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**PERIODIC ORBITS OF HAMILTONIAN SYSTEMS
LINEAR AND HYPERBOLIC AT INFINITY**

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We consider Hamiltonian diffeomorphisms of symplectic Euclidean spaces, generated by compactly supported time-dependent perturbations of hyperbolic quadratic forms. We prove that, under some natural assumptions, such a diffeomorphism must have simple periodic orbits of arbitrarily large period when it has fixed points which are not necessary from a homological perspective.

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1. Introduction and main results

Introduction. In this paper we consider time-dependent Hamiltonians H on \mathbb{R}^{2n} which, outside a compact set, are autonomous and coincide with a hyperbolic quadratic form (i.e., a nondegenerate quadratic form whose Hamiltonian vector field has no purely imaginary eigenvalues). We prove that, under some additional conditions, the Hamiltonian diffeomorphism φ_H must have simple (i.e., uniterated) periodic orbits of arbitrarily large (prime) period when it has certain “homologically unnecessary” fixed points. In particular, φ_H then has infinitely many periodic orbits. To be more precise, this result holds provided that φ_H has at least one nondegenerate (or even homologically nontrivial) fixed point with nonzero mean index, and the quadratic form (i.e., the corresponding linear Hamiltonian vector field) has only real eigenvalues. (See Remark 3.3 for the case of complex eigenvalues.)

Our main motivation for studying this question comes from a variant of the Conley conjecture applicable to manifolds for which the standard Conley conjecture fails. Recall in this connection that the latter asserts the existence of infinitely

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many periodic orbits for every Hamiltonian diffeomorphism of a closed symplectic manifold. This is the case for manifolds with spherically vanishing first Chern class (of the tangent bundle) and also for negative monotone manifolds; see [Chance et al. 2013; Ginzburg and Gürel 2009a; Hein 2012] and also [Franks and Handel 2003; Ginzburg 2010; Ginzburg and Gürel 2012; Hingston 2009; Le Calvez 2006; Salamon and Zehnder 1992]. However the Conley conjecture, as stated, fails for some simple manifolds, such as S^2 : an irrational rotation of S^2 about the z -axis has only two periodic orbits, which are also the fixed points; these are the poles. In fact, any manifold that admits a Hamiltonian torus action with isolated fixed points also admits a Hamiltonian diffeomorphism with finitely many periodic orbits. In particular, $\mathbb{C}\mathbb{P}^n$, the Grassmannians and, more generally, most of the coadjoint orbits of compact Lie groups as well as symplectic toric manifolds all admit Hamiltonian diffeomorphisms with finitely many periodic orbits.

A viable alternative to the Conley conjecture for such manifolds is the conjecture that a Hamiltonian diffeomorphism with more fixed points than necessarily required by the (weak) Arnold conjecture has infinitely many periodic orbits. (It is possible that in this conjecture one might need to impose some kind of nondegeneracy condition (e.g., homological nontriviality) on the fixed points, as is the case for the version considered in this paper.) For $\mathbb{C}\mathbb{P}^n$, the expected threshold is $n + 1$. This conjecture is inspired by a celebrated theorem of Franks [1992; 1996] stating that a Hamiltonian diffeomorphism (or even an area preserving homeomorphism) of S^2 with at least three fixed points must have infinitely many periodic orbits; see also [Franks and Handel 2003; Le Calvez 2006] for further refinements and [Bramham and Hofer 2012; Collier et al. 2012; Kerman 2012] for symplectic topological proofs. We will refer to this analogue of the Conley conjecture as the *HZ conjecture* since, to the best of our knowledge, the first written account of the assertion is in [Hofer and Zehnder 1994, p. 263].

We find it useful to view the HZ conjecture in a broader context. Namely it appears that the presence of a fixed point that is unnecessary from a homological or geometrical perspective is already sufficient to force the existence of infinitely many periodic orbits. For instance, a theorem from [Ginzburg and Gürel 2014] asserts that for a certain class of closed monotone symplectic manifolds including $\mathbb{C}\mathbb{P}^n$, any Hamiltonian diffeomorphism with a hyperbolic fixed point must necessarily have infinitely many periodic orbits. (Note that the original HZ conjecture, at least for non-degenerate Hamiltonian diffeomorphisms of $\mathbb{C}\mathbb{P}^n$, would follow if one could replace a hyperbolic fixed point with a nonelliptic one in this theorem.) Furthermore there are obvious analogues of the HZ conjecture for symplectomorphisms or noncontractible periodic orbits of Hamiltonian diffeomorphisms. These analogues are also of interest and in some instances more accessible than the original HZ conjecture; see, for example, [Batoreo 2013; Ginzburg and Gürel 2009b; 2014; Gürel 2013].

The generalized HZ conjecture is also the central theme of this paper, although here we focus on a different aspect of the problem. Our main result, Theorem 1.1, can be viewed as a “local version” of this conjecture, and it holds in all dimensions. Namely we prove a variant of the HZ conjecture for Hamiltonians on \mathbb{R}^{2n} which are compactly supported perturbations of certain quadratic forms. Working with \mathbb{R}^{2n} allows us to circumvent a number of symplectic topological obstacles to proving the HZ conjecture and concentrate on what we interpret as the dynamical part of the problem, which is still quite nontrivial. This is a key difference, technical and conceptual, between the present work and the approach taken in [Ginzburg and Gürel 2014], where the symplectic topology of the ambient manifold plays a central role. We use Floer-theoretical techniques in the proofs. Deferring a more detailed discussion of our method to Section 1, we merely mention at this point that for technical reasons the quadratic form needs to be hyperbolic. Finally it should also be noted that Hamiltonian systems on \mathbb{R}^{2n} with a controlled (e.g., asymptotically linear) behavior at infinity have been extensively studied in the context of Hamiltonian mechanics by classical variational methods; see, for example, [Abbondandolo 2001; Amann and Zehnder 1980; Antonacci 1997; Cornea 2001; Mawhin and Willem 1989; Rabinowitz 1980; Zhang and Liu 2011; Zou 2001] and references therein. However, to the best of our knowledge, there is no overlap between that approach and the present work, including the results.

Main results. To state the main results of the paper, recall that the *mean index* $\Delta_H(x) \in \mathbb{R}$ of a periodic orbit x of the Hamiltonian flow of H measures, roughly speaking, the total angle swept out by certain eigenvalues with absolute value one of the linearized flow $d\varphi_H^t$ along x ; see [Long 2002; Salamon and Zehnder 1992] and also [Entov and Polterovich 2009, Section 3.3] and references therein for a more detailed discussion. For instance, the mean index is zero when $d\varphi_H^t$ has no eigenvalues on the unit circle for any $t \neq 0$, and hence the orbit is hyperbolic. Finally denote by $\text{Fix}(\varphi_H)$ the collection of fixed points of φ_H .

Theorem 1.1. *Let $H : S^1 \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be a Hamiltonian which is equal to a hyperbolic quadratic form Q at infinity (i.e., outside a compact set) such that Q has only real eigenvalues. Assume that φ_H has a nondegenerate fixed point with nonzero mean index and $\text{Fix}(\varphi_H)$ is finite. Then φ_H has simple, that is, uniterated, periodic orbits of arbitrarily large period.*

As a consequence, φ_H has infinitely many simple periodic orbits regardless of whether $\text{Fix}(\varphi_H)$ is finite or not. In fact, the nondegeneracy condition in Theorem 1.1 can be relaxed and replaced by a much weaker, albeit more technical, condition that the point is isolated and homologically nontrivial, that is, its local Floer homology is nonzero. This is Theorem 4.1.

Remark 1.2. Theorems 1.1 and 1.4 (below), and their generalizations discussed

in Section 4, also hold when the quadratic form Q has complex eigenvalues σ , provided that $|\operatorname{Re} \sigma| > |\operatorname{Im} \sigma|$; see Remark 3.3.

Remark 1.3. From the perspective of the generalized HZ-conjecture, observe that the nondegenerate (or homologically nontrivial) fixed point with nonzero mean index in Theorem 1.1 is the “unnecessary” point. Moreover the presence of one such point x implies the existence of at least two other (homologically nontrivial) orbits. Indeed the Floer homology for all iterations of H is concentrated in degree zero (see Section 3), and once k is so large that the index of the iterated orbit x^k is outside the range $[-n, n]$, another orbit must take over generating the homology. Furthermore there should be at least one more periodic orbit to cancel out the contribution of x^k to the homology in higher degrees.

Hypothetically results similar to Theorem 1.1 and other theorems discussed in this section hold when a hyperbolic quadratic form is replaced by any (autonomous) quadratic form Q without nontrivial periodic orbits. For instance in this case, one can expect to have infinitely many periodic orbits whenever φ_H has a nondegenerate fixed point with mean index different from $\Delta_Q(0)$ or has at least two nondegenerate fixed points; cf. Remark 4.6. (The latter conjecture, which was the starting point of this work, is due to Alberto Abbondandolo.)

