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ON THE CLASSIFICATION OF COMPLETE AREA-STATIONARY AND STABLE SURFACES IN THE SUBRIEMANNIAN SOL MANIFOLD

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We study the classification of area-stationary and stable C^2 regular surfaces in the space of the rigid motions of the Minkowski plane $E(1, 1)$, equipped with its subriemannian structure. We construct examples of area-stationary surfaces that are not foliated by subriemannian geodesics. We also prove that there exist an infinite number of C^2 area-stationary surfaces with a singular curve. Finally we show the stability of C^2 area-stationary surfaces foliated by subriemannian geodesics.

1. Introduction

The study of the subriemannian area functional in three-dimensional pseudohermitian manifolds and in other subriemannian spaces has been largely investigated in the last years, see [Ambrosio et al. 2006; Barbieri and Citti 2011; Barone Adesi et al. 2007; Bigolin and Cassano 2010; Capogna et al. 2009; Cheng and Hwang 2010; Cheng et al. 2012; 2005; 2007; Danielli et al. 2007; 2008; 2009; Galli 2013; Galli and Ritoré 2013; Garofalo and Nhieu 1996; Hladky and Pauls 2008; Hurtado et al. 2010; Hurtado and Rosales 2008; ≥ 2014; Ritoré 2009; Ritoré and Rosales 2008; Rosales 2012; Shcherbakova 2009], among others.

One of the more interesting questions concerning the subriemannian area functional is this:

Problem 1. Which are the area-minimizing surfaces in a given three-dimensional contact subriemannian manifold?

A surface Σ is *area-minimizing* if $A(\Sigma) \leqslant A(\tilde{\Sigma})$, for any compact deformation $\tilde{\Sigma}$ of Σ. To answer the previous question, a natural preliminary step is to study the *area-stationary* surfaces, the critical points of the area functional.

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Problem 2. Which are the area-stationary surfaces in a given three-dimensional contact subriemannian manifold?

We will consider these questions in the class of C^2 regular surfaces. For a general introduction about the study of the area functional in subriemannian spaces, we refer the interested reader to [Capogna et al. 2007] and [Galli 2012], which treat the case of $\mathbb{H}^n$ and the contact subriemannian manifolds respectively.

In Sasakian space forms, the classification of C^2 area stationary surfaces was given in [Hurtado et al. 2010] in the case of the Heisenberg group $\mathbb{H}^1$ and in [Rosales 2012] for the Sasakian structures of S^3 and $\widetilde{\mathrm{SL}}_2(\mathbb{R})$. In the case of pseudohermitian three-manifolds that are not Sasakian, the only known results concerning Problem 1 and Problem 2 are given in [Galli 2013], where the group of the rigid motions of the Euclidean plane $E(2)$ is studied.

Concerning the three-dimensional pseudohermitian manifolds, we have the following classification result, [Perrone 1998, Theorem 3.1], in terms of the Webster scalar curvature W and of the pseudohermitian torsion τ:

Proposition 1.1. *Let M be a simply connected contact 3-manifold that is homogeneous* (in the sense of [Boothby and Wang 1958]). *The following possibilities arise*:

1. *If M is unimodular, it is*
   - (i) *the first Heisenberg group $\mathbb{H}^1$ when $W = |\tau| = 0$;*
   - (ii) *the three-sphere group $\mathrm{SU}(2)$ when $W > 2|\tau|$;*
   - (iii) *the group $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ when $-2|\tau| \neq W < 2|\tau|$;*
   - (iv) *the group $\widetilde{E}(2)$, universal cover of the group of rigid motions of the Euclidean plane, when $W = 2|\tau| > 0$;*
   - (v) *the group $E(1, 1)$ of rigid motions of Minkowski 2-space, when $W = -2|\tau| < 0$;*
2. *If M is not unimodular, the Lie algebra is given by*

$$[X, Y] = \alpha Y + 2T, \quad [X, T] = \gamma Y, \quad [Y, T] = 0, \quad \alpha \neq 0,$$

where $\{X, Y\}$ is an orthonormal basis of $\mathcal{H}$, $J(X) = Y$ and T is the Reeb vector field. In this case $W < 2|\tau|$ and when $\gamma = 0$ the structure is Sasakian and $W = -\alpha^2$.

The only case for which Problems 1 and 2 have not been investigated is that of Sol geometry, modeled by the space $E(1, 1)$. Its study is the aim of the present work.

After some preliminaries, the paper is organized as follows.

In Section 3, we compute explicitly the coordinates of the characteristic curves with given initial conditions. These curves play an important role in the study of area-stationary surfaces, since the regular part $\Sigma - \Sigma_0$ of a surface Σ is foliated by characteristic curves, that are not in general subriemannian geodesics, since $E(1, 1)$ is characterized by a nonvanishing pseudohermitian torsion.

Section 4 is the core of the paper. We first characterize the C^2 complete, area-stationary surfaces immersed in $E(1, 1)$ with singular points or singular curves that are subriemannian geodesics. On the other hand, for the first time in the three-dimensional pseudohermitian setting, we also find examples of area-stationary surfaces that are not foliated by subriemannian geodesics. We stress that these examples form an infinite family; that is, given an horizontal curve Γ, we can construct an area-stationary surface having Γ as singular set Σ_0.

