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BIHARMONIC SURFACES OF CONSTANT MEAN CURVATURE

ERIC LOUBEAU AND CEZAR ONICIUC

We compute a Simons-type formula for the stress-energy tensor of biharmonic maps from surfaces. Specializing to Riemannian immersions, we prove several rigidity results for biharmonic CMC surfaces, putting in evidence the influence of the Gaussian curvature on pseudoumbilicity. Finally the condition of biharmonicity is shown to enable an extension of the classical Hopf theorem to CMC surfaces in any ambient Riemannian manifold.

1. Introduction

While harmonic maps between abstract Riemannian manifolds are a generalization of minimal submanifolds, their study on two-dimensional domains remained nonetheless very valuable and brought new light to both theories. When, for topological or geometrical reasons, harmonic maps are nonexistent or unsatisfactory, one can then measure the failure of harmonicity with the *bienergy functional*

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2\, v_g,$$

where M is compact, $\phi : (M, g) \to (N, h)$ is a smooth map and $\tau(\phi) = \operatorname{trace} \nabla d\phi$ is the tension field. Usual arguments (see [Jiang 1986]) show that critical points of E_2, called *biharmonic maps*, are solutions of

$$\tau_2(\phi) = -\Delta\tau(\phi) - \operatorname{trace} R^N(d\phi(\cdot), \tau(\phi))d\phi(\cdot) = 0,$$

and we will use the adjective *proper* to designate nonharmonic biharmonic maps.

Whilst the interconnections between harmonic maps and minimal surfaces are clear and well-established, in many cases, but not always, biharmonic Riemannian immersions have constant mean curvature (CMC). However, this link is not as clear as harmonicity and minimality, and the principal objective of this article is to explain how biharmonicity constrains CMC surfaces in an abstract ambient

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manifold. This is particularly well-illustrated on compact biharmonic CMC surfaces whose Gaussian curvature has constant sign. They must be flat or pseudoumbilical if K^M is nonnegative (Corollary 11); otherwise they have pseudoumbilical points (Theorem 12). The role of pseudoumbilical points in relaxing curvature constraints is further felt in the noncompact case, as their absence forces the CMC surface to be conformally flat (Theorem 12).

For complete surfaces, nonnegative Gaussian curvature and an upper bound on the sectional curvature of the ambient space will cause the surface to be flat or pseudoumbilical, but note that both can occur simultaneously (Proposition 18). When the ambient manifold is a three-dimensional space form, the surface must be umbilical (Corollary 14); consult [Montaldo and Oniciuc 2006] for the classification.

Our approach is to derive, in Proposition 3, a Simons-type formula for the biharmonic stress-energy tensor, valid for all smooth maps. As cumbersome as this equation is in the general case, on biharmonic maps from surfaces it simplifies enough (Proposition 5) to enable the use of a divergence argument (Theorem 6) and draw some consequences (Corollaries 8 and 9). However, the main consequences are for CMC biharmonic surfaces.

To close the article, we show that, in any ambient space, the condition of biharmonicity preserves the holomorphicity of the Hopf differential of CMC surfaces (Theorem 20).

Biharmonic CMC surfaces were also studied in [Fetcu and Pinheiro 2013; Ou and Wang 2011] and [Sasahara 2007].

The conventions we adopt are that the Riemann curvature tensor is

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

while its $(0, 4)$ counterpart is

$$R(X, Y, Z, W) = \langle R(X, Y)W, Z\rangle.$$

The choice of sign for the Laplacians on sections and functions is the same, and on the real line $\Delta f = -f''$.

All objects, unless specified, are smooth and we assume summation on repeated indices, when apt.

2. The biharmonic stress-energy tensor on surfaces

Since biharmonic maps stem from a variational problem, one can apply the general principle of studying the same functional but under variations of the domain metric. This idea taken up on the bienergy leads to the biharmonic stress-energy tensor, which is symmetric and of type $(0, 2)$; see [Loubeau et al. 2008].

Definition 1. Let (M, g) and (N, h) be Riemannian manifolds and $\phi : M \to N$ a smooth map. The biharmonic stress-energy tensor of ϕ is

$$S_2(X, Y) = \left\{ \frac{|\tau(\phi)|^2}{2} + \langle d\phi, \nabla\tau(\phi)\rangle \right\} g(X, Y) - T(X, Y),$$

where $T(X, Y) = \langle d\phi(X), \nabla_Y \tau(\phi)\rangle + \langle d\phi(Y), \nabla_X \tau(\phi)\rangle$.

The main feature of S_2 is satisfying Hilbert's principle of being divergence-free at critical points [Loubeau et al. 2008; Jiang 1987]; that is, $\operatorname{div} S_2 = -\langle d\phi, \tau_2(\phi)\rangle$.

In order to exploit the biharmonicity of the map ϕ, we compute the rough Laplacian of its biharmonic stress-energy tensor. This second-order operator on $(0, 2)$-tensors will reveal curvature terms which combine with the bitension field, and formulas will involve swapping vector positions in the third fundamental form of ϕ, with curvature appearing according to a lemma we quote separately, without proof.

Lemma 2. *Let $\phi : (M^m, g) \to (N^n, h)$ be a smooth map. Then*

$$(\nabla^2 d\phi)(X, Y, Z) - (\nabla^2 d\phi)(Z, Y, X) = R(X, Z)d\phi(Y) - d\phi(R^M(X, Z)Y)$$

for any $X, Y, Z \in C(TM)$.