As has been pointed out above, the proof of Theorem 1.1 is based on Floer theory. However for a general quadratic form Q , even when the Floer homology exists, continuation maps fail to have the desired properties and the homology is not invariant under iterations. This is the case, for instance, for positive or negative definite Q (see Remark 3.6) and the main reason why we restrict our attention to hyperbolic quadratic forms. Even for such forms some foundational aspects of Floer theory have to be reexamined. We do this in Section 3, using, as one could expect, a version of the maximum principle.

The condition that the fixed point is nondegenerate (and that it has nonzero mean index) is essential in Theorem 1.1. For instance, starting with the flow of $Q(x, y) = xy$ on \mathbb{R}^2 , it is easy to introduce degenerate (homologically trivial) fixed points by slightly perturbing the flow away from the saddle. This way one can create an arbitrarily large number of fixed points without generating infinitely many periodic orbits. In fact, we expect some form of nondegeneracy (e.g., homological nontriviality) to be essential in the HZ conjecture beyond the case of S^2 .

In low dimensions, Theorem 1.1 combined with simple index analysis implies the HZ conjecture in its original form for Hamiltonians in question. To state the result, recall first that φ_H is said to be strongly nondegenerate if all iterations of φ_H are nondegenerate.

Theorem 1.4. *Let $H : S^1 \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$, with $2n = 2$ or 4 , be a Hamiltonian which is equal to a hyperbolic quadratic form Q at infinity such that Q has only real*

eigenvalues. Assume that φ_H is strongly nondegenerate and has at least two fixed points, and $\text{Fix}(\varphi_H)$ is finite. Then φ_H has simple periodic orbits of arbitrarily large period.

Note that strong nondegeneracy is a C^∞ -generic condition in the class of Hamiltonians under consideration. Let us also point out that, in contrast with many closed manifolds (see, e.g., [Ginzburg and Gürel 2009b]), the existence of infinitely many periodic orbits is obviously not a C^∞ - or even C^2 -generic property of Hamiltonians in Theorem 1.4: one has to have an extra periodic orbit which serves as a seed eventually “spawning an infinitude of offspring”.

In dimension two, the strong nondegeneracy requirement can be relaxed. It suffices to just assume that φ_H has at least two isolated homologically nontrivial fixed points; see Theorem 4.5. (Also note that in this case the eigenvalues of a hyperbolic quadratic form are automatically real.) However in dimension four, nondegeneracy enters the proof in a crucial way. Finally note that the two-dimensional case of Theorem 1.4 is intimately related to Franks’s theorem; see Remarks 4.6 and 4.7.

Remark 1.5. A more general version of Theorem 1.4 for $2n = 2$ was proved in [Abbondandolo 2001, Theorem 5.1.9].

Organization of the paper. In Section 2, we set conventions and notation, and briefly recall some of the tools used in the paper and provide relevant references. We establish a version of the maximum principle and show that the Floer homology, as well as the relevant continuation maps, are defined for the class of Hamiltonians in question in Section 3. Finally in Section 4, we prove Theorems 1.1 and 1.4.

2. Conventions and notation

Throughout the paper, we will be working with the symplectic manifold $(\mathbb{R}^{2n}, \omega)$, where ω is the standard symplectic form. All Hamiltonians H considered here are assumed to be one-periodic in time, that is, $H : S^1 \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$, and we set $H_t = H(t, \cdot)$ for $t \in S^1 = \mathbb{R}/\mathbb{Z}$. The Hamiltonian vector field X_H of H is defined by $i_{X_H}\omega = -dH$. The (time-dependent) flow of X_H is denoted by φ_H^t and its time-one map by φ_H . Such time-one maps are referred to as *Hamiltonian diffeomorphisms*. The action of a one-periodic Hamiltonian H on a loop $\gamma : S^1 \rightarrow \mathbb{R}^{2n}$ is defined by

$$\mathcal{A}_H(\gamma) = - \int_z \omega + \int_{S^1} H_t(\gamma(t)) dt,$$

where $z : D^2 \rightarrow M$ is such that $z|_{S^1} = \gamma$. The least action principle asserts that the critical points of \mathcal{A}_H on the space of all smooth maps $\gamma : S^1 \rightarrow \mathbb{R}^{2n}$ are exactly the one-periodic orbits of φ_H^t .

Let K and H be two one-periodic Hamiltonians. The *composition* $K \natural H$ is defined by the formula

$$(2-1) \quad (K \natural H)_t = K_t + H_t \circ (\varphi_K^t)^{-1},$$

and the flow of $K \natural H$ is $\varphi_K^t \circ \varphi_H^t$. We set $H^{\natural k} = H \natural \dots \natural H$ (k times). Abusing terminology, we will refer to $H^{\natural k}$ as the k -th iteration of H . Clearly $H^{\natural k} = kH$ when H is autonomous. (Note that the flow $\varphi_{H^{\natural k}}^t = (\varphi_H^t)^k$, $t \in [0, 1]$, is homotopic with fixed end-points to the flow φ_H^t , $t \in [0, k]$. Also, in general, $H^{\natural k}$ is not one-periodic, even when H is.) Furthermore, setting

$$(2-2) \quad \|F\|_B = \int_{S^1} \sup_B |F| dt$$

for a bounded set $B \subset \mathbb{R}^{2n}$, we have $\|H^{\natural k}\|_B = k\|H\|_B$ when H is autonomous. Note that $\|F\|_B$ is a variant of the Hofer norm. (When F is compactly supported on \mathbb{R}^{2n} , we will also use the notation $\|F\|_{\mathbb{R}^{2n}}$ with the obvious meaning.)

The k -th iteration of a one-periodic orbit γ of H will be denoted by γ^k . More specifically, $\gamma^k(t) = \varphi_{H^{\natural k}}^t(\gamma(0))$, where $t \in [0, 1]$. We can think of γ^k as the k -periodic orbit $\gamma(t)$, $t \in [0, k]$, of H . Hence there is an action-preserving one-to-one correspondence between one-periodic orbits of $H^{\natural k}$ and k -periodic orbits of H .

The *action spectrum* $\mathcal{S}(H)$ of H is the set of critical values of \mathcal{A}_H . This is a zero measure, closed (hence nowhere dense) set; see, for example, [Hofer and Zehnder 1994]. Clearly the action functional is homogeneous with respect to iteration:

$$\mathcal{A}_{H^{\natural k}}(\gamma^k) = k\mathcal{A}_H(\gamma).$$

A periodic orbit γ of H is said to be *nondegenerate* if the linearized return map $d\varphi_H : T_{\gamma(0)}M \rightarrow T_{\gamma(0)}M$ has no eigenvalues equal to one. A Hamiltonian is called nondegenerate if all its one-periodic orbits are nondegenerate and strongly nondegenerate if all k -periodic orbits (for all k) are nondegenerate.

Let γ be a nondegenerate periodic orbit. The *Conley–Zehnder index*, denoted by $\mu_{\text{CZ}}(H, \gamma) \in \mathbb{Z}$, is defined, up to a sign, as in [Salamon 1999; Salamon and Zehnder 1992]. (When H is clear from the context we use the notation $\mu_{\text{CZ}}(\gamma)$.) More specifically, in this paper, the Conley–Zehnder index is the negative of that in [Salamon 1999]. In other words, we normalize μ_{CZ} so that $\mu_{\text{CZ}}(\gamma) = n$ when γ is a nondegenerate maximum of an autonomous Hamiltonian with small Hessian. Furthermore recall that the mean index $\Delta_H(\gamma)$ is defined regardless of whether γ is degenerate or not, and $\Delta_H(\gamma)$ depends continuously on H and γ in the obvious sense. When γ is nondegenerate, we have

$$0 \leq |\Delta_H(\gamma) - \mu_{\text{CZ}}(H, \gamma)| < n.$$

Furthermore the mean index is also homogeneous with respect to iteration:

$$\Delta_{H \circ k}(\gamma^k) = k \Delta_H(\gamma).$$

3. Maximum principle and Floer homology

Our goal in this section is to show that the Floer homology is defined and has the standard properties for the class of Hamiltonians in question. In our setting, essentially the only issue to deal with is the compactness of moduli spaces of Floer trajectories, which we establish by proving a version of the maximum principle.

Floer homology. Let Q be a hyperbolic quadratic form on \mathbb{R}^{2n} , that is, Q is nondegenerate and has no eigenvalues on $i\mathbb{R}$. (Recall that throughout the paper by eigenvalues of Q we mean the eigenvalues of the linear Hamiltonian vector field X_Q .) Assume further that all eigenvalues of Q are real. (See Remark 3.3 for a variant of the maximum principle when Q has complex eigenvalues.) Denote by \mathcal{H}_Q the set of one-periodic Hamiltonians $H : S^1 \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ which are compactly supported time-dependent perturbations of Q . Let $J = J_t$ be a time-dependent almost complex structure compatible with ω . We are interested in solutions $u : \mathbb{R} \times S^1 \rightarrow \mathbb{R}^{2n}$ of the Floer equation

$$(3-1) \quad \partial_s u + J(u) \partial_t u = -\nabla H_t(u),$$

where $u = u(s, t)$ with coordinates (s, t) on $\mathbb{R} \times S^1$ and the gradient is taken with respect to the one-periodic in time metric $\langle \cdot, \cdot \rangle = \omega(\cdot, J \cdot)$ on \mathbb{R}^{2n} .

