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PERTURBATIONS OF A CRITICAL FRACTIONAL EQUATION

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We deal with the following fractional critical problem:

$$\begin{cases} (-\Delta)^{\alpha/2} u = |u|^{2\alpha/(N-\alpha)} u + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a regular bounded domain, $0 < \alpha < 2$ and $N > \alpha$. Under appropriate conditions on the size of f, we prove existence and multiplicity of solutions.

1. Introduction

It is well known, using the Pohozhaev identity [1970], that the critical problem

$$\begin{cases} -\Delta u = |u|^{4/(N-2)} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1-1}$$

has no positive solution whenever Ω is a star-shaped domain. Starting from this nonexistence result, in the last decades several perturbations of this problem have been investigated in order to obtain a solution and understand the criticality of the problem. A pioneering work in that sense is the one performed by Brézis and Nirenberg [1983], in which the authors study the existence of positive solutions of the problem

$$\begin{cases} -\Delta u = |u|^{4/(N-2)} u + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-2}$$

where $f(x,u) = f(u) = \lambda u$, with $\lambda \in \mathbb{R}$ and $N > 2$. Among other extensions, we highlight the work [Ambrosetti et al. 1994], where the authors studied the case $f(u) = \lambda |u|^{q-2} u$ with $1 < q < 2$, as well as [Tarantello 1992], in which the case $f(x,u) = f(x)$ was investigated; see also [Rey 1992].

Our purpose here is to study the similar situation that occurs for the fractional Laplacian in a bounded domain and the corresponding critical power.

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We define the fractional Laplacian in a bounded domain Ω via its spectral decomposition, namely

$$(-\Delta)^{\alpha/2}u = \sum a_j \rho_j^{\alpha/2} \varphi_j,$$

where $\{\rho_j, \varphi_j\}$ is the spectral decomposition of the operator $-\Delta$ in Ω under zero Dirichlet boundary conditions and the a_j are the coefficients of u for the base $\{\varphi_j\}$ in $L^2(\Omega)$. A more precise notation would be $(-\Delta)_\Omega^{\alpha/2}$, because the operator strongly depends on the domain Ω, but we omit the subscript since the domain is fixed throughout the paper. We also recall that in the case where the domain under consideration is the whole space $\mathbb{R}^N$, the associated fractional Laplacian operator $(-\Delta)_{\mathcal{F}}^{\alpha/2}$ is defined via Fourier transformation for functions in the Schwartz class:

$$\big[(-\Delta)_{\mathcal{F}}^{\alpha/2} g\big]^{\wedge}(\xi) = |\xi|^\alpha \hat{g}(\xi),$$

which gives a different operator.

The critical problem corresponding to (1-2) with the fractional Laplacian is

$$\begin{cases} (-\Delta)^{\alpha/2}u = |u|^{2\alpha/(N-\alpha)}u + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1-3}$$

First, there is again a (fractional) Pohozhaev-type identity, which in the case $f \equiv 0$ yields, as for the classical problem (1-1), the nonexistence of positive solutions whenever Ω is a star-shaped domain; see [Brändle et al. 2013]. Note that

$$\frac{2\alpha}{N-\alpha} = 2_\alpha^* - 2, \quad \text{where} \quad 2_\alpha^* := \frac{2N}{N-\alpha}$$

is the critical Sobolev exponent associated to α.

Next, in the case $f(x,u) = f(u)$, we point out [Barrios et al. 2012], in which an existence and multiplicity result was proved for positive solutions when

$$f(u) = \lambda |u|^{q-2}u, \quad \lambda > 0, \quad 0 < \alpha < 2, \quad N > \alpha, \quad \text{and} \quad 1 < q < \frac{2N}{N-\alpha}.$$

The case $\alpha = 1$ and $q = 2$ was studied previously in [Tan 2011].

In this paper we investigate zero order perturbations, $f(x,u) = f(x)$ small in (1-3), of the critical problem $f \equiv 0$, in relation to the results of [Tarantello 1992] for the classical Laplace operator. Thus, we consider the following problem:

$$\begin{cases} (-\Delta)^{\alpha/2}u = |u|^{p-2}u + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $0 < \alpha < 2$, $N > \alpha$, $p = \dfrac{2N}{N-\alpha}$ and f belongs to a suitable space.

In order to establish the functional setting, we define the function space $H_0^{\alpha/2}(\Omega)$ as the completion of $\mathscr{C}_0^\infty(\Omega)$ endowed with the norm

$$\|u\|_{H_0^{\alpha/2}} = \|(-\Delta)^{\alpha/4}u\|_2 = \Big(\sum a_j^2 \rho_j^{\alpha/2}\Big)^{1/2}.$$

The operator $L(u) = (-\Delta)^{\alpha/2}u - |u|^{p-2}u$ is well defined from $H_0^{\alpha/2}(\Omega)$ into its dual $H^{-\alpha/2}(\Omega)$ by the Sobolev inequality; see (2-3) below. Thus it is natural to consider data f in that space: we have $f \in H^{-\alpha/2}(\Omega)$ if and only if $f = (-\Delta)^{\alpha/2}g$ with $g \in H_0^{\alpha/2}(\Omega)$; the associated norm is given by $\|f\|_{H^{-\alpha/2}} = \|g\|_{H_0^{\alpha/2}}$.

Throughout, we will consider solutions of the problem (P) in the following sense:

Definition 1.1. Let $f \in H^{-\alpha/2}(\Omega)$. We say that $u \in H_0^{\alpha/2}(\Omega)$ is an energy solution to the problem (P) if

$$\int_\Omega (-\Delta)^{\alpha/4}u(x)(-\Delta)^{\alpha/4}\psi(x)\,dx = \int_\Omega (|u(x)|^{p-2}u(x) + f(x))\psi(x)\,dx \tag{1-4}$$

for every $\psi \in H_0^{\alpha/2}(\Omega)$.

In the sequel we use the simplified notation $\int f = \int f(x)\,dx$ when no confusion can arise.

The paper is organized as follows: In Section 2 we state the main existence results and establish some preliminaries, and Sections 3 and 4 contain the proofs.

2. Main results and preliminaries

We will focus on functions $f \in H^{-\alpha/2}(\Omega)$ that are small in the following sense:

$$\int_\Omega f\varphi < c(\alpha, N)\|\varphi\|_{H_0^{\alpha/2}}^{(N+\alpha)/\alpha} \quad \text{for all } \varphi \in H_0^{\alpha/2}(\Omega) \text{ with } \|\varphi\|_p = 1, \tag{2-1}$$

where

$$c(\alpha, N) = \frac{2\alpha}{N-\alpha}\Big(\frac{N-\alpha}{N+\alpha}\Big)^{\frac{N+\alpha}{2\alpha}}.$$

The main result of the paper is the following:

Theorem 2.1. *Assume $f \not\equiv 0$ satisfies* (2-1). *Then the problem* (P) *has at least two solutions. Moreover*, *if $f \geq 0$ a.e. in Ω*, *then these solutions are nonnegative a.e. in Ω.*

We will also prove that, if we relax the strict inequality in condition (2-1) by replacing it with the condition

$$\int_\Omega f\varphi \leq c(\alpha, N)\|\varphi\|_{H_0^{\alpha/2}}^{(N+\alpha)/\alpha} \quad \text{for all } \varphi \in H_0^{\alpha/2}(\Omega) \text{ with } \|\varphi\|_p = 1, \tag{2-2}$$

then we still obtain the existence of at least one solution:

Theorem 2.2. *Assume $f \not\equiv 0$ satisfies* (2-2). *Then the problem* (P) *has at least one solution. Moreover*, *if f is nonnegative a.e. in Ω then this solution is nonnegative a.e. in Ω.*

For the fractional Laplacian defined above and $N > \alpha$, the following Sobolev inequality holds:

$$\int_\Omega |(-\Delta)^{\alpha/4}\varphi|^2 \geq S(\alpha, N)\left(\int_\Omega |\varphi|^{2N/(N-\alpha)}\right)^{\frac{N-\alpha}{N}} \quad \text{for all } \varphi \in H_0^{\alpha/2}(\Omega). \tag{2-3}$$

See, for example, [Brändle et al. 2013], where the inequality is proved as a consequence of the Hardy–Littlewood–Sobolev inequality [Hardy and Littlewood 1928; Sobolev 1938]. In the case of $\mathbb{R}^N$ and $(-\Delta)_{\mathcal{F}}^{\alpha/4}$, it takes the form

$$\int_{\mathbb{R}^N} |(-\Delta)_{\mathcal{F}}^{\alpha/4}\varphi|^2 \geq S(\alpha, N)\left(\int_{\mathbb{R}^N} |\varphi|^{2N/(N-\alpha)}\right)^{\frac{N-\alpha}{N}} \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^N). \tag{2-4}$$

The value of $S(\alpha, N)$ can be seen, for instance, in [Lieb 1983]. It is independent of the domain and is not attained in any bounded domain, although it is attained in $\mathbb{R}^N$.