Proposition 3 (the rough Laplacian of S_2). *Let (M, g) and (N, h) be Riemannian manifolds and $\phi : M \to N$ a smooth map; then the (rough) Laplacian of S_2 is the symmetric $(0, 2)$-tensor*

$$\begin{aligned}
(\Delta^R S_2)(X, Y)&\\
= \Big(&\langle \Delta\tau(\phi), \tau(\phi)\rangle - 2|\nabla\tau(\phi)|^2 - 2\textstyle\sum\langle R(X_i, X_j)d\phi(X_i), \nabla_{X_j}\tau(\phi)\rangle\\
&- 2\langle d\phi(\mathrm{Ricci}^M(\,\cdot\,)), \nabla_{(\cdot)}\tau(\phi)\rangle - 2\langle \nabla d\phi, \nabla^2\tau(\phi)\rangle + \langle d\phi, \nabla(\Delta\tau(\phi))\rangle\\
&- \langle \nabla(\operatorname{trace} R^N(d\phi(\,\cdot\,), \tau(\phi))d\phi(\,\cdot\,)), d\phi\rangle\\
&- \langle \operatorname{trace} R^N(d\phi(\,\cdot\,), \tau(\phi))d\phi(\,\cdot\,), \tau(\phi)\rangle\Big) g(X, Y)\\
&+ 2\langle \nabla_X\tau(\phi), \nabla_Y\tau(\phi)\rangle + \textstyle\sum\langle R(X_i, X)d\phi(X_i), \nabla_Y\tau(\phi)\rangle\\
&+ \textstyle\sum\langle R(X_i, Y)d\phi(X_i), \nabla_X\tau(\phi)\rangle + \langle d\phi(\mathrm{Ricci}^M(X)), \nabla_Y\tau(\phi)\rangle\\
&+ \langle d\phi(\mathrm{Ricci}^M(Y)), \nabla_X\tau(\phi)\rangle + 2\textstyle\sum\langle \nabla d\phi(X_i, X), (\nabla^2\tau(\phi))(X_i, Y)\rangle\\
&+ 2\textstyle\sum\langle \nabla d\phi(X_i, Y), (\nabla^2\tau(\phi))(X_i, X)\rangle - \langle d\phi(X), \nabla_Y(\Delta\tau(\phi))\rangle\\
&- \langle d\phi(Y), \nabla_X(\Delta\tau(\phi))\rangle + \textstyle\sum\langle d\phi(X), R(X_i, Y)\nabla_{X_i}\tau(\phi)\rangle\\
&+ \textstyle\sum\langle d\phi(Y), R(X_i, X)\nabla_{X_i}\tau(\phi)\rangle + \textstyle\sum\langle d\phi(X), \nabla_{X_i}R(X_i, Y)\tau(\phi)\rangle\\
&+ \textstyle\sum\langle d\phi(Y), \nabla_{X_i}R(X_i, X)\tau(\phi)\rangle + \langle d\phi(X), \nabla_{\mathrm{Ricci}^M(Y)}\tau(\phi)\rangle\\
&+ \langle d\phi(Y), \nabla_{\mathrm{Ricci}^M(X)}\tau(\phi)\rangle,
\end{aligned}$$

where $\{X_i\}$ is a geodesic frame around the point $p \in M$.

Proof. Let $\phi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds. We will work with a geodesic frame $\{X_i\}$ around the point $p \in M$ and evaluate at p.

Writing out the Laplacian in the geodesic frame yields

$$\begin{aligned}
&\Delta(\langle d\phi, \nabla\tau(\phi)\rangle)\\
&\quad= -\sum\big\{\langle \nabla_{X_i}[\nabla d\phi(X_i, X_j)], \nabla_{X_j}\tau(\phi)\rangle + 2\langle \nabla d\phi(X_i, X_j), (\nabla^2\tau(\phi))(X_i, X_j)\rangle\\
&\qquad\qquad + \langle d\phi(X_j), \nabla_{X_i}\nabla_{X_i}\nabla_{X_j}\tau(\phi)\rangle - \langle d\phi(X_j), \nabla_{X_i}\nabla_{\nabla_{X_i}X_j}\tau(\phi)\rangle\big\},
\end{aligned}$$

and by the symmetry formula of the third fundamental form, we have

$$\sum \nabla_{X_i}[\nabla d\phi(X_i, X_j)] = \nabla_{X_j}\tau(\phi) + \sum R(X_i, X_j)d\phi(X_i) + d\phi(\mathrm{Ricci}^M(X_j)),$$

and

$$\begin{aligned}
&\sum\big(\nabla_{X_i}\nabla_{X_i}\nabla_{X_j}\tau(\phi) - \nabla_{X_i}\nabla_{\nabla_{X_i}X_j}\tau(\phi)\big)\\
&\qquad= \sum\big\{\nabla_{X_j}\nabla_{X_i}\nabla_{X_i}\tau(\phi) + \nabla_{[X_i,X_j]}\nabla_{X_i}\tau(\phi) + R(X_i, X_j)\nabla_{X_i}\tau(\phi)\\
&\qquad\qquad + \nabla_{X_i}R(X_i, X_j)\tau(\phi)\\
&\qquad\qquad - \big(\nabla_{\nabla_{X_j}X_i}\nabla_{X_i}\tau(\phi) + \nabla_{[X_i,\nabla_{X_j}X_i]}\tau(\phi) + R(X_i, \nabla_{X_j}X_i)\tau(\phi)\big)\big\}\\
&\qquad= -\nabla_{X_j}\big(\Delta\tau(\phi)\big) + \sum\big\{(\nabla^2\tau(\phi))(X_j, \nabla_{X_i}X_i) + R(X_i, X_j)\nabla_{X_i}\tau(\phi)\\
&\qquad\qquad + \nabla_{X_i}R(X_i, X_j)\tau(\phi)\big\}\\
&\qquad\qquad + \nabla_{\mathrm{Ricci}^M(X_j)}\tau(\phi),
\end{aligned}$$

since

$$\sum[X_i, \nabla_{X_j}X_i] = \sum \nabla_{X_j}\nabla_{X_i}X_i - \mathrm{Ricci}^M(X_j).$$

Therefore

$$\begin{aligned}
&\Delta\big(\langle d\phi, \nabla\tau(\phi)\rangle\big)\\
&\quad= -\sum\langle \nabla_{X_j}\tau(\phi), \nabla_{X_j}\tau(\phi)\rangle - \sum\langle R(X_i, X_j)d\phi(X_i), \nabla_{X_j}\tau(\phi)\rangle\\
&\qquad - \sum\langle d\phi(\mathrm{Ricci}^M(X_j)), \nabla_{X_j}\tau(\phi)\rangle\\
&\qquad - 2\langle \nabla d\phi, \nabla^2\tau(\phi)\rangle + \sum\langle d\phi(X_j), \nabla_{X_j}(\Delta\tau(\phi))\rangle\\
&\qquad - \sum\big(\langle d\phi(X_j), R(X_i, X_j)\nabla_{X_i}(\tau(\phi))\rangle + \langle d\phi(X_j), \nabla_{X_i}R(X_i, X_j)\tau(\phi)\rangle\\
&\qquad\qquad + \langle d\phi(X_j), \nabla_{\mathrm{Ricci}^M(X_j)}\tau(\phi)\rangle\big),
\end{aligned}$$