In this setting we have:

Theorem 3.1. *Let Q be a hyperbolic quadratic form on $(\mathbb{R}^{2n}, \omega)$ with only real eigenvalues. Then there exists a linear complex structure J_Q compatible with ω such that whenever $J \equiv J_Q$ and $H \equiv Q$ outside an open ball B with respect to the metric $\langle \cdot, \cdot \rangle_Q := \omega(\cdot, J_Q \cdot)$, any solution of (3-1) for the pair (H, J) that is asymptotic to periodic orbits of H in B is necessarily contained in B .*

More generally, consider now solutions $u : \Omega \rightarrow \mathbb{R}^{2n}$ of (3-1), where $\Omega \subset \mathbb{R} \times S^1$ is an open connected subset. Theorem 3.1 is an immediate consequence of the following proposition:

Proposition 3.2 (maximum principle). *Let Q be a hyperbolic quadratic form on $(\mathbb{R}^{2n}, \omega)$ with only real eigenvalues. Then there exists a linear complex structure J_Q compatible with ω such that for any solution u (with domain $\Omega \subset \mathbb{R} \times S^1$) of the Floer equation (3-1) for (Q, J_Q) , the function $\rho = \|u\|^2/2$, where the norm is induced by the metric $\langle \cdot, \cdot \rangle = \omega(\cdot, J_Q \cdot)$, cannot attain a maximum at an interior point of Ω unless ρ is constant.*

Proof of Proposition 3.2. Below we first introduce J_O and then prove that ρ is subharmonic on Ω , that is, $\Delta\rho \geq 0$, where the Laplacian is taken with respect to metric $\omega(\cdot, J_O\cdot)$.

Since we will be changing the basis and the inner product on the ambient space throughout the proof, it is more convenient to work with a hyperbolic quadratic form Q on a finite-dimensional symplectic vector space (V^{2n}, ω) . Equip V with a symplectic basis

$$(\partial_p, \partial_q) = (\partial_{p_1}, \dots, \partial_{p_n}, \partial_{q_1}, \dots, \partial_{q_n})$$

such that in the corresponding coordinates (p, q) on \mathbb{R}^{2n} , the quadratic form Q is expressed as

$$(3-2) \quad Q(p, q) = \langle Ap, q \rangle.$$

Here A is a nondegenerate lower triangular $n \times n$ matrix, and $\langle \cdot, \cdot \rangle$ is the standard inner product on \mathbb{R}^n . Indeed, since Q is nondegenerate with only real eigenvalues, it can be expressed in some symplectic basis (∂_p, ∂_q) as the direct sum of the normal forms

$$\sigma \sum_{i=1}^m p_i q_i - \sum_{i=1}^{m-1} p_i q_{i+1},$$

where σ ranges over the positive eigenvalues of Q and m is the multiplicity of σ ; see [Arnold 1978; Williamson 1936]. We emphasize that p 's and q 's are treated here as vectors in \mathbb{R}^n using the bases (∂_p) and (∂_q) , respectively. (It is clear from this formula that A is indeed lower triangular.) Note that with this choice all diagonal entries of A , that is, the eigenvalues of A , are positive.

Let $A = D + E$, where D is the diagonal part of A and E is the strictly lower triangular part. By rescaling the basis vectors (∂_p, ∂_q) , while still keeping the basis symplectic and keeping (3-2), we can make E arbitrarily small. (We will specify shortly how small E has to be. Here we merely note that the rescaling does not affect D and that, in fact, E is required to be small compared to D .) We keep the notation (∂_p, ∂_q) for the new basis and (p, q) for the resulting linear coordinates.

The complex structure J_O is defined by the requirement $J_O\partial_p = -\partial_q$. This structure is compatible with ω , and we denote by $\langle \cdot, \cdot \rangle_Q$ the resulting inner product $\omega(\cdot, J_O\cdot)$ on V , that is, $\langle \cdot, \cdot \rangle_Q := \omega(\cdot, J_O\cdot)$. From now on we identify (V, ω) with the standard symplectic \mathbb{R}^{2n} using the basis (∂_p, ∂_q) . Under this identification, J_O becomes the standard complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$, and $\langle \cdot, \cdot \rangle_Q$ turns into the standard inner product. Note also that the restriction of $\langle \cdot, \cdot \rangle_Q$ to the subspaces generated by ∂_p and ∂_q , respectively, is the inner product $\langle \cdot, \cdot \rangle$ on \mathbb{R}^n . Finally we emphasize that all these structures, except for ω , depend on the choice of the basis (∂_p, ∂_q) which is to be finalized below (after we state how small E needs to be).

In what follows, we calculate the Laplacian with respect to the metric $\langle \cdot, \cdot \rangle_Q$, where we set $u(s, t) = (p, q)$ and use the Floer equation (3-1):

$$\begin{aligned}
 (3-3) \quad \Delta \rho &= \rho_{ss} + \rho_{tt} \\
 &= \|u_s\|^2 + \|u_t\|^2 - \langle u, \partial_s \nabla Q(u) \rangle_Q + \langle u, J_Q \partial_t \nabla Q(u) \rangle_Q \\
 &= \|u_s\|^2 + \|u_t\|^2 + \langle A^2 p, p \rangle + \langle A^2 q, q \rangle \\
 &\quad + \langle p, (A - A^T)q_s \rangle - \langle q, (A - A^T)p_s \rangle \\
 &= \|u_s\|^2 + \|u_t\|^2 + \|Dp\|^2 + \|Dq\|^2 + \langle E^2 p, p \rangle + \langle E^2 q, q \rangle \\
 &\quad + \langle (DE + ED)p, p \rangle + \langle (DE + ED)q, q \rangle \\
 &\quad + \langle p, (E - E^T)q_s \rangle - \langle q, (E - E^T)p_s \rangle.
 \end{aligned}$$

Next we specify the requirements on E . To this end, let $\lambda := \min \lambda_i > 0$, where the λ_i are the eigenvalues of A (or D). Then we have

$$(3-4) \quad \|Dx\|^2 \geq \lambda^2 \|x\|^2 \quad \text{for any } x \in \mathbb{R}^n.$$

Now E is required to be so small that:

- (i) $|\langle E^2 x, x \rangle| \leq \lambda^2 \|x\|^2 / 10$ for any $x \in \mathbb{R}^n$,
- (ii) $|\langle DE x, x \rangle| \leq \lambda^2 \|x\|^2 / 20$ and $|\langle ED x, x \rangle| \leq \lambda^2 \|x\|^2 / 20$ for any $x \in \mathbb{R}^n$,
- (iii) $|\langle x, (E - E^T)y \rangle| \leq \lambda \|x\| \|y\| / 8$ for any x and $y \in \mathbb{R}^n$.

Using (3-4), and (i) and (ii) for $x = p$ and $x = q$, and (iii) for $(x, y) = (p, q_s)$ and $(x, y) = (q, p_s)$ in (3-3), it is straightforward to show that

$$\begin{aligned}
 (3-5) \quad \Delta \rho &\geq \frac{3\lambda^2}{10} \|u\|^2 + \|u_s\|^2 + \frac{\lambda^2}{2} \|u\|^2 - \frac{\lambda}{4} \|u\| \|u_s\| \\
 &\geq \frac{3\lambda^2}{10} \|u\|^2 + \left(\|u_s\| - \frac{\lambda}{\sqrt{2}} \|u\| \right)^2 \geq \frac{3\lambda^2}{10} \|u\|^2 \geq 0. \quad \square
 \end{aligned}$$

Remark 3.3. It is not hard to see that Proposition 3.2 still holds when the quadratic form Q has complex eigenvalues σ , provided that $|\operatorname{Re} \sigma| > |\operatorname{Im} \sigma|$ or, equivalently, $\operatorname{Re} \sigma^2 > 0$ for all eigenvalues. However in general without this assumption (or when Q is elliptic but not positive definite), there seems to be no reason to expect the maximum principle to hold. There are also several other variants of the maximum principle which hold for solutions of the Floer equation for Q . For instance, it holds for the functions $\|p\|^2$ and $\|q\|^2$ separately.

As a consequence of Theorem 3.1, the total and filtered Floer homology groups of $H \in \mathcal{H}_Q$, denoted by $\operatorname{HF}(H)$ and $\operatorname{HF}^{(a,b)}(H)$, respectively, are defined and have properties similar to those for closed symplectically aspherical manifolds; see, for example, [Hofer and Zehnder 1994; McDuff and Salamon 2004]. (For the sake

of simplicity all homology groups are taken over \mathbb{Z}_2 .) Likewise the local Floer homology $\text{HF}(H, \gamma)$ of H at an isolated periodic orbit γ is also defined and has the usual properties; see, for example, [Floer 1989a; 1989b; Ginzburg 2010; Ginzburg and Gürel 2010]. (Here J is an ω -compatible almost complex structure which is generic within the class of almost complex structures equal outside a compact set to some J_Q , as in Theorem 3.1. We will discuss the dependence of the Floer homology on J shortly.)