The condition (2-1) is equivalent to

$$\int_\Omega f\varphi < c(\alpha, N)\frac{\|\varphi\|_{H_0^{\alpha/2}}^{(N+\alpha)/\alpha}}{\|\varphi\|_p^{N/\alpha}} \quad \text{for all } \varphi \in H_0^{\alpha/2}(\Omega)\backslash\{0\}. \tag{2-5}$$

Moreover, since

$$\int_\Omega f\varphi \leq \|f\|_{H^{-\alpha/2}}\|\varphi\|_{H_0^{\alpha/2}}, \tag{2-6}$$

using the Sobolev inequality (2-3) we obtain the following sufficient condition for f to satisfy (2-1):

$$\|f\|_{H^{-\alpha/2}} \leq c(\alpha, N)S(\alpha, N)^{N/2\alpha}. \tag{2-7}$$

Remarks. (1) An assumption on the size of f like (2-1) is necessary in order to find solutions of problem (P). For example, if f is a sufficiently large positive constant, then problem (P) has no solutions.

(2) Condition (2-7) seems not to be sharp, in view of the result in [Castro and Zuluaga 1993] for the case $\alpha = 2$, which could also be proved in our functional framework.

The associated energy functional to problem (P) is given by

$$I(u) = \frac{1}{2}\int_\Omega \left|(-\Delta)^{\alpha/4}u\right|^2 - \frac{1}{p}\int_\Omega |u|^p - \int_\Omega fu.$$

Clearly, *critical points* of I correspond to solutions of (P) in the sense of (1-4). Indeed, one of the solutions we will construct in the proof of Theorem 2.1 is a local minimum of I in $H_0^{\alpha/2}(\Omega)$.

3. Proof of Theorem 2.1

First solution. We start with the definition of the Nehari manifold associated to problem (P):

$$\mathcal{N} = \{u \in H_0^{\alpha/2}(\Omega) : u \not\equiv 0,\ \langle I'(u), u\rangle = 0\}.$$

It is natural to look for solutions in this manifold. Note that the condition $u \in \mathcal{N}$ is equivalent to the identity

$$\|u\|^2_{H_0^{\alpha/2}} = \|u\|_p^p + \int_\Omega fu. \tag{3-1}$$

Therefore the functional I restricted to $\mathcal{N}$ takes the equivalent forms

$$I(u) = \frac{\alpha}{2N}\|u\|^2_{H_0^{\alpha/2}} - \frac{N+\alpha}{2N}\int_\Omega fu = \frac{\alpha}{2N}\|u\|_p^p - \frac{1}{2}\int_\Omega fu. \tag{3-2}$$

We will use both expressions in the sequel. In particular, using the first one we deduce that the functional I is bounded from below on $\mathcal{N}$:

$$I(u) \ge \frac{\alpha}{2N}\|u\|^2_{H_0^{\alpha/2}} - \frac{N+\alpha}{2N}\|f\|_{H^{-\alpha/2}}\|u\|_{H_0^{\alpha/2}} \ge -\frac{(N+\alpha)^2}{8N\alpha}\|f\|^2_{H^{-\alpha/2}}, \tag{3-3}$$

where the last step is a consequence of the minimization of the function

$$\alpha t^2 - (N+\alpha)\|f\|_{H^{-\alpha/2}}\, t.$$

Remark. Taking (3-3) into account, it makes sense to define

$$c_0 = \inf_{\mathcal{N}} I > -\infty, \tag{3-4}$$

although the functional is not bounded from below in the whole space $H_0^{\alpha/2}(\Omega)$.

Note that if u_0 is a local minimum of I in $H_0^{\alpha/2}(\Omega)$, then necessarily

$$\|u_0\|^2_{H_0^{\alpha/2}} - (p-1)\|u_0\|_p^p \ge 0.$$

In fact, as we will prove in Lemma 3.4, this inequality is strict; namely,

$$\|u_0\|^2_{H_0^{\alpha/2}} - (p-1)\|u_0\|_p^p > 0. \tag{3-5}$$

In the same way, if u_0 is a local maximum of I, we have

$$\|u_0\|^2_{H_0^{\alpha/2}} - (p-1)\|u_0\|_p^p < 0. \tag{3-6}$$

Thus, we first minimize the restriction of the functional I to $\mathcal{N}$ in order to find a critical point and therefore a solution to the problem (P). As we will see, c_0 is achieved. To prove that, we start with some preliminary results.

Lemma 3.1. *Let $f \not\equiv 0$ satisfy* (2-1). *Given $u \in H_0^{\alpha/2}(\Omega)$, assume $\int_\Omega fu > 0$. Then there exist unique numbers $\sigma = \sigma(u) > 0$ and $\tau = \tau(u) > \sigma$ with σu, $\tau u \in \mathcal{N}$ and such that* (3-5) *is satisfied with $u_0 = \sigma u$ and* (3-6) *with $u_0 = \tau u$.*

Proof. Let $\theta(t) = t\|u\|^2_{H_0^{\alpha/2}} - t^{p-1}\|u\|_p^p$. The maximum value of this function occurs at

$$t_M = \left(\frac{(N-\alpha)\|u\|^2_{H_0^{\alpha/2}}}{(N+\alpha)\|u\|_p^p}\right)^{\frac{N-\alpha}{2\alpha}},$$

and

$$\theta(t_M) = \frac{2\alpha}{N-\alpha}\left(\frac{N-\alpha}{N+\alpha}\right)^{\frac{N+\alpha}{2\alpha}} \frac{\|u\|^{(N+\alpha)/\alpha}_{H_0^{\alpha/2}}}{\|u\|_p^{N/\alpha}} = c(\alpha, N)\frac{\|u\|^{(N+\alpha)/\alpha}_{H_0^{\alpha/2}}}{\|u\|_p^{N/\alpha}}.$$

Note that θ is a concave function, increasing on $(0, t_M)$ and decreasing on (t_M, ∞), with $\lim_{t\to\infty}\theta(t) = -\infty$. By (2-5) we get $0 < \int_\Omega fu < \theta(t_M)$. Thus there exist two unique values $0 < \sigma < t_M < \tau$ such that

$$\theta(\tau) = \int_\Omega fu = \theta(\sigma), \quad \theta'(\tau) < 0 < \theta'(\sigma). \tag{3-7}$$

Multiplying in the previous expression by τ, we have

$$0 = \tau\theta(\tau) - \tau\int_\Omega fu = \|\tau u\|^2_{H_0^{\alpha/2}} - \|\tau u\|_p^p - \int_\Omega \tau f u;$$

thus $\tau u \in \mathcal{N}$. Moreover,

$$\|\tau u\|^2_{H_0^{\alpha/2}} - (p-1)\|\tau u\|_p^p = \tau^2\theta'(\tau) < 0.$$

Arguing in a similar way for σ, we obtain $\sigma u \in \mathcal{N}$ and

$$\|\sigma u\|^2_{H_0^{\alpha/2}} - (p-1)\|\sigma u\|_p^p = \sigma^2\theta'(\sigma) > 0. \qquad \square$$

Observe that without the condition $\int_\Omega fu > 0$ we still can find a value $\tau > 0$ with $\tau u \in \mathcal{N}$ satisfying (3-5). Conversely, the condition $\int_\Omega fu > 0$ is guaranteed for any function $u \in \mathcal{N}$ that satisfies (3-5).

