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AN EXPLICIT MAJORANA REPRESENTATION OF THE GROUP $3^2{:}2$ OF $3C$-PURE TYPE

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We study a coset vertex operator algebra (VOA) $\widetilde{W}$ in the lattice VOA $V_{E_8^3}$. We show that the coset VOA $\widetilde{W}$ is generated by nine Ising vectors such that any two Ising vectors generate a $3C$ subVOA U_{3C}, and the group generated by the corresponding Miyamoto involutions has shape $3^2{:}2$. This gives an explicit example for Majorana representations of the group $3^2{:}2$ of $3C$-pure type.

1. Introduction

A vertex operator algebra (VOA) $V = \bigoplus_{n=0}^{\infty} V_n$ is said to be of *moonshine type* if $\dim(V_0) = 1$ and $V_1 = 0$. In this case, the weight-2 subspace V_2 has a commutative nonassociative product defined by $a \cdot b = a_1 b$ for $a, b \in V_2$ and it has a symmetric invariant bilinear form $\langle\,\cdot\,,\,\cdot\,\rangle$ given by $\langle a, b\rangle \mathbb{1} = a_3 b$ for $a, b \in V_2$ [Frenkel et al. 1988]. The algebra $(V_2, \cdot, \langle\,\cdot\,,\,\cdot\,\rangle)$ is often called the *Griess algebra* of V. An element $e \in V_2$ is called an *Ising vector* if $e \cdot e = 2e$ and the subVOA generated by e is isomorphic to the simple Virasoro VOA $L(\frac{1}{2}, 0)$ of central charge $\frac{1}{2}$. In [Miyamoto 1996], the basic properties of Ising vectors have been studied. Miyamoto also gave a simple method to construct involutive automorphisms of a VOA V from Ising vectors. These automorphisms are often called Miyamoto involutions. When V is the famous Moonshine VOA $V^\natural$, Miyamoto [2004] showed that there is a one-to-one correspondence between the $2A$-involutions of the Monster group and Ising vectors in $V^\natural$ (see also [Höhn 2010]). This correspondence is very useful for studying some mysterious phenomena of the Monster group and many problems about $2A$-involutions in the Monster group may also be translated into questions about Ising vectors. For example, McKay's observation on the affine E_8-diagram was studied in [Lam et al. 2007] using Miyamoto involutions and certain VOAs generated by two Ising vectors were constructed. Nine VOAs were constructed, denoted by U_{1A}, U_{2A}, U_{2B}, U_{3A}, U_{3C}, U_{4A}, U_{4B}, U_{5A}, and U_{6A} because of their connection to the 6-transposition property of the Monster group (see [ibid., Introduction]),

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where $1A, 2A, \ldots, 6A$ are the labels for certain conjugacy classes of the Monster as denoted in [Conway et al. 1985]. In [Sakuma 2007], Griess algebras generated by two Ising vectors contained in a moonshine-type VOA over $\mathbb{R}$ with a positive definite invariant form are classified. There are also nine possible cases, and they correspond exactly to the Griess algebras $\mathcal{G}U_{nX}$ of the nine VOAs U_{nX}, for nX in $\{1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A, 6A\}$. Therefore, there is again a correspondence between the dihedral subgroups generated by two $2A$-involutions, up to conjugacy and the Griess subalgebras generated by two Ising vectors in $V^\natural$, up to isomorphism. It is also conjectured that the subVOA generated by two Ising vectors is isomorphic to one of the U_{nX}, for $nX \in \{1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A, 6A\}$. However, this conjecture is still open except for the cases $1A$, $2A$, $2B$, $3A$, and $4B$.

Motivated by [Sakuma 2007], Ivanov [2009] axiomatized the properties of Ising vectors and introduced the notion of Majorana representations for finite groups. Ivanov and his research group also initiated a program on classifying the Majorana representations for various finite groups [Ivanov et al. 2010; Ivanov 2011a; 2011b; Ivanov and Seress 2012]. In particular, the famous 196884-dimensional Monster Griess algebra constructed by Griess [1982] is a Majorana representation of the Monster simple group. In fact, most known examples of Majorana representations are constructed as certain subalgebras of this Monster Griess algebra.

In this article, we construct explicitly a moonshine-type VOA $\widetilde{W}$ in the lattice VOA $V_{E_8^3}$. We show that the VOA $\widetilde{W}$ is generated by nine Ising vectors such that (1) any two of them generate a $3C$ subVOA U_{3C}; and (2) the group generated by the corresponding Miyamoto involutions has the shape $3^2{:}2$. Thus, we obtain an example for a Majorana representation of the group $3^2{:}2$ of $3C$-pure type. Recall that the centralizer of a $3C$-element in the Monster is isomorphic to $3 \times \mathrm{Th}$, where Th is the Thompson simple group [Conway et al. 1985]. The Thompson group Th has exactly three conjugacy classes of order 3 and by the character table, one can show that the Th conjugacy classes $3A$, $3B$, $3C$ are of the classes $3A$, $3B$, $3B$ in the Monster, respectively (see [ibid.] and [Wilson 1988, Section 4]). Therefore, there are no $3C$-pure 3^2 subgroups in the Monster and hence the VOA that we constructed cannot be embedded into the Moonshine VOA.

Our method is essentially a combination of the construction of the so-called dihedral subVOA from [Lam et al. 2007] and the construction of EE_8 pairs from [Griess and Lam 2011]. In fact, it is quite straightforward to find Ising vectors satisfying our hypotheses. The main difficulty is to show that the subVOA generated by these Ising vectors has zero weight-1 subspace.

The organization of this article is as follows. In Section 2, we recall some basic definitions and notation. We also review the structure of the so-called $3C$-algebra from [Lam et al. 2005; 2007]. In Section 3, we give an explicit construction of a coset subVOA $\widetilde{W}$ in the lattice $V_{E_8^3}$. We also construct explicitly several

Ising vectors satisfying our main hypotheses and show that the subVOA W they generate is of moonshine type. In Section 4, we show that the VOA W is isomorphic to the commutant subVOA $\widetilde{W} = \mathrm{Com}_{V_{E_8^3}}(L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0))$ using the theory of parafermion VOA. The decomposition of W as a sum of irreducible modules of the parafermion VOA $K(\mathfrak{sl}_3(\mathbb{C}), 9)$ is also obtained. In Section 5, we give several structural results about Griess algebras generated by Ising vectors. We show that the Griess algebra generated by Ising vectors such that the subgroup generated by the corresponding Miyamoto involutions has the shape $3^2{:}2$ and is of $3C$-pure type is uniquely determined, up to isomorphisms. We also show that the VOA generated by these Ising vectors has central charge 4 and has a full subVOA isomorphic to $L(\frac{1}{2},0)\otimes L(\frac{21}{22},0)\otimes L(\frac{28}{11},0)$. In the Appendix, we explain several results which are used to show that $\dim(\widetilde{W}_2) = 9$.

2. Preliminaries

First we will recall some definitions and review several basic facts.

Definition 2.1. Let V be a VOA. A bilinear $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle$ form on V is said to be *invariant* (or *contragredient*; see [Frenkel et al. 1993]) if

$$\langle\langle Y(a,z)u, v\rangle\rangle = \langle\langle u, Y(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})v\rangle\rangle \tag{2-1}$$

for any $a, u, v \in V$.

Definition 2.2. Let V be a VOA over $\mathbb{C}$. A *real form* of V is a subVOA $V_{\mathbb{R}}$ of V over $\mathbb{R}$ (with the same vacuum and Virasoro elements) such that $V = V_{\mathbb{R}} \otimes \mathbb{C}$. A real form $V_{\mathbb{R}}$ is said to be *positive definite* if the invariant form $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle$ restricted to $V_{\mathbb{R}}$ is real-valued and positive definite.

Definition 2.3. Let V be a VOA. An element $v \in V_2$ is called a *simple Virasoro vector* of central charge c if the subVOA $\mathrm{Vir}(v)$ generated by e is isomorphic to the simple Virasoro VOA $L(c,0)$ of central charge c.

Definition 2.4. A simple Virasoro vector of central charge $\frac{1}{2}$ is called an *Ising vector*.

Remark 2.5. It is well known that the VOA $L(\frac{1}{2},0)$ is rational and has exactly three irreducible modules $L(\frac{1}{2},0)$, $L(\frac{1}{2},\frac{1}{2})$, and $L(\frac{1}{2},\frac{1}{16})$ (see [Dong et al. 1994; Miyamoto 1996]).

Remark 2.6. Let V be a VOA and let $e \in V$ be an Ising vector. Then we have the decomposition

$$V = V_e(0) \oplus V_e(\tfrac{1}{2}) \oplus V_e(\tfrac{1}{16}),$$

where $V_e(h)$ denotes the sum of all irreducible $\mathrm{Vir}(e)$-submodules of V isomorphic to $L(\frac{1}{2},h)$ for $h \in \{0, \frac{1}{2}, \frac{1}{16}\}$.

Theorem 2.7 [Miyamoto 1996]. *The linear map $\tau_e : V \to V$ defined by*

$$\tau_e := \begin{cases} 1 & \text{on } V_e(0) \oplus V_e(\frac{1}{2}), \\ -1 & \text{on } V_e(\frac{1}{16}), \end{cases} \tag{2-2}$$

is an automorphism of V.

Remark 2.8. On the fixed point subspace V^{τ_e} of τ_e, we have $V^{\tau_e} = V_e(0) \oplus V_e(\frac{1}{2})$. The linear map $\sigma_e : V^{\tau_e} \to V^{\tau_e}$ which acts as 1 on $V_e(0)$ and -1 on $V_e(\frac{1}{2})$ also defines an automorphism of V^{τ_e} [ibid.]. Nevertheless, we do not need this fact in this article.

The 3C-algebra. We recall the properties of the $3C$-algebra U_{3C} from [Lam et al. 2005, Section 3.9] (see also [Sakuma 2007]).

Lemma 2.9. *Let $U = U_{3C}$ be the $3C$-algebra. Then*:

(1) *$U_1 = 0$ and U is generated by its weight-2 subspace U_2 as a VOA.*

(2) *$\dim U_2 = 3$ and it is spanned by three Ising vectors.*

(3) *There exist exactly three Ising vectors in U_2, say, e^0, e^1, e^2. Moreover, we have*

$$(e^i)_1(e^j) = \tfrac{1}{32}(e^i + e^j - e^k) \quad \text{and} \quad \langle e^i, e^j \rangle = \tfrac{1}{2^8}$$

for $i \neq j$ and $\{i, j, k\} = \{0, 1, 2\}$.

(4) *Let $g = \tau_{e^0}\tau_{e^1}$. Then g has order 3. Moreover, $e^1 = ge^0$ and $e^2 = g^2e^0 = ge^1$.*

(5) *The Virasoro element of U is given by*

$$\tfrac{32}{33}(e^0 + e^1 + e^2).$$

(6) *Let $a = \frac{32}{33}(e^0 + e^1 + e^2) - e^0$. Then a is a simple Virasoro vector of central charge $\frac{21}{22}$. Moreover, the subVOA generated by e^0 and a is isomorphic to $L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0)$.*

3. Commutant subVOAs in $V_{E_8 \perp E_8 \perp E_8}$

In this section, we shall construct explicitly a VOA $\widetilde{W}$ inside the lattice VOA $V_{E_8 \perp E_8 \perp E_8}$ such that (1) $\widetilde{W}$ is generated by nine Ising vectors and any two Ising vectors generate a $3C$ subVOA U_{3C}; and (2) the group generated by the corresponding Miyamoto involutions has the shape $3^2{:}2$.