Our next goal is to define the continuation maps induced by homotopies of Hamiltonians in \mathcal{H}_Q . To this end, we say that a homotopy $F_s = Q + f_s$ in \mathcal{H}_Q from H_0 to H_1 is compactly supported if $\bigcup_s \text{supp } f_s$ is bounded. (Observe that this is not automatically the case.) Then we have a continuation map

$$(3-6) \quad \Psi : \text{HF}^{(a,b)}(H_0) \rightarrow \text{HF}^{(a,b)+C}(H_1)$$

for any H_0 and H_1 in \mathcal{H}_Q , induced by a homotopy F_s in \mathcal{H}_Q . Here $(a, b) + C$ stands for $(a + C, b + C)$, and

$$(3-7) \quad C \geq \int_{-\infty}^{\infty} \int_{S^1} \sup_{\mathbb{R}^{2n}} \partial_s F_s \, dt \, ds = \int_{-\infty}^{\infty} \int_{S^1} \sup_{\mathbb{R}^{2n}} \partial_s f_s \, dt \, ds,$$

with $\partial_s F_s \equiv 0$ when $|s|$ is large; see [Ginzburg 2007, Section 3.2.2]. Note that the suprema in (3-7) exist since the homotopy is compactly supported.

We now have $\text{HF}(H) = \text{HF}(Q) \cong \mathbb{Z}_2$, concentrated in degree $\mu_{\text{cz}}(Q, 0) = 0$. It is clear that the filtered Floer homology $\text{HF}^{(a,b)}(H)$ is independent of the almost complex structure J as long as J_Q is fixed. However it is not obvious at all whether this homology is independent of the choice of J_Q . In what follows, we will always have J_Q fixed and suppress this hypothetical dependence in the notation.

Continuation maps beyond \mathcal{H}_Q . The class \mathcal{H}_Q is not closed under iteration. For instance, $H^{12} \in \mathcal{H}_{2Q}$ when $H \in \mathcal{H}_Q$. To incorporate iteration into the picture, we consider a broader class $\widehat{\mathcal{H}}_Q$ which is the union of the classes \mathcal{H}_{kQ} for all real $k > 0$. Clearly this class is now closed under iteration. Moreover one can see from the proof of Proposition 3.2 that there exists a common almost complex structure, J_Q , for which Theorem 3.1 holds for all Hamiltonians in $\widehat{\mathcal{H}}_Q$ or, to be more precise, any J_Q can also be taken as J_{kQ} . (The reason is that J_Q is determined by the requirement that the off-diagonal part E of A is small compared to D , rather than just small. Thus if conditions (i), (ii) and (iii) are satisfied for Q , they are also automatically satisfied in the same basis for kQ for any $k > 0$.) From now on we fix J_Q .

As above, we say that a homotopy $F_s = k(s)Q + f_s$ in $\widehat{\mathcal{H}}_Q$ is compactly supported if $\bigcup_s \text{supp } f_s$ is bounded and call the closure of this union the support

of the homotopy. A homotopy is called *slow* if it is compactly supported and, say,

$$(3-8) \quad \frac{|k'(s)|}{(k(s))^2} \leq \frac{3\lambda^2}{20} \inf_{x \in \mathbb{R}^{2n} \setminus \{0\}} \frac{\|x\|^2}{|Q(x)|},$$

where $\lambda = \min \lambda_i$ is as in Section 3; cf. [Cieliebak et al. 1995]. Clearly the right-hand side in (3-8) is positive. (In fact, the infimum in (3-8) is equal to $1/\lambda_{\max}$, where λ_{\max} is the largest of the absolute values of the eigenvalues of Q with respect to $\|x\|^2$. This follows from the Courant–Fischer minimax theorem; see, for example, [Demmel 1997, Chapter 5].) Recall also that a homotopy F_s from H_0 to H_1 is called linear if $F_s = (1 - g(s))H_0 + g(s)H_1$, where $g : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing smooth function equal to zero for $s \ll 0$ and one for $s \gg 1$.

Theorem 3.4. *Let F_s be a slow homotopy in $\widehat{\mathcal{H}}_O$ from H_0 to H_1 , supported in a ball B with respect to the metric $\langle \cdot, \cdot \rangle_Q = \omega(\cdot, J_O \cdot)$. The continuation map Ψ as in (3-6) is defined, where C satisfies*

$$(3-9) \quad C \geq \int_{-\infty}^{\infty} \int_{S^1} \sup_B \partial_s F_s \, dt \, ds.$$

This map is independent of the slow homotopy. Furthermore, for a linear slow homotopy, we can take $C = \|H_1 - H_0\|_B := \int_{S^1} \sup_B |H_1 - H_0| \, dt$.

It is not hard to see that any compactly supported homotopy in $\widehat{\mathcal{H}}_O$ can be reparametrized to make it slow without changing the right-hand side in (3-9). Note also that although the notion of a slow homotopy is independent of the size of the support, the lower bound in (3-9) does depend in general on the ball B containing the support and increases with the size of B . However one can show that the continuation map Ψ is independent of the ball B in the following sense: whenever for a fixed homotopy and two different balls C satisfies (3-9) for both of the balls, the resulting continuation map is independent of the ball. (In what follows, we will not use this fact.)

The continuation maps Ψ have properties similar to their counterparts in the ordinary Floer homology. For instance, the continuation map induced by a concatenation of homotopies is equal to the composition of the continuation maps, and continuation maps commute with the maps in the long exact sequence in filtered Floer homology. (See [Ginzburg 2007] for a detailed account on the so-called C -bounded homotopies in filtered Floer homology.) Note however that here, as in (3-6), the almost complex structure J_s is independent of s outside B .

Remark 3.5. We emphasize that the continuation map Ψ is not necessarily defined when the homotopy is not slow.

Proof of Theorem 3.4. To prove that Ψ is well-defined, it suffices to show that the maximum principle, Proposition 3.2, extends to solutions of the Floer equation for slow homotopies $Q_s = k(s)Q$ connecting $Q_0 = k_0Q$ and $Q_1 = k_1Q$, where $k_0 = k(0)$ and $k_1 = k(1)$. To this end, note that $Q_s = k(s)\langle Ap, q \rangle$ and recall that, as was noted above, we can take J_{kQ} to be J_Q . Calculating the Laplacian with respect to the metric $\langle \cdot, \cdot \rangle_Q = \omega(\cdot, J_Q \cdot)$ in this setting, we obtain

$$\begin{aligned} \Delta\rho &= \|u_s\|^2 + \|u_t\|^2 + k(s)(\langle p, (A - A^T)q_s \rangle - \langle q, (A - A^T)p_s \rangle) \\ &\quad + (k(s))^2(\langle A^2 p, p \rangle + \langle A^2 q, q \rangle) - 2k'(s)\langle Ap, q \rangle \\ &= \|u_s\|^2 + \|u_t\|^2 + (k(s))^2(\|Dp\|^2 + \|Dq\|^2 + \langle E^2 p, p \rangle + \langle E^2 q, q \rangle) \\ &\quad + (k(s))^2(\langle (DE + ED)p, p \rangle + \langle (DE + ED)q, q \rangle) \\ &\quad + k(s)(\langle p, (E - E^T)q_s \rangle - \langle q, (E - E^T)p_s \rangle) - 2k'(s)\langle Ap, q \rangle \\ &\geq \frac{3\lambda^2}{10}(k(s))^2\|u\|^2 + \left(\|u_s\| - \frac{\lambda|k(s)|}{\sqrt{2}}\|u\|\right)^2 - 2|k'(s)| \cdot |Q(u)|. \end{aligned}$$

Hence $\Delta\rho \geq 0$ by (3-8).

That we can take C satisfying (3-9) for a general slow homotopy and that $C = \|H_1 - H_0\|_B$ satisfies (3-9) for a linear slow homotopy is established by a standard calculation (see, e.g., [Ginzburg 2007; Schwarz 2000]) combined with the observation that homotopy trajectories (i.e., solutions of (3-1) for the pair (F_s, J_s)) are confined to B due to the maximum principle. \square

Remark 3.6. The maximum principle is also known to hold for positive definite quadratic Hamiltonians; see [McDuff 1991; Viterbo 1999] and also [Seidel 2008]. This fact underlies the definition of symplectic homology, and in fact, it was the motivation of our approach in this paper. However it is worth pointing out that in this case the continuation map between a Hamiltonian equal to kQ at infinity and the one equal to $(k + 1)Q$ at infinity is defined only in one direction and this map, depending on Q , may be zero. This is the main reason why our approach to the proof of Theorem 1.1 does not carry over to positive definite quadratic Hamiltonians.

