Therefore, Corollary 1.7 is implied by Theorem 1.8.

If there exists a compact orientable n-dimensional Alexandrov space X, without boundary, of curvature $\ge \kappa$, with $\kappa < 0$, which has nonnegative Ricci curvature with respect to some reference measure m such that $m \ll \mathcal{H}^n$ and $\mathrm{supp}(m) = X$, then Theorem 1.8 yields $\|X\| = 0$.

Acknowledgements

Mitsuishi is supported by Research Fellowship for Young Scientists of the Japan Society for the Promotion of Science.

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Received February 17, 2013. Revised August 21, 2013.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 270 No. 2 August 2014

0030-8730(201408)270:2;1-2