We notice that the purpose of the strict condition (2-1) on f is just to obtain $\int_\Omega fu < \theta(t_M)$. It also appears to be of importance in Lemma 3.3 below. It is known that, when one deals with the problem associated to the standard Laplacian, and under certain hypotheses, the condition (2-1) is not sharp; see [Castro and Zuluaga 1993]. We suspect that a similar fact can occur in our case.

Corollary 3.2. *Under the hypotheses of Lemma 3.1, we have*

$$I(\tau u)=\max_{t\geq\sigma} I(tu) \quad \textit{and} \quad I(\sigma u)=\min_{0\leq t\leq\tau} I(tu).$$

Proof. It is straightforward once we notice that the function $g(t)=I(tu)$ satisfies $g'(t)=\theta(t)-\int_\Omega fu$. □

The next property uses a technical result analogous to Lemma 2.2 in [Tarantello 1992]. The proof follows almost word by word the one in that paper; see also [Brézis and Nirenberg 1989]. We only have to adapt the calculations to the functional framework of the fractional Laplacian. We leave the details to the interested reader.

Lemma 3.3. *Let $f\not\equiv 0$ satisfy* (2-1). *Then*

$$\mu_0:=\inf_{\substack{u\in H_0^{\alpha/2}(\Omega)\\ \|u\|_p=1}}\left(c(\alpha,N)\|u\|_{H_0^{\alpha/2}}^{(N+\alpha)/\alpha}-\int_\Omega fu\right) \tag{3-8}$$

is achieved, and moreover $\mu_0>0$.

The proof of this lemma is a straightforward adaptation to our setting of the similar one in the classical case; see Lemma 2.2 in [Tarantello 1992], which is inspired by the corresponding result in [Brézis and Nirenberg 1989].

The following lemma establishes a crucial property for minima of the functional; see inequality (3-5).

Lemma 3.4. *Let $f\not\equiv 0$ satisfy* (2-1) *and let $u\in\mathcal{N}$. Then*

$$\|u\|_{H_0^{\alpha/2}}^2-(p-1)\|u\|_p^p\neq 0.$$

Proof. Consider the functional, defined for $u\in H_0^{\alpha/2}(\Omega)$, $u\not\equiv 0$, by

$$\phi(u)=c(\alpha,N)\frac{\|u\|_{H_0^{\alpha/2}}^{(N+\alpha)/\alpha}}{\|u\|_p^{N/\alpha}}-\int_\Omega fu.$$

If $\|u\|_p=1$, we have

$$\phi(tu)=t\left(c(\alpha,N)\|u\|_{H_0^{\alpha/2}}^{(N+\alpha)/\alpha}-\int_\Omega fu\right).$$

Thus, given $\gamma>0$ (to be chosen later), by Lemma 3.3 we have

$$\inf_{\|u\|_p\geq\gamma}\phi(u)\geq\gamma\mu_0. \tag{3-9}$$

Note that this infimum is also positive.

Now we suppose, for a contradiction, that there exists $u\in\mathcal{N}$ such that

$$\|u\|_{H_0^{\alpha/2}}^2-(p-1)\|u\|_p^p=0. \tag{3-10}$$

By the Sobolev inequality (2-3), we obtain

$$S(\alpha, N)\|u\|_p^2 - (p-1)\|u\|_p^p \le 0,$$

which implies

$$\|u\|_p \ge \Big(\frac{S(\alpha, N)}{p-1}\Big)^{1/(p-2)} =: \gamma.$$

Now, substituting (3-10) into (3-1), we get

$$0 = \|u\|^2_{H_0^{\alpha/2}} - \|u\|_p^p - \int_\Omega fu = (p-2)\|u\|_p^p - \int_\Omega fu\,. \tag{3-11}$$

Finally, by (3-9) and (3-11), we conclude that

$$\begin{aligned}
0 < \gamma\mu_0 \le \phi(u) &= (p-2)\Big(\frac{N-\alpha}{N+\alpha}\Big)^{\frac{N+\alpha}{2\alpha}} \frac{\|u\|_{H_0^{\alpha/2}}^{(N+\alpha)/\alpha}}{\|u\|_p^{N/\alpha}} - \int_\Omega fu \\
&= (p-2)\Bigg[\Big(\frac{N-\alpha}{N+\alpha}\Big)^{\frac{N+\alpha}{2\alpha}} \frac{\|u\|_{H_0^{\alpha/2}}^{(N+\alpha)/\alpha}}{\|u\|_p^{N/\alpha}} - \|u\|_p^p\Bigg] \\
&= (p-2)\|u\|_p^p\Bigg[\bigg(\frac{(N-\alpha)\|u\|^2_{H_0^{\alpha/2}}}{(N+\alpha)\|u\|_p^p}\bigg)^{\frac{N-\alpha}{2\alpha}} - 1\Bigg] = 0,
\end{aligned}$$

which is a contradiction. □

Lemma 3.5. *Let $f \not\equiv 0$ be a function satisfying* (2-1). *Given $u \in \mathcal{N}$, there exists a positive function $\mu_u : H_0^{\alpha/2}(\Omega) \to \mathbb{R}$, differentiable in a neighborhood $\mathcal{U}_0$ of the origin in $H_0^{\alpha/2}(\Omega)$, such that*

$$\mu_u(0) = 1, \quad \mu_u(z)(u-z) \in \mathcal{N},$$

and

$$\langle \mu_u'(0), z\rangle = \frac{2\int_\Omega (-\Delta)^{\alpha/4}u(-\Delta)^{\alpha/4}z - p\int_\Omega |u|^{p-2}uz - \int_\Omega fz}{\|u\|^2_{H_0^{\alpha/2}} - (p-1)\|u\|_p^p} \tag{3-12}$$

for all $z \in \mathcal{U}_0$.

Proof. Consider the function

$$F(\mu, z) = \mu\|u-z\|^2_{H_0^{\alpha/2}(\Omega)} - \mu^{p-1}\|u-z\|_p^p - \int_\Omega f(u-z).$$

By Lemma 3.4, we have

$$\frac{\partial F}{\partial \mu}(1,0) = \|u\|^2_{H_0^{\alpha/2}} - (p-1)\|u\|_p^p \ne 0.$$

We complete the proof by applying the implicit function theorem to the function F at the point $(1, 0)$. □

We are now in a position to prove one of the main results of the paper.