4. Proofs and generalizations

Proof of Theorem 1.1. As has been mentioned in the introduction, we establish a more general result. To state it, recall again that an isolated periodic orbit x is said to be homologically nontrivial if the local Floer homology of H at x is nonzero. For instance, a nondegenerate fixed point is homologically nontrivial. More generally, an isolated fixed point with nonvanishing topological index is homologically nontrivial, for this index is equal, up to a sign, to the Euler characteristic of the local Floer homology. The notion of homological nontriviality seems to be particularly well

suites for use in the context of HZ and Conley conjectures; see, for example, Remark 1.3. Theorem 1.1 is an immediate consequence of the following result.

Theorem 4.1. *Let $H : S^1 \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be a Hamiltonian which is equal to a hyperbolic quadratic form Q at infinity (i.e., outside a compact set) such that Q has only real eigenvalues. Assume that φ_H has an isolated homologically nontrivial fixed point x with nonzero mean index and $\text{Fix}(\varphi_H)$ is finite. Then φ_H has simple periodic orbits of arbitrarily large period.*

Proof. In what follows, for the sake of brevity, we suppress the t -dependence when taking a supremum or specifying the support of a function. For instance, when we say that a function is supported in $Y \subset \mathbb{R}^{2n}$, we mean that the support is in $S^1 \times Y$. Likewise two functions being equal on Y means that they are equal on $S^1 \times Y$, etc. Finally the supremum, without a set specified, will stand for the supremum over \mathbb{R}^{2n} .

Let $H = Q + f$ as in the statement of the theorem. Pick a polyball $P = B^n \times B^n$ containing $\text{supp } f$ and a ball $V \supset P$. Throughout the proof, as in Section 3, we assume that the off-diagonal part E of A is small enough when compared to the diagonal part D . In particular, every integral curve of the flow of Q intersects P along a connected set. Before we actually turn to the proof of the theorem, we need to first modify H , without essentially changing its dynamics, to control the energy shift resulting from the homotopy between different iterations of H .

Lemma 4.2. *There exist constants $C_1 > 0$ and $C_2 > 0$, depending only on the quadratic form Q and the ball V , such that for every $\epsilon \in (0, 1]$, there exists an autonomous Hamiltonian \tilde{Q} with the following properties:*

- (i) $\tilde{Q} = Q$ on V .
- (ii) $\tilde{Q} = \epsilon Q$ outside a ball $V_\epsilon \supset V$ of radius $R = C_1/\sqrt{\epsilon}$.
- (iii) $\sup_{V_\epsilon} |\tilde{Q}| = C_2$.
- (iv) *The Hamiltonian flow of \tilde{Q} has no periodic orbits other than the origin, and every integral curve of its flow intersects P along a connected set.*

The essential point here is that the constants C_1 and C_2 are independent of ϵ while $R = C_1/\sqrt{\epsilon}$ (but not, say, of order $1/\epsilon$). We will prove this lemma by giving an explicit construction of \tilde{Q} after the proof of Theorem 4.1. One can think of \tilde{Q} as a family of Hamiltonians smoothly parametrized by ϵ with $\tilde{Q} = Q$ for $\epsilon = 1$.

Consider now the Hamiltonian

$$\tilde{H} = \tilde{Q} + f = \epsilon Q + (\tilde{Q} - \epsilon Q) + f = \epsilon Q + h,$$

where $h = (\tilde{Q} - \epsilon Q) + f$ is supported in V_ϵ . Observe that $\tilde{H} \in \hat{\mathcal{H}}_Q$. Furthermore $\tilde{H} = H$ in V , the ball where the Hamiltonians have nontrivial dynamics. Moreover for every period, \tilde{H} and H have exactly the same periodic orbits by Lemma 4.2(iv),

and the orbits have the same actions and indices. In fact, one might expect these Hamiltonians to have exactly the same filtered Floer homology with isomorphism induced by a slow linear homotopy. However we have not been able to prove this fact.

The next lemma concerns the iterations $\tilde{H}^{\natural k}$ and an estimate, independent of k , of the difference $\tilde{H}^{\natural(k+\ell)} - \tilde{H}^{\natural k}$, which will be essential for the proof of Theorem 4.1.

Lemma 4.3. *The Hamiltonian $\tilde{H}^{\natural k}$ satisfies the following conditions:*

- (i) $\tilde{H}^{\natural k} \in \widehat{\mathcal{H}}_Q$ and is equal to $k\epsilon Q$ outside the ball B_k of radius $\|\varphi_Q^{\epsilon(k-1)}\| R$ centered at the origin, where $\varphi_Q^{\epsilon(k-1)}$ is viewed as a linear operator.
- (ii) Assume that k, ℓ and ϵ are such that $\|\varphi_Q^{\epsilon(k+\ell-1)}\| \leq 2$. Then

$$(4-1) \quad \|\tilde{H}^{\natural(k+\ell)} - \tilde{H}^{\natural k}\|_{B_{k+\ell}} \leq C_3 \ell,$$

where C_3 is independent of k, ℓ and ϵ , and the norm is as defined in (2-2).

Proof of Lemma 4.3. Denote by B_1 the ball V_ϵ from Lemma 4.2, that is, B_1 is the ball of radius $R = C_1/\sqrt{\epsilon}$ centered at the origin. Consider the nested sets

$$Y_k = \bigcup_{t \in [0, 1]} \varphi_{\epsilon Q}^{(k-1)t}(B_1) \quad \text{for } k \in \mathbb{N}.$$

Let $B_k = B(R_k)$ be the ball of radius $R_k = \|\varphi_{\epsilon Q}^{(k-1)}\| R$. Clearly $B_k \supset Y_k$.

Recall that $\tilde{H} = \epsilon Q + h$, where $h = (\tilde{Q} - \epsilon Q) + f$ is supported in B_1 . Observe that $\tilde{H}^{\natural k}$ can be expressed as

$$\tilde{H}^{\natural k} = k\epsilon Q + \sum_{j=0}^{k-1} h \circ (\varphi_{\tilde{H}}^t)^{-j} + \epsilon \sum_{j=0}^{k-1} (Q \circ (\varphi_{\tilde{H}}^t)^{-j} - Q) = k\epsilon Q + h_k.$$

We now show that $\text{supp } h_k \subset Y_k$, which settles (i). Since $\text{supp } h \subset B_1$, a point x can be in $\text{supp}(h \circ (\varphi_{\tilde{H}}^t)^{-j})$ only if $(\varphi_{\tilde{H}}^\tau)^{-j}(x) \in B_1$ for some $\tau \in [0, t]$. This implies that

$$x \in ((\varphi_{\tilde{H}}^\tau)^{-j})^{-1}(B_1) = \varphi_{\tilde{H}}^{j\tau}(B_1) \subset \bigcup_{t \in [0, 1]} \varphi_{\tilde{H}}^{jt}(B_1) = Y_{j+1}.$$

Hence the first term in h_k is supported in Y_k . Dealing with the second term in h_k , we first note that

$$\epsilon(Q \circ (\varphi_{\tilde{H}}^t)^{-j} - Q) = \epsilon(Q \circ (\varphi_{\tilde{H}}^t)^{-j} - Q \circ (\varphi_{\epsilon Q}^t)^{-j})$$

since Q is autonomous. Now it is clear that $(\varphi_{\tilde{H}}^t)^{-j}(x) \neq (\varphi_{\epsilon Q}^t)^{-j}(x)$ only when the integral curve of ϵQ through x for $[-jt, 0]$ enters B_1 , that is, $(\varphi_{\epsilon Q}^\tau)^{-j}(x) \in B_1$ for some $\tau \in [0, t]$. Hence, similarly to the first term, the second term in h_k is also supported in Y_k , and we have $\text{supp } h_k \subset Y_k$.