Finally in Section 5 we prove that complete area-stationary surfaces with nonempty singular set, whose characteristic curves are subriemannian geodesics, are stable. We also find three families of nonsingular planes that are area-minimizing, using a calibration argument.

We remark that Section 5 opens two interesting questions. Is a stable complete area-stationary surface in $E(1, 1)$ with a singular curve always foliated by subriemannian geodesics in $\Sigma - \Sigma_0$? Do some other complete stable area-stationary surfaces in $E(1, 1)$ with empty singular set exist?

2. Preliminaries

The group $E(1, 1)$ of rigid motions of the Minkowski plane. We consider the group of rigid motions of the Minkowski plane $E(1, 1)$, a unimodular Lie group with a natural subriemannian structure. As a model of $E(1, 1)$ we choose the underlying manifold $\mathbb{R}^3$ with the following orthonormal basis of left-invariant vector fields:

$$X = \frac{\partial}{\partial z}, \quad Y = \frac{1}{\sqrt{2}}\left(-e^z \frac{\partial}{\partial x} + e^{-z}\frac{\partial}{\partial y}\right), \quad T = \frac{1}{\sqrt{2}}\left(e^z \frac{\partial}{\partial x} + e^{-z}\frac{\partial}{\partial y}\right). \tag{2-1}$$

We have that $\{X, Y\}$ is an orthonormal basis of the horizontal distribution $\mathcal{H}$ and T is the Reeb vector field. The scalar product of two vector fields W and V with respect to the metric induced by the basis $\{X, Y, T\}$ will be often denoted by $\langle W, V\rangle$. This structure of $E(1, 1)$ is characterized by the following Lie brackets, [Milnor 1976],

$$[X, Y] = -T, \quad [X, T] = -Y, \quad [Y, T] = 0. \tag{2-2}$$

In fact, applying [Galli 2013, (9.1) and (9.3)] we obtain that the Webster scalar curvature is $W = -\frac{1}{2}$ and the matrix of the pseudohermitian torsion τ in the X, Y, T

basis is

$$\begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The following derivatives can be easily computed:

$$\begin{aligned} &\nabla_X X = 0, \quad \nabla_Y X = 0, \quad \nabla_T X = \tfrac{1}{2} Y, \\ &\nabla_X Y = 0, \quad \nabla_Y Y = 0, \quad \nabla_T Y = -\tfrac{1}{2} X, \end{aligned} \tag{2-3}$$

where ∇ denotes the pseudohermitian connection; see [Dragomir and Tomassini 2006]. Moreover we have $-2|\tau|^2 = W < 0$, which characterizes $E(1, 1)$; see [Perrone 1998]. We also define the involution J on $\mathcal{H}$, called the complex structure, by $J(X) = Y$ and $J(Y) = -X$.

The geometry of regular surfaces in $E(1,1)$. Consider a C^1 surface Σ immersed in $E(1, 1)$. We define the *subriemannian area* of Σ as

$$A(\Sigma) = \int_\Sigma |N_h|\, d\Sigma,$$

where N_h denotes the projection of the Riemannian unit normal N to $\mathcal{H}$ and $d\Sigma$ denotes the Riemannian area element on Σ. In the sequel we always denote by N the inner unit normal. The singular set Σ_0 is composed of the points in which $T\Sigma$ coincides with $\mathcal{H}$. Outside Σ_0, we can define the *horizontal unit normal* as

$$\nu_h := \frac{N_h}{|N_h|}$$

and the *characteristic vector field* as $Z := J(\nu_h)$. It is straightforward to verify that $\{Z, S\}$ is an orthonormal basis of $T\Sigma$ outside Σ_0, where

$$S := \langle N, T\rangle \nu_h - |N_h| T.$$

Finally, outside Σ_0, we define the *mean curvature* of Σ by

$$H := -\langle \nabla_Z \nu_h, Z\rangle. \tag{2-4}$$

Given a surface Σ as the zero level set of a function $u : \Omega \subset E(1, 1) \to \mathbb{R}$, we can express

$$\nu_h = -\frac{u_z X + \frac{1}{\sqrt{2}}(-e^z u_x + e^{-z} u_y) Y}{\sqrt{u_z^2 + \frac{1}{2}(-e^z u_x + e^{-z} u_y)^2}} \tag{2-5}$$

and

$$Z = \frac{\frac{1}{\sqrt{2}}(-e^z u_x + e^{-z} u_y) X - u_z Y}{\sqrt{u_z^2 + \frac{1}{2}(-e^z u_x + e^{-z} u_y)^2}}. \tag{2-6}$$

We define a *minimal surface* as a surface with vanishing mean curvature H.