Proposition 3.6. *The functional I possesses a local minimum in $H_0^{\alpha/2}(\Omega)$. In particular*, (P) *has a solution. Moreover*, *if f is nonnegative a.e. in Ω*, *this solution is nonnegative a.e. in Ω.*

Proof. Consider v, the unique solution to the equation $(-\Delta)^{\alpha/2}v = f$ in $H_0^{\alpha/2}(\Omega)$. Let $\sigma = \sigma(v)$ be as defined in Lemma 3.1. Since $\sigma(v)v \in \mathcal{N}$, we have

$$
\begin{aligned}
I(\sigma v) &= \frac{\sigma^2}{2}\|v\|^2_{H_0^{\alpha/2}} - \frac{\sigma^p}{p}\|v\|_p^p - \sigma\|v\|^2_{H_0^{\alpha/2}} \\
&= -\frac{\sigma^2}{2}\|v\|^2_{H_0^{\alpha/2}} + \frac{N+\alpha}{2N}\sigma^p\|v\|_p^p \\
&< -\frac{\alpha\sigma^2}{2N}\|v\|^2_{H_0^{\alpha/2}} = -\frac{\alpha\sigma^2}{2N}\|f\|^2_{H^{-\alpha/2}}.
\end{aligned} \tag{3-13}
$$

Then, by (3-3) and (3-13), the infimum in (3-4) satisfies the estimate

$$
-\frac{(N+\alpha)^2}{8N\alpha}\|f\|^2_{H^{-\alpha/2}} \le c_0 < -\frac{\alpha\sigma^2}{2N}\|f\|^2_{H^{-\alpha/2}} < 0. \tag{3-14}
$$

The expression (3-2) shows that the restriction of the functional I to $\mathcal{N}$ is weakly lower semicontinuous. Therefore, by Ekeland's variational principle [1974], we obtain a minimizing sequence of the functional I constrained to $\mathcal{N}$, i.e., $\{u_n\} \subset \mathcal{N}$ such that, for every $n \in \mathbb{N}$,

$$
\text{(i)}\ \ I(u_n) < c_0 + \frac{1}{n} \quad \text{and} \quad \text{(ii)}\ \ \frac{1}{n}\|u_n - v\|_{H_0^{\alpha/2}} \ge I(u_n) - I(v) \ \text{ for all } v \in \mathcal{N}.
$$

Combining (i), (3-14) and (3-2), we have

$$
I(u_n) = \frac{\alpha}{2N}\|u_n\|^2_{H_0^{\alpha/2}} - \frac{N+\alpha}{2N}\int_\Omega f u_n < c_0 + \frac{1}{n} < -\frac{\alpha\sigma^2}{2N}\|f\|^2_{H^{-\alpha/2}}
$$

for n large enough. Therefore

$$
\frac{\alpha\sigma^2}{N+\alpha}\|f\|^2_{H^{-\alpha/2}} \le \int_\Omega f u_n \quad \text{and} \quad \|u_n\|^2_{H_0^{\alpha/2}} \le \frac{N+\alpha}{\alpha}\int_\Omega f u_n. \tag{3-15}
$$

These inequalities, together with (2-6), give

$$
\frac{\alpha\sigma^2}{N+\alpha}\|f\|_{H^{-\alpha/2}} \le \|u_n\|_{H_0^{\alpha/2}} \le \frac{N+\alpha}{\alpha}\|f\|_{H^{-\alpha/2}}. \tag{3-16}
$$

Thus, we have (for a subsequence) that $u_n \rightharpoonup u_0$ weakly in $H^{\alpha/2}(\Omega)$, with $u_0 \not\equiv 0$. We claim that $\|I'(u_0)\|_{H^{-\alpha/2}} = 0$. Take $z \in H_0^{\alpha/2}(\Omega)$ with $\|z\|_{H_0^{\alpha/2}} = 1$. By Lemma 3.5, for every $n \in \mathbb{N}$, there exists a positive function μ_{u_n} such that

$$
w_\delta = \mu_{u_n}(\delta z)(u_n - \delta z) \in \mathcal{N}
$$

for $\delta > 0$ small enough. Set $t_n(\delta) = \mu_{u_n}(\delta z)$. Putting $v = w_\delta$ in (ii) and using the mean value theorem, we have

$$\frac{1}{n}\|w_\delta - u_n\|_{H_0^{\alpha/2}} \geq (1-t_n(\delta))\langle I'(w_\delta), u_n\rangle + \delta t_n(\delta)\langle I'(w_\delta), z\rangle + o(\delta).$$

Dividing by δ and taking the limit as δ goes to 0, we have

$$\frac{1}{n}\big(1+|t_n'(0)|\,\|u_n\|_{H_0^{\alpha/2}}\big) \geq \|I'(u_n)\|_{H^{-\alpha/2}}$$

with $|t_n'(0)| = \langle \mu_{u_n}'(0), z\rangle$, so that, by (3-16), we get

$$\|I'(u_n)\|_{H^{-\alpha/2}} \leq \frac{1}{n}\Big(1+\frac{N+\alpha}{\alpha}|t_n'(0)|\,\|f\|_{H^{-\alpha/2}}\Big). \tag{3-17}$$

Thus we are done once we prove that $|t_n'(0)|$ is uniformly bounded. By Lemma 3.5 and (3-16) we obtain

$$|t_n'(0)| \leq \frac{C}{\big|\|u_n\|^2_{H_0^{\alpha/2}} - (p-1)\|u_n\|_p^p\big|}$$

for some constant C. Assume, for a contradiction, that

$$\|u_n\|^2_{H_0^{\alpha/2}} - (p-1)\|u_n\|_p^p \to 0 \quad \text{as } n\to\infty. \tag{3-18}$$

From (3-18) and (3-1) we deduce the estimate

$$\int_\Omega f u_n = (p-2)\|u_n\|_p^p + o(1).$$

Moreover, from (3-16) we derive that $\|u_n\|_p \geq \gamma$ for some constant $\gamma > 0$. Thus, reasoning as in Lemma 3.4, we get

$$\begin{aligned} 0 < \gamma^{(N+\alpha)/2}\mu_0 &\leq \|u_n\|^{\alpha/N}_{H_0^{\alpha/2}}\phi(u_n) \\ &= (p-2)\left[\left(\frac{(N-\alpha)\|u_n\|^2_{H_0^{\alpha/2}}}{N+\alpha}\right)^{\frac{N-\alpha}{2\alpha}} - (\|u_n\|_p^p)^{(N-\alpha)/2\alpha}\right] \to 0, \end{aligned}$$

which leads to a contradiction. Therefore $\|I'(u_0)\|_{H^{-\alpha/2}} = 0$, and we have obtained a weak solution of (P).

To obtain strong convergence, we proceed as usual. Recalling that I is weakly lower semicontinuous in $\mathcal{N}$, we get

$$c_0 \leq I(u_0) \leq \lim_{n\to\infty} I(u_n) = c_0.$$

This implies, using (3-2), the limits

$$\lim_{n\to\infty}\|u_n\|_{H_0^{\alpha/2}} = \|u_0\|_{H_0^{\alpha/2}}, \qquad \lim_{n\to\infty}\|u_n\|_p = \|u_0\|_p.$$

To see that u_0 is a local minimum in $H_0^{\alpha/2}(\Omega)$, we first show that (3-5) holds. In fact, since $u_0 \in \mathcal{N}$ and also $\int_\Omega f u_0 > 0$ by (3-15), it is clear that one of the values $\sigma(u_0)$

or $\tau(u_0)$ given by Lemma 3.1 equals 1. Assume, for a contradiction (see Lemma 3.4), that u_0 satisfies (3-6), i.e., $\sigma(u_0) < \tau(u_0) = 1$. By Corollary 3.2, $I(\sigma(u_0)u_0) < I(u_0)$, which contradicts the fact that u_0 is the infimum in $\mathcal{N}$. Hence u_0 satisfies (3-5) and $\sigma(u_0) = 1$. We remark that having the strict inequality in (2-5) is crucial in the present argument. In particular, we have obtained $1 = \sigma(u_0) < t_M < \tau(u_0)$, or

$$1 < \left(\frac{(N-\alpha)\|u_0\|^2_{H_0^{\alpha/2}}}{(N+\alpha)\|u_0\|_p^p} \right)^{\frac{N-\alpha}{2\alpha}}, \tag{3-19}$$

which is the same. Take $\varepsilon > 0$ small enough such that

$$1 < \left(\frac{(N-\alpha)\|u_0 - z\|^2_{H_0^{\alpha/2}}}{(N+\alpha)\|u_0 - z\|_p^p} \right)^{\frac{N-\alpha}{2\alpha}} =: t_{M,\varepsilon} \tag{3-20}$$

for $\|z\|_{H_0^{\alpha/2}} < \varepsilon$. By Lemma 3.5, there exists a positive function $\mu_{u_0} : H_0^{\alpha/2}(\Omega) \to \mathbb{R}$ such that $\mu_{u_0}(z)(u_0 - z) \in \mathcal{N}$ for every $\|z\|_{H_0^{\alpha/2}} < \varepsilon$, with ε smaller if necessary. Indeed, by continuity we have $\mu_{u_0}(z) < t_{M,\varepsilon}$ for $\varepsilon > 0$ sufficiently small. Thus we get that $\mu_{u_0}(z)(u_0 - z)$ satisfies (3-5), and as a consequence of Lemma 3.1 and Corollary 3.2 applied to $u_0 - z$, we obtain

$$I(s(u_0 - z)) \geq I(\mu_{u_0}(z)(u_0 - z)) \geq I(u_0) \quad \text{for all } s \in (0, t_{M,\varepsilon}).$$