but

$$\begin{aligned}
&\sum\langle d\phi(X_j), \nabla_{X_i}R(X_i, X_j)\tau(\phi)\rangle\\
&\qquad= \sum X_i\langle \mathrm{trace}\, R^N(d\phi(\cdot), \tau(\phi))d\phi(\cdot), d\phi(X_i)\rangle - \langle \nabla d\phi, R(\cdot,\cdot)\tau(\phi)\rangle\\
&\qquad= \langle \nabla(\mathrm{trace}\, R^N(d\phi(\cdot), \tau(\phi))d\phi(\cdot)), d\phi\rangle\\
&\qquad\qquad + \langle \mathrm{trace}\, R^N(d\phi(\cdot), \tau(\phi))d\phi(\cdot), \tau(\phi)\rangle,
\end{aligned}$$

and

$$\sum\langle R(X_i, X_j)d\phi(X_i), \nabla_{X_j}\tau(\phi)\rangle = \sum\langle R(X_i, X_j)\nabla_{X_i}\tau(\phi), d\phi(X_j)\rangle,$$

whilst

$$\sum\langle d\phi(\mathrm{Ricci}^M(\,\cdot\,)), \nabla\tau(\phi)\rangle = \sum\langle d\phi(\,\cdot\,), \nabla_{\mathrm{Ricci}^M(\,\cdot\,)}\tau(\phi)\rangle,$$

so the Laplacian of the scalar term is

$$\begin{aligned}
\Delta\Bigl(\frac{|\tau(\phi)|^2}{2} + \langle d\phi, \nabla\tau(\phi)\rangle\Bigr)&\\
&= \langle\Delta\tau(\phi), \tau(\phi)\rangle - 2|\nabla\tau(\phi)|^2\\
&\quad - 2\sum\langle R(X_i, X_j)d\phi(X_i), \nabla_{X_j}\tau(\phi)\rangle - 2\langle d\phi(\mathrm{Ricci}^M(\,\cdot\,)), \nabla_{(\cdot)}\tau(\phi)\rangle\\
&\quad - 2\langle\nabla d\phi, \nabla^2\tau(\phi)\rangle + \langle d\phi(\,\cdot\,), \nabla_{(\cdot)}(\Delta\tau(\phi))\rangle\\
&\quad - \langle\nabla(\mathrm{trace}\, R^N(d\phi(\,\cdot\,), \tau(\phi))d\phi(\,\cdot\,)), d\phi\rangle\\
&\quad - \langle\mathrm{trace}\, R^N(d\phi(\,\cdot\,), \tau(\phi))d\phi(\,\cdot\,), \tau(\phi)\rangle.
\end{aligned}$$

On the other hand, to compute the (rough) Laplacian of the symmetric two-tensor

$$T(X, Y) = \langle d\phi(X), \nabla_Y\tau(\phi)\rangle + \langle d\phi(Y), \nabla_X\tau(\phi)\rangle,$$

we put $X = X_k$ and $Y = X_j$ and obtain, still evaluating expressions at the point p,

$$\begin{aligned}
-(\Delta^R T)(X, Y) = \sum\bigl(&\langle\nabla_{X_i}\nabla_{X_i}d\phi(X), \nabla_Y\tau(\phi)\rangle + 2\langle\nabla_{X_i}d\phi(X), \nabla_{X_i}\nabla_Y\tau(\phi)\rangle\\
&+ \langle d\phi(X), \nabla_{X_i}\nabla_{X_i}\nabla_Y\tau(\phi)\rangle + \langle\nabla_{X_i}\nabla_{X_i}d\phi(Y), \nabla_X\tau(\phi)\rangle\\
&+ 2\langle\nabla_{X_i}d\phi(Y), \nabla_{X_i}\nabla_X\tau(\phi)\rangle + \langle d\phi(Y), \nabla_{X_i}\nabla_{X_i}\nabla_X\tau(\phi)\rangle\\
&- \langle d\phi(Y), \nabla_{X_i}\nabla_{\nabla_{X_i}X}\tau(\phi)\rangle - \langle d\phi(X), \nabla_{X_i}\nabla_{\nabla_{X_i}Y}\tau(\phi)\rangle\bigr),
\end{aligned}$$

since $\nabla_{X_i}\nabla_{X_i}X_j$ vanishes at the point p. This last expression simplifies further if we use the symmetries properties of the third fundamental form of ϕ to obtain

$$\begin{aligned}
\sum\nabla_{X_i}\nabla_{X_i}d\phi(X) &= \sum(\nabla^2 d\phi)(X_i, X_i, X)\\
&= \nabla_X\tau(\phi) + \sum R(X_i, X)d\phi(X_i) + d\phi(\mathrm{Ricci}^M(X)),
\end{aligned}$$

and the curvature tensor of the pullback bundle for

$$\begin{aligned}
\sum\bigl(\langle d\phi(X), \nabla_{X_i}\nabla_{X_i}\nabla_Y\tau(\phi)\rangle - \langle d\phi(X), \nabla_{X_i}\nabla_{\nabla_{X_i}Y}\tau(\phi)\rangle\bigr)&\\
= -\langle d\phi(X), \nabla_Y(\Delta\tau(\phi))\rangle + \sum\langle d\phi(X), R(X_i, Y)\nabla_{X_i}\tau(\phi)\rangle&\\
+ \sum\langle d\phi(X), \nabla_{X_i}R(X_i, Y)\tau(\phi)\rangle + \langle d\phi(X), \nabla_{\mathrm{Ricci}^M(Y)}\tau(\phi)\rangle&.
\end{aligned}$$