Proposition 2.1. *Let Σ be a minimal surface defined as the zero level set of a C^2 function $u : \Omega \subset E(1,1) \to \mathbb{R}$. Then u satisfies the equation*

$$u_{zz}(-e^z u_x + e^{-z} u_y)^2 + u_z^2(-e^{2z} u_{xx} - 2u_{xy} + e^{-2z} u_{yy}) \\ - u_z(-e^z u_x + e^{-z} u_y)(-2e^z u_{xz} - e^z u_x + 2e^{-z} u_{yz} - e^{-z} u_y) = 0 \tag{2-7}$$

on Ω.

Proof. From (2-4), (2-5) and (2-6) we find that u has to satisfy

$$Y(u)^2 X(X(u)) - Y(u)X(u)Y(X(u)) - Y(u)X(u)X(Y(u)) + X(u)^2 Y(Y(u)) = 0 \tag{2-8}$$

on Ω. Now, using (2-1), we can transform (2-8) into (2-7). □

We will call (2-7) the *minimal surface equation*.

Remark 2.2. From (2-8), we immediately note that a surface Σ satisfying $u_z \equiv 0$ or $-e^z u_x + e^{-z} u_y \equiv 0$ is always minimal.

In the following lemma, we compute some important quantities related to the torsion and the geometry of a surface. The lemma follows from [Galli 2013, (9.8)].

Lemma 2.3. *Let Σ be a C^1 surface in $E(1,1)$. Then we have*

$$\langle \tau(Z), Z\rangle = -\langle Z, X\rangle\langle Z, Y\rangle = \langle \nu_h, X\rangle\langle \nu_h, Y\rangle = -\langle \tau(\nu_h), \nu_h\rangle,$$
$$\langle \tau(Z), \nu_h\rangle = \tfrac{1}{2}(\langle Z, Y\rangle^2 - \langle Z, X\rangle^2).$$

3. Characteristic curves in $E(1,1)$

In this section we will study the equation of the integral curves of Z on Σ, known as *characteristic curves*. It is well-known that a surface with constant mean curvature H is foliated by characteristic curves in $\Sigma - \Sigma_0$. In general, a *characteristic curve* is an arc-length parametrized horizontal curve γ in $E(1,1)$ that satisfies the equation

$$\nabla_{\dot\gamma}\dot\gamma + HJ(\dot\gamma) = 0, \tag{3-1}$$

where $\dot\gamma$ denotes the tangent vector along γ and H is the (constant) curvature of γ. We stress that a curve γ satisfying (3-1) is not a subriemannian geodesic. In fact a characteristic curve γ is a subriemannian geodesic if and only if $H = 0$ and $\dot\gamma$ satisfies the additional equation

$$\langle \tau(\dot\gamma), \dot\gamma\rangle = 0, \tag{3-2}$$

see [Rumin 1994, Proposition 15], which forces γ to be an integral curve of X or Y, by Lemma 2.3.

Proposition 3.1. *Let γ be a characteristic curve in $E(1,1)$ with curvature $H=0$. Then γ belongs to the family of curves*

$$\gamma(t)=(x_0+\dot{x}_0t,\, y_0+\dot{y}_0t,\, z_0) \tag{3-3}$$

or to the family

$$\gamma(t)=\left(x_0+\frac{\dot{x}_0}{\dot{z}_0}(e^{\dot{z}_0t}-1),\, y_0-\frac{\dot{y}_0}{\dot{z}_0}(e^{-\dot{z}_0t}-1),\, z_0+\dot{z}_0t\right), \tag{3-4}$$

where $\gamma(0)=(x_0,y_0,z_0)$ and $\dot{\gamma}(0)=(\dot{x}_0,\dot{y}_0,\dot{z}_0)$.

Proof. We consider the curve $\gamma: I\to\Sigma$, where I denotes an interval. We express $\gamma(t)=(x(t),y(t),z(t))$ and we get

$$\begin{aligned}\dot{\gamma}(t)&=\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\dot{z}\frac{\partial}{\partial z}\\ &=\dot{z}X+\frac{1}{\sqrt{2}}(\dot{y}e^{z}-\dot{x}e^{-z})Y+\frac{1}{\sqrt{2}}(\dot{y}e^{z}+\dot{x}e^{-z})T,\end{aligned} \tag{3-5}$$

since

$$\frac{\partial}{\partial x}=\frac{1}{\sqrt{2}}e^{-z}(T-Y),\qquad \frac{\partial}{\partial y}=\frac{1}{\sqrt{2}}e^{z}(Y+T).$$

From (3-5) and the fact that γ is horizontal, we have

$$\dot{y}e^{z}+\dot{x}e^{-z}=0. \tag{3-6}$$

Now $\nabla_{\dot{\gamma}}\dot{\gamma}=0$ is equivalent to the system

$$\begin{cases}\dot{z}=\dot{z}_0,\\ \dot{y}e^{z}-\dot{x}e^{-z}=c_0,\end{cases} \tag{3-7}$$

where $\dot{z}_0$ and c_0 are constants. We distinguish two cases. The first one corresponds to $\dot{z}_0=0$. This means which $z=z_0$, with $z_0\in\mathbb{R}$, and so (3-6) and (3-7) reduce to

$$\begin{cases}2\dot{y}=e^{-z_0}c_0,\\ 2\dot{x}=-e^{z_0}c_0,\end{cases} \tag{3-8}$$

which implies $\gamma(t)=(x_0-e^{z_0}(c_0/2)t,\, y_0+e^{-z_0}(c_0/2)t,\, z_0)$, where $c_0\neq 0$ and $x_0, y_0\in\mathbb{R}$.