Since by (3-20) we can take $s = 1$, we conclude that $I(u_0 - z) \geq I(u_0)$ for every $\|z\|_{H_0^{\alpha/2}} < \varepsilon$, i.e, u_0 is a local minimum in $H_0^{\alpha/2}(\Omega)$.

To finish, we assume that $f \geq 0$. Then it follows that $\int_\Omega f|u_0| > 0$. Take $\sigma = \sigma(|u_0|) > 0$ and $\tau = \tau(|u_0|) > \sigma$. We have

$$\|u_0\|_p^p + \int_\Omega f u_0 = \|u_0\|^2_{H_0^{\alpha/2}} > (p-1)\|u_0\|_p^p$$

and, since $\tau|u_0|$ satisfies (3-6), we get

$$\tau^p \|u_0\|_p^p + \tau \int_\Omega f|u_0| = \tau^2 \big\| |u_0| \big\|^2_{H_0^{\alpha/2}} < (p-1)\tau^p \|u_0\|_p^p.$$

Thus,

$$(p-2)\|u_0\|_p^p < \int_\Omega f u_0 \leq \int_\Omega f|u_0| \leq (p-2)\tau^{p-1}\|u_0\|_p^p,$$

which implies $\tau > 1$. Therefore, by Corollary 3.2, we have

$$I(u_0) \leq I(\sigma|u_0|) \leq I(|u_0|).$$

On the other hand, by the generalized Stroock–Varopoulos inequality [de Pablo et al. 2012], we have

$$\int_\Omega \big|(-\Delta)^{\alpha/4}|u_0|\big|^2 \leq \int_\Omega |(-\Delta)^{\alpha/4} u_0|^2,$$

which implies $I(|u_0|) \le I(u_0)$. As a consequence, $I(u_0) = I(|u_0|)$, $\sigma = 1$, and thus $|u_0| \in \mathcal{N}$ is a solution. □

Second solution. We will look for the second solution using a classical approach that relies on the well-known mountain pass theorem; see [Ambrosetti and Rabinowitz 1973]. Recall that $\{u_n\} \subset H_0^{\alpha/2}(\Omega)$ is a Palais–Smale (PS for short) sequence of level c for I if $I(u_n) \to c$ and $\|I'(u_n)\|_{H^{-\alpha/2}} \to 0$ as $n \to \infty$. Moreover, we say that I satisfies a PS *condition of level* c (PS_c for short) if every PS sequence of level c for I has a convergent subsequence in $H_0^{\alpha/2}(\Omega)$. As is usual in critical problems, the functional I does not satisfy a global PS condition, i.e., a PS_c condition for every c. Our aim is to prove that I satisfies a PS_c condition for c below a precise critical level c^*. We define

$$c^* = c_0 + \frac{\alpha}{2N} S(\alpha, N)^{N/\alpha}. \tag{3-21}$$

This value, which is obtained in the next lemma, also appears in several other contexts, for instance when one applies the concentration-compactness principle to critical problems; see [Ambrosetti et al. 1994; Brézis and Nirenberg 1983; Hardy and Littlewood 1928; Lions 1985] for the standard case, and, for example, [Barrios et al. 2012] for the fractional case, and [Barrios et al. 2014; Servadei and Valdinoci $\ge$ 2014] for different nonlocal operators which include a different fractional Laplacian.

Lemma 3.7. *The functional I satisfies a local PS_c condition for any $c < c^*$.*

Proof. Let $\{u_n\} \subset H_0^{\alpha/2}(\Omega)$ be a PS sequence of level $c < c^*$. It is easy to check that the u_n are uniformly bounded in $H^{\alpha/2}(\Omega)$. Thus, there exists a subsequence (still denoted by u_n) such that $u_n \rightharpoonup z_0$ weakly in $H_0^{\alpha/2}(\Omega)$. As a consequence, $z_0 \in H_0^{\alpha/2}(\Omega)$ is a solution of (P).

We rewrite u_n as $u_n = u_0 + \phi_n$ with $\phi_n \to 0$. Applying the Brézis–Lieb lemma [1983] we get

$$\|u_n\|_p^p = \|u_0\|_p^p + \|\phi_n\|_p^p + o(1). \tag{3-22}$$

On one hand, by (3-22) and taking n large enough we have

$$\begin{aligned} c^* > I(u_n) &= I(u_0) + \frac{1}{2}\|\phi_n\|^2_{H_0^{\alpha/2}} - \frac{1}{p}\|\phi_n\|_p^p + o(1) \\ &\ge c_0 + \frac{1}{2}\|\phi_n\|^2_{H_0^{\alpha/2}} - \frac{1}{p}\|\phi_n\|_p^p + o(1). \end{aligned}$$

Hence, by the definition of c^* in (3-21), we obtain

$$\frac{1}{2}\|\phi_n\|^2_{H_0^{\alpha/2}} - \frac{1}{p}\|\phi_n\|_p^p < \frac{\alpha}{2N} S(\alpha, N)^{N/\alpha} + o(1). \tag{3-23}$$

Taking into account that $\{u_n\}$ is a PS sequence, we have

$$\begin{aligned} o(1) &= \langle I'(u_n), u_n\rangle = \|u_n\|^2_{H_0^{\alpha/2}} - \|u_n\|_p^p - \int_\Omega f u_n \\ &= \|u_0\|^2_{H_0^{\alpha/2}} - \|u_0\|_p^p - \int_\Omega f u_0 + \|\phi_n\|^2_{H_0^{\alpha/2}} - \|\phi_n\|_p^p + o(1) \\ &= \langle I'(u_0), u_0\rangle + \|\phi_n\|^2_{H_0^{\alpha/2}} - \|\phi_n\|_p^p + o(1) \\ &= \|\phi_n\|^2_{H_0^{\alpha/2}} - \|\phi_n\|_p^p + o(1). \end{aligned} \tag{3-24}$$

Now we want to prove that $\{\phi_n\}$ has a subsequence strongly converging to 0 in $H_0^{\alpha/2}(\Omega)$. Suppose, on the contrary, that there are $C, k > 0$ such that $\|\phi_n\|_{H_0^{\alpha/2}} \geq C$ for all $n \geq k$. Using (2-3) in (3-24), we get $\|\phi_n\|_p^{p-2} \geq S(\alpha, N) + o(1)$ and hence

$$\|\phi_n\|_p^p \geq S(\alpha, N)^{N/\alpha} + o(1). \tag{3-25}$$

From (3-23) and (3-25), we have

$$\begin{aligned} \frac{\alpha}{2N} S(\alpha, N)^{N/\alpha} &\leq \frac{\alpha}{2N}\|\phi_n\|_p^p + o(1) = \frac{1}{2}\|\phi_n\|^2_{H_0^{\alpha/2}} - \frac{1}{p}\|\phi_n\|_p^p + o(1) \\ &< \frac{\alpha}{2N} S(\alpha, N)^{N/\alpha}, \end{aligned}$$

which is a contradiction. □

It is known (see, for instance, [Chen et al. 2006]) that the minimizers for the Sobolev inequality (2-4) are given by the two-parameter family of functions

$$u_{\varepsilon,x_0}(x) = \frac{\varepsilon^{(N-\alpha)/2}}{(|x-x_0|^2+\varepsilon^2)^{(N-\alpha)/2}}, \tag{3-26}$$

where $x_0 \in \mathbb{R}^N$, $\varepsilon > 0$. In what follows we will use the notation

$$A = \|u_{\varepsilon,x_0}\|_p, \quad B = \|(-\Delta)^{\alpha/4}_{\mathscr{F}} u_{\varepsilon,x_0}\|_2 = \left(\int_{\mathbb{R}^N} |\xi|^\alpha |\hat{u}_{\varepsilon,x_0}(\xi)|^2\, d\xi\right)^{1/2}. \tag{3-27}$$

Note that the last quantity defines a norm in the homogeneous fractional Sobolev space $\dot{H}^{\alpha/2}(\mathbb{R}^N)$. Both numbers A and B are clearly independent of ε and x_0, and moreover $B^2 = S(\alpha, N)A^2$.