Our notation for the lattice vertex operator algebra

$$V_L = M(1) \otimes \mathbb{C}\{L\} \tag{3-1}$$

associated with a positive definite even lattice L is standard [Frenkel et al. 1988]. In particular, $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ is an abelian Lie algebra and we extend the bilinear form to $\mathfrak{h}$ by $\mathbb{C}$-linearity. Also, $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}k$ is the corresponding affine algebra and $\mathbb{C}k$ is the one-dimensional center of $\hat{\mathfrak{h}}$. The subspace $M(1)$ given by $\mathbb{C}[\alpha_i(n) \mid 1 \le i \le d, n < 0]$ for a basis $\{\alpha_1, \ldots, \alpha_d\}$ of $\mathfrak{h}$, where $\alpha(n) = \alpha \otimes t^n$, is the unique irreducible $\hat{\mathfrak{h}}$-module such that $\alpha(n) \cdot 1 = 0$ for all $\alpha \in \mathfrak{h}$ and n nonnegative, and $k = 1$. Also, $\mathbb{C}\{L\} = \mathrm{span}\{e^\beta \mid \beta \in L\}$ is the twisted group algebra of the additive group L such that $e^\beta e^\alpha = (-1)^{\langle \alpha, \beta \rangle} e^\alpha e^\beta$ for any $\alpha, \beta \in L$. The vacuum vector $\mathbb{1}$ of V_L is $1 \otimes e^0$ and the Virasoro element ω_L is $\frac{1}{2} \sum_{i=1}^{d} \beta_i(-1)^2 \cdot \mathbb{1}$ where $\{\beta_1, \ldots, \beta_d\}$ is an orthonormal basis of $\mathfrak{h}$. For the explicit definition of the corresponding vertex operators, we shall refer to [ibid.] for details.

Definition 3.1. Let A and B be integral lattices with the inner products $\langle\, , \rangle_A$ and $\langle\, , \rangle_B$, respectively. The *tensor product of the lattices* A and B is defined to be the integral lattice which is isomorphic to $A \otimes_{\mathbb{Z}} B$ as a $\mathbb{Z}$-module and has the inner product given by

$$\langle \alpha \otimes \beta, \alpha' \otimes \beta' \rangle = \langle \alpha, \alpha' \rangle_A \cdot \langle \beta, \beta' \rangle_B, \quad \text{for any } \alpha, \alpha' \in A,\ \beta, \beta' \in B.$$

We simply denote the tensor product of the lattices A and B by $A \otimes B$.

$\sqrt{2}E_8$-sublattices. Let $L = E_8 \perp E_8 \perp E_8$ be the orthogonal sum of 3 copies of the root lattice of type E_8. Set

$$\begin{aligned} M &= \{(\alpha, -\alpha, 0) \mid \alpha \in E_8\} < L, \\ N &= \{(0, \alpha, -\alpha) \mid \alpha \in E_8\} < L. \end{aligned} \tag{3-2}$$

Then $M \cong N \cong \sqrt{2}E_8$ and $M + N \cong A_2 \otimes E_8$ (see [Griess and Lam 2011]). We also define

$$E := \mathrm{Ann}_L(M + N) = \{\beta \in L \mid \langle \beta, \beta' \rangle = 0 \text{ for all } \beta' \in M + N\}. \tag{3-3}$$

Note that $E = \{(\alpha, \alpha, \alpha) \mid \alpha \in E_8\} < L$ and there is a third $\sqrt{2}E_8$-sublattice

$$\widetilde{N} = \{(\alpha, 0, -\alpha) \mid \alpha \in E_8\} < M + N.$$

We shall fix a (bilinear) 2-cocycle $\varepsilon_0 : E_8 \times E_8 \to \mathbb{Z}_2$ such that

$$\begin{aligned} \varepsilon_0(\alpha, \alpha) &\equiv \tfrac{1}{2} \langle \alpha, \alpha \rangle \bmod 2, \\ \varepsilon_0(\alpha, \beta) - \varepsilon_0(\beta, \alpha) &\equiv \langle \alpha, \beta \rangle \bmod 2, \end{aligned} \tag{3-4}$$

for all $\alpha, \beta \in E_8$. Note that such a 2-cocycle exists (see [Frenkel et al. 1988, (6.1.27)–(6.1.29)]). Moreover, $e^\alpha e^{-\alpha} = -e^0$ for any $\alpha \in E_8$ such that $\langle \alpha, \alpha \rangle = 2$.

We shall extend ε_0 to L by defining

$$\varepsilon_0((\alpha, \alpha', \alpha''), (\beta, \beta', \beta'')) = \varepsilon_0(\alpha, \beta) + \varepsilon_0(\alpha', \beta') + \varepsilon_0(\alpha'', \beta'').$$

It is easy to check by direct calculations that ε_0 is trivial on M, N, or $\widetilde{N}$.

Affine vertex operator algebras. We recall the notion of affine vertex operator algebras [Frenkel and Zhu 1992; Dong and Lepowsky 1993]. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra and $\hat{\mathfrak{g}}$ the affine Kac–Moody Lie algebra associated with $\mathfrak{g}$. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a set of simple roots and θ the highest root. Let Q be the root lattice of $\mathfrak{g}$. For any positive integer k, we set

$$P_+^k(\mathfrak{g}) = \{\Lambda \in \mathbb{Q} \otimes_{\mathbb{Z}} Q \mid \langle \alpha_i, \Lambda\rangle \in \mathbb{Z}_{\geq 0} \text{ for all } i = 1, \dots, n \text{ and } \langle \theta, \Lambda\rangle \leq k\},$$

the set of dominant integral weights for $\mathfrak{g}$ with level k.

Let $L_{\hat{\mathfrak{g}}}(k, \Lambda)$ be the irreducible module of $\hat{\mathfrak{g}}$ with highest weight Λ and level k. Then $L_{\hat{\mathfrak{g}}}(k, 0)$ forms a simple VOA with the Virasoro element given by the Sugawara construction

$$\Omega_{\mathfrak{g},k} = \frac{1}{2(k+h^\vee)} \sum (u_i)_{-1} u^i, \tag{3-5}$$

where $h^\vee$ is the dual Coxeter number, $\{u_i\}$ is a basis of $\mathfrak{g}$ and $\{u^i := (u_i)^*\}$ is the dual basis of $\{u_i\}$ with respect to the normalized Killing form (see [Frenkel and Zhu 1992]). Moreover, the central charge of $L_{\hat{\mathfrak{g}}}(k, 0)$ is

$$\frac{k \dim \mathfrak{g}}{k+h^\vee}. \tag{3-6}$$

A commutant subVOA. Consider the lattice VOA

$$V_L \cong V_{E_8} \otimes V_{E_8} \otimes V_{E_8}$$

and let $\boldsymbol{a}$ be an element of E_8 such that

$$K := \{\beta \in E_8 \mid \langle \beta, \boldsymbol{a}\rangle \in 3\mathbb{Z}\} \cong A_8.$$

Then, we have an embedding

$$V_{K\perp K\perp K} \cong V_K \otimes V_K \otimes V_K \hookrightarrow V_L.$$

It is also well known that $V_K \cong V_{A_8}$ is an irreducible level-1 representation of the affine Lie algebra $\widehat{\mathfrak{sl}}_9(\mathbb{C})$ [Frenkel et al. 1988]. Moreover, the weight-1 subspace $(V_K)_1$ is a simple Lie algebra isomorphic to $\mathfrak{sl}_9(\mathbb{C})$.

Let $\eta_i : K \to K \perp K \perp K$, $i = 1, 2, 3$, be the embedding of K into the i-th direct summand of $K \perp K \perp K$, i.e.,

$$\eta_1(\alpha) = (\alpha, 0, 0), \quad \eta_2(\alpha) = (0, \alpha, 0), \quad \eta_3(\alpha) = (0, 0, \alpha),$$

for any $\alpha \in K$.

Notation 3.2. For any $\alpha \in K(2) := \{\alpha \in K \mid \langle \alpha, \alpha\rangle = 2\}$, set

$$H_\alpha = (\alpha, \alpha, \alpha)(-1) \cdot \mathbb{1},$$
$$E_\alpha = e^{\eta_1(\alpha)} + e^{\eta_2(\alpha)} + e^{\eta_3(\alpha)}.$$

Then $\{H_\alpha, E_\alpha \mid \alpha \in K(2)\}$ generates a subVOA isomorphic to the affine VOA $L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0)$ in V_L (see [Frenkel and Zhu 1992; Dong and Lepowsky 1993, Proposition 13.1]). Moreover, the Virasoro element of $L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0)$ is given by

$$\Omega = \frac{1}{2(3+9)}\Bigg[\sum_{k=1}^{8}(h^k,h^k,h^k)(-1)^2\cdot\mathbb{1} + \sum_{\alpha\in K(2)}(E_\alpha)_{-1}(-E_{-\alpha})\Bigg],$$

where $\{h^1,\dots,h^8\}$ is an orthonormal basis of $K\otimes\mathbb{C} = E_8\otimes\mathbb{C}$. Note that the dual vector of E_α is $-E_{-\alpha}$.

Lemma 3.3. *Let M, N and E be defined as in (3-2) and (3-3) and denote the Virasoro element of a lattice VOA V_S by ω_S. Then we have*

$$\Omega = \omega_E + \tfrac{3}{4}\omega_{M+N} - \tfrac{1}{12}\sum_{\substack{\alpha\in K(2)\\ 1\le i,j\le 3,\, i\ne j}} e^{\eta_i(\alpha)-\eta_j(\alpha)}.$$

Proof. Let $\{h^1,\dots,h^8\}$ be an orthonormal basis of $A_8\otimes\mathbb{C} = E_8\otimes\mathbb{C}$. Then

$$\begin{aligned}\Omega &= \frac{1}{2(3+9)}\Bigg[\sum_{k=1}^{8}(h^k,h^k,h^k)(-1)^2\cdot\mathbb{1} \\ &\qquad - \sum_{\alpha\in K(2)}\big(e^{\eta_1(\alpha)}+e^{\eta_2(\alpha)}+e^{\eta_3(\alpha)}\big)_{-1}\big(e^{-\eta_1(\alpha)}+e^{-\eta_2(\alpha)}+e^{-\eta_3(\alpha)}\big)\Bigg] \\ &= \frac{1}{24}\Bigg[6\omega_E + \sum_{\alpha\in K(2)}\sum_{i=1}^{3}\tfrac{1}{2}\big(\eta_i(\alpha)(-2)\cdot\mathbb{1}+\eta_i(\alpha)(-1)^2\cdot\mathbb{1}\big) - 2\sum_{\substack{\alpha\in K(2)\\ 1\le i,j\le 3\\ i\ne j}} e^{\eta_i(\alpha)-\eta_j(\alpha)}\Bigg] \\ &= \tfrac{1}{4}\omega_E + \tfrac{18}{24}\omega_L - \tfrac{1}{12}\sum_{\substack{\alpha\in K(2)\\ 1\le i,j\le 3,\, i\ne j}} e^{\eta_i(\alpha)-\eta_j(\alpha)}.\end{aligned}$$

Since $\omega_L = \omega_{M+N} + \omega_E$, we have

$$\Omega = \omega_E + \tfrac{3}{4}\omega_{M+N} - \tfrac{1}{12}\sum_{\substack{\alpha\in K(2)\\ 1\le i,j\le 3,\, i\ne j}} e^{\eta_i(\alpha)-\eta_j(\alpha)}$$

as desired. □

Theorem 3.4. *Let*

$$\widetilde{W} = \mathrm{Com}_{V_L}(L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0)) = \{v\in V_L \mid x_n v = 0 \textit{ for all } x\in L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0),\, n\ge 0\}$$

be the commutant subVOA of $L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0)$ in V_L. Then the central charge of $\widetilde{W}$ is 4. Moreover, $\widetilde{W}_1 = 0$.

Proof. By (3-6), the central charge of $L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0)$ is $3(80)/(3+9) = 20$. Hence, the central charge of $\widetilde{W} = \mathrm{Com}_{V_L}(L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0))$ is 4 $(= 24-20)$.

We now show that $\widetilde{W}_1 = 0$. Since $h(-1)\cdot\mathbb{1} \in L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0)$ for all $h \in E$,

$$\widetilde{W} = \mathrm{Com}_{V_L}\big(L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0)\big) \subset V_{M+N}.$$

Therefore, it suffices to show $\widetilde{W} \cap (V_{M+N})_1 = 0$.

Recall that $M+N \cong A_2 \otimes E_8$ has no roots. Thus,

$$(V_{M+N})_1 = \mathrm{span}_{\mathbb{C}}\{h(-1)\cdot\mathbb{1} \mid h \in (M+N)\otimes\mathbb{C}\}.$$

However, by Lemma 3.3,

$$\Omega_1 h(-1)\cdot\mathbb{1} = \Bigg(\omega_E + \tfrac{3}{4}\omega_{M+N} - \tfrac{1}{12}\sum_{\substack{\alpha\in K(2)\\ 1\le i,j\le 3,\, i\ne j}} e^{\eta_i(\alpha)-\eta_j(\alpha)}\Bigg)_1 h(-1)\cdot\mathbb{1} = \tfrac{3}{4}h(-1)\cdot\mathbb{1} \neq 0$$

for any $0 \neq h \in (M+N)\otimes\mathbb{C}$. Thus, $\widetilde{W} \cap (V_{M+N})_1 = 0$ and we have $\widetilde{W}_1 = 0$. □

Ising vectors. Next we shall define explicitly some Ising vectors in V_L.