The Laplacian of the tensor T is then equal to

$$
\begin{aligned}
-(\Delta^R T)(X, Y)& \\
=&\langle\nabla_X \tau(\phi), \nabla_Y \tau(\phi)\rangle + \sum\langle R(X_i, X)d\phi(X_i), \nabla_Y \tau(\phi)\rangle \\
&+ \langle d\phi(\mathrm{Ricci}^M(X)), \nabla_Y \tau(\phi)\rangle + 2\sum\langle\nabla d\phi(X_i, X), (\nabla^2\tau(\phi))(X_i, Y)\rangle \\
&- \langle d\phi(X), \nabla_Y(\Delta\tau(\phi))\rangle + \sum\langle d\phi(X), R(X_i, Y)\nabla_{X_i}\tau(\phi)\rangle \\
&+ \sum\langle d\phi(X), \nabla_{X_i} R(X_i, Y)\tau(\phi)\rangle + \langle d\phi(X), \nabla_{\mathrm{Ricci}^M(Y)}\tau(\phi)\rangle \\
&+ \langle\nabla_Y \tau(\phi), \nabla_X \tau(\phi)\rangle + \sum\langle R(X_i, Y)d\phi(X_i), \nabla_X \tau(\phi)\rangle \\
&+ \langle d\phi(\mathrm{Ricci}^M(Y)), \nabla_X \tau(\phi)\rangle + 2\sum\langle\nabla d\phi(X_i, Y), (\nabla^2\tau(\phi))(X_i, X)\rangle \\
&- \langle d\phi(Y), \nabla_X(\Delta\tau(\phi))\rangle + \sum\langle d\phi(Y), R(X_i, X)\nabla_{X_i}\tau(\phi)\rangle \\
&+ \sum\langle d\phi(Y), \nabla_{X_i} R(X_i, X)\tau(\phi)\rangle + \langle d\phi(Y), \nabla_{\mathrm{Ricci}^M(X)}\tau(\phi)\rangle.
\end{aligned}
$$

Summing the various parts together yields the proposition. □

Remark 4. In order to see the geometric meaning of the term $\sum \nabla_{X_i} R(X_i, X)\tau(\phi)$, we can rewrite it as $\sum(\nabla R)(X_i, X_i, X, \tau(\phi)) + \sum R(X_i, X)\nabla_{X_i}\tau(\phi)$.

While the general expression for the rough Laplacian of S_2 at first seems unwieldy, in a manner reminiscent of its harmonic counterpart (see [Baird et al. 2011]) it becomes amenable when the domain is a surface and the map biharmonic. The final formula only involves three ingredients: the tensor S_2 itself, the Gaussian curvature and the norm of the tension field of the map. This paves the way for a series of propositions and corollaries for both maps and Riemannian immersions, where topological and curvature conditions restrict the existence of biharmonic maps.

Proposition 5. *Let $\phi : (M^2, g) \to (N, h)$ be a biharmonic map defined on a surface M^2. The Laplacian of its biharmonic stress-energy tensor is*

$$\Delta^R S_2 = -2K^M S_2 + \nabla d(|\tau(\phi)|^2) + \{K^M|\tau(\phi)|^2 + \Delta|\tau(\phi)|^2\}g,$$

where K^M is the Gaussian curvature of (M^2, g).

Proof. Since $\dim M = 2$, its Ricci curvature is $\mathrm{Ricci}^M = K^M I$, with $K^M \in C^\infty(M)$. We will work with a geodesic frame $\{X_1, X_2\}$ around a point $p \in M^2$ and evaluate all expressions at this point.

As $\Delta^R S_2$ is a symmetric $(0, 2)$-tensor, there are only two cases to consider, and, from the previous proposition, combined with basic symmetries of the curvature tensor and the biharmonicity condition, we have

$$
\begin{aligned}
&(\Delta^R S_2)(X_1, X_2) \\
&= 2\langle \nabla_{X_1}\tau(\phi), \nabla_{X_2}\tau(\phi)\rangle - \langle d\phi(X_2), \nabla_{X_1}(\Delta\tau(\phi))\rangle \\
&\quad + 2K^M\big\{\langle d\phi(X_1), \nabla_{X_2}\tau(\phi)\rangle + \langle d\phi(X_2), \nabla_{X_1}\tau(\phi)\rangle\big\} \\
&\quad + 2\langle \nabla d\phi(X_1, X_2), -\Delta\tau(\phi)\rangle + 2\langle \nabla d\phi(X_1, X_1), (\nabla^2\tau(\phi))(X_1, X_2)\rangle \\
&\quad + 2\langle \nabla d\phi(X_2, X_2), (\nabla^2\tau(\phi))(X_2, X_1)\rangle - \langle d\phi(X_1), \nabla_{X_2}(\Delta\tau(\phi))\rangle \\
&\quad + \langle d\phi(X_1), \nabla_{X_1} R(X_1, X_2)\tau(\phi)\rangle + \langle d\phi(X_2), \nabla_{X_2} R(X_2, X_1)\tau(\phi)\rangle \\
&= 2\langle \nabla_{X_1}\tau(\phi), \nabla_{X_2}\tau(\phi)\rangle + 2K^M\big\{\langle d\phi(X_1), \nabla_{X_2}\tau(\phi)\rangle + \langle d\phi(X_2), \nabla_{X_1}\tau(\phi)\rangle\big\} \\
&\quad + 2\langle \nabla d\phi(X_1, X_1), (\nabla^2\tau(\phi))(X_1, X_2)\rangle + 2\langle \nabla d\phi(X_2, X_2), (\nabla^2\tau(\phi))(X_2, X_1)\rangle \\
&\quad - \langle \nabla_{X_1} d\phi(X_1), R(X_1, X_2)\tau(\phi)\rangle - \langle \nabla_{X_2} d\phi(X_2), R(X_2, X_1)\tau(\phi)\rangle.
\end{aligned}
$$

But

$$
\begin{aligned}
&2\langle \nabla d\phi(X_1, X_1), (\nabla^2\tau(\phi))(X_1, X_2)\rangle - \langle \nabla d\phi(X_1, X_1), R(X_1, X_2)\tau(\phi)\rangle \\
&\qquad = \langle \nabla d\phi(X_1, X_1), 2\nabla_{X_1}\nabla_{X_2}\tau(\phi) - \nabla_{X_1}\nabla_{X_2}\tau(\phi) + \nabla_{X_2}\nabla_{X_1}\tau(\phi)\rangle,
\end{aligned}
$$

so

$$
\begin{aligned}
(\Delta^R S_2)(X_1, X_2) &= 2K^M\big\{\langle d\phi(X_1), \nabla_{X_2}\tau(\phi)\rangle + \langle d\phi(X_2), \nabla_{X_1}\tau(\phi)\rangle\big\} \\
&\quad + 2\langle \nabla_{X_1}\tau(\phi), \nabla_{X_2}\tau(\phi)\rangle + \langle \tau(\phi), \nabla_{X_1}\nabla_{X_2}\tau(\phi) + \nabla_{X_2}\nabla_{X_1}\tau(\phi)\rangle.
\end{aligned}
$$