Without loss of generality we may assume that $0 \in \Omega$. We define a cut-off function $\theta \in \mathscr{C}^\infty(\mathbb{R}^N)$ by $\theta(x) = \theta_0(|x|/\rho)$ with $\rho > 0$, where $\theta_0 \in \mathscr{C}^\infty(\mathbb{R})$ is a nonincreasing function satisfying

$$\theta_0(s) = 1 \quad \text{if } s \leq \tfrac{1}{2}, \qquad \theta_0(s) = 0 \quad \text{if } s \geq 1.$$

We now recall that, if u_0 is the solution constructed in the previous subsection, we can find a set $\Sigma \subset \Omega$ of positive Lebesgue measure such that $u_0 \geq \nu > 0$ a.e.

in Σ (replace u_0 with $-u_0$ and f with $-f$ if necessary). For $x_0 \in \Sigma$, we set $\tilde{u}_{\varepsilon,x_0} = \theta u_{\varepsilon,x_0} \in H_0^{\alpha/2}(\Omega)$.

Proposition 3.8. *In the above notation, for a.e.* $x_0 \in \Sigma$ *there exists* $\varepsilon^* = \varepsilon^*(x_0) > 0$ *sufficiently small such that*

$$\sup_{t\geq 0} I(u_0 + t\tilde{u}_{\varepsilon,x_0}) < c^* \quad \textit{for all } 0 < \varepsilon < \varepsilon^*. \tag{3-28}$$

We observe that when one evaluates the functional in (3-28), one needs to evaluate $\|\tilde{u}_{\varepsilon,x_0}\|_{H_0^{\alpha/2}}$; i.e., one needs to evaluate the fractional Laplacian of a product of functions. This requires the use of a different, but equivalent, norm which does not involve directly the fractional Laplacian. It uses the so-called *α-harmonic extension* of [Caffarelli and Silvestre 2007] for $(-\Delta)_{\mathcal{F}}^{\alpha/4}$, adapted to the bounded domain setting in [Brändle et al. 2013; Cabré and Tan 2010; Stinga and Torrea 2010].

Consider the semi-infinite cylinder $\mathscr{C}_\Omega = \{(x, y) : x \in \Omega,\ y > 0\} \subset \mathbb{R}^{N+1}_+$ and its lateral boundary $\partial_L\mathscr{C}_\Omega = \partial\Omega \times (0, \infty)$. For a function $u \in H_0^{\alpha/2}(\Omega)$, we denote its α-harmonic extension to $\mathscr{C}_\Omega$ by $w = \mathrm{E}_\alpha(u)$, defined as the solution to the problem

$$\begin{cases} \operatorname{div}(y^{1-\alpha}\nabla w) = 0 & \text{in } \mathscr{C}_\Omega, \\ w = 0 & \text{on } \partial_L\mathscr{C}_\Omega, \\ w = u & \text{on } \Omega \times \{y = 0\}. \end{cases} \tag{3-29}$$

Then the equation

$$-\kappa_\alpha \lim_{y\searrow 0} \frac{\partial w}{\partial y} = (-\Delta)^{\alpha/2} u$$

holds, with κ_α a positive constant. Let $X_0^\alpha(\mathscr{C}_\Omega)$ be the completion of $\mathscr{C}_0^\infty(\Omega\times[0, \infty))$ under the norm

$$\|\phi\|_{X_0^\alpha} = \left(\kappa_\alpha \int_{\mathscr{C}_\Omega} y^{1-\alpha} |\nabla\phi(x, y)|^2 \, dx\, dy\right)^{1/2}.$$

In the case $\Omega = \mathbb{R}^N$ the corresponding space for functions in the upper half-space $\mathbb{R}^{N+1}_+$ is $X^\alpha(\mathbb{R}^{N+1}_+)$, which can be defined in the same way with the integral extended to $\mathbb{R}^{N+1}_+$. The extension operator can be characterized also as a minimization of the X_0^α-norm (equivalently, the X^α norm) for all the functions with common trace at $y = 0$. Note that the extension operator is an isometry from $H_0^{\alpha/2}(\Omega)$ to $X_0^\alpha(\mathscr{C}_\Omega)$ and from $H^{\alpha/2}(\mathbb{R}^N)$ to $X^\alpha(\mathbb{R}^{N+1}_+)$; that is,

$$\|\mathrm{E}_\alpha(\psi)\|_{X_0^\alpha} = \|\psi\|_{H_0^{\alpha/2}} \quad \text{for all } \psi \in H_0^{\alpha/2}(\Omega), \tag{3-30}$$

$$\|\mathrm{E}_\alpha(\psi)\|_{X^\alpha} = \|\psi\|_{\dot{H}^{\alpha/2}} \quad \text{for all } \psi \in \dot{H}^{\alpha/2}(\mathbb{R}^N). \tag{3-31}$$

This means that

$$\|w\|_{X_0^\alpha} \geq \|w(\,\cdot\,, 0)\|_{H_0^{\alpha/2}} \quad \text{for all } w \in X_0^\alpha(\mathscr{C}_\Omega) \tag{3-32}$$

and

$$\|w\|_{X^\alpha} \geq \|w(\,\cdot\,,0)\|_{H^{\alpha/2}} \quad \text{for all } w \in X^\alpha(\mathbb{R}^N). \tag{3-33}$$

See [Brändle et al. 2013] for the details.

We define the family $w_{\varepsilon,x_0} = E_\alpha(u_{\varepsilon,x_0})$, with u_{ε,x_0} given in (3-26). We want to find a family of modified minimizers in the extended space, by using a cutoff function in $\mathscr{C}_\Omega$. To do that we take

$$\phi(x,y) = \theta_0\left(\frac{(|x-x_0|^2+y^2)^{1/2}}{\rho}\right),$$

where θ_0 is defined above. With this notation we define $\tilde{w}_{\varepsilon,\,x_0} = \phi w_{\varepsilon,x_0} \in X_0^\alpha(\mathscr{C}_\Omega)$ and $\tilde{w}_{\varepsilon,\,x_0}(\,\cdot\,,0) = \tilde{u}_{\varepsilon,\,x_0}(\,\cdot\,)$.

In [Barrios et al. 2012, Lemma 3.8] the following estimates for $\tilde{w}_{\varepsilon,x_0}$ are proved:

$$\|\tilde{w}_{\varepsilon,x_0}\|^2_{X_0^\alpha} = \|w_{\varepsilon,x_0}\|^2_{X^\alpha} + O(\varepsilon^{N-\alpha}). \tag{3-34}$$

In view of (3-30), (3-32) and (3-34), we have

$$\|\tilde{u}_{\varepsilon,x_0}\|^2_{H_0^{\alpha/2}} \leq B^2 + O(\varepsilon^{N-\alpha}). \tag{3-35}$$

Moreover,

$$\|\tilde{u}_{\varepsilon,x_0}\|_p^p \geq A^p + O(\varepsilon^N). \tag{3-36}$$

We now state a result that will be useful in the proof of Proposition 3.8. Its proof follows the same arguments as in [Brézis and Nirenberg 1989], with obvious changes for our setting, so we omit the details.