The second possibility is $\dot{z}_0\neq 0$, which implies $z(t)=z_0+\dot{z}_0t$, with $z_0\in\mathbb{R}$. In this case integrating (3-8) we obtain

$$\gamma(t)=\left(x_0+\frac{c_0e^{z_0}}{2\dot{z}_0}-\frac{c_0}{2\dot{z}_0}e^{z_0+\dot{z}_0t},\, y_0+\frac{c_0e^{-z_0}}{2\dot{z}_0}-\frac{c_0}{2\dot{z}_0}e^{-(z_0+\dot{z}_0)t},\, z_0+\dot{z}_0t\right),$$

where $\gamma(0)=(x_0,y_0,z_0)$. Finally, to conclude the result, we note that

$$\frac{c_0}{2}=\dot{y}_0e^{z_0}=-\dot{x}_0e^{-z_0}.$$

□

4. Complete area-stationary surfaces with nonempty singular set in $E(1,1)$

Complete area-stationary surfaces containing isolated singular points. The local structure of a C^1 surface Σ with prescribed mean curvature $H \in C$, in a neighborhood of an isolated singular point, is well understood [Cheng et al. 2012, Theorem D and Corollary E]. In the less general case of a bounded mean curvature surface of class C^2, applying [Cheng et al. 2005, Theorem B and Section 7], we have:

Lemma 4.1. *Let Σ be a C^2 oriented immersed surface with constant mean curvature H in $E(1,1)$. If $p \in \Sigma_0$ is an isolated singular point, then there exists $r > 0$ and $\lambda \in \mathbb{R}$ such that the set*

$$D_r(p) = \big\{\gamma^H_{p,v}(s) : v \in T_p\Sigma,\ |v| = 1,\ s \in [0, r)\big\},$$

is an open neighborhood of p in Σ, where $\gamma^H_{p,v}$ denotes the characteristic curve starting from p in the direction v with curvature H.

First we construct the unique example, up to contact isometries, of a minimal surface with isolated singular points.

Proposition 4.2. *Let Σ be a C^2 complete, area-stationary surface immersed in $E(1,1)$ with $H = 0$ and with an isolated singular point $p_0 = (x_0, y_0, z_0)$. Then $\Sigma = \{(x, y, z) \in E(1,1) : e^{z-z_0}(y - y_0) + x - x_0 = 0\}$.*

Proof. By Lemma 4.1, the only possible way to construct a complete area-stationary surface, with a singular point p_0, is to consider the union of all characteristic curves γ of curvature 0 with initial conditions $\gamma(0) = p_0$ and $\dot\gamma(0) \in T_{p_0}\Sigma = \mathcal{H}_{p_0}$, $|\dot\gamma(0)| = 1$. We can suppose $p_0 = 0$, since $E(1,1)$ is homogeneous.

We consider the initial velocities

$$\begin{aligned}\dot\gamma_\alpha(0) &= \cos\alpha\, X(0) + \sin\alpha\, Y(0)\\ &= \cos\alpha\,\frac{\partial}{\partial z}(0) + \frac{\sin\alpha}{\sqrt{2}}\Big(-\frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0)\Big),\end{aligned}$$

for $\alpha \in [0, 2\pi[$. In this way we obtain as characteristic curves

$$\begin{aligned}\gamma_\alpha(t) &\\ &= \Big(-\frac{\sin\alpha}{\sqrt{2}\cos\alpha}(e^{\cos(\alpha)t} - 1),\ -\frac{\sin\alpha}{\sqrt{2}\cos\alpha}(e^{-\cos(\alpha)t} - 1),\ \cos(\alpha)t\Big),\end{aligned} \tag{4-1}$$

for $\alpha \in \,]0, 2\pi[$ and $\gamma_0(t) = (0, 0, t)$ when $\alpha = 0$. At this point it is easy show that Σ is the zero level set of the function $e^z y + x$ (or equivalently $e^{-z}x + y$), which satisfies (2-7). □

Complete area-stationary surfaces containing singular curves.