Since

$$
\begin{aligned}
&(\nabla d|\tau(\phi)|^2)(X_1, X_2) \\
&\qquad = \langle \nabla_{X_1}\nabla_{X_2}\tau(\phi) + \nabla_{X_2}\nabla_{X_1}\tau(\phi), \tau(\phi)\rangle + 2\langle \nabla_{X_1}\tau(\phi), \nabla_{X_2}\tau(\phi)\rangle,
\end{aligned}
$$

we deduce that

$$
(\Delta^R S_2)(X_1, X_2) = -2K^M S_2(X_1, X_2) + (\nabla d|\tau(\phi)|^2)(X_1, X_2).
$$

The other case to look at is when the two vectors are the same, and then Proposition 3 shows that, using the symmetries of R^N,

$$
\begin{aligned}
&(\Delta^R S_2)(X_1, X_1) \\
&= -2\langle R^N(d\phi(X_1), \tau(\phi))d\phi(X_1), \tau(\phi)\rangle \\
&\quad - 2\langle R^N(d\phi(X_2), \tau(\phi))d\phi(X_2), \tau(\phi)\rangle - 2\langle \nabla_{X_2}\tau(\phi), \nabla_{X_2}\tau(\phi)\rangle \\
&\quad - 2K^M\langle d\phi(X_2), \nabla_{X_2}\tau(\phi)\rangle - 2\langle \nabla d\phi(X_2, X_2), (\nabla^2\tau(\phi))(X_2, X_2)\rangle \\
&\quad - 2\langle \nabla d\phi(X_1, X_2), (\nabla^2\tau(\phi))(X_1, X_2)\rangle + 2\langle d\phi(X_1), \nabla_{X_1}(\Delta\tau(\phi))\rangle \\
&\quad + 2\langle d\phi(X_2), \nabla_{X_2}(\Delta\tau(\phi))\rangle + 2K^M\langle d\phi(X_1), \nabla_{X_1}\tau(\phi)\rangle \\
&\quad + 2\langle \nabla d\phi(X_1, X_1), (\nabla^2\tau(\phi))(X_1, X_1)\rangle + 2\langle \nabla d\phi(X_2, X_1), (\nabla^2\tau(\phi))(X_2, X_1)\rangle \\
&\quad - 2\langle d\phi(X_1), \nabla_{X_1}(\Delta\tau(\phi))\rangle + 2\langle d\phi(X_1), \nabla_{X_2} R(X_2, X_1)\tau(\phi)\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= -2|\nabla_{X_2}\tau(\phi)|^2 - 2K^M\langle d\phi(X_2), \nabla_{X_2}\tau(\phi)\rangle + 2K^M\langle d\phi(X_1), \nabla_{X_1}\tau(\phi)\rangle \\
&\qquad - 2\langle R^N(d\phi(X_1), \tau(\phi))d\phi(X_1), \nabla d\phi(X_1, X_1)\rangle \\
&\qquad - 2X_2\langle d\phi(X_2), R^N(d\phi(X_1), \tau(\phi))d\phi(X_1)\rangle \\
&\qquad - 2\langle R^N(d\phi(X_2), \tau(\phi))d\phi(X_2), \nabla d\phi(X_1, X_1)\rangle \\
&\qquad - 2\langle \nabla d\phi(X_2, X_2), (\nabla^2\tau(\phi))(X_2, X_2)\rangle \\
&\qquad + 2\langle \nabla d\phi(X_1, X_1), (\nabla^2\tau(\phi))(X_1, X_1)\rangle - 2\langle \nabla d\phi(X_1, X_2), R(X_1, X_2)\tau(\phi)\rangle \\
&\qquad + 2\langle d\phi(X_1), \nabla_{X_2} R(X_2, X_1)\tau(\phi)\rangle,
\end{aligned}
$$

since

i) $$
\begin{aligned}
&-2\langle \nabla d\phi(X_1, X_2), (\nabla^2\tau(\phi))(X_1, X_2) \\
&\quad +2\langle \nabla d\phi(X_2, X_1), (\nabla^2\tau(\phi))(X_2, X_1)\rangle \\
&\qquad\qquad = -2\langle \nabla d\phi(X_1, X_2), R(X_1, X_2)\tau(\phi)\rangle,
\end{aligned}
$$

ii) $$
\begin{aligned}
&-2\langle R^N(d\phi(X_1), \tau(\phi))d\phi(X_1), \tau(\phi)\rangle \\
&\quad -2\langle d\phi(X_2), \nabla_{X_2} R^N(d\phi(X_1), \tau(\phi))d\phi(X_1)\rangle \\
&\qquad = -2\langle R^N(d\phi(X_1), \tau(\phi))d\phi(X_1), \nabla_{X_1} d\phi(X_1)\rangle \\
&\qquad\quad - 2\langle R^N(d\phi(X_1), \tau(\phi))d\phi(X_1), \nabla_{X_2} d\phi(X_2)\rangle \\
&\qquad\quad - 2X_2\langle d\phi(X_2), R^N(d\phi(X_1), \tau(\phi))d\phi(X_1)\rangle \\
&\qquad\quad + 2\langle \nabla_{X_2} d\phi(X_2), R^N(d\phi(X_1), \tau(\phi))d\phi(X_1)\rangle,
\end{aligned}
$$

iii) $$
\begin{aligned}
&-2\langle R^N(d\phi(X_2), \tau(\phi))d\phi(X_2), \tau(\phi)\rangle \\
&\quad -2\langle d\phi(X_2), \nabla_{X_2} R^N(d\phi(X_2), \tau(\phi))d\phi(X_2)\rangle \\
&\qquad = -2\langle R^N(d\phi(X_2), \tau(\phi))d\phi(X_2), \nabla d\phi(X_1, X_1)\rangle.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&-X_2\langle d\phi(X_2), R^N(d\phi(X_1), \tau(\phi))d\phi(X_1)\rangle + \langle d\phi(X_1), \nabla_{X_2} R(X_2, X_1)\tau(\phi)\rangle \\
&= -X_2 R^N(d\phi(X_2), d\phi(X_1), d\phi(X_1), \tau(\phi)) \\
&\quad + X_2 R^N(d\phi(X_2), d\phi(X_1), d\phi(X_1), \tau(\phi)) + \langle \nabla d\phi(X_1, X_2), R(X_1, X_2)\tau(\phi)\rangle
\end{aligned}
$$