Lemma 3.9. *Assume that $a, b > 0$ and that u_0, $\tilde{u}_{\varepsilon,x_0}$ are defined as above. For $t \in [a,b]$, we have*

$$\begin{aligned}\|u_0 + t\tilde{u}_{\varepsilon,x_0}\|_p^p = \|u_0\|_p^p + t^p\|\tilde{u}_{\varepsilon,x_0}\|_p^p + pt\int_\Omega |u_0|^{p-2}u_0\tilde{u}_{\varepsilon,x_0}\\ + pt^{p-1}\int_\Omega |\tilde{u}_{\varepsilon,x_0}|^{p-2}\tilde{u}_{\varepsilon,x_0}u_0 + o(\varepsilon^{(N-\alpha)/2}).\end{aligned} \tag{3-37}$$

Proof of Proposition 3.8. On the one hand, since $I(u_0 + t\tilde{u}_{\varepsilon,x_0})|_{t=0} = c_0 < c^*$, by a continuity argument we can find t_0, $\varepsilon_0 > 0$ both small enough such that

$$I(u_0 + t\tilde{u}_{\varepsilon,x_0}) < c^* \quad \text{for all } t \in (0,t_0) \text{ and all } \varepsilon \in (0,\varepsilon_0).$$

On the other hand, by Lemma 3.9, together with (3-36) and the fact that A and B are independent of ε, we have

$$I(u_0 + t\tilde{u}_{\varepsilon,x_0}) \to -\infty \text{ as } t \to \infty \quad \text{for all } \varepsilon > 0.$$

Hence there exists $t_1 > 0$ large enough that

$$I(u_0 + t\tilde{u}_{\varepsilon,x_0}) < c_0 < c^* \quad \text{for all } t \geq t_1 \text{ and all } \varepsilon \in (0, \varepsilon_0).$$

Thus, we just need to prove that there exists $\varepsilon^* \in (0, \varepsilon_0)$ such that

$$\sup_{t_0 \leq t \leq t_1} I(u_0 + t\tilde{u}_{\varepsilon,x_0}) < c^*$$

for every $0 < \varepsilon < \varepsilon^*$.

Take $t \in [t_0, t_1]$. Clearly, we have

$$I(u_0+t\tilde{u}_{\varepsilon,x_0}) = \frac{1}{2}\|u_0\|^2_{H_0^{\alpha/2}} + t\int_\Omega (-\Delta)^{\alpha/4}u_0(-\Delta)^{\alpha/4}\tilde{u}_{\varepsilon,x_0} + \frac{t^2}{2}\|\tilde{u}_{\varepsilon,x_0}\|^2_{H_0^{\alpha/2}} - \frac{1}{p}\|u_0 + t\tilde{u}_{\varepsilon,x_0}\|_p^p - \int_\Omega f u_0 - t\int_\Omega f\tilde{u}_{\varepsilon,x_0}. \tag{3-38}$$

Since $S(\alpha, N)$ is attained for the function u_{ε,x_0}, substituting (3-35), (3-36) and (3-37) in (3-38) we have

$$\begin{aligned} I(u_0 + t\tilde{u}_{\varepsilon,x_0}) \leq \frac{1}{2}\|u_0\|^2_{H_0^{\alpha/2}} + t\int_\Omega (-\Delta)^{\alpha/4}u_0(-\Delta)^{\alpha/4}\tilde{u}_{\varepsilon,x_0} + \frac{t^2}{2}B^2 \\ - \frac{1}{p}\|u_0\|_p^p - \frac{t^p}{p}A^p - t\int_\Omega |u_0|^{p-2}u_0\tilde{u}_{\varepsilon,x_0} - t^{p-1}\int_\Omega |\tilde{u}_{\varepsilon,x_0}|^{p-1}u_0 \\ - \int_\Omega f u_0 - t\int_\Omega f\tilde{u}_{\varepsilon,x_0} + o(\varepsilon^{(N-\alpha)/2}). \end{aligned}$$

On the other hand, since u_0 is solution of (P), we get

$$I(u_0+t\tilde{u}_{\varepsilon,\,x_0}) \leq I(u_0) + \frac{t^2}{2}B^2 - t^{p-1}\int_\Omega |\tilde{u}_{\varepsilon,\,x_0}|^{p-1}u_0 - \frac{t^p}{p}A^p + o(\varepsilon^{(N-\alpha)/2}). \tag{3-39}$$

Extending u_0 by zero outside Ω, we get

$$\begin{aligned} \int_\Omega |\tilde{u}_{\varepsilon,x_0}|^{p-1}u_0 &= \int_{\mathbb{R}^N} u_0(x)\theta^{p-1}(x)\frac{\varepsilon^{(N+\alpha)/2}}{(|x-x_0|^2+\varepsilon^2)^{(N+\alpha)/2}} \\ &= \varepsilon^{(N-\alpha)/2}\int_{\mathbb{R}^N} u_0(x)\theta^{p-1}(x)\,\frac{1}{\varepsilon^N}\eta\Big(\frac{x-x_0}{\varepsilon}\Big), \end{aligned}$$

with $\eta(x) = (|x|^2+1)^{-(N+\alpha)/2}$. Thus, there exists a constant $\nu > 0$ such that

$$\int_{\mathbb{R}^N} u_0(x)\theta^{p-1}(x)\,\frac{1}{\varepsilon^N}\eta\Big(\frac{x-x_0}{\varepsilon}\Big) \geq K\nu$$

for every $\varepsilon > 0$ sufficiently small, $x_0 \in \Sigma$ and $K = \int_{\mathbb{R}^N}\eta(x) < \infty$. Therefore

$$\int_\Omega |\tilde{u}_{\varepsilon,x_0}|^{p-1}u_0 = \varepsilon^{(N-\alpha)/2}K\nu + o(\varepsilon^{(N-\alpha)/2}). \tag{3-40}$$

Substituting (3-40) in (3-39), we have

$$I(u_0 + t\tilde{u}_{\varepsilon,x_0}) \leq c_0 + \frac{t^2}{2}B^2 - t^{p-1}\varepsilon^{(N-\alpha)/2}K\nu - \frac{t^p}{p}A^p + o(\varepsilon^{(N-\alpha)/2}). \tag{3-41}$$

Let us now define the function

$$g(s)=\frac{s^2}{2}B^2-s^{p-1}\varepsilon^{(N-\alpha)/2}K\nu-\frac{s^p}{p}A^p \quad \text{for } s>0,$$

and let $s_\varepsilon>0$ be the point of global maximum, i.e.,

$$0=g'(s_\varepsilon)=s_\varepsilon B^2-(p-1)s_\varepsilon^{p-2}\varepsilon^{(N-\alpha)/2}K\nu-s_\varepsilon^{p-1}A^p. \tag{3-42}$$

We denote $S_0=(B^2/A^p)^{1/(p-2)}$. Note that $0<s_\varepsilon<S_0$ and $s_\varepsilon\to S_0$ as $\varepsilon\searrow 0$. Let $\delta_\varepsilon>0$ be such that $s_\varepsilon=S_0(1-\delta_\varepsilon)$. Since $B^2/A^p=S_0^{p-2}$, by (3-42) we have

$$\Big(\frac{B^{2(p-1)}}{A^p}\Big)^{1/(p-2)}(1-\delta_\varepsilon-(1-\delta_\varepsilon)^{p-1})-(p-1)S_0^{p-2}(1-\delta_\varepsilon)^{p-2}\varepsilon^{(N-\alpha)/2}K\nu=0,$$

which implies

$$(p-2)\Big(\frac{B^{2(p-1)}}{A^p}\Big)^{1/(p-2)}\delta_\varepsilon=(p-1)S_0^{p-2}\varepsilon^{(N-\alpha)/2}K\nu+o(\varepsilon^{(N-\alpha)/2}). \tag{3-43}$$