Definition 3.5. Let $\boldsymbol{a}$ be an element of E_8 such that

$$K = \{\beta \in E_8 \mid \langle \beta, \boldsymbol{a}\rangle \in 3\mathbb{Z}\} \cong A_8.$$

Set $\tilde{\boldsymbol{a}} = (\boldsymbol{a}, -\boldsymbol{a}, 0)$ and define an automorphism ρ of V_L by

$$\rho = \exp\Big(\frac{2\pi i}{3}\tilde{\boldsymbol{a}}(0)\Big).$$

Then ρ has order 3 and the fixed point subspace $V_M^{\rho} \cong V_{\sqrt{2}A_8}$.

Notation 3.6. Let M and N be defined as in (3-2). Set

$$\begin{aligned}
e &:= e_M = \tfrac{1}{16}\omega_M + \tfrac{1}{32}\sum_{\alpha\in M(4)} e^{\alpha},\\
f &:= e_N = \tfrac{1}{16}\omega_N + \tfrac{1}{32}\sum_{\alpha\in N(4)} e^{\alpha},\\
e_{\widetilde{N}} &:= \tfrac{1}{16}\omega_{\widetilde{N}} + \tfrac{1}{32}\sum_{\alpha\in \widetilde{N}(4)} e^{\alpha},\\
e' &:= \rho(e).
\end{aligned}$$

It is shown in [Dong et al. 1998] that e, f and $e_{\widetilde{N}}$ are Ising vectors and hence $e' = \rho(e)$ is also an Ising vector (see also [Lam et al. 2005; 2007]).

The following lemma can be proved by direct calculations (see [Lam et al. 2005; 2007; Griess and Lam 2011]).

Lemma 3.7. *We have* $\langle e, f\rangle = \langle e, e'\rangle = \langle f, e'\rangle = 1/2^8$. *Moreover, the subVOAs* $\mathrm{VOA}(e, f)$, $\mathrm{VOA}(e, g)$, $\mathrm{VOA}(f, g)$ *generated by* $\{e, f\}$, $\{e, e'\}$, *and* $\{f, e'\}$, *are isomorphic to the* $3C$*-algebra* U_{3C}. *We also have* $e_M \cdot e_N = \frac{1}{32}(e_M + e_N - e_{\widetilde{N}})$, *and hence* $\tau_e(f) = e_{\widetilde{N}}$.

Notation 3.8. Let $W := \mathrm{VOA}(e, f, e')$ be the subVOA generated by e, f, and e'. We also denote $h = \tau_e\tau_f$ and $g = \tau_e\tau_{e'}$. Then g and h both have order 3. Note also that $e, f, e' \in V_{M+N}$ and thus $W < V_{M+N} \cong V_{A_2\otimes E_8}$.

Lemma 3.9. *The elements* g *and* h *commute as automorphisms of* W.

Proof. Recall that $g = \tau_e\tau_{e'} = \rho$ on V_L (see [Lam et al. 2007]). Also, $h(e) = f = e_N$ and $h^2(e) = e_{\widetilde{N}}$.

By a direct calculation, we have

$$hg(e) = hgh^{-1}h(e) = \rho^h(e_N),$$

where $\rho^h = h\rho h^{-1} = \exp\big(\frac{2\pi i}{3}(0, \boldsymbol{a}, -\boldsymbol{a})(0)\big)$.

Since $\langle(0, \beta, -\beta), (0, \boldsymbol{a}, -\boldsymbol{a})\rangle = 2\langle\beta, \boldsymbol{a}\rangle$ and $\langle(0, \beta, -\beta), (\boldsymbol{a}, -\boldsymbol{a}, 0)\rangle = -\langle\beta, \boldsymbol{a}\rangle$, we have

$$gh(e) = \rho(e_N) = \rho^h(e_N) = hg(e).$$

Similarly, we have

$$hg(e') = hg^2(e) = (\rho^h)^2(e_N), \quad gh(e') = ghg(e) = g(\rho^h(e_N)) = (\rho^h)^2(e_N)$$

and

$$hg(f) = hgh(e) = (hgh^2)h^2(e) = \rho^h(e_{\widetilde{N}}), \quad gh(f) = g(e_{\widetilde{N}}) = \rho(e_{\widetilde{N}}).$$

Hence $gh = hg$ on W. □

Notation 3.10. For any $0 \le i, j \le 2$, denote

$$e^{i,j} = g^i h^j(e).$$

In particular, we have

$$\begin{aligned} e^{0,0} &= e_M, & e^{0,1} &= e_N, & e^{0,2} &= e_{\widetilde{N}}, \\ e^{1,0} &= \rho e_M, & e^{1,1} &= \rho e_N, & e^{1,2} &= \rho e_{\widetilde{N}}, \\ e^{2,0} &= \rho^2 e_M, & e^{2,1} &= \rho^2 e_N, & e^{2,2} &= \rho^2 e_{\widetilde{N}}. \end{aligned}$$

Remark 3.11. By the same methods as in [Lam et al. 2007; Griess and Lam 2011], it is quite straightforward to verify that $\langle e^{i,j}, e^{i'j'}\rangle = \frac{1}{2^8}$ whenever $(i, j) \neq (i', j')$.

Lemma 3.12. *Let G be the subgroup of* $\mathrm{Aut}(W)$ *generated by τ_e, τ_f and $\tau_{e'}$. Then $G = \langle g, h\rangle : \langle \tau_e\rangle$, where $\langle g, h\rangle$ is elementary abelian of order* 3^2 *and τ_e inverts g and h.*

Proof. By Lemma 3.9, we know that the group $\langle g, h\rangle$ generated by g and h is elementary abelian of order 3^2. Also, τ_e inverts g and h because $\tau_e g\tau_e = \tau_e(\tau_e\tau_{e'})\tau_e = \tau_{e'}\tau_e = g^{-1}$ and $\tau_e h\tau_e = \tau_e(\tau_e\tau_f)\tau_e = \tau_f\tau_e = h^{-1}$.

First, we shall prove that $\langle g, h\rangle$ is normal in G. By Lemma 3.9 we have $gh = hg$ and hence $\tau_f\tau_e\tau_{e'} = \tau_{e'}\tau_e\tau_f$. Thus $\tau_f h\tau_f = \tau_f\tau_e\tau_{e'}\tau_f = \tau_{e'}\tau_e\tau_f^2 = \tau_{e'}\tau_e = h^2 \in \langle g, h\rangle$. Similar computation gives that $\langle g, h\rangle$ is normal in G.

Next we show that $G = \langle g, h\rangle\langle\tau_e\rangle$. Recall that τ_e, τ_f and $\tau_{e'}$ are involutions. Thus every nonidentity element in G has the form

$$\tau_{a_1}\tau_{a_2}\cdots\tau_{a_k},$$

where $a_i = e$, f, or e' and $a_i \neq a_{i+1}$ for $i = 1, \ldots, k-1$.

Note also that $\tau_e\tau_{e'} = g$, $\tau_e\tau_f = h$, $\tau_f\tau_{e'} = h^{-1}g$, and g and h have order 3. Hence, $\tau_a\tau_{a'} \in \langle g, h\rangle$ for any $a, a' \in \{e, f, e'\}$. Therefore, $\tau_{a_1}\tau_{a_2}\cdots\tau_{a_k} \in \langle g, h\rangle$ if k is even and $\tau_{a_1}\tau_{a_2}\cdots\tau_{a_k} = (\tau_{a_1}\tau_{a_2}\cdots\tau_{a_k}\tau_e)\tau_e \in \langle g, h\rangle\langle\tau_e\rangle$ if k is odd. Thus we have $G = \langle g, h\rangle\langle\tau_e\rangle$.

Since $|\langle g, h\rangle| = 3^2$ and $|\langle\tau_e\rangle| = 2$, we get $\langle g, h\rangle\cap\langle\tau_e\rangle = 1$. Hence $G = \langle g, h\rangle : \langle\tau_e\rangle$ as desired. □

Lemma 3.13. *Let Ω be the Virasoro element of $L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3, 0)$. Then*

$$\Omega = \omega_L - \tfrac{8}{9}\sum_{0\le i,j\le 2} e^{i,j}.$$

Proof. By Lemma 3.3, we have

$$\Omega = \omega_E + \tfrac{3}{4}\omega_{M+N} - \tfrac{1}{12}\sum_{\substack{\alpha\in K(2)\\ 1\le i,j\le 3,\, i\neq j}} e^{\eta_i(\alpha)-\eta_j(\alpha)}.$$

Now let us set $\Delta^i := \{\beta \in E_8(2) \mid \langle \boldsymbol{a}, \beta\rangle = i \mod 3\mathbb{Z}\}$ for $i = 0, 1, 2$. Note that $\Delta^0 = K(2)$. Then we have

$$e^{0,0} = e_M = \tfrac{1}{16}\omega_M + \tfrac{1}{32}\sum_{i=0}^{2}\sum_{\alpha\in\Delta^i} e^{(\alpha,-\alpha,0)},$$

$$e^{1,0} = \rho e_M = \tfrac{1}{16}\omega_M + \tfrac{1}{32}\sum_{i=0}^{2}\sum_{\alpha\in\Delta^i} \xi^{2i} e^{(\alpha,-\alpha,0)},$$

$$e^{2,0} = \rho^2 e_M = \tfrac{1}{16}\omega_M + \tfrac{1}{32}\sum_{i=0}^{2}\sum_{\alpha\in\Delta^i} \xi^{i} e^{(\alpha,-\alpha,0)}.$$

Hence

$$\sum_{i=0}^{2} e^{i,0} = \big(1+\rho+\rho^2\big)e_M = \tfrac{3}{16}\omega_M + \tfrac{3}{32}\sum_{\alpha\in K(2)} e^{(\alpha,-\alpha,0)}.$$

A similar computation gives

$$\sum_{0\le i,j\le 2} e^{i,j} = \tfrac{3}{16}(\omega_M+\omega_N+\omega_{\widetilde{N}}) + \tfrac{3}{32}\sum_{\substack{\alpha\in K(2)\\ 1\le i,j\le 3,\, i\ne j}} e^{\eta_i(\alpha)-\eta_j(\alpha)}.$$

Recall that $M+N \cong A_2\otimes E_8$. It contains a full rank sublattice isometric to $(\sqrt{2}A_2)^8$ and hence ω_{M+N} is the sum of the conformal elements of each tensor copy of $V_{\sqrt{2}A_2}^{\otimes 8}$. We also note that the conformal element of the lattice VOA $V_{\sqrt{2}A_2}$ is given by

$$\begin{aligned}\omega_{\sqrt{2}A_2} &= \tfrac{1}{6}\big(\alpha_1(-1)^2+\alpha_2(-1)^2+\alpha_3(-1)^2\big)\cdot\mathbb{1}\\ &= \tfrac{2}{3}\Big(\tfrac{1}{2}\Big(\frac{\alpha_1(-1)}{\sqrt{2}}\Big)^2+\tfrac{1}{2}\Big(\frac{\alpha_2(-1)}{\sqrt{2}}\Big)^2+\tfrac{1}{2}\Big(\frac{\alpha_3(-1)}{\sqrt{2}}\Big)^2\Big)\cdot\mathbb{1},\end{aligned}$$

where $\alpha_1, \alpha_2, \alpha_3$ are positive roots of a root lattice type A_2 [Dong et al. 1998].

Thus $\omega_{M+N} = \frac{2}{3}(\omega_M+\omega_N+\omega_{\widetilde{N}})$ and we get

$$\Omega = \omega_L - \tfrac{8}{9}\sum_{0\le i,j\le 2} e^{i,j},$$

as desired. Note that $\omega_L = \omega_E + \omega_{M+N}$. □

Lemma 3.14. *For any $0\le i, j\le 2$, we have $e^{i,j}\in \widetilde{W} = \mathrm{Com}_{V_L}\big(L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0)\big)$. Hence $W\subset \widetilde{W}$ and $W_1 = 0$.*

Proof. Since $e^{i,j}\in V_{M+N}$ and $E = \{(\alpha,\alpha,\alpha)\mid \alpha\in E_8\}$ is orthogonal to $M+N$, it is clear that $(H_\alpha)_n e^{i,j} = 0$ for all $n\ge 0$. It is also clear that $(E_\alpha)_n e^{i,j} = 0$ for any root $\alpha\in K$ and $n\ge 2$.