Lemma 4.3 [Galli 2013, Corollary 5.4]. *Let Σ be a C^2 minimal surface with nonempty singular set Σ_0 immersed in $E(1,1)$. Then Σ is area stationary if and only if the characteristic curves meet the singular curves orthogonally with respect the metric $\langle\ ,\ \rangle$, induced by the orthonormal basis (2-1).*

A minimal area-stationary surface cannot contain more than one singular curve:

Lemma 4.4. *Let Σ be a C^2 complete, minimal, area-stationary surface, containing a singular curve Γ immersed in $E(1,1)$. Then Σ cannot contain more singular curves.*

Proof. We consider a singular curve

$$\Gamma(\varepsilon)=\big(x(\varepsilon),y(\varepsilon),z(\varepsilon)\big)$$

in Σ. Since Σ is foliated by characteristic curves, we can parametrize it by the map

$$F(\varepsilon,t)=\gamma_\varepsilon(t)=\big(x(\varepsilon,t),y(\varepsilon,t),z(\varepsilon,t)\big),$$

where $\gamma_\varepsilon(t)$ is the characteristic curve with initial data $\gamma_\varepsilon(0)=\Gamma(\varepsilon)$ and

$$\begin{aligned}\dot\gamma_\varepsilon(0)&=J(\dot\Gamma(\varepsilon))=\dot z(\varepsilon)J(X)+\tfrac{1}{\sqrt2}(\dot y(\varepsilon)e^{z(\varepsilon)}-\dot x(\varepsilon)e^{-z(\varepsilon)})J(Y)\\&=\tfrac{1}{\sqrt2}\big(-\dot z(\varepsilon)e^{z(\varepsilon)},\ \dot z(\varepsilon)e^{-z(\varepsilon)},\ \dot x(\varepsilon)e^{-z(\varepsilon)}-\dot y(\varepsilon)e^{z(\varepsilon)}\big).\end{aligned}\tag{4-2}$$

We define

$$V_\varepsilon(t):=\frac{\partial F}{\partial\varepsilon}(t,\varepsilon),$$

which is a smooth Jacobi-like vector field along $\gamma_\varepsilon(t)$; see [Galli 2013, Section 4]. At a singular point (ε,t), the vertical component of V_ε vanishes:

$$\langle V_\varepsilon,T\rangle(\varepsilon,t)=\frac{\partial x}{\partial\varepsilon}(\varepsilon,t)e^{-z(\varepsilon,t)}+\frac{\partial y}{\partial\varepsilon}(\varepsilon,t)e^{z(\varepsilon,t)}=0.$$

We suppose that Γ is not an integral curve of X or Y. Then from the expressions of the component of $F(\varepsilon,t)$, which are

$$\begin{aligned}x(\varepsilon,t)&=x(\varepsilon)+\frac{\dot z(\varepsilon)e^{z(\varepsilon)}}{\dot x(\varepsilon)e^{-z(\varepsilon)}-\dot y(\varepsilon)e^{z(\varepsilon)}}(e^{(\dot x(\varepsilon)e^{-z(\varepsilon)}-\dot y(\varepsilon)e^{z(\varepsilon)})t/\sqrt2}-1),\\y(\varepsilon,t)&=y(\varepsilon)-\frac{\dot z(\varepsilon)e^{-z(\varepsilon)}}{\dot x(\varepsilon)e^{-z(\varepsilon)}-\dot y(\varepsilon)e^{z(\varepsilon)}}(e^{-(\dot x(\varepsilon)e^{-z(\varepsilon)}-\dot y(\varepsilon)e^{z(\varepsilon)})t/\sqrt2}-1),\\z(\varepsilon,t)&=z(\varepsilon)+\frac{\dot x(\varepsilon)e^{-z(\varepsilon)}-\dot y(\varepsilon)e^{z(\varepsilon)}}{\sqrt2}t,\end{aligned}\tag{4-3}$$

we have

$$
\begin{aligned}
&\langle V_\varepsilon, T\rangle(\varepsilon,t)\\
&=\left(\dot x(\varepsilon)e^{-z(\varepsilon)}+\frac{\ddot z(\varepsilon)}{\dot x(\varepsilon)e^{-z(\varepsilon)}-\dot y(\varepsilon)e^{z(\varepsilon)}}-\frac{\dot z(\varepsilon)\dfrac{\partial}{\partial\varepsilon}(\dot x(\varepsilon)e^{-z(\varepsilon)}-\dot y(\varepsilon)e^{z(\varepsilon)})}{(\dot x(\varepsilon)e^{-z(\varepsilon)}-\dot y(\varepsilon)e^{z(\varepsilon)})^2}\right)\\
&\qquad\qquad\cdot\left(e^{-(\dot x(\varepsilon)e^{-z(\varepsilon)}-\dot y(\varepsilon)e^{z(\varepsilon)})\frac{t}{\sqrt2}}-e^{(\dot x(\varepsilon)e^{-z(\varepsilon)}-\dot y(\varepsilon)e^{z(\varepsilon)})\frac{t}{\sqrt2}}\right)\\
&\quad+\frac{\dot z(\varepsilon)^2}{\left(\dot x(\varepsilon)e^{-z(\varepsilon)}-\dot y(\varepsilon)e^{z(\varepsilon)}\right)^2}\left(e^{-(\dot x(\varepsilon)e^{-z(\varepsilon)}-\dot y(\varepsilon)e^{z(\varepsilon)})\frac{t}{\sqrt2}}+e^{(\dot x(\varepsilon)e^{-z(\varepsilon)}-\dot y(\varepsilon)e^{z(\varepsilon)})\frac{t}{\sqrt2}}-2\right),
\end{aligned}
$$