To establish (ii), denote by $B(1)$ the unit ball and observe that

$$\begin{aligned} \sup_{B_{k+\ell}} |h \circ (\varphi_{\tilde{H}}^t)^{-j}| &= \sup_{B_1} |\tilde{Q} - \epsilon Q + f| \\ &\leq \sup_{B_1} |\tilde{Q}| + \epsilon \sup_{B_1} |Q| + \sup |f| \\ &\leq C_2 + \epsilon \frac{C_1^2}{\epsilon} \sup_{B(1)} |Q| + \sup |f| \\ &= C_2 + C_1^2 \sup_{B(1)} |Q| + \sup |f| \end{aligned}$$

for any $j = k, \dots, k + \ell - 1$. Furthermore, by the energy conservation law, we have

$$\begin{aligned} \sup_{B_{k+\ell}} |Q \circ (\varphi_{\tilde{H}}^t)^{-j} - Q| &= \sup_{B_{k+\ell}} |Q \circ (\varphi_{\tilde{H}}^t)^{-j} - Q \circ (\varphi_{\epsilon Q}^t)^{-j}| \\ &= \sup_{B_1} |Q \circ (\varphi_{\tilde{H}}^t)^{-j} - Q \circ (\varphi_{\epsilon Q}^t)^{-j}| \\ &\leq 2 \sup_{B_1} |Q| \leq 2 \frac{C_1^2}{\epsilon} \sup_{B(1)} |Q|. \end{aligned}$$

Now recall that k, ℓ and ϵ are such that $\|\varphi_Q^{\epsilon(k+\ell-1)}\| \leq 2$. Thus $B_{k+\ell} \subset 2B_1$. Setting $M = \sup_{B(1)} |Q|$ and using the above estimates, we have

$$\begin{aligned} \sup_{B_{k+\ell}} |\tilde{H}^{\natural(k+\ell)} - \tilde{H}^{\natural k}| &= \sup_{B_{k+\ell}} \left| \ell \epsilon Q + \sum_{j=k}^{k+\ell-1} h \circ (\varphi_{\tilde{H}}^t)^{-j} + \epsilon \sum_{j=k}^{k+\ell-1} (Q \circ (\varphi_{\tilde{H}}^t)^{-j} - Q) \right| \\ &\leq \ell \epsilon \sup_{2B_1} |Q| \\ &\quad + \sum_{j=k}^{k+\ell-1} \sup_{B_{k+\ell}} |h \circ (\varphi_{\tilde{H}}^t)^{-j}| + \epsilon \sum_{j=k}^{k+\ell-1} \sup_{B_{k+\ell}} |(Q \circ (\varphi_{\tilde{H}}^t)^{-j} - Q)| \\ &\leq 4\ell \epsilon \frac{C_1^2}{\epsilon} M + \ell(C_2 + C_1^2 M + \sup |f|) + 2\ell \epsilon \frac{C_1^2}{\epsilon} M \leq (7C_1^2 M + C_2 + \sup |f|)\ell. \end{aligned}$$

Setting $C_3 := 7C_1^2 M + C_2 + \sup |f|$, we then have

$$(4-2) \quad \|\tilde{H}^{\natural(k+\ell)} - \tilde{H}^{\natural k}\|_{B_{k+\ell}} = \int_{S^1} \sup_{B_{k+\ell}} |\tilde{H}^{\natural(k+\ell)} - \tilde{H}^{\natural k}| dt \leq C_3 \ell,$$

with C_3 independent of k, ℓ and ϵ , as required. \square

From now on we will work with the Hamiltonians \tilde{H} , and at this stage we prefer not to specify the parameter ϵ yet. These Hamiltonians have the same periodic orbits with the same actions and indices, and up to the point when the homotopy between the iterated Hamiltonians is considered, the argument applies to any of the Hamiltonians \tilde{H} .

It is worth mentioning again that the Hamiltonians $\tilde{H}^{\natural k}$ are not one-periodic in time even though \tilde{H} is. This issue however is quite standard and can be dealt with in a straightforward way. Namely consider a Hamiltonian $G = K + g$, where $g = g_t$ is time-dependent for $t \in [0, 1]$ and K is any autonomous Hamiltonian. The Hamiltonian diffeomorphism φ_G can be generated by a one-periodic Hamiltonian

$$\bar{G} = K + \lambda'(t)g_{\lambda(t)} \circ \varphi_K^{\lambda(t)-t},$$

where $\lambda : [0, 1] \rightarrow [0, 1]$ is an increasing function equal to zero for $t \approx 0$ and one for $t \approx 1$. We apply this procedure to $\tilde{H}^{\natural k}$ with $K = k \in Q$ and $g = h_k$. The actions, the Conley–Zehnder indices and the mean indices of the periodic orbits do not change. The change of the set B_k can be made arbitrarily small, of the order $\|1 - \lambda'(t)\|_{L^1}$. As a consequence, the upper bound (4-1) can also be adjusted by an arbitrarily small amount independent of k . In what follows, we will treat the Hamiltonians $\tilde{H}^{\natural k}$ as one-periodic in time, allowing for these straightforward modifications.

Now we are in a position to proceed with the proof of Theorem 4.1. It suffices to show that there exist arbitrarily large primes which occur as periods of simple periodic orbits. Arguing by contradiction, assume that only finitely many prime numbers are attained as the periods. From now on, we always denote by p or p_i a prime number greater than the largest period. Let $\tilde{H} = \tilde{Q} + f$, where \tilde{Q} is any Hamiltonian from Lemma 4.2. Then for any such prime p , all p -periodic orbits of $\varphi_{\tilde{H}}$ are iterations of fixed points of φ_H , and hence $\mathcal{S}(\tilde{H}^{\natural p}) = p \mathcal{S}(H)$. Recall in this connection that φ_H is assumed to have finitely many fixed points. Next let us note that all sufficiently large prime numbers are admissible in the sense of [Ginzburg and Gürel 2010]. Thus under such iterations of \tilde{H} , the orbit x stays isolated, and

$$\text{HF}(\tilde{H}^{\natural p}, x^p) = \text{HF}(H^{\natural p}, x) = \text{HF}(H, x)$$

up to, in the second equality, a shift of degree determined by the order of iteration p ; see [Ginzburg and Gürel 2010, Theorem 1.1]. In particular, in our case, $\text{HF}(\tilde{H}^{\natural p}, x^p) \neq 0$ since $\text{HF}(H, x) \neq 0$.

As has been mentioned above, $\Delta_{\tilde{H}}(x) = \Delta_H(x)$, and let us assume that $\Delta_H(x) > 0$, for the argument is similar if $\Delta_H(x) < 0$. Moreover let us assume for the sake of simplicity that $\mathcal{A}_H(x) = 0$ and hence $\mathcal{A}_{\tilde{H}}(x) = 0$. (The general case can be dealt with in a similar fashion and requires only notational modifications.) Consequently $\mathcal{A}_{\tilde{H}^{\natural p}}(x^p) = 0$ for all iterations p . Let $a > 0$ be outside $\mathcal{S}(\tilde{H}) = \mathcal{S}(H)$ such that 0 is the only point in $(-a, a) \cap \mathcal{S}(H)$ and therefore in $(-ap, ap) \cap \mathcal{S}(\tilde{H}^{\natural p})$. Then we have

$$(4-3) \quad \text{HF}_*^{(-ap, ap)}(\tilde{H}^{\natural p}) = \text{HF}_*(H^{\natural p}, x^p) \oplus \dots,$$

where the dots represent the local Floer homology contributions from the fixed points with zero action other than x . Furthermore we henceforth focus on degrees $*$

such that $|*| > n$. This guarantees that the fixed points with zero mean index do not contribute to $\text{HF}_*^{(-ap, ap)}(\tilde{H}^{\natural p})$, for their local Floer homology groups are supported in $[-n, n]$, where the support is by definition the set of degrees for which the local Floer homology groups are nonzero. Thus all terms on the right-hand side of (4-3) come from fixed points with nonzero mean index. Moreover we can further restrict $*$ so that only the fixed points γ having the same mean index as x contribute to the right-hand side of (4-3). This is possible since the supports of local Floer homology groups coming from fixed points with other nonzero mean indices are separated from $\text{supp HF}_*(H^{\natural p}, x^p) \subset [p\Delta_H(x) - n, p\Delta_H(x) + n]$ whenever p is sufficiently large.

From now on, we work with primes $p > 2$ which are as large as is needed above. Let us order these prime numbers as $p_1 < p_2 < \dots$. In what follows, p_i always denotes a prime from this sequence.

Next recall that $\Delta_H(x) > 0$ and let $m \in \mathbb{N}$ be such that $m > n/\Delta_H(x)$. Then using the fact that $p_{i+m} - p_i \geq 2m$, we see that the supports of $\text{HF}(H^{\natural p_i}, \gamma^{p_i})$ and $\text{HF}(H^{\natural p_{i+m}}, \gamma^{p_{i+m}})$ are disjoint for all i and for all fixed points γ of φ_H with $\mathcal{A}_H(\gamma) = 0$ and $\Delta_H(\gamma) = \Delta_H(x)$. This is because

$$[p_i \Delta_H(x) - n, p_i \Delta_H(x) + n] \cap [p_{i+m} \Delta_H(x) - n, p_{i+m} \Delta_H(x) + n] = \emptyset,$$

where the first interval contains $\text{supp HF}(H^{\natural p_i}, \gamma^{p_i})$ and the second one contains $\text{supp HF}(H^{\natural p_{i+m}}, \gamma^{p_{i+m}})$. Moreover for any p_i , there exists an integer s_i such that $\text{HF}_{s_i}(H^{\natural p_i}, x^{p_i}) \neq 0$, as is mentioned earlier and proved in [Ginzburg and Gürel 2010]. Thus we see that

$$(4-4) \quad \text{HF}_{s_i}(H^{\natural p_i}, x^{p_i}) \neq 0 \quad \text{and} \quad \text{HF}_{s_i}(H^{\natural p_{i+m}}, \gamma^{p_{i+m}}) = 0$$

for all fixed points γ as above since s_i is outside $\text{supp HF}(H^{\natural p_{i+m}}, \gamma^{p_{i+m}})$ for all such γ .