Recall from [Frenkel et al. 1988] that

$$Y(e^\alpha, z) = \exp\Big(\sum_{n\in\mathbb{Z}^+}\frac{\alpha(-n)}{n}z^n\Big)\exp\Big(\sum_{n\in\mathbb{Z}^+}\frac{\alpha(n)}{-n}z^{-n}\Big)e^\alpha z^\alpha.$$

Now let $\sigma = (123)$ be a 3-cycle. Then by direct calculation, we have

$$\begin{aligned}(E_\alpha)_1 e^{i,j} &= (E_\alpha)_1(\rho^i h^j e_M)\\ &= \big(e^{\eta_1(\alpha)}+e^{\eta_2(\alpha)}+e^{\eta_3(\alpha)}\big)_1\\ &\qquad\times\Big(\tfrac{1}{16}\omega_{h^j(M)} + \tfrac{1}{32}\sum_{\alpha\in\Delta^+(E_8)}\rho^i\big(e^{(\eta_{\sigma^j(1)}-\eta_{\sigma^j(2)})(\alpha)} + e^{-(\eta_{\sigma^j(1)}-\eta_{\sigma^j(2)})(\alpha)}\big)\Big)\\ &= \tfrac{1}{16}\langle\alpha,\alpha\rangle^2\tfrac{1}{8}\big(e^{\eta_{\sigma^j(1)}(\alpha)}+e^{\eta_{\sigma^j(2)}(\alpha)}\big) + \tfrac{1}{32}\varepsilon(\alpha,-\alpha)\big(e^{\eta_{\sigma^j(1)}(\alpha)}+e^{\eta_{\sigma^j(2)}(\alpha)}\big)\\ &= 0,\end{aligned}$$

and

$$
\begin{aligned}
(E_\alpha)_0 e^{i,j} &= \big(e^{\eta_1(\alpha)}+e^{\eta_2(\alpha)}+e^{\eta_3(\alpha)}\big)_0 \\
&\qquad \times\Big(\tfrac{1}{16}\omega_{h^j(M)}+\tfrac{1}{32}\sum_{\alpha\in\Delta^+(E_8)}\rho^i\big(e^{(\eta_{\sigma^j(1)}-\eta_{\sigma^j(2)})(\alpha)}+e^{-(\eta_{\sigma^j(1)}-\eta_{\sigma^j(2)})(\alpha)}\big)\Big)\\
&= \tfrac{1}{16}\Big(\langle\alpha,\alpha\rangle^2\tfrac{1}{8}\big(\eta_{\sigma^j(1)}(\alpha)(-1)e^{\eta_{\sigma^j(1)}(\alpha)}+\eta_{\sigma^j(2)}(\alpha)(-1)e^{\eta_{\sigma^j(2)}(\alpha)}\big)\\
&\qquad -2\langle\alpha,\alpha\rangle\tfrac{1}{8}\big((\eta_{\sigma^j(1)}-\eta_{\sigma^j(2)})(\alpha)(-1)e^{\eta_{\sigma^j(1)}(\alpha)}\\
&\qquad\qquad -(\eta_{\sigma^j(1)}-\eta_{\sigma^j(2)})(\alpha)(-1)e^{\eta_{\sigma^j(2)}(\alpha)}\big)\Big)\\
&\quad +\tfrac{1}{32}\varepsilon(\alpha,-\alpha)\big(\eta_{\sigma^j(2)}(\alpha)(-1)e^{\eta_{\sigma^j(1)}(\alpha)}+\eta_{\sigma^j(1)}(\alpha)(-1)e^{\eta_{\sigma^j(2)}(\alpha)}\big)\\
&=0
\end{aligned}
$$

for any root $\alpha\in K$. Therefore, $(E_\alpha)_n e^{i,j}=0$ for all $n\ge 0$. Since $L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0)$ is generated by E_α and H_α, we have the desired conclusion. □

Remark 3.15. Note that the lattice VOA V_L also contains a subVOA isomorphic to $L_{\hat{E}_8}(3,0)$, the level-3 affine VOA associated to the Kac–Moody Lie algebra of type $E_8^{(1)}$. The central charge of $\mathrm{Com}_{V_L}(L_{\hat{E}_8}(3,0))$ is $\frac{16}{11}$, which is the same as U_{3C}. In fact, it can be shown by the similar calculation as Lemma 3.14 that e_M and e_N defined in Notation 3.6 are contained in $\mathrm{Com}_{V_L}(L_{\hat{E}_8}(3,0))$. Moreover,

$$U_{3C}\cong \mathrm{VOA}(e_M,e_N)=\mathrm{Com}_{V_L}(L_{\hat{E}_8}(3,0)).$$

4. Parafermion VOA and W

In this section, we shall show that the VOA W defined in Notation 3.8 is, in fact, equal to the commutant subVOA $\widetilde{W}=\mathrm{Com}_{V_L}(L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0))$. Recall that the lattice VOA $V_{A_3^8}$ contains a full subVOA $K(\mathfrak{sl}_3(\mathbb{C}),9)\otimes L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0)$ (see [Lam 2014]), where $K(\mathfrak{sl}_3(\mathbb{C}),9)$ is the parafermion VOA associated to the affine VOA $L_{\widehat{\mathfrak{sl}}_3(\mathbb{C})}(9,0)$. Therefore, the VOA $\widetilde{W}$ contains a full subVOA isomorphic to the parafermion VOA $K(\mathfrak{sl}_3(\mathbb{C}),9)$.

Parafermion VOA. First, we recall the definition of parafermion VOA from [Dong and Wang 2010], henceforth abbreviated [DW] (cf. [Dong et al. 2009; 2010]).

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra and $\hat{\mathfrak{g}}$ the affine Kac–Moody Lie algebra associated with $\mathfrak{g}$. The level-k affine vertex operator algebra $L_{\hat{\mathfrak{g}}}(k,0)$ contains a Heisenberg vertex operator algebra corresponding to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let $M_{\hat{\mathfrak{h}}}(k,0)$ be the vertex operator subalgebra of $L_{\hat{\mathfrak{g}}}(k,0)$ generated by $h(-1)\cdot\mathbb{1}$ for $h\in\mathfrak{h}$. The commutant $K(\mathfrak{g},k)$ of $M_{\hat{\mathfrak{h}}}(k,0)$ in $L_{\hat{\mathfrak{g}}}(k,0)$ is called a parafermion vertex operator algebra.

The VOA $L_{\hat{\mathfrak{g}}}(k,0)$ is completely reducible as an $M_{\hat{\mathfrak{h}}}(k,0)$-module and we have a decomposition (see [DW]).

Lemma 4.1. *For any $\lambda \in \mathfrak{h}^*$, let $M_{\hat{\mathfrak{h}}}(k,\lambda)$ be the irreducible highest weight module for $\hat{\mathfrak{h}}$ with a highest weight vector v_λ such that $h(0)v_\lambda = \lambda(h)v_\lambda$ for $h \in \mathfrak{h}$. Set*

$$K_{\mathfrak{g},k}(\lambda) = K_{\mathfrak{g},k}(0,\lambda) = \left\{ v \in L_{\hat{\mathfrak{g}}}(k,0) \,\middle|\, h(m)v = \lambda(h)\delta_{m,0}v \text{ for } h \in \mathfrak{h}, m \geq 0 \right\}.$$

Then we have

$$L_{\hat{\mathfrak{g}}}(k,0) = \bigoplus_{\lambda \in Q} K_{\mathfrak{g},k}(\lambda) \otimes M_{\hat{\mathfrak{h}}}(k,\lambda),$$

where Q is the root lattice of $\mathfrak{g}$.

Similarly, for any dominant integral weight $\Lambda \in P_+^k(\mathfrak{g})$, we also have the decomposition.

Lemma 4.2. *Set*

$$K_{\mathfrak{g},k}(\Lambda,\lambda) = \left\{ v \in L_{\hat{\mathfrak{g}}}(k,\Lambda) \,\middle|\, h(m)v = \lambda(h)\delta_{m,0}v \text{ for } h \in \mathfrak{h}, m \geq 0 \right\}.$$

Then

$$L_{\hat{\mathfrak{g}}}(k,\Lambda) = \bigoplus_{\lambda \in \Lambda + Q} K_{\mathfrak{g},k}(\Lambda,\lambda) \otimes M_{\hat{\mathfrak{h}}}(k,\lambda).$$

A generating set. In [DW], it is shown that the parafermion VOA $K(\mathfrak{g},k)$ is generated by subVOAs isomorphic to $K(\mathfrak{sl}_2(\mathbb{C}),k)$. We first give a brief review of their work.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let Δ_+ be the set of all positive roots of $\mathfrak{g}$. Then

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_+} (\mathbb{C}x_\alpha \oplus \mathbb{C}x_{-\alpha}),$$

where $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha} = \{u \in \mathfrak{g} \mid [h,u] = \pm\alpha(h)u \text{ for all } h \in \mathfrak{h}\}$.

Notation 4.3. For any $\alpha \in \Delta_+$, let $h_\alpha = [x_\alpha, x_{-\alpha}]$. Then $S_\alpha = \operatorname{span}\{h_\alpha, x_\alpha, x_{-\alpha}\}$ is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Define

$$\omega_\alpha = \frac{1}{2k(k+2)}(kh_\alpha(-2)1 - h_\alpha(-1)2\mathbb{1} + 2kx_\alpha(-1)x_{-\alpha}(-1)\mathbb{1})$$

and

$$\begin{aligned} W_\alpha^3 = {}& k^2h_\alpha(-3)\mathbb{1} + 3kh_\alpha(-2)h_\alpha(-1)\mathbb{1} + 2h_\alpha(-1)^3\mathbb{1} \\ & - 6kh_\alpha(-1)x_\alpha(-1)x_\alpha(-1)\mathbb{1} + 3k^2x_\alpha(-2)x_\alpha(-1)\mathbb{1} - 3k^2x_\alpha(-1)x_\alpha(-2)\mathbb{1}. \end{aligned}$$

We use P_α to denote the vertex operator subalgebra of $K(\mathfrak{g},k)$ generated by ω_α and W_α^3 for $\alpha \in \Delta_+$.

Theorem 4.4 [DW, Theorem 4.2]. *The simple vertex operator algebra $K(\mathfrak{g}, k)$ is generated by P_α, $\alpha \in \Delta_+$ and P_α is a simple vertex operator algebra isomorphic to the parafermion vertex operator algebra $K(\mathfrak{sl}_2(\mathbb{C}), k)$ associated to $\mathfrak{sl}_2(\mathbb{C})$.*

The lattice VOA $V_{A_n^{k+1}}$. Next we recall an embedding of the VOA

$$K(\mathfrak{sl}_{k+1}, n+1) \otimes L_{\widehat{\mathfrak{sl}}_{n+1}(\mathbb{C})}(k+1, 0)$$

into the lattice VOA $V_{A_n^{k+1}}$ from [Lam 2014].

We use the standard model for the root lattice of type A_ℓ. In particular,

$$A_\ell = \Big\{ \sum a_i \epsilon_i \in \mathbb{Z}^{\ell+1} \;\Big|\; a_i \in \mathbb{Z} \text{ and } \sum_{i=1}^{\ell+1} a_i = 0 \Big\},$$

where ϵ_i is the row vector whose i-th entry is 1 and all other entries are 0. The dual lattice

$$A_\ell^* = \bigcup_{i=0}^{\ell} (\gamma_{A_\ell}(i) + A_\ell),$$

where $\gamma_{A_\ell}(i) = \dfrac{1}{\ell+1}\Big(\sum_{j=1}^{\ell+1-i} i\epsilon_j - \sum_{j=\ell+1-i+1}^{\ell+1} (\ell+1-i)\epsilon_j\Big)$ for $i = 0, \dots, \ell$.