which, when t is positive, vanishes only for the values $(\varepsilon, 0)$. On the other hand, if Γ is an integral curve of Y we get

$$
x(\varepsilon,t)=x(\varepsilon),\quad y(\varepsilon,t)=y(\varepsilon),\quad z(\varepsilon,t)=z(\varepsilon)+\frac{\dot x(\varepsilon)e^{-z(\varepsilon)}-\dot y(\varepsilon)e^{z(\varepsilon)}}{\sqrt2}t, \tag{4-4}
$$

and if Γ is an integral curve of X we have

$$
x(\varepsilon,t)=x(\varepsilon)-\frac{\dot z(\varepsilon)e^{z(\varepsilon)}}{\sqrt2}t,\quad y(\varepsilon,t)=y(\varepsilon)+\frac{\dot z(\varepsilon)e^{-z(\varepsilon)}}{\sqrt2}t,\quad z(\varepsilon,t)=z(\varepsilon). \tag{4-5}
$$

In both cases, the singular set is only the curve $\Gamma(\varepsilon)$. □

The vertical component of V_ε can be computed more directly using [Galli 2013, Proposition 4.3], since $H = 0$. On the other hand, the explicit computation of the components of the parametrization $F(\varepsilon, t)$ allows us to characterize all C^2 area-stationary complete surfaces with a singular curve that is a characteristic curve of curvature 0. We stress that, when the characteristic curves are subriemannian geodesics, these examples can also be constructed from Remark 2.2.

Proposition 4.5. *Let Σ be an area-stationary surface with $H = 0$, with a singular curve Γ that is a characteristic curve of curvature 0. Then, if Γ is a subriemannian geodesic, Σ belongs to one of the following families*:

(i) $\{ax + by + c = 0 : (x, y, z) \in E(1, 1), a, b, c \in \mathbb{R}\}$;

(ii) $\{e^{z-z_0}(y - y_0) + e^{z_0-z}(x - x_0) = 0 : (x, y, z) \in E(1, 1), x_0, y_0, z_0 \in \mathbb{R}\}$.

Otherwise, we suppose that Γ is a characteristic curve passing through (x_0, y_0, z_0) with velocity $(\dot x_0, \dot y_0, \dot z_0)$, $\dot x_0, \dot y_0, \dot z_0 \neq 0$. We can parametrize Σ by $F : \mathbb{R}^2 \to E(1, 1)$, with $F(\varepsilon, t) = (x(\varepsilon, t), y(\varepsilon, t), z(\varepsilon, t))$ and

$$
\begin{aligned}
x(\varepsilon,t) &= x_0 + \frac{\dot{x}_0}{\dot{z}_0}(e^{\dot{z}_0\varepsilon}-1) + \frac{\dot{z}_0 e^{z_0+\dot{z}_0\varepsilon}}{\dot{x}_0 e^{-z_0}-\dot{y}_0 e^{z_0}}(e^{(\dot{x}_0 e^{-z_0}-\dot{y}_0 e^{z_0})t/\sqrt{2}}-1),\\
y(\varepsilon,t) &= y_0 - \frac{\dot{y}_0}{\dot{z}_0}(e^{-\dot{z}_0\varepsilon}-1) - \frac{\dot{z}_0 e^{-z_0-\dot{z}\varepsilon}}{\dot{x}_0 e^{-z_0}-\dot{y}_0 e^{z_0}}(e^{-(\dot{x}_0 e^{-z_0}-\dot{y}_0 e^{z_0})t/\sqrt{2}}-1),\\
z(\varepsilon,t) &= z_0 + \dot{z}_0\varepsilon + \frac{\dot{x}_0 e^{-z_0}-\dot{y}_0 e^{z_0}}{\sqrt{2}}t.
\end{aligned}
\tag{4-6}
$$

Remark 4.6. The surfaces parametrized by (4-6) are the first examples of area-stationary surfaces that are not foliated by subriemannian geodesics in three-dimensional contact subriemannian manifolds, up to our knowledge. In fact this phenomenon does not appear in the group of rigid motions [Galli 2013, Lemma 10.4], even if its pseudohermitian torsion is nonvanishing. In that case, the presence of two singular curves forces the surface to be foliated by subriemannian geodesics or to be not area-stationary. On the other hand, it is well-known that a minimal surface is foliated by subriemannian geodesics in any three-dimensional Sasakian manifold.

Remark 4.7. Given any horizontal curve $\Gamma = (x(\varepsilon), y(\varepsilon), z(\varepsilon))$ in $E(1,1)$, we stress that (4-3) provides a parametrization $F(\varepsilon,t) : \mathbb{R}^2 \to \Sigma \subset E(1,1)$ of a complete area-stationary surface Σ with $\Sigma_0 = \Gamma$.