so

$$
\begin{aligned}
&(\Delta^R S_2)(X_1, X_1) \\
&= -2|\nabla_{X_2}\tau(\phi)|^2 - 2K^M\langle d\phi(X_2), \nabla_{X_2}\tau(\phi)\rangle \\
&\qquad + 2K^M\langle d\phi(X_1), \nabla_{X_1}\tau(\phi)\rangle - 2\langle R^N(d\phi(X_1), \tau(\phi))d\phi(X_1), \nabla d\phi(X_1, X_1)\rangle \\
&\qquad + 2\langle \nabla d\phi(X_1, X_1), (\nabla^2\tau(\phi))(X_1, X_1)\rangle \\
&\qquad - 2\langle R^N(d\phi(X_2), \tau(\phi))d\phi(X_2), \nabla d\phi(X_1, X_1)\rangle \\
&\qquad - 2\langle \nabla d\phi(X_2, X_2), (\nabla^2\tau(\phi))(X_2, X_2)\rangle.
\end{aligned}
$$

But

$$\langle \tau(\phi), \nabla_{X_2}\nabla_{X_2}\tau(\phi)\rangle = -\tfrac{1}{2}\Delta|\tau(\phi)|^2 - \tfrac{1}{2}X_1X_1(|\tau(\phi)|^2) - |\nabla_{X_2}\tau(\phi)|^2,$$

so

$$\begin{aligned}(\Delta^R S_2)(X_1, X_1) = -2K^M S_2(X_1, X_1) + \big\{K^M|\tau(\phi)|^2 + \Delta|\tau(\phi)|^2\big\}g(X_1, X_1)\\ + (\nabla d|\tau(\phi)|^2)(X_1, X_1),\end{aligned}$$

with a similar expression for $(\Delta^R S_2)(X_2, X_2)$. Therefore

$$\Delta^R S_2 = -2K^M S_2 + \nabla d(|\tau(\phi)|^2) + \{K^M|\tau(\phi)|^2 + \Delta|\tau(\phi)|^2\}g. \qquad \square$$

The expression for the Laplacian of the biharmonic stress-energy tensor on a surface is simple enough to be contracted with S_2 itself and combined with the divergence theorem, if the domain is assumed to be compact. The ensuing integral formula tightly binds the tensor S_2, the Gaussian curvature and the norm of the tension field together, and conditions on two of them determine the third.

More geometrical applications will be found for Riemannian immersions in the next section.

Theorem 6. *Let $\phi : M^2 \to N^n$ be a biharmonic map and assume M^2 is compact. Then*

$$\int_M |\nabla S_2|^2\, v_g + 2\int_M K^M\left(|S_2|^2 - \frac{|\tau(\phi)|^4}{2}\right) v_g = \int_M |d(|\tau(\phi)|^2)|^2\, v_g,$$

where K^M is the Gaussian curvature of (M^2, g).

Proof. Observe that

$$\operatorname{div}\langle S_2, d(|\tau(\phi)|^2)\rangle = \langle \operatorname{div} S_2, d(|\tau(\phi)|^2)\rangle + \langle S_2, \operatorname{Hess}(|\tau(\phi)|^2)\rangle.$$

As $\operatorname{div} S_2 = 0$, we have

$$\int_M \langle S_2, \operatorname{Hess}(|\tau(\phi)|^2)\rangle\, v_g = \int_M \operatorname{div}\{\langle S_2, d(|\tau(\phi)|^2)\rangle\}\, v_g = 0,$$

which combined with the classical equality

$$\int_M \langle \Delta^R S_2, S_2\rangle\, v_g = \int_M |\nabla S_2|^2\, v_g$$

gives the theorem. $\square$

Remark 7. Note that the term $2|S_2|^2 - |\tau(\phi)|^4$ is always nonnegative since it is equal to $(S_2(X_1, X_1) - S_2(X_2, X_2))^2 + 4S_2^2(X_1, X_2)$, and $|S_2|^2 = |\tau(\phi)|^4/2$ if and only if $S_2 = |\tau(\phi)|^2 g/2$.

A biharmonic map with parallel stress-energy tensor must have a tension field of constant norm [Loubeau et al. 2008], but Proposition 5 shows greater restrictions for two-dimensional domains.

Corollary 8. *Let $\phi : M^2 \to N^n$ be a biharmonic map, and assume M is compact and $\nabla S_2 = 0$. Then $|\tau(\phi)|$ is constant and $\int_M K^M v_g = 0$ or $S_2 = |\tau(\phi)|^2 g/2$.*

Proof. If $\nabla S_2 = 0$, then its norm and trace, $|\tau(\phi)|^2$, are constant, hence

$$\left(|S_2|^2 - \frac{|\tau(\phi)|^4}{2}\right)\int_M K^M v_g = 0. \qquad \square$$

If the norm of the tension field is constant, we can deduce a partial converse for nonnegative curvature.

Corollary 9. *Let $\phi : (M^2, g) \to (N^n, h)$ be a proper -biharmonic map with $|\tau(\phi)|^2$ constant. Assume M is compact and $K^M \geq 0$. Then S_2 is parallel and M is flat or $S_2 = |\tau(\phi)|^2 g/2$.*

3. Constant mean curvature surfaces

To be able to offer conditions with greater geometrical content, we concentrate our applications on Riemannian immersions. The recurrent condition on the map is pseudoumbilicity, as an equality between the shape operator A_H in the direction of the mean curvature vector field H and the metric.

The pivotal role of pseudoumbilical immersions, already observed in the study of the biharmonic stress-energy tensor (see [Loubeau et al. 2008]), emerges again in connection with the curvature of the domain surface, sometimes to the extent of determining its topology.

In the absence of compactness, the divergence theorem is substituted with a parabolicity argument on constant mean curvature surfaces, associated with a bound on the curvature tensor of the target space.

Finally, working with complex coordinates on a Riemann surface, the (2, 0)-part of the H-component of the second fundamental form B is shown to be holomorphic if and only if the mean curvature is constant.