Notation 4.5. Let n and k be positive integers. We shall consider two injective maps $\eta_i : \mathbb{Z}^{n+1} \to \mathbb{Z}^{(n+1)(k+1)}$ and $\iota_i : \mathbb{Z}^{k+1} \to \mathbb{Z}^{(n+1)(k+1)}$ defined by

$$\eta_i(\epsilon_j) = \epsilon_{(n+1)(i-1)+j} \quad \text{and} \quad \iota_i(\epsilon_j) = \epsilon_{(n+1)(j-1)+i}$$

for $i = 1, \dots, k+1$, $j = 1, \dots, n+1$.

Let

$$d_{k+1} = \sum_{j=1}^{k+1} \eta_j : \mathbb{Z}^{n+1} \to \mathbb{Z}^{(n+1)(k+1)} \quad \text{and} \quad \mu_{n+1} = \sum_{j=1}^{n+1} \iota_j : \mathbb{Z}^{k+1} \to \mathbb{Z}^{(n+1)(k+1)}.$$

Then we have

$$d_{k+1}(a_1, \dots, a_{n+1}) = (a_1, \dots, a_{n+1}, a_1, \dots, a_{n+1}, \dots, a_1, \dots, a_{n+1}),$$
$$\mu_{n+1}(a_1, \dots, a_{k+1}) = (a_1, \dots, a_1, a_2, \dots, a_2, \dots, a_{k+1}, \dots, a_{k+1}).$$

Set $X = d_{k+1}(A_n)$ and $Y = \mu_{n+1}(A_k)$. Then $X \cong \sqrt{k+1}A_n$ and $Y \cong \sqrt{n+1}A_k$. Moreover, we have

$$\begin{aligned} \operatorname{Ann}_{A_{(n+1)(k+1)-1}}(Y) &= \bigoplus_{i=1}^{k+1} \eta_i(A_n) \cong A_n^{k+1}, \\ \operatorname{Ann}_{A_{(n+1)(k+1)-1}}(X) &= \bigoplus_{j=1}^{n+1} \iota_j(A_k) \cong A_k^{n+1}, \end{aligned} \tag{4-1}$$

where $\mathrm{Ann}_A(B) = \{x \in A \mid \langle x, y\rangle = 0 \text{ for all } y \in B\}$ is the annihilator of a sublattice B in an integral lattice A.

By the same construction as in Notation 3.2 (see also [Dong and Lepowsky 1993, Chapter 13]), one can obtain subVOAs isomorphic to $L_{\widehat{\mathfrak{sl}}_{n+1}(\mathbb{C})}(k+1, 0)$ and $L_{\widehat{\mathfrak{sl}}_{k+1}(\mathbb{C})}(n+1, 0)$ in the lattice VOA $V_{A_{(k+1)(n+1)-1}}$.

The next proposition is well known in the literature [Kac and Wakimoto 1988; Nakanishi and Tsuchiya 1992; Lam 2014].

Proposition 4.6. *The VOAs $L_{\widehat{\mathfrak{sl}}_{n+1}(\mathbb{C})}(k+1, 0)$ and $L_{\widehat{\mathfrak{sl}}_{k+1}(\mathbb{C})}(n+1, 0)$ are mutually commutative in the lattice VOA $V_{A_{(k+1)(n+1)-1}}$. Moreover,*

$$L_{\widehat{\mathfrak{sl}}_{n+1}(\mathbb{C})}(k+1, 0) \otimes L_{\widehat{\mathfrak{sl}}_{k+1}(\mathbb{C})}(n+1, 0)$$

is a full subVOA of $V_{(n+1)(k+1)-1}$.

Remark 4.7. It is also known that the VOA $V_{\mu_{n+1}(A_k)}$ is contained in the affine VOA $L_{\widehat{\mathfrak{sl}}_{k+1}(\mathbb{C})}(n+1, 0)$ and $K(\mathfrak{sl}_{k+1}(\mathbb{C}), n+1) = \mathrm{Com}_{L_{\widehat{\mathfrak{sl}}_{k+1}(\mathbb{C})}(n+1,0)}(V_{\mu_{n+1}(A_k)})$ (see [Lam 2014, Lemma 4.1]). Moreover, for any $\Lambda \in P^+_{n+1}(\mathfrak{sl}_{k+1}(\mathbb{C}))$, we have the decomposition

$$L_{\widehat{\mathfrak{sl}}_{k+1}}(n+1, \Lambda) = \bigoplus_{\lambda \in \frac{\frac{1}{n+1}\mu_{n+1}(\Lambda + A_k)}{\mu_{n+1}(A_k)}} K_{\mathfrak{sl}_{k+1}(\mathbb{C}), n+1}(\Lambda, (n+1)\bar{\lambda}) \otimes V_{\lambda + \mu_{n+1}(A_k)} \tag{4-2}$$

as a module of $V_{\mu_{n+1}(A_k)} \otimes K(\mathfrak{sl}_{k+1}(\mathbb{C}), n+1)$ ($\bar{\lambda} \in \frac{1}{n+1} A_k^*$) such that $\mu_{n+1}(\bar{\lambda}) = \lambda$ (see [ibid., Lemma 4.3]).

Note that it is shown in [Dong and Lepowsky 1993, Theorem 14.20] that $K_{\mathfrak{sl}_{k+1}(\mathbb{C}), n+1}(\Lambda, (n+1)\bar{\lambda})$, for $\Lambda \in P_+^{n+1}(\mathfrak{sl}_{k+1}(\mathbb{C}))$, $\lambda \in (\mu_{n+1}(A_k))^*$, are irreducible $K(\mathfrak{sl}_{k+1}(\mathbb{C}), n+1)$-modules.

Next we consider the case $n = 8$, $k = 2$. Then $(n+1)(k+1) - 1 = 26$. We shall study the decomposition of $\widetilde{W} = \mathrm{Com}_{V_{E_8^3}}(L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3, 0))$ as a $K(\mathfrak{sl}_3(\mathbb{C}), 9)$-module.

Set

$$\nu_1 = \eta_1 - \eta_2, \qquad \nu_2 = \eta_2 - \eta_3,$$

and define $\mu = \mu_3 : \mathbb{Z}^3 \to \mathbb{Z}^{27}$ by

$$\mu(a_1, a_2, a_3) = (a_1, \ldots, a_1, a_2, \ldots, a_2, a_3, \ldots a_3).$$

Note that $Y = \mu(A_2) \cong 3A_2$ and

$$\mathrm{Ann}_{A_{26}}(Y) = \{\alpha \in A_{26} \mid \langle \alpha, \beta\rangle = 0 \text{ for any } \beta \in Y\} \cong A_8^3.$$

Next we discuss the coset decomposition $Y + A_8^3$ in A_{26}.

Lemma 4.8. *Let* $\alpha_1 = (1, -1, 0)$ *and* $\alpha_2 = (0, 1, -1)$ *be roots of* A_2. *Then we have*

$$A_{26} = \bigcup_{0 \le i, j \le 8} \Big(\big(-\tfrac{1}{9}(i\mu(\alpha_1) + j\mu(\alpha_2)) + Y\big) + \big(\nu_1(\gamma_{A_8}(i)) + \nu_2(\gamma_{A_8}(j)) + A_8^3\big)\Big).$$

Proof. First we note that $[A_{26} : Y + A_8^3] = \sqrt{(9^2 \cdot 3) \cdot 9^3/27} = 9^2$. Moreover, we have

$$-\tfrac{1}{9}(i\mu(\alpha_1) + j\mu(\alpha_2)) + \nu_1(\gamma_{A_8}(i)) + \nu_2(\gamma_{A_8}(j)) = \sum_{k=10-i}^{9} \iota_k(\alpha_1) + \sum_{k'=10-i}^{9} \iota_{k'}(\alpha_2).$$

Note that $\sum_{k=10-i}^{9} \iota_k(\alpha_p) \notin Y + A_8^3$ for any $i \neq 0$, $p = 1, 2$. Therefore,

$$\big(-\tfrac{1}{9}(i\mu(\alpha_1) + j\mu(\alpha_2)) + Y\big) + \big(\nu_1(\gamma_{A_8}(i)) + \nu_2(\gamma_{A_8}(j)) + A_8^3\big),$$

for $i, j = 0, \dots, 8$, give 9^2 distinct cosets in $A_{26}/(Y + A_8^3)$. Thus, we have the desired conclusion. □

Lemma 4.9. *Let* $\delta = \gamma_{A_8}(3) = \frac{1}{3}(1^6, (-2)^3) \in A_8^*$. *Then for any* $k, \ell = 0, \pm 1$, *we have*

$$\begin{aligned}\mathrm{Com}_{V_{(k\nu_1+\ell\nu_2)(\delta)+A_8^3}}(L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0)) &= \big\{v \in V_{(k\nu_1+\ell\nu_2)(\delta)+A_8^3} \;\big|\; \Omega_n v = 0 \textit{ for all } n \ge 0\big\}\\ &\cong K_{\mathfrak{sl}_3(\mathbb{C}),9}(0, -3(k\alpha_1 + \ell\alpha_2)).\end{aligned}$$

Proof. By Lemma 4.8,

$$V_{A_{26}} = \bigoplus_{0 \le i, j \le 8} V_{-\frac{1}{9}(i\mu(\alpha_1)+j\mu(\alpha_2))+Y} \otimes V_{\nu_1(\gamma_{A_8}(i))+\nu_2(\gamma_{A_8}(j))+A_8^3}.$$

Moreover, by (4-2),

$$L_{\widehat{\mathfrak{sl}}_3}(9,0) = \mathrm{Com}_{V_{A_{26}}}(L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0)) = \bigoplus_{\lambda \in \frac{1}{9}Y/Y} V_{\lambda+Y} \otimes K_{\mathfrak{sl}_3(\mathbb{C}),9}(0, 9\bar{\lambda}).$$

Take $i = 3k$ and $j = 3\ell$. Then we have

$$\begin{aligned}\mathrm{Com}_{V_{(k\nu_1+\ell\nu_2)(\delta)+A_8^3}}(L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0)) &\cong K_{\mathfrak{sl}_3(\mathbb{C}),9}(0, 9 \cdot -\tfrac{1}{9}(3k\alpha_1 + 3\ell\alpha_2))\\ &= K_{\mathfrak{sl}_3(\mathbb{C}),9}(0, -3(k\alpha_1 + \ell\alpha_2))\end{aligned}$$

as desired. □

Lemma 4.10. *We have the decomposition*

$$\widetilde{W} = \mathrm{Com}_{V_{E_8^3}}(L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0)) = \bigoplus_{i,j=0,\pm 1} K_{\mathfrak{sl}_3(\mathbb{C}),9}(0, 3(i\alpha_1 + j\alpha_2)).$$

Proof. First we note that $M + N \cong A_2 \otimes E_8$ and

$$M + N = \bigcup_{0 \le k, \ell \le 2} ((k\nu_1 + \ell\nu_2)(\delta) + A_2 \otimes A_8).$$

Since $A_2 \otimes A_8 \cong \mathrm{Ann}_{A_8^3}(d_3(A_8))$ and $V_{d_3(A_8)} \subset L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0)$, we have

$$\widetilde{W} = \mathrm{Com}_{V_{E_8^3}}(L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0)) < V_{M+N}.$$

The conclusion now follows from Lemma 4.9. □

Now let $\alpha \in A_2$ be a root. Then $\mathbb{Z}\alpha \cong A_1$ and

$$L(\alpha) = \bigoplus_{j=1}^{9} \iota_j(\mathbb{Z}\alpha) \cong A_1^9 \subset A_{26}.$$

Let H_α and E_α be defined as in Notation 3.2. Then $\{H_\alpha, E_\alpha, -E_{-\alpha}\}$ forms a $\mathfrak{sl}_2$-triple in the lattice VOA $V_{A_1^9} < V_{A_{26}}$. Moreover, it generates a subVOA $\mathscr{L}_\alpha$ isomorphic to the affine VOA $L_{\widehat{\mathfrak{sl}}_2(\mathbb{C})}(9,0)$. Let $M_\alpha(9,0)$ be the subVOA generated by H_α. Then

$$\mathscr{K}_\alpha := \mathrm{Com}_{\mathscr{L}_\alpha}(M_\alpha(9,0)) \cong K(\mathfrak{sl}_2(\mathbb{C}), 9).$$

Note also that $\mathscr{K}_\alpha = \mathrm{Com}_{\mathscr{L}_\alpha}(M_\alpha(9,0)) < \mathrm{Com}_{L_{\widehat{\mathfrak{sl}}_3(\mathbb{C})}(9,0)}(V_{\mu(A_2)}) = K(\mathfrak{sl}_3(\mathbb{C}), 9)$.