Choose p_i so large that $p_i a > 6C_3(p_{i+m} - p_i)$, where C_3 is introduced in Lemma 4.3. That one can do so is guaranteed by the equality $p_{i+1} - p_i = o(p_i)$; see [Baker et al. 2001]. (Obviously one can write $p_{i+m} - p_i$ as a telescoping sum of the differences of two consecutive primes, and hence by a simple inductive argument, $p_{i+m} - p_i = o(p_i)$.) Now pick $\alpha > 0$, depending on m and i , such that

$$-p_i a < -\alpha < -\alpha + 2C_3(p_{i+m} - p_i) < 0 < \alpha < \alpha + 2C_3(p_{i+m} - p_i) < p_i a.$$

For instance, α satisfying $p_i a - 4C_3(p_{i+m} - p_i) < \alpha < p_i a - 2C_3(p_{i+m} - p_i)$ would work. As a consequence, we also have

$$-p_{i+m} a < -\alpha + C_3(p_{i+m} - p_i) < 0 < \alpha + C_3(p_{i+m} - p_i) < p_{i+m} a.$$

Finally let us specify \tilde{Q} , and in turn \tilde{H} . To this end, we choose $\epsilon > 0$ so small that $\|\varphi_Q^{\epsilon(p_{i+m}-1)}\| \leq 2$ and hence (4-1) is satisfied with $k = p_i$ and $k + l = p_{i+m}$

in the second assertion of Lemma 4.3. Set $\delta := C_3(p_{i+m} - p_i)$. Then for a linear homotopy, which we may assume to be slow in the sense of Section 3, from $\tilde{H}^{\natural p_i}$ to $\tilde{H}^{\natural p_{i+m}}$, we have the induced map

$$\mathrm{HF}^{(-\alpha, \alpha)}(\tilde{H}^{\natural p_i}) \rightarrow \mathrm{HF}^{(-\alpha, \alpha) + \delta}(\tilde{H}^{\natural p_{i+m}}).$$

Here the fact that δ is the correct action shift follows from Theorem 3.4 and Lemma 4.3. Likewise the linear-homotopy map from $\tilde{H}^{\natural p_{i+m}}$ to $\tilde{H}^{\natural p_i}$ results in another action shift in δ . Consider now the following commutative diagram:

$$\begin{array}{ccc} & \mathrm{HF}_{s_i}^{(-\alpha, \alpha) + \delta}(\tilde{H}^{\natural p_{i+m}}) = 0 & \\ & \nearrow & \searrow \\ 0 \neq \mathrm{HF}_{s_i}^{(-\alpha, \alpha)}(\tilde{H}^{\natural p_i}) & \xrightarrow{\cong} & \mathrm{HF}_{s_i}^{(-\alpha, \alpha) + 2\delta}(\tilde{H}^{\natural p_i}) \end{array}$$

Here the top group is zero due to our choice of the degree s_i . On the other hand,

$$\mathrm{HF}_{s_i}^{(-\alpha, \alpha)}(\tilde{H}^{\natural p_i}) = \mathrm{HF}_{s_i}(H^{\natural p_i}, x^{p_i}) \oplus \dots \neq 0,$$

and the horizontal arrow is induced by the natural quotient-inclusion map; see, for example, [Ginzburg 2007]. This is indeed an isomorphism by the stability of filtered Floer homology (see, e.g., [Ginzburg and Gürel 2010]) because 0 is the only action value in the intervals $(-\alpha, \alpha)$ and $(-\alpha, \alpha) + 2\delta$. To summarize, a nonzero isomorphism factors through a zero group in the diagram. This contradiction completes the proof of Theorem 4.1, modulo a proof of Lemma 4.2, which is given below. \square

Remark 4.4. Notice that we have actually established the existence of a simple periodic orbit of either $H^{\natural p_i}$ or $H^{\natural p_{i+m}}$. In particular, starting with a sufficiently large prime number, among every m consecutive primes, there exists at least one prime which is the period of a simple periodic orbit of φ_H .

Furthermore for an infinite sequence of simple p_l -periodic orbits x_l of φ_H found this way, where $p_l \rightarrow \infty$, we have $\Delta_{H^{\natural p_l}}(x_l)/p_l \rightarrow \Delta_H(x)$. Hence in some sense, the mean index $\Delta_H(x)$ is an accumulation point in the union of normalized index spectra for H and its all iterations. (Of course, $\Delta_H(x)$ could possibly be isolated, but then $\Delta_{H^{\natural p_l}}(x_l)/p_l = \Delta_H(x)$.) A similar fact also holds for the action.

Finally note that the condition that the eigenvalues σ of Q are real can be relaxed and replaced by the requirement that $|\mathrm{Re} \sigma| > |\mathrm{Im} \sigma|$; cf. Remark 3.3.

Proof of Lemma 4.2. We construct the function \tilde{Q} in three steps and then show that \tilde{Q} has the required properties.

Step 1. Set $c = \sup_V |Q|$. Let $\eta : \mathbb{R} \rightarrow \mathbb{R}$ be an *odd* smooth function that equals the identity in $[-c, c]$ and satisfies $\eta' \geq \epsilon/2$ everywhere and

$$\eta(x) = \epsilon x \quad \text{when } |x| \geq c' = 2c/\epsilon.$$

It is easy to see that such a function exists. Note that to have a monotone function η such that $\eta(x) = x$ when $|x| \leq c$ and $\eta(x) = \epsilon x$ when $|x| \geq c'$, we must have $c' > c/\epsilon$. This is the main reason why the radius R in the statement of the lemma must be of order $1/\sqrt{\epsilon}$. To start off the construction of \tilde{Q} , we replace Q by $\eta \circ Q$.

Step 2. In the second step, we appropriately cut off $\eta \circ Q$ and define a new Hamiltonian \hat{Q} which is a linear transition from $\eta \circ Q$ to ϵQ in the q -direction. Namely let $r > 0$ be the radius of the ball V and set $a_0 = r/\sqrt{\epsilon}$ and $a_1 = 2a_0$. (So $r < a_0 < a_1$.) The modification of $\eta \circ Q$ takes place on the domain $a_0 \leq \|q\| \leq a_1$. To this end, choose a smooth monotone increasing function $\phi : [0, \infty) \rightarrow \mathbb{R}$ such that $\phi(x) = 0$ when $x \leq a_0$, $\phi(x) = 1$ when $x \geq a_1$, and $|\phi'| \leq 2/|a_1 - a_0| = 2/a_0$. We then set

$$\hat{Q} = \phi(\|q\|)\epsilon Q + (1 - \phi(\|q\|))\eta \circ Q.$$

Step 3. In the third step, we suitably cut off \hat{Q} and finally define the desired Hamiltonian \tilde{Q} . To this end, let $b_0 = \max\{r, 32c/\lambda r \sqrt{\epsilon}\}$ and $b_1 = 2b_0$. Here, as in Section 3, $\lambda = \min \lambda_i > 0$, where λ_i 's are the eigenvalues of A . (The reason for this choice of b_0 will be clear at the end of the proof.) Choose a smooth monotone increasing function $\psi : [0, \infty) \rightarrow \mathbb{R}$ such that $\psi(x) = 0$ when $x \leq b_0$ and $\psi(x) = 1$ when $x \geq b_1$. Define

$$\tilde{Q} = \psi(\|p\|)\epsilon Q + (1 - \psi(\|p\|))\hat{Q}.$$

Checking conditions (i)–(iv). Since $\eta \circ Q = Q$ on V by the definition of c , and $a_0 \geq r$ and $b_0 \geq r$, we clearly have $\tilde{Q} = Q$ on the ball V . Furthermore $\tilde{Q} = \epsilon Q$ outside the ball of radius $R = \sqrt{a_1^2 + b_1^2}$. Let V_ϵ be this ball. It is clear from our choice of a_1 and b_1 that R has the form $C_1/\sqrt{\epsilon}$, where C_1 is independent of ϵ . This proves (i) and (ii).

To establish (iii), observe first that

$$(4-5) \quad \begin{aligned} \sup |\eta(x) - \epsilon x| &= \sup_{[0, c']} |\eta(x) - \epsilon x| \leq \eta(c') + \epsilon c' \\ &= 2\epsilon c' = 4\epsilon c/\epsilon = 4c. \end{aligned}$$

Thus

$$\begin{aligned} \sup_{V_\epsilon} |\tilde{Q}| &\leq \sup_{V_\epsilon} |\epsilon Q| + \sup_{V_\epsilon} |\hat{Q}| \leq 2 \sup_{V_\epsilon} |\epsilon Q| + \sup_{V_\epsilon} |\eta \circ Q| \\ &\leq 3 \sup_{V_\epsilon} |\epsilon Q| + \sup_{V_\epsilon} |\eta \circ Q - \epsilon Q| \\ &\leq 3\epsilon \frac{C_1^2}{\epsilon} \sup_{B(1)} |Q| + 4c = 3C_1^2 \sup_{B(1)} |Q| + 4c =: C_2, \end{aligned}$$

with C_2 independent of ϵ . This is where replacing Q by $\eta \circ Q$ in Step 1 is essential.