Recall that if $\phi : M^2 \to N$ is a pseudoumbilical proper-biharmonic Riemannian immersion then it is CMC. As a consequence, and since $S_2 = -2|H|^2 g + 4A_H$, a rewording of Corollaries 8 and 9 is as follows:

Corollary 10. *Let $\phi : (M^2, g) \to (N^n, h)$ be a proper-biharmonic Riemannian immersion from a compact oriented surface, with $\nabla A_H = 0$. Then M is topologically a torus or pseudoumbilical.*

Corollary 11. *Let* $\phi : (M^2, g) \to (N^n, h)$ *be a CMC proper-biharmonic Riemannian immersion. Assume* M *is compact and* $K^M \geq 0$*. Then* $\nabla A_H = 0$ *and* M *is flat or pseudoumbilical.*

The next result shows that pseudoumbilical points allow some flexibility of the curvature; since away from these points special coordinates exist in which the metric is conformally flat (with a globally defined factor), the shape operator has a simple expression, while its eigenvalues can be computed from the mean curvature vector field (see [Hasanis and Vlachos 1996] for a similar result).

Theorem 12. *Let* $\phi : (M^2, g) \to (N^n, h)$ *be a CMC proper-biharmonic Riemannian immersion. We denote by* λ_1 *and* λ_2 *the principal curvatures of* M *corresponding to* A_H*, with* $\lambda_1 \geq \lambda_2$*, and let* $\mu = \lambda_1 - \lambda_2$*. Consider* $p \in M$ *such that* $\mu(p) > 0$*; that is,* p *is a nonpseudoumbilical point. Then, around* p *there is a local chart* $(U; x, y)$ *which is both isothermal and a line of curvature coordinate system for* A_H*. We have, on* U,

$$g = \frac{1}{\mu}(dx^2 + dy^2), \qquad \langle A_H(\cdot), \cdot \rangle = \frac{1}{\mu}(\lambda_1\, dx^2 + \lambda_2\, dy^2),$$

$$\sum_{i=1}^{2} R^N(X_i, H, X_i, H) - |\nabla^\perp H|^2 - 2|H|^4 > 0,$$

and

$$\lambda_1 = |H|^2 + \frac{\sqrt{2}}{2}\sqrt{\sum\nolimits_{i=1}^{2} R^N(X_i, H, X_i, H) - |\nabla^\perp H|^2 - 2|H|^4}\,,$$

$$\lambda_2 = |H|^2 - \frac{\sqrt{2}}{2}\sqrt{\sum\nolimits_{i=1}^{2} R^N(X_i, H, X_i, H) - |\nabla^\perp H|^2 - 2|H|^4}\,,$$

with $X_1 = \sqrt{\mu}\,\partial x$, $X_2 = \sqrt{\mu}\,\partial y$*. Moreover*

$$\Delta \ln\biggl(\sum_{i=1}^{2} R^N(X_i, H, X_i, H) - |\nabla^\perp H|^2 - 2|H|^4\biggr) = -4K^M,$$

and, in codimension one, the Gauss equation becomes

$$\mathrm{Riem}^N(X_1, X_2) = K^M - 2|H|^2 + \frac{1}{2|H|^2}\,\mathrm{Ricci}^N(H, H).$$

Proof. Let λ_1 and λ_2 be the principal curvatures in the direction of H; that is, λ_1 and λ_2 are the eigenvalues of A_H. In an open neighborhood U around a nonpseudoumbilical point p, $\lambda_1 > \lambda_2$ on U and $\lambda_1, \lambda_2 \in C^\infty(U)$ (in general they are only continuous), and therefore $\mu = \lambda_1 - \lambda_2$ is a positive smooth function on U.

Let $\{X_1, X_2\}$ be a local orthonormal frame on U such that $A_H(X_1) = \lambda_1 X_1$ and $A_H(X_2) = \lambda_2 X_2$. We consider $\omega_1^2, \omega_2^1 \in \bigwedge^1(U)$ defined by

$$\nabla X_1 = \omega_1^2 X_2 \quad \text{and} \quad \nabla X_2 = \omega_2^1 X_1.$$

Clearly $\omega_1^2 = -\omega_2^1$. If we put $X = Z = X_1$ and $Y = X_2$, the Codazzi equation becomes

$$R^N(X_1, H, X_2, X_1) = -\omega_2^1(X_1)\mu - X_2\lambda_1 - \langle B(X_2, X_1), \nabla^\perp_{X_1} H\rangle + \langle B(X_1, X_1), \nabla^\perp_{X_2} H\rangle.$$

Recall that, since $|H|$ is constant, the tangent part of the biharmonic equation is

$$\operatorname{trace} A_{\nabla^\perp_{\cdot} H}(\,\cdot\,) + \operatorname{trace}(R^N(d\phi(\,\cdot\,), H)d\phi(\,\cdot\,))^T = 0.$$

Taking the inner product with X_2, we have

$$\langle B(X_2, X_1), \nabla^\perp_{X_1} H\rangle + \langle B(X_2, X_2), \nabla^\perp_{X_2} H\rangle + R^N(X_1, H, X_2, X_1) = 0;$$

thus

$$\omega_2^1(X_1)\mu + X_2\lambda_1 = 2\langle H, \nabla^\perp_{X_2} H\rangle = 0$$

and

$$\omega_2^1(X_1) = -\frac{X_2\lambda_1}{\mu}.$$

Note that

$$X_2(\lambda_2) = X_2\langle A_H(X_2), X_2\rangle = X_2\langle B(X_2, X_2), H\rangle = -X_2\lambda_1,$$

therefore

$$\omega_2^1(X_1) = \frac{1}{2}\left(-\frac{X_2\lambda_1}{\mu} + \frac{X_2\lambda_2}{\mu}\right) = -\frac{1}{2}\frac{X_2\mu}{\mu}.$$

Exchanging X_1 and X_2, we similarly obtain

$$\omega_2^1(X_2) = \frac{1}{2}\frac{X_1\mu}{\mu},$$

therefore

$$\omega_2^1 = -\frac{1}{2}\frac{X_2\mu}{\mu}\omega_1 + \frac{1}{2}\frac{X_1\mu}{\mu}\omega_2.$$