Set $h_\alpha = H_\alpha$, $x_\alpha = E_\alpha$ and $x_{-\alpha} = -E_{-\alpha}$. Then the elements ω_α and W_α^3 defined in Notation 4.3 are contained in $\mathscr{K}_\alpha$. In fact, $\mathscr{K}_\alpha$ is generated by ω_α and W_α^3 (see [Dong et al. 2009]).

Theorem 4.11. *The VOA W defined in Notation 3.8 contains a full subVOA isomorphic to $K(\mathfrak{sl}_3(\mathbb{C}), 9)$.*

Proof. Recall that $W = (e^{i,j} \mid 0 \le i, j \le 2)$. We also have

$$M = (\eta_1 - \eta_2)(E_8), \quad N = (\eta_2 - \eta_3)(E_8), \quad \widetilde{N} = (\eta_1 - \eta_3)(E_8).$$

Let $\alpha_1 = (1,-1,0)$, $\alpha_2 = (0,1,-1)$ and $\alpha_3 = \alpha_1 + \alpha_2 = (1,0,-1)$ be the positive roots of A_2. Then by the same calculations as in [Lam et al. 2007], it is straightforward to verify that

$$\mathscr{K}_{\alpha_1} < \mathrm{VOA}(e_M, \rho e_M), \quad \mathscr{K}_{\alpha_2} < \mathrm{VOA}(e_N, \rho e_N), \quad \mathscr{K}_{\alpha_3} < \mathrm{VOA}(e_{\widetilde{N}}, \rho e_{\widetilde{N}}),$$

where $e_M, e_N, e_{\widetilde{N}}$ and ρ are defined as in Notation 3.6.

Now by Theorem 4.4, $\mathscr{K}_{\alpha_1}, \mathscr{K}_{\alpha_2}$ and $\mathscr{K}_{\alpha_3}$ generate a subVOA isomorphic to $K(\mathfrak{sl}_3(\mathbb{C}), 9)$ in W. It is a full subVOA of W because they have the same central charge. □

Theorem 4.12. *We have $W = \widetilde{W} = \mathrm{Com}_{V_{E_8^3}}(L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0))$.*

Proof. By the previous lemma, the subVOA W contains $K(\mathfrak{sl}_3(\mathbb{C}), 9)$ as a full subVOA.

Therefore, it suffices to show that $K_{\mathfrak{sl}_3(\mathbb{C}),9}(0, 3(i\alpha_1 + j\alpha_2))$ is contained in W for any $i, j = 0, \pm 1$.

By [Lam et al. 2007, Proposition 2.2],

$$X^+_{\nu_1} = \frac{1}{32} \sum_{\substack{\gamma\in\nu_1(\delta)+\nu_1(A_8)\\ \langle\gamma,\gamma\rangle=4}} e^\gamma \quad\text{and}\quad X^-_{\nu_1} = \frac{1}{32} \sum_{\substack{\gamma\in-\nu_1(\delta)+\nu_1(A_8)\\ \langle\gamma,\gamma\rangle=4}} e^\gamma$$

are contained in $\mathrm{VOA}(e_M, \rho e_M) < W$. Moreover, it is straightforward to verify that

$$X^+_{\nu_1} \in \mathrm{Com}_{V_{\nu_1(\delta)+A_8^3}}(L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0)) \cong K_{\mathfrak{sl}_3(\mathbb{C}),9}(0,-3\alpha_1)$$

and

$$X^-_{\nu_1} \in \mathrm{Com}_{V_{-\nu_1(\delta)+A_8^3}}(L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0)) \cong K_{\mathfrak{sl}_3(\mathbb{C}),9}(0,3\alpha_1).$$

Therefore, W contains $K_{\mathfrak{sl}_3(\mathbb{C}),9}(0,\pm 3\alpha_1)$ as $K(\mathfrak{sl}_3(\mathbb{C}),9)$-submodules. Similarly, W also contains $K_{\mathfrak{sl}_3(\mathbb{C}),9}(0,\pm 3\alpha_2)$ and $K_{\mathfrak{sl}_3(\mathbb{C}),9}(0,\pm 3(\alpha_1+\alpha_2))$ as $K(\mathfrak{sl}_3(\mathbb{C}),9)$-submodules.

Moreover, it is clear that $0 \neq (X^+_{\nu_1})_{-3}(X^-_{\nu_2}) \in V_{(\nu_1-\nu_2)(\delta)+A_8^3}$. Since $X^+_{\nu_1}$ and $X^-_{\nu_2}$ are contained in the commutant of $L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0)$, we have

$$(X^+_{\nu_1})_{-3}(X^-_{\nu_2}) \in \mathrm{Com}_{V_{(\nu_1-\nu_2)(\delta)+A_8^3}}(L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3,0)).$$

Hence W contains $K_{\mathfrak{sl}_3(\mathbb{C}),9}(0,3(\alpha_1-\alpha_2))$. Similarly, $K_{\mathfrak{sl}_3(\mathbb{C}),9}(0,3(\alpha_2-\alpha_1))$ is contained in W, also. □

Remark 4.13. Recall that $e^{i,j} \in V_{E_8^3}$, $0 \le i, j \le 2$, are fixed by the diagonal action of the Weyl group of K. Therefore, the VOA $\widetilde{W}$ is fixed by the Weyl group of K pointwise. Using this fact and Lemma 4.9, it is straightforward to show that $\dim(K_{\mathfrak{sl}_3(\mathbb{C}),9}(0,3\alpha)_2) = 1$ for any root α of A_8, $\dim(K_{\mathfrak{sl}_3(\mathbb{C}),9}(0,0)_2) = 3$, and $\dim(K_{\mathfrak{sl}_3(\mathbb{C}),9}(0,\pm 3(\alpha_1-\alpha_2))_2) = 0$ (see the Appendix). Thus, $\dim(\widetilde{W}_2) = 9$ and $\widetilde{W}_2$ is spanned by $\{e^{i,j} \mid 0 \le i, j \le 2\}$.

A positive definite real form. Next we shall show that the Ising vectors $e^{i,j}$, for $0 \le i, j \le 2$, are contained in a positive definite real form of $V_{E_8^3}$.

First we recall that the lattice VOA constructed in [Frenkel et al. 1988] can be defined over $\mathbb{R}$. Let $V_{L,\mathbb{R}} = S(\hat{\mathfrak{h}}^-_{\mathbb{R}}) \otimes \mathbb{R}\{L\}$ be the real lattice VOA associated to an even positive definite lattice, where $\mathfrak{h} = \mathbb{R} \otimes_{\mathbb{Z}} L$, $\hat{\mathfrak{h}}^- = \oplus_{n\in\mathbb{Z}^+} \mathfrak{h} \otimes \mathbb{R}t^{-n}$. As usual, we use $x(-n)$ to denote $x \otimes t^{-n}$ for $x \in \mathfrak{h}$ and $n \in \mathbb{Z}^+$.

Notation 4.14. Let $\theta : V_{L,\mathbb{R}} \to V_{L,\mathbb{R}}$ be defined by

$$\theta(x(-n_1)\cdots x(-n_k) \otimes e^\alpha) = (-1)^k x(-n_1)\cdots x(-n_k) \otimes e^{-\alpha}.$$

Then θ is an automorphism of $V_{L,\mathbb{R}}$, which is a lift of the (-1)-isometry of L [ibid.]. We shall denote the (± 1)-eigenspaces of θ on $V_{L,\mathbb{R}}$ by $V^\pm_{L,\mathbb{R}}$.

The following result is well-known [Frenkel et al. 1988; Miyamoto 2004].

Proposition 4.15 (cf. Proposition 2.7 of [Miyamoto 2004]). *Let L be an even positive definite lattice. Then the real subspace $\widetilde{V}_{L,\mathbb{R}} = V^+_{L,\mathbb{R}} \oplus \sqrt{-1}V^-_{L,\mathbb{R}}$ is a real form of V_L. Furthermore, the invariant form on $\widetilde{V}_{L,\mathbb{R}}$ is positive definite.*

Now apply the above theorem to the case $L = E_8^3$. We have the following result.

Proposition 4.16. *Let $\widetilde{V}_{E_8^3,\mathbb{R}} = V^+_{E_8^3,\mathbb{R}} \oplus \sqrt{-1}V^-_{E_8^3,\mathbb{R}}$. Then $\widetilde{V}_{E_8^3,\mathbb{R}}$ is a positive definite real form of $V_{E_8^3}$.*

The next lemma is clear by the definitions of e_N, e_N, and $e_{\widetilde{N}}$.

Lemma 4.17. *The Ising vectors e_M, e_N and $e_{\widetilde{N}}$ defined in Notation 3.6 lie in $V^+_{E_8^3,\mathbb{R}}$.*

Recall the automorphism $\rho = \exp\left(\frac{2\pi i}{3}(\boldsymbol{a}, -\boldsymbol{a}, 0)(0)\right)$ defined in Definition 3.5, where $\boldsymbol{a}$ is an element of E_8 such that $K = \{\beta \in E_8 \mid \langle \beta, \boldsymbol{a}\rangle \in 3\mathbb{Z}\} \cong A_8$. Then we have the coset decomposition

$$E_8 = A_8 \cup (b + A_8) \cup (-b + A_8),$$

where b is a root of E_8 such that $\langle b, \boldsymbol{a}\rangle \equiv 1 \mod 3$.

Note that

$$M = \{(\alpha, -\alpha, 0) \mid \alpha \in E_8\} \cong \sqrt{2}E_8,$$
$$\widetilde{K} = \{(\alpha, -\alpha, 0) \mid \alpha \in K\} \cong \sqrt{2}A_8.$$

Set

$$X^0 := \tfrac{1}{3}(e_M + \rho e_M + \rho^2 e_M),$$
$$X^1 := \tfrac{1}{3}(e_M + \xi\rho e_M + \xi^2\rho^2 e_M),$$
$$X^2 := \tfrac{1}{3}(e_M + \xi^2\rho e_M + \xi\rho^2 e_M),$$

where $\xi = \exp\frac{2\pi i}{3} = \frac{1}{2}(-1 + \sqrt{-3})$.

The next lemma can be proved by the same calculations as in [Lam et al. 2007]. Note that $\rho X^0 = X^0$, $\rho X^1 = \xi^2 X^1$ and $\rho X^2 = \xi X^2$.

Lemma 4.18. *The vector X^0 is contained in $V^+_{M,\mathbb{R}}$. Moreover,*

$$X^1 = \frac{1}{32}\sum_{\substack{\gamma \in (b,-b,0)+\widetilde{K}\\ \langle\gamma,\gamma\rangle=4}} e^\gamma \quad \text{and} \quad X^2 = \frac{1}{32}\sum_{\substack{\gamma \in -(b,-b,0)+\widetilde{K}\\ \langle\gamma,\gamma\rangle=4}} e^\gamma.$$

Therefore, $X^1 + X^2 \in V^+_{M,\mathbb{R}}$ and $X^1 - X^2 \in V^-_{M,\mathbb{R}}$.

Lemma 4.19. *The Ising vectors $e^{i,j}$, $0 \le i, j \le 2$, are all contained in $\widetilde{V}_{E_8^3,\mathbb{R}}$.*

Proof. By the discussion above, we have

$$\rho e_M = X^0 - \tfrac{1}{2}(X^1 + X^2) + \tfrac{1}{2}\sqrt{-3}(X^1 - X^2).$$

Since $X^1 + X^2 \in V^+_{M,\mathbb{R}}$ and $X^1 - X^2 \in V^-_{M,\mathbb{R}}$, we have $\rho e_M \in \widetilde{V}_{E_8^3,\mathbb{R}}$. Similarly, we have $\rho^2 e_M$, ρe_N, $\rho^2 e_N$, $\rho e_{\widetilde{N}}$, $\rho^2 e_{\widetilde{N}} \in \widetilde{V}_{E_8^3,\mathbb{R}}$ as desired. □

5. Griess algebras generated by Ising vectors

In this section, we shall give few structural results about Griess algebras generated by Ising vectors in a moonshine-type VOA V over $\mathbb{R}$ such that the invariant bilinear form is positive definite. Our setting is as follows.