To verify condition (iv), note that without loss of generality we may assume that the off-diagonal part of A is so small that

$$\mathcal{L}_{X_Q} \|p\|^2 \leq -\lambda \|p\|^2 \quad \text{and} \quad \mathcal{L}_{X_Q} \|q\|^2 \leq -\lambda \|q\|^2.$$

(Here we dropped the factor of 2 on the right-hand side of the inequalities to account for the off-diagonal terms.) In particular, every integral curve of Q enters the polyball P through the “side” part, $\|p\| = \text{const}$, of the boundary ∂P and leaves it through the “top”, $\|q\| = \text{const}$, of ∂P .

We will show that

- (a) the flow of \tilde{Q} is equal to the flow of Q on P and on the disk $q = 0$, $\|p\| \leq b_0$,
- (b) $\mathcal{L}_{X_{\tilde{Q}}} \|q\|^2 \geq 0$ when $\|p\| \leq b_0$, with strict inequality when $q \neq 0$,
- (c) $\mathcal{L}_{X_{\tilde{Q}}} \|p\|^2 < 0$ when $\|p\| \geq b_0$.

It is not hard to see that (iv) readily follows from these assertions.

Assertion (a) is obvious since $\tilde{Q} = Q$ in the region where $|Q| \leq c$ and $\|q\| \leq a_0$ and $\|p\| \leq b_0$, containing $V \supset P$ and the disk $q = 0$, $\|p\| \leq b_0$. It remains to check (b) and (c), that is, that the function $\|q\|^2$ increases along the flow of \tilde{Q} when $\|p\| \leq b_0$, and the function $\|p\|^2$ decreases along the flow of \tilde{Q} when $\|p\| \geq b_0$, unless $q = 0$.

To prove (b), first note that $\tilde{Q} = \hat{Q}$ in the region where $\|p\| \leq b_0$. Also recall that $\eta' \geq \epsilon/2$, and hence

$$\epsilon \phi(\|q\|) + (1 - \phi(\|q\|))(\eta' \circ Q) \geq \epsilon/2.$$

Therefore since $\phi(\|q\|)$ is independent of p , we have

$$\begin{aligned} \mathcal{L}_{X_{\tilde{Q}}} \|q\|^2 &= \mathcal{L}_{X_Q} \|q\|^2 = \epsilon \phi(\|p\|) \mathcal{L}_{X_Q} \|q\|^2 + (1 - \phi(\|p\|))(\eta' \circ Q) \mathcal{L}_{X_Q} \|q\|^2 \\ &\geq \frac{\epsilon \lambda}{2} \|q\|^2 \geq 0. \end{aligned}$$

Let us now establish (c), which is somewhat more involved than (b) due to the q -dependence of ϕ . Writing ψ for $\psi(\|p\|)$ and ϕ for $\phi(\|q\|)$ and ϕ' for $\phi'(\|q\|)$, we have

$$\begin{aligned} \mathcal{L}_{X_{\tilde{Q}}} \|p\|^2 &= \epsilon \psi \mathcal{L}_{X_Q} \|p\|^2 + (1 - \psi) \mathcal{L}_{X_{\tilde{Q}}} \|p\|^2 \\ &= \epsilon \psi \mathcal{L}_{X_Q} \|p\|^2 + (1 - \psi) (\epsilon \phi \mathcal{L}_{X_Q} \|p\|^2 + (1 - \phi)(\eta' \circ Q) \mathcal{L}_{X_Q} \|p\|^2) \\ &\quad + (1 - \psi) (\epsilon Q - \eta \circ Q) \phi' \mathcal{L}_{X_{\|q\|}} \|p\|^2 \\ &= [\epsilon \psi + (1 - \psi) (\epsilon \phi + (1 - \phi)(\eta' \circ Q))] \mathcal{L}_{X_Q} \|p\|^2 \\ &\quad + (1 - \psi) (\epsilon Q - \eta \circ Q) \phi' \mathcal{L}_{X_{\|q\|}} \|p\|^2. \end{aligned}$$

To bound from above the first term in this expression, note that the coefficient of $\mathcal{L}_{X_Q} \|p\|^2$ is positive and satisfies

$$\epsilon \psi + (1 - \psi) (\epsilon \phi + (1 - \phi)(\eta' \circ Q)) \geq \epsilon \psi + \frac{\epsilon}{2} (1 - \psi) = \frac{\epsilon}{2} (1 + \psi) \geq \frac{\epsilon}{2}.$$

Therefore

$$[\epsilon\psi + (1 - \psi)(\epsilon\phi + (1 - \phi)(\eta' \circ Q))] \mathcal{L}_{X_Q} \|p\|^2 \leq -\epsilon \frac{\lambda}{2} \|p\|^2.$$

Bounding from above the second term, our choices of $a_0 = r/\sqrt{\epsilon}$ and $a_1 = 2a_0$ and also the requirement that $|\phi'| \leq 2/|a_1 - a_0| = 2/a_0$ enter the picture. Then by (4-5), we have

$$2|\epsilon Q - \eta \circ Q| |\phi'| \|p\| \leq 8c \frac{2}{|a_1 - a_0|} \|p\| = \frac{16c}{a_0} \|p\| \leq \frac{16c\sqrt{\epsilon}}{r} \|p\|.$$

Thus in the region where $\|p\| > b_0 = 32c/\lambda r \sqrt{\epsilon}$, we have

$$\mathcal{L}_{X_Q} \|p\|^2 < -\epsilon \frac{\lambda}{2} \|p\|^2 + \frac{16c\sqrt{\epsilon}}{r} \|p\| < 0,$$

completing the proof of (c). (Note that b_0 is chosen exactly to make (c) hold.) \square

Proof of Theorem 1.4. As pointed out in the introduction, we have a more general result in dimension two:

Theorem 4.5. *Let $H : S^1 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a Hamiltonian which is equal to a hyperbolic quadratic form at infinity. Assume that φ_H has at least two isolated homologically nontrivial fixed points and $\text{Fix}(\varphi_H)$ is finite. Then φ_H has simple periodic orbits of arbitrarily large period.*

Remark 4.6. A similar two-dimensional result holds when H is elliptic quadratic at infinity. (In fact, the requirement that the fixed points be homologically nontrivial is not needed in this case.) Indeed since the Hamiltonian is elliptic outside a compact set, a sufficiently large sublevel will be invariant under the flow. Then Franks’s theorem [1992] stating that an area-preserving map of the two-disk has either one or infinitely many periodic points implies the result.

Remark 4.7. Using Theorem 4.5, we can also prove a weaker version of Franks’s theorem on S^2 , asserting that a Hamiltonian diffeomorphism of S^2 with a hyperbolic fixed point must necessarily have infinitely many periodic orbits. However the argument is somewhat involved, and we omit it since this result also follows from the main theorem of [Ginzburg and Gürel 2014], and as has been mentioned in the introduction, at least two other symplectic proofs of Franks’s theorem are available; see [Bramham and Hofer 2012; Collier et al. 2012; Kerman 2012].

Proof of Theorem 4.5. Observe that if φ_H has a homologically nontrivial fixed point with nonzero mean index, then the theorem follows from Theorem 4.1. So let us assume that there are at least two isolated homologically nontrivial fixed points with zero mean index. Notice that all of these points cannot have nonzero local Floer homology concentrated in degree zero: $\text{HF}_*(H) = 0$ when $* \neq 0$ and

$\text{HF}_0(H) = \mathbb{Z}_2$. Thus φ_H must have at least one fixed point with zero mean index and nonzero local Floer homology in degree ± 1 . Such an orbit is a symplectically degenerate maximum, and its presence implies that φ_H has simple periodic orbits of arbitrarily large prime period; see [Ginzburg and Gürel 2009a; 2010] and also [Ginzburg 2010; Hein 2012; Hingston 2009]. (Strictly speaking, the latter fact has been established only for (a broad class of) closed symplectic manifolds. However since the Hamiltonian in our case is a compactly supported perturbation of a hyperbolic quadratic form on \mathbb{R}^{2n} , having periodic orbits only within the support of the perturbation, the proof in the case of closed manifolds, for instance, the one in [Ginzburg and Gürel 2009a], goes through word for word.) \square

Proof of Theorem 1.4 in dimension four. Recall that the homology is concentrated in degree zero and $\text{HF}_0(H) = \mathbb{Z}_2$. Hence one of the fixed points of φ_H must have nonzero Conley–Zehnder index. By a straightforward index analysis, it is easy to see that in dimension four such an orbit must necessarily have nonzero mean index. (Indeed observe that in dimension four the mean index of a nondegenerate fixed point is zero if and only if the linearization is hyperbolic or its eigenvalues comprise two pairs “conjugate” to each other. It is clear that in both cases the Conley–Zehnder index is zero.) Finally applying Theorem 4.1, we obtain the existence of simple periodic orbits with arbitrarily large prime period. \square

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
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