By (3-41) with $t=s_\varepsilon$ and (3-43), we have

$$\begin{aligned} I(u_0+s_\varepsilon\tilde u_{\varepsilon,x_0}) &\le c_0+\frac{s_\varepsilon^2}{2}B^2-s_\varepsilon^{p-1}\varepsilon^{(N-\alpha)/2}K\nu-\frac{s_\varepsilon^p}{p}A^p+o(\varepsilon^{(N-\alpha)/2})\\ &=c_0+\frac{S_0^2}{2}B^2-S_0^{p-1}\varepsilon^{(N-\alpha)/2}K\nu-\frac{S_0^p}{p}A^p+o(\varepsilon^{(N-\alpha)/2})\\ &=c_0+\frac{\alpha}{2N}S(\alpha,N)^{N/\alpha}-S_0^{p-1}\varepsilon^{(N-\alpha)/2}K\nu+o(\varepsilon^{(N-\alpha)/2})\\ &=c^*-S_0^{p-1}\varepsilon^{(N-\alpha)/2}K\nu+o(\varepsilon^{(N-\alpha)/2}). \end{aligned}$$

Taking ε sufficiently small, this finishes the proof. □

Lemma 3.10. *Assume $f\not\equiv 0$ satisfies* (2-1). *Then the functional I possesses a critical point different from u_0. In particular*, (P) *has a second solution. Moreover, if $f\ge 0$ a.e. in Ω then this solution is nonnegative a.e. in Ω.*

Proof. Set $\eta_{\varepsilon,M}=u_0+M\tilde u_{\varepsilon,x_0}$, with $0<\varepsilon<\varepsilon^*$ and $x_0\in\Sigma$ so that (3-28) holds. Assume that $M>0$ is large enough such that $I(\eta_{\varepsilon,M})<c_0$.

Now we set

$$\Gamma=\{\gamma:[0,1]\to H_0^{\alpha/2}(\Omega) \text{ such that } \gamma(0)=u_0,\ \gamma(1)=\eta_{\varepsilon,M}\}.$$

By Proposition 3.8 we have

$$c_0<c_1=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I(\gamma(t))<c^*.$$

Thus, using the mountain pass theorem, we obtain a PS sequence of level c_1, and as a consequence of Lemma 3.7 we can find a critical point u_1 in $H_0^{\alpha/2}(\Omega)$ with energy level $c_1>c_0$, i.e., u_1 is a solution of (P) with $u_1\not\equiv u_0$.

To prove that the solution is positive in the case that $f \geq 0$, we set

$$\tilde{\mathcal{N}} := \{u \in \mathcal{N} : u \text{ satisfies (3-6)}\}$$

and $c_2 = \inf_{\tilde{\mathcal{N}}} I$. Is easy to see that, taking a larger M if necessary, we can assume

$$c_0 < c_2 \leq c_1 < c^*. \tag{3-44}$$

Now, using Ekeland's variational principle and following the steps of the proof of Proposition 3.6, we can obtain a PS sequence of level c_2. Again, Lemma 3.7 implies the existence of a solution $u_2 \in \mathcal{N}$ such that $I(u_2) = c_2$. Put $\tau = \tau(|u_2|) > 0$. Then $\tau|u_2| \in \tilde{\mathcal{N}}$. Finally, by Corollary 3.2,

$$\inf_{\tilde{\mathcal{N}}} I = I(u_2) = \max_{t \geq t_M} I(tu_2) \geq I(\tau u_2) \geq I(\tau|u_2|),$$

which finishes the proof. □

Remark. Note that u_2 could coincide with u_1.

4. Proof of Theorem 2.2

When f satisfies condition (2-2) instead of (2-1), we use an approximation argument.

Proof of Theorem 2.2. Consider a sequence of numbers $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0, 1)$ such that $\varepsilon_k \searrow 0$ as $k \to \infty$, and define $f_k = (1 - \varepsilon_k)f$. Clearly f_k satisfies condition (2-1) for every $k \in \mathbb{N}$. We define I_k and $\mathcal{N}_k$ in a natural way:

$$I_k(u) = \frac{1}{2}\int_\Omega |(-\Delta)^{\alpha/4}u|^2 - \frac{1}{p}\int_\Omega |u|^p - \int_\Omega f_k u,$$
$$\mathcal{N}_k = \{u \in H_0^{\alpha/2}(\Omega) : u \not\equiv 0,\ \langle I_k'(u), u\rangle = 0\}.$$

Let $u_k \in \mathcal{N}_k$ be the local minimum found via Theorem 2.1, namely,

$$I_k(u_k) = \inf_{\mathcal{N}_k} I_k := c_k.$$

In particular, we have

$$\langle I_k'(u_k), z\rangle = 0 \quad \text{for all } z \in H_0^{\alpha/2}(\Omega), \tag{4-1}$$

and moreover

$$\|u_k\|^2_{H_0^{\alpha/2}} - \|u_k\|_p^p - \int_\Omega f_k u_k = 0, \tag{4-2}$$

which, by (2-3) and (2-6), implies that $\|u_k\|^2_{H_0^{\alpha/2}} < C$ for any $k \in \mathbb{N}$ and some constant $C > 0$ independent of k. Take $u \in \mathcal{N}$ satisfying (3-5). Then

$$\int_\Omega f_k u > 0 \quad \text{for all } k \in \mathbb{N}.$$

Applying Lemma 3.1 with $f = f_k$ and $\mathcal{N} = \mathcal{N}_k$, we find the values $0 < \sigma_k < t_{M_k} < \tau_k$ such that $\sigma_k u, \tau_k u \in \mathcal{N}_k$. Since u satisfies the inequality (3-5), we have $\tau_k > 1$. Thus, by Corollary 3.2 we have $I_k(\sigma_k u) \leq I_k(u)$, which leads to

$$c_k \leq I_k(\sigma_k u) \leq I_k(u) \leq I(u) + \varepsilon_k \|f\|_{H^{-\alpha/2}} \|u\|_{H_0^{\alpha/2}} \leq I(u) + C\varepsilon_k.$$

In particular, $c_k \leq c_0 + C\varepsilon_k$. Finally, reasoning as in (3-13) with $f = f_k$, we obtain

$$-\frac{(N+\alpha)^2}{8N\alpha}\|f\|^2_{H^{-\alpha/2}} < -\frac{(N+\alpha)^2}{8N\alpha}\|f_k\|^2_{H^{-\alpha/2}} \leq c_k \leq c_0 + C\varepsilon_k.$$

After passing to a subsequence, we can assume that c_k converges to some value c' such that

$$-\frac{(N+\alpha)^2}{8N\alpha}\|f\|^2_{H^{-\alpha/2}} \leq c' \leq c_0.$$

Moreover, since $\|u_k\|^2_{H_0^{\alpha/2}}$ is uniformly bounded, again for a subsequence if necessary, we have $u_k \rightharpoonup u^*$ weakly in $H_0^{\alpha/2}(\Omega)$. Then by (4-1) we have

$$\langle I'(u^*), z\rangle = 0 \quad \text{for all } z \in H_0^{\alpha/2}(\Omega),$$

and $I(u^*) \leq c_0$. This implies $u^* \in \mathcal{N}$ and $I(u^*) = c_0$, which finishes the proof. The positivity of the solution when the datum f is taken nonnegative follows from the same argument as in the proof of Theorem 2.1. □

We finally remark that the solution constructed in this way is not necessarily a minimum of the functional. Therefore we cannot prove the mountain pass geometry in order to find a second solution.

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