Notation 5.1. Let e, e', e'' be three distinct Ising vectors in V. Assume that

(I) $\langle e, e'\rangle = \langle e, e'\rangle = \langle e', e''\rangle = 1/2^8$ and $\tau_e\tau_{e'}$, $\tau_e\tau_{e''}$, $\tau_{e'}\tau_{e''}$ are of order 3.

Then each of $\{\tau_e, \tau_{e'}\}$, $\{\tau_e, \tau_{e''}\}$, and $\{\tau_{e'}, \tau_{e''}\}$ generates a dihedral group of order 6 and the Griess algebras generated by $\{e, e'\}$, $\{e, e''\}$, and $\{e', e''\}$ are isomorphic to the Griess algebra $\mathcal{G}U_{3C}$ of the $3C$-algebra U_{3C}.

Let $g = \tau_e\tau_{e'}$ and $h = \tau_e\tau_{e''}$. We shall assume that

(II) the subgroup H generated by g and h is elementary abelian of order 3^2.

For any $0 \le i, j \le 2$, denote $e^{i,j} := g^i h^j e$. Note that $e' = ge = e^{1,0}$ and $e'' = he = e^{0,1}$ by Lemma 2.9(4). Furthermore, we assume that

(III) $\langle e^{0,0}, e^{1,1}\rangle = 1/2^8$.

Therefore, the Griess subalgebra generated by $e^{0,0}$, $e^{1,1}$ is also isomorphic to $\mathcal{G}U_{3C}$.

Lemma 5.2. *Let G be the subgroup generated by τ_e, $\tau_{e'}$, and $\tau_{e''}$. Then $G = H : \langle\tau_e\rangle$, where $H \cong 3^2$ is normal in G and τ_e inverts every element in H, i.e., $\tau_e y \tau_e = y^{-1}$ for all $y \in H$.*

Proof. The proof is essentially the same as Lemma 3.12 because $H = \langle g, h\rangle$ is elementary abelian of order 3^2 by our assumption. □

Lemma 5.3. *For any $(i, j) \neq (i', j')$, we have $\langle e^{i,j}, e^{i',j'}\rangle = 1/2^8$.*

Proof. By definition, $\langle e^{i,j}, e^{i',j'}\rangle = \langle g^i h^j e, g^{i'} h^{j'} e\rangle = \langle e, g^{i'-i} h^{j'-j} e\rangle$.

By our assumption, we have

$$\langle e, ge\rangle = \langle e, g^{-1}e\rangle = \langle e, he\rangle = \langle e, h^{-1}e\rangle = 1/2^8,$$
$$\langle e, gh^{-1}e\rangle = \langle e, g^{-1}he\rangle = \langle ge, he\rangle = \langle e', e''\rangle = 1/2^8,$$
$$\langle e, ghe\rangle = \langle e, g^{-1}h^{-1}e\rangle = 1/2^8.$$

Thus, $\langle e^{i,j}, e^{i',j'}\rangle = 1/2^8$ if $(i, j) \neq (i', j')$. □

Lemma 5.4. *Let $\mathcal{G}$ be the Griess subalgebra generated by $\{e, e', e''\}$. Then $\mathcal{G}$ is spanned by $\{e^{i,j} \mid 0 \le i, j \le 2\}$ and* $\dim \mathcal{G} = 9$. *The algebra structure of $\mathcal{G}$ is unique.*

Proof. Recall that g commutes with h and for any (i, j) and (i', j'), we have

$$\begin{aligned}\tau_{e^{i,j}}\tau_{e^{i',j'}} &= g^i h^j \tau_e g^{-i} h^{-j} g^{i'} h^{j'} \tau_e g^{-i'} h^{-j'} = g^i h^j g^i h^j \tau_e \tau_e g^{-i'} h^{-j'} g^{-i'} h^{-j'}\\ &= g^{i'-i} h^{j'-j}.\end{aligned}$$

By Lemma 2.9(3) and (4), we know that

$$e^{-i-i',-j-j'} = g^{i'-i} h^{j'-j}(e^{i,j}) = e^{i,j} + e^{i',j'} - 32 e^{i,j} \cdot e^{i',j'} \tag{5-1}$$

if $(i, j) \neq (i'j')$. Therefore, $e^{-i-i',-j-j'} \in \mathcal{G}\{e^{i,j}, e^{i',j'}\}$, the Griess subalgebra generated by $\{e^{i,j}, e^{i',j'}\}$. Hence,

$$e^{2,0} = e^{0,0} + e^{1,0} - 32 e^{0,0} \cdot e^{1,0},\ e^{0,2} = e^{0,0} + e^{0,1} - 32 e^{0,0} \cdot e^{0,1},$$

and $e^{2,2} = e^{1,0} + e^{0,1} - 32 e^{1,0} \cdot e^{0,1}$ are in $\mathcal{G}$.

Similarly, we also have $e^{1,1} \in \mathcal{G}\{e^{0,0}, e^{2,2}\} < \mathcal{G}$, $e^{1,2} \in \mathcal{G}\{e^{0,2}, e^{2,2}\} < \mathcal{G}$, and $e^{2,1} \in \mathcal{G}\{e^{2,0}, e^{2,2}\} < \mathcal{G}$. Thus, all $e^{i,j}$, $0 \le i, j \le 2$, are in $\mathcal{G}$. In addition, by (5-1), we have

$$e^{i,j} \cdot e^{i',j'} = \begin{cases} \frac{1}{32}\left(e^{i,j} + e^{i',j'} - e^{i'',j''}\right) & \text{if } (i, j) \neq (i'j'),\\ 2e^{i,j} & \text{if } (i, j) = (i'j'),\end{cases}$$

where $i + i' + i'' = j + j' + j'' = 0 \bmod 3$. Therefore, $\operatorname{span}\{e^{i,j} \mid 0 \le i, j \le 2\}$ is closed under the Griess algebra product and $\mathcal{G} = \operatorname{span}\{e^{i,j} \mid 0 \le i, j \le 2\}$. By our assumption, we have the Gram matrix

$$\left(\langle e^{i,j}, e^{i',j'}\rangle\right)_{0 \le i,j,i',j' \le 2} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2^8} & \cdots & \frac{1}{2^8}\\ \frac{1}{2^8} & \frac{1}{4} & \cdots & \frac{1}{2^8}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1}{2^8} & \frac{1}{2^8} & \cdots & \frac{1}{4}\end{pmatrix}.$$

It has rank 9 and hence $\{e^{i,j} \mid 0 \le i, j \le 2\}$ is a linearly independent set and $\dim \mathcal{G} = 9$. □

Next, we shall give some information about the VOA W generated by $\{e^{i,j}\}$.

Lemma 5.5. *Let*

$$\omega := \tfrac{8}{9} \sum_{0 \le i, j \le 2} e^{i,j}.$$

Then ω is a Virasoro vector of central charge 4. Moreover, $\omega \cdot e^{i,j} = e^{i,j} \cdot \omega = 2e^{i,j}$ for any $0 \le i, j \le 2$. In other words, $\omega/2$ is the identity element in $\mathcal{G}$.

Proof. This follows from a straightforward calculation using Lemma 2.9. □

Lemma 5.6. *Let*

$$b^1 = \tfrac{8}{9} \sum_{0 \le i,j \le 2} e^{i,j} - \tfrac{32}{33}(e^{0,0} + e^{0,1} + e^{0,2}).$$

Then b^1 is a Virasoro vector of central charge $\frac{28}{11}$. Moreover, $e^{0,0}$, a^1, and b^1 are mutually orthogonal and $\omega = e^{0,0} + a^1 + b^1$. Therefore, W has a full subVOA isomorphic to the tensor product of Virasoro VOA

$$L\big(\tfrac{1}{2}, 0\big) \otimes L\big(\tfrac{21}{22}, 0\big) \otimes L\big(\tfrac{28}{11}, 0\big).$$

Proof. It follows from (4) and (5) of Lemma 2.9 and Lemma 5.5. □

Remark 5.7. Because of Lemma 4.10 and Theorem 4.12, we conjecture that the subVOA VOA(e, e', e'') generated by $\{e, e', e''\}$ is isomorphic to

$$\widetilde{W} = \bigoplus_{i,j=0,\pm 1} K_{\mathfrak{sl}_3(\mathbb{C}),9}(0, 3(i\alpha_1 + j\alpha_2)).$$

Recall from [Lam 2014] that the parafermion VOA $K_{\mathfrak{sl}_3(\mathbb{C}),9}(0,0)$ contains a full subVOA $W_9(1,1) \otimes W_9(2,1)$, where $W_9(1,1)$ has central charge $\frac{32}{11}$ and $W_9(2,1)$ has central charge $\frac{28}{11}$. Therefore, we believe that the subVOA VOA(e, e', e'') also contains a full subVOA isomorphic to $L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0) \otimes W_9(2,1)$, which is expected to be rational. However, we are not aware of any uniqueness results of the parafermion VOA $K_{\mathfrak{sl}_3(\mathbb{C}),9}(0,0)$ nor the W-algebra $W_9(2,1)$ in terms of generators and relations. Therefore, it is unclear if VOA(e, e', e'') contains $K_{\mathfrak{sl}_3(\mathbb{C}),9}(0,0)$ or $L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0) \otimes W_9(2,1)$ as a full subVOA.

Finally, we describe explicitly several highest weight vectors of the subVOA $\mathrm{vir}(e^{0,0}) \otimes \mathrm{vir}(a^1) \otimes \mathrm{vir}(b^1)$.

Lemma 5.8. *With respect to the subVOA $\mathrm{vir}(e^{0,0}) \otimes \mathrm{vir}(a^1) \otimes \mathrm{vir}(b^1)$, we have the following highest weight vectors.*

(1) *The vectors $a^i - a^j$, $i, j \in \{2,3,4\}$, $i \neq j$, are highest weight vectors of weight $(0, \frac{1}{11}, \frac{21}{11})$, where*

$$a^1 = \tfrac{32}{33}(e^{0,0} + e^{0,1} + e^{0,2}) - e^{0,0}, \quad a^2 = \tfrac{32}{33}(e^{0,0} + e^{1,0} + e^{2,0}) - e^{0,0},$$
$$a^3 = \tfrac{32}{33}(e^{0,0} + e^{1,1} + e^{2,2}) - e^{0,0}, \quad a^4 = \tfrac{32}{33}(e^{0,0} + e^{1,2} + e^{2,1}) - e^{0,0}.$$

(2) *The vector $e^{0,1} - e^{0,2}$ is a highest weight vector of weight $(\frac{1}{16}, \frac{31}{16}, 0)$.*

(3) *The vector $(e^{1,0} + e^{1,1} + e^{1,2}) - (e^{2,0} + e^{2,1} + e^{2,2})$ is a highest weight vector of weight $(\frac{1}{16}, \frac{21}{176}, \frac{20}{11})$.*

(4) *The vectors $(e^{1,1} - e^{2,2}) - (e^{1,2} - e^{2,1})$ and $(e^{1,0} - e^{2,0}) - (e^{1,1} - e^{2,2})$ are highest weight vectors of weight $(\frac{1}{16}, \frac{5}{176}, \frac{21}{11})$.*

Proof. (1) By Lemma 2.9, it is straightforward to show that

$$a^i \cdot a^j = \tfrac{1}{33}(2a^i + 2a^j - a^k - a^\ell),$$

for any $i \neq j$ and $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$. Thus,

$$a^1_1\big(a^i - a^j\big) = \tfrac{1}{33}\big[\big(2a^1 + 2a^i - a^j - a^k\big) - \big(2a^1 + 2a^j - a^i - a^k\big)\big] = \tfrac{1}{11}\big(a^i - a^j\big),$$

where $\{i, j, k\} = \{2, 3, 4\}$.

Since $e^{0,0}_1 a^i = 0$ and $\omega_1 a^i = 2a^i$ for all i, $a^i - a^j$ is a highest weight vector of weight $(0, \frac{1}{11}, \frac{21}{11})$ with respect to $\mathrm{vir}(e^{0,0}) \otimes \mathrm{vir}(a^1) \otimes \mathrm{vir}(b^1)$.