5. Complete area-minimizing surfaces in $E(1,1)$

Complete area-minimizing surfaces with empty singular set. Proposition 9.8 of [Galli 2013] gave a general necessary condition for the stability of a nonsingular surface in pseudohermitian Lie groups. This condition states that the quantity

$$
W - \langle \tau(Z), \nu_h \rangle = \langle \nu_h, Y \rangle^2 - 1 = \langle Z, X \rangle^2 - 1
$$

must be always nonpositive. This condition is trivial in $E(1,1)$ due to the negativity of the Webster scalar curvature. On the other hand it has been used crucially in the classification of the stable, area-stationary surfaces without singular points in the manifolds $\mathbb{H}^1$, SU(2) and $\widetilde{E(2)}$, see [Galli 2013; Hurtado et al. 2010; Rosales 2012]. In any case, we can prove:

Proposition 5.1. *The families of planes*

(i) $\{x + c = 0 : (x,y,z) \in E(1,1), c \in \mathbb{R}\}$,

(ii) $\{y + c = 0 : (x,y,z) \in E(1,1), c \in \mathbb{R}\}$,

(iii) $\{z + c = 0 : (x,y,z) \in E(1,1), c \in \mathbb{R}\}$,

are area-stationary, foliated by subriemannian geodesics, and area-minimizing.

Proof. We prove the result for $\Sigma = \{x = 0 : (x,y,z) \in E(1,1)\}$, since all the cases are similar. In this case, from (2-5) and (2-6) we have

$$
\nu_h = Y, \quad Z = -X.
$$

So the integral curves of Z are subriemannian geodesics and $\Sigma_0 = \varnothing$. Now Remark 2.2 implies that Σ is area-stationary. Finally we can foliate a neighborhood of Σ in $E(1, 1)$ by translating Σ. We obtain a foliation by area-stationary surfaces, and a standard calibration argument implies that Σ is area-minimizing; see, for example, [Barone Adesi et al. 2007; Ritoré 2009; Ritoré and Rosales 2008, § 5]. □

Remark 5.2. The planes in the family

$$\{ax + by + cz + d = 0 : (x, y, z) \in E(1, 1), a, b, c, d \in \mathbb{R}\}$$

are not minimal, since they do not satisfy (2-7).

A very natural question is: are the planes in Proposition 5.1 the unique complete area-minimizing surfaces with empty singular set in $E(1, 1)$? We have only been able to find the following sufficient condition:

Lemma 5.3. *Let Σ be a C^2 complete oriented minimal surface immersed in $E(1, 1)$, with empty singular set Σ_0. If $\langle N, T\rangle \leqslant 0$ holds on Σ, then Σ is stable.*

Proof. Taking into account the expression of the stability operator for nonsingular surfaces in [Galli 2013, Lemma 8.3], we only need to show that

$$2Z(G) + G^2 \leqslant 0 \quad \text{on } \Sigma, \qquad \text{where } G := \frac{\langle N, T\rangle}{|N_h|}.$$

Given a point p in Σ, let I be an open interval containing the origin and let $\alpha : I \to \Sigma$ be a piece of the integral curve of S passing through p. Consider the characteristic curve $\gamma_\varepsilon(s)$ of Σ with $\gamma_\varepsilon(0) = \alpha(\varepsilon)$. We define the map $F : I \times \mathbb{R} \to \Sigma$ by $F(\varepsilon, s) = \gamma_\varepsilon(s)$ and set $V(s) := (\partial F/\partial\varepsilon)(0, s)$, which is a Jacobi-like vector field along γ_0; see [ibid., Proposition 4.3]. Let $'$ represent differentiation with respect to s. Using [ibid., Lemma 3.1, (4.4) and (4.5)] we get

$$\langle V, T\rangle(0) = -|N_h|, \tag{5-1}$$

$$\langle V, T\rangle'(0) = -\langle N, T\rangle, \tag{5-2}$$

$$\langle V, T\rangle''(0) = -|N_h|(Z(G) + G^2). \tag{5-3}$$

It is easy to show that $g(V, T)$ never vanishes along γ_0 since Σ_0 is empty; see [ibid., proof of Proposition 9.5]. On the other hand, by [ibid., Proposition 4.3] and Lemma 2.3, we have that $\langle V, T\rangle$ satisfies the ordinary differential equation

$$\langle V, T\rangle'''(s) - \langle Z, X\rangle^2 \langle V, T\rangle'(s) = 0$$

along γ_0. We suppose that $\langle Z, X\rangle \neq 0$. Taking into account the initial conditions (5-1), (5-2) and (5-3), we obtain

$$\langle V, T\rangle(s) = a \cosh(|\langle Z, X\rangle| s) + b \sinh(|\langle Z, X\rangle| s) + c,$$

where

$$a = \frac{|N_h|(Z(G)+G^2)}{\langle X, Z\rangle^2}, \quad b = -\frac{\langle N, T\rangle}{|\langle Z, X\rangle|}, \quad c = -|N_h| - a.$$

We have that $\langle V, T\rangle(s) \neq 0$ implies

$$a+b = \frac{|N_h|(Z(G)+G^2)}{\langle X, Z\rangle^2} - \frac{\langle N, T\rangle}{|\langle Z, X\rangle|} \leqslant 0.$$

Then we can conclude that

$$2Z(G)+G^2 \leqslant 2(Z(G)+G^2) \leqslant 2|\langle Z, X\rangle| \frac{\langle N, T\rangle}{|N_h|} \leqslant 0$$

on γ_0. Now since the choice of p is arbitrary, we get the statement.