(2) By direct calculations, we have

$$e^{0,0}_1\big(e^{0,1} - e^{0,2}\big) = \tfrac{1}{32}\big[\big(e^{0,0} + e^{0,1} - e^{0,2}\big) - \big(e^{0,0} + e^{0,2} - e^{0,1}\big)\big] = \tfrac{1}{16}\big(e^{0,1} - e^{0,2}\big)$$

and

$$\tfrac{32}{33}\big(e^{0,0} + e^{0,1} + e^{0,2}\big)_1\big(e^{0,1} - e^{0,2}\big) = 2\big(e^{0,1} - e^{0,2}\big).$$

Since $a^1 = \frac{32}{33}(e^{0,0} + e^{0,1} + e^{0,2}) - e^{0,0}$ and $b^1 = \omega - \frac{32}{33}(e^{0,0} + e^{0,1} + e^{0,2})$, we have

$$a^1_1\big(e^{0,1} - e^{0,2}\big) = \tfrac{31}{16}\big(e^{0,1} - e^{0,2}\big) \quad \text{and} \quad b^1_1\big(e^{0,1} - e^{0,2}\big) = 0.$$

(3), (4) By the same calculations as in (2), $(e^{1,0} - e^{2,0})$, $(e^{1,1} - e^{2,2})$, and $(e^{1,2} - e^{2,1})$ are $\frac{1}{16}$-eigenvectors of $e^{0,0}_1$. By Lemma 2.9, we also have

$$\tfrac{32}{33}\big(e^{0,0} + e^{0,1} + e^{0,2}\big)_1\big(e^{1,1} - e^{2,2}\big) = \tfrac{1}{33}\big(4\big(e^{1,1} - e^{2,2}\big) + \big(e^{1,0} - e^{2,0}\big) + \big(e^{1,2} - e^{2,1}\big)\big).$$

Let $v = (e^{1,0} + e^{1,1} + e^{1,2}) - (e^{2,0} + e^{2,1} + e^{2,2})$. Then

$$\tfrac{32}{33}\big(e^{0,0} + e^{0,1} + e^{0,2}\big)_1 v = \tfrac{1}{33}(4 + 1 + 1)v = \tfrac{2}{11}v.$$

Thus, $a^1_1 v = (\frac{2}{11} - \frac{1}{16})v = \frac{21}{176}v$ and $b^1_1 v = (2 - \frac{2}{11})v = \frac{20}{11}v$.

Moreover,

$$\begin{aligned}
&\tfrac{32}{33}\big(e^{0,0} + e^{0,1} + e^{0,2}\big)_1\big(\big(e^{1,1} - e^{2,2}\big) - \big(e^{1,2} - e^{2,1}\big)\big) \\
&\qquad = \tfrac{1}{33}(4 - 1)\big(\big(e^{1,1} - e^{2,2}\big) - \big(e^{1,2} - e^{2,1}\big)\big) = \tfrac{1}{11}\big(\big(e^{1,1} - e^{2,2}\big) - \big(e^{1,2} - e^{2,1}\big)\big).
\end{aligned}$$

Thus, we have

$$a^1_1\big(\big(e^{1,1} - e^{2,2}\big) - \big(e^{1,2} - e^{2,1}\big)\big) = \tfrac{5}{176}\big(\big(e^{1,1} - e^{2,2}\big) - \big(e^{1,2} - e^{2,1}\big)\big)$$

and

$$b^1_1\big(\big(e^{1,1} - e^{2,2}\big) - \big(e^{1,2} - e^{2,1}\big)\big)v = \tfrac{21}{11}\big(\big(e^{1,1} - e^{2,2}\big) - \big(e^{1,2} - e^{2,1}\big)\big).$$

The remaining cases can be proved similarly. □

Appendix: Dimensions of $K_{\mathfrak{sl}_3(\mathbb{C}),9}(0, 3(i\alpha_1 + j\alpha_2))_2$

In this appendix, we shall compute the dimension of $K_{\mathfrak{sl}_3(\mathbb{C}),9}(0, 3(i\alpha_1 + j\alpha_2))_2$ for all $0 \le i, j \le 2$. First we recall a result from [Frenkel et al. 1988, Chapter 8].

Let α, β have norm 4 in a lattice L. Then

$$e^{\alpha}_1 e^{\beta} = \begin{cases} \frac{1}{2}\alpha(-1)^2 \cdot \mathbb{1} & \text{if } \beta = -\alpha, \\ \varepsilon(\alpha, \beta) e^{\alpha+\beta} & \text{if } \langle \beta, \alpha\rangle = -2, \\ 0 & \text{otherwise.} \end{cases} \tag{A-1}$$

Lemma A.1. *For any* $k, \ell = 0, \pm 1$, *we have* $\dim(K_{\mathfrak{sl}_3(\mathbb{C}),9}(0, 3(k\alpha_1 + \ell\alpha_2))_2) = 1$ *if* $(k, \ell) = (0, \pm 1)$, $(\pm 1, 0)$, *or* $(\pm 1, \pm 1)$ *and* $\dim(K_{\mathfrak{sl}_3(\mathbb{C}),9}(0, 3(k\alpha_1 + \ell\alpha_2))_2) = 0$ *if* $(k, \ell) = (\pm 1, \mp 1)$.

Proof. Recall that

$$K_{\mathfrak{sl}_3(\mathbb{C}),9}(0, -3(k\alpha_1 + \ell\alpha_2)) \cong \mathrm{Com}_{V_{(k\nu_1+\ell\nu_2)(\delta)+A_8^3}}(L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3, 0)) < V_{M+N}.$$

Moreover, by Lemma 3.3, the conformal vector Ω of $L_{\widehat{\mathfrak{sl}}_9(\mathbb{C})}(3, 0)$ is given by

$$\Omega = \omega_E + \tfrac{3}{4}\omega_{M+N} - \tfrac{1}{12} \sum_{\substack{\alpha \in A_8(2) \\ 1 \le i, j \le 3,\, i \ne j}} e^{\eta_i(\alpha) - \eta_j(\alpha)}.$$

Thus, for any $X \in (V_{M+N})_2$, $\Omega_1 X = 0$ if and only if

$$\Bigg(\sum_{\substack{\alpha \in A_8(2) \\ 1 \le i, j \le 3,\, i \ne j}} e^{\eta_i(\alpha) - \eta_j(\alpha)} \Bigg)_1 X = 18X. \tag{A-2}$$

Let $\Psi = \{\gamma \in (k\nu_1 + \ell\nu_2)(\delta) + A_8^3 \mid \langle \gamma, \gamma\rangle = 4\}$. Then

$$(V_{(k\nu_1+\ell\nu_2)(\delta)+A_8^3})_2 = \mathrm{span}\{e^{\gamma} \mid \gamma \in \Psi\}.$$

Moreover, $\Psi = \{(k\nu_1 + \ell\nu_2)(\delta + \beta) \mid \beta \in A_8(2), \langle \beta, \delta\rangle = -1\}$ if $(k, \ell) = (0, \pm 1)$, $(\pm 1, 0)$, or $(\pm 1, \pm 1)$ and $\Psi = \varnothing$ if $(k, \ell) = (\pm 1, \mp 1)$.

Suppose $X = \sum_{\gamma \in \Psi} a_\gamma e^{\gamma}$ be an element in $K_{\mathfrak{sl}_3(\mathbb{C}),9}(0, -3(k\alpha_1 + \ell\alpha_2))_2$. Then by (A-1) and (A-2), we must have $a_\gamma = a_{\gamma'}$ for all $\gamma, \gamma' \in \Psi$. Note that there are exactly 18 roots in A_8 such that $\langle \delta, \beta\rangle = -1$ and $\widetilde{W}$ is fixed pointwise by the Weyl group of $K \cong A_8$.

Hence, $K_{\mathfrak{sl}_3(\mathbb{C}),9}(0, -3(k\alpha_1 + \ell\alpha_2))_2$ is spanned by $\sum_{\gamma \in \Psi} e^{\gamma}$ if $(k, \ell) = (0, \pm 1)$, $(\pm 1, 0)$, or $(\pm 1, \pm 1)$ and is zero if $(k, \ell) = (\pm 1, \mp 1)$. □

Next we consider the Griess algebra $K_{\mathfrak{sl}_3(\mathbb{C}),9}(0, 0)_2$. The next lemma follows immediately from (A-1) and the choice of the 2-cocycle $\varepsilon_0(\ ,\ )$.

Lemma A.2. *Let β be a root of A_8 and $1 \le k, \ell \le 3$. Then*

$$\Bigg(\sum_{\substack{\alpha\in A_8(2)\\ i\neq j}} e^{(\eta_i-\eta_j)(\alpha)}\Bigg)_1 e^{(\eta_k-\eta_\ell)(\beta)}$$
$$= \sum_{\substack{\alpha\in A_8(2)\\ \langle\alpha,\beta\rangle=-1}} e^{(\eta_k-\eta_\ell)(\alpha+\beta)} + \tfrac{1}{2}(\eta_k(\beta)-\eta_\ell(\beta))(-1)^2\cdot\mathbb{1} - \sum_{i\neq k} e^{(\eta_i-\eta_\ell)(\beta)} - \sum_{j\neq \ell} e^{(\eta_k-\eta_j)(\beta)}.$$

The next lemma can also be proved easily by the definition of vertex operators [Frenkel et al. 1988].

Lemma A.3. *Let $\beta \in A_8$. Then*

$$\Bigg(\sum_{\substack{\alpha\in A_8(2)\\ i\neq j}} e^{(\eta_i-\eta_j)(\alpha)}\Bigg)_1 (\eta_k-\eta_\ell)(\beta))(-1)^2\cdot\mathbb{1}$$
$$= \sum_{\substack{\alpha\in A_8(2)\\ i\neq j}} \langle(\eta_i-\eta_j)(\alpha), (\eta_k-\eta_\ell)(\beta)\rangle^2 e^{(\eta_i-\eta_j)(\alpha)}.$$

Lemma A.4. *The Griess algebra $K_{\mathfrak{sl}_3(\mathbb{C}),9}(0,0)_2$ has dimension 3 and is spanned by $\{\omega_{\alpha_1}, \omega_{\alpha_2}, \omega_{\alpha_1+\alpha_2}\}$.*

Proof. Let

$$X = \sum_{\substack{1\le i<j\le 3,\\ \alpha\in A_8(2)}} \big(a_{i,j,\alpha}(\eta_i(\alpha)-\eta_j(\alpha))(-2)\cdot\mathbb{1} + b_{i,j,\alpha}(\eta_i(\alpha)-\eta_j(\alpha))(-1)^2\cdot\mathbb{1} + c_{i,j,\alpha}e^{\eta_i(\alpha)-\eta_j(\alpha)}\big)$$

be an element in $K_{\mathfrak{sl}_3(\mathbb{C}),9}(0,0)_2$.

Since X is fixed by the Weyl group of A_8, we have $a_{i,j,\alpha} = a_{i,j,\beta}$, $b_{i,j,\alpha} = b_{i,j,\beta}$, and $c_{i,j,\alpha} = c_{i,j,\beta}$ for any roots $\alpha, \beta \in A_8$. Set $a_{i,j} = a_{i,j,\alpha}$, $b_{i,j} = b_{i,j,\alpha}$, and $c_{i,j} = c_{i,j,\alpha}$ for any root $\alpha \in A_8$. Then, for any $1 \le i < j \le 3$,

$$\sum_{\alpha\in A_8(2)} a_{i,j,\alpha}(\eta_i(\alpha)-\eta_j(\alpha))(-2) = a_{i,j}\sum_{\alpha\in A_8(2)}(\eta_i(\alpha)-\eta_j(\alpha))(-2) = 0$$

and

$$X = \sum_{1\le i<j\le 3}\Bigg(b_{i,j}\sum_{\alpha\in A_8(2)}(\eta_i(\alpha)-\eta_j(\alpha))(-1)^2\cdot\mathbb{1} + c_{i,j}\sum_{\alpha\in A_8(2)} e^{\eta_i(\alpha)-\eta_j(\alpha)}\Bigg).$$

Moreover, $\Bigg(\sum_{\substack{\alpha\in A_8(2)\\ 1\le i,j\le 3,\, i\neq j}} e^{\eta_i(\alpha)-\eta_j(\alpha)}\Bigg)_1 X = 18X$ since $X \in K_{\mathfrak{sl}_3(\mathbb{C}),9}(0,0)_2$.

By Lemmas A.2 and A.3, it is straightforward to show $X \in \mathrm{span}\{\omega_{\alpha_1}, \omega_{\alpha_2}, \omega_{\alpha_1+\alpha_2}\}$ and $\dim(K_{\mathfrak{sl}_3(\mathbb{C}),9}(0,0)_2) = 3$. □

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