If $\langle Z, X\rangle = 0$, we conclude that Σ is stable if and only if $\langle N, T\rangle = 0$, by [Galli 2013, Proposition 9.8]. □

Remark 5.4. The surfaces described in the points (i), (ii) and (iii) of Proposition 5.1 are characterized by $\langle N, T\rangle = -e^z/\sqrt{2}$, $\langle N, T\rangle = -e^z/\sqrt{2}$ and $\langle N, T\rangle \equiv 0$, respectively, where N denotes the inward unit normal on Σ. In the third family the planes are vertical surfaces and they satisfy $W - \langle \tau(Z), \nu_h\rangle \equiv 0$.

Taking into account the geometric invariants of $E(1, 1)$, we expect the existence of other examples of complete oriented minimal surface with empty singular set.

Complete area-minimizing surfaces with nonempty singular set. We consider the stability operator constructed in [Galli 2013, Theorem 8.6].

Lemma 5.5. *Let Σ be a C^2 oriented minimal surface immersed in E(1,1), with singular set Σ_0 and $\partial\Sigma = \varnothing$. If Σ is stable then, for any function $u \in C_0^1(\Sigma)$ such that $Z(u) = 0$ in a tubular neighborhood of a singular curve and constant in a tubular neighborhood of an isolated singular point, we have $Q(u) \geqslant 0$, where*

$$Q(u) := \int_\Sigma \{|N_h|^{-1} Z(u)^2 + |N_h|\big((1+\langle Z, Y\rangle^2) - \big(|N_h|(\tfrac{1}{2} - \langle Z, Y\rangle^2) - \langle \nabla_S \nu_h, Z\rangle\big)^2\big)u^2\}\, d\Sigma$$
$$+ 4\int_{(\Sigma_0)_c} \langle N, T\rangle \langle Z, Y\rangle^2 \langle Z, \nu\rangle u^2 \, d(\Sigma_0)_c + \int_{(\Sigma_0)_c} S(u)^2 \, d(\Sigma_0)_c.$$

Here $d(\Sigma_0)_c$ is the Riemannian length measure on $(\Sigma_0)_c$ and ν is the external unit normal to $(\Sigma_0)_c$.

Corollary 5.6. *Let Σ be a plane in the family*

$$\{ax+by+c = 0 : (x, y, z) \in E(1, 1), a, b, c \in \mathbb{R}\}.$$

Then Σ is stable.

Proof. We know that Σ is area-stationary with a singular line, obtained intersecting Σ with the plane $z = \log\sqrt{b/a}$. From (2-6) we get

$$Z = \frac{-be^z + ae^{-z}}{|-be^z + ae^{-z}|}X,$$

which is orthogonal to the singular line. Since $\langle \nabla_S \nu_h, Z\rangle = \langle \nabla_S Y, X\rangle = \dfrac{|N_h|}{2}$, the stability operator

$$Q(u) = \int_\Sigma \{|N_h|^{-1} Z(u)^2 + |N_h|\langle N, T\rangle^2 u^2\}\, d\Sigma + \int_{\Sigma_0} S(u)^2\, d\Sigma_0$$

is always nonnegative for any admissible test function u. □

Remark 5.7. The planes $\{ax + by + c = 0 : (x, y, z) \in E(1, 1), a, b, c \in \mathbb{R}\}$ are also area-minimizing, by calibration arguments.

Corollary 5.8. *The surface $\Sigma = \{e^z y + e^{-z} x = 0 : (x, y, z) \in E(1, 1)\}$ is stable.*

Proof. From (2-6) we get

$$Z = -\frac{(e^z y - e^{-z} x)Y}{|e^z y - e^{-z} x|}$$

and $\Sigma_0 = \{(0, 0, z) : (x, y, z) \in E(1, 1)\}$. From (2-3) we have

$$\langle \nabla_S \nu_h, Z\rangle = \langle \nabla_S Y, X\rangle = -\frac{|N_h|}{2},$$

which implies

$$Q(u) = \int_\Sigma \{|N_h|^{-1} Z(u)^2 + 2|N_h|^2 u^2\}\, d\Sigma + \int_{\Sigma_0} S(u)^2\, d\Sigma_0 + 4\int_{\Sigma_0} u^2\, d\Sigma_0 \geqslant 0,$$

for all admissible u. □

Corollary 5.9. *The surfaces defined in Proposition 4.2 are stable.*

Proof. For simplicity we will prove the statement in the case of $x_0 = y_0 = z_0 = 0$. We note that, since $\Sigma_0 = (0, 0, 0)$, the argument in the proof of Lemma 5.3 works and the condition $\langle N, T\rangle = -(1 + e^z)/\sqrt{2} \leqslant 0$ is sufficient for the stability in the complement of any tubular neighborhood of Σ_0. Finally we observe that the stability operator in Lemma 5.5 makes no contribution to the singular set in the case of isolated singular points. □

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