The Gauss equation implies that

$$d\omega_2^1(X_1, X_2) = K^M;$$

that is,

$$K^M = \tfrac{1}{2}(X_1X_1 \ln\mu + X_2X_2 \ln\mu) - (\omega_2^1(X_1))^2 - (\omega_2^1(X_2))^2,$$

but

$$\nabla_{X_1}X_1 = \tfrac{1}{2}(X_2 \ln \mu)X_2, \qquad (\nabla_{X_1}X_1)(\ln \mu) = \tfrac{1}{2}(X_2 \ln \mu)^2,$$
$$(\omega_2^1(X_1))^2 = \tfrac{1}{4}(X_2 \ln \mu)^2 = \tfrac{1}{2}(\nabla_{X_1}X_1)(\ln \mu),$$

while

$$\nabla_{X_2}X_2 = \tfrac{1}{2}(X_1 \ln \mu)X_1, \qquad (\nabla_{X_2}X_2)(\ln \mu) = \tfrac{1}{2}(X_1 \ln \mu)^2,$$
$$(\omega_2^1(X_2))^2 = \tfrac{1}{4}(X_1 \ln \mu)^2 = \tfrac{1}{2}(\nabla_{X_2}X_2)(\ln \mu).$$

Therefore

$$\Delta \ln \mu = -2K^M.$$

Since

$$\left[\frac{1}{\sqrt{\mu}}X_1, \frac{1}{\sqrt{\mu}}X_2\right] = 0,$$

there exist coordinate functions (x, y) on U such that $\partial/\partial x = X_1/\sqrt{\mu}$ and $\partial/\partial y = X_2/\sqrt{\mu}$. Moreover, the normal part of the biharmonicity equation

$$\Delta^\perp H + \operatorname{trace} B(\cdot, A_H \cdot) + \operatorname{trace}(R^N(\cdot, H)\cdot)^\perp = 0$$

implies, when H is constant,

$$|\nabla^\perp H|^2 + |A_H|^2 - \sum_{i=1}^{2} R^N(X_i, H, X_i, H) = 0,$$

and, since

$$\lambda_1 + \lambda_2 = 2|H|^2 \quad \text{and} \quad \lambda_1^2 + \lambda_2^2 = |A_H|^2,$$

we deduce that

$$|A_H|^2 - 2|H|^4 = \frac{(\lambda_1 - \lambda_2)^2}{2},$$

hence

$$\lambda_1 - \lambda_2 = \sqrt{2}\sqrt{\textstyle\sum_{i=1}^{2} R^N(X_i, H, X_i, H) - |\nabla^\perp H|^2 - 2|H|^4}. \qquad \square$$

Remark 13. i) If $n = 3$, we can replace $\sum_{i=1}^{2} R^N(X_i, H, X_i, H)$ by $\mathrm{Ricci}^N(H, H)$.
ii) Let $\phi : (M^2, g) \to (N, h)$ be a CMC proper-biharmonic Riemannian immersion. If (M^2, g) is complete and has no pseudoumbilical point then its universal cover is (globally) conformally equivalent to $\mathbb{R}^2$.

Corollary 14. *Let $\phi : (M^2, g) \to N^3(c)$ be a CMC proper-biharmonic Riemannian immersion in a three-dimensional real space form. Then it is umbilical.*

Proof. If there exists a nonumbilical point $p_0 \in M$, then, around p_0, we have

$$\operatorname{Riem}^N(X_1, X_2) = K^M - 2|H|^2 + \frac{1}{2|H|^2} \operatorname{Ricci}^N(H, H)$$

and

$$K^M = -\tfrac{1}{4} \Delta \ln(\operatorname{Ricci}^N(H, H) - 2|H|^4),$$

but $\operatorname{Ricci}^N(H, H) = 2c|H|^2$ is constant, so K^M is zero. On the other hand, the first equation implies that $c = K^M - 2|H|^2 + c$, which contradicts $K^M = 0$. □

As the formulas for λ_1 and λ_2 in Theorem 12 remain valid also for pseudoumbilical points, we deduce:

Corollary 15. *Let $\phi : (M^2, g) \to (N^3, h)$ be a CMC proper-biharmonic Riemannian immersion. Assume that there exists $c > 0$ such that* $\operatorname{Ricci}^N(U, U) \geq c|U|^2$ *with* $|H|^2 \in (0, c/2)$. *Then M^2 has no pseudoumbilical point.*

Corollary 16. *Let $\phi : (M^2, g) \to (N^n, h)$ be a CMC proper-biharmonic Riemannian immersion. Assume M is compact, oriented and has no pseudoumbilical point; then M is topologically a torus.*

Corollary 17. *Let $\phi : (M^2, g) \to (N^n, h)$ be a proper-biharmonic Riemannian immersion. Assume that λ_1 and λ_2 are constant; then* $\nabla A_H = 0$, *and M is flat or pseudoumbilical.*

If M is not compact, we need some assumption on the curvature of the target space (see also [Fetcu and Pinheiro 2013, Proposition 4.6 and 4.7]).

Proposition 18. *Let $\phi : (M^2, g) \to (N^n, h)$ be a CMC proper-biharmonic Riemannian immersion. Assume M is noncompact and complete and K^M is nonnegative. Assume that* $\operatorname{Riem}^N \leq K_0$, *where* $K_0 > 0$ (*in the sense that* $R^N(U, V, U, V) \leq K_0$ *for all* $\{U, V\}$ *orthonormal*). *Then* $\nabla A_H = 0$, *and M is flat or pseudoumbilical.*

Proof. By the previous formulas for the Laplacian of S_2, we have

$$\begin{aligned} -\tfrac{1}{2}\Delta|S_2|^2 &= -\langle \Delta^R S_2, S_2\rangle + |\nabla S_2|^2 \\ &= K^M(2|S_2|^2 - |\tau(\phi)|^4) + |\nabla S_2|^2, \end{aligned}$$

which must be nonnegative (Remark 7); therefore $|S_2|^2$ is a subharmonic function and bounded from above since, for Riemannian immersions,

$$|S_2|^2 = 8(2|A_H|^2 - 3|H|^4)$$

and $|A_H|^2$ is itself bounded from above. Indeed if ϕ is biharmonic, then

$$\Delta^\perp H + \operatorname{trace} B(\cdot, A_H \cdot) + \operatorname{trace}(R^N(\cdot, H)\cdot)^\perp = 0;$$

thus

$$|A_H|^2 = -|\nabla^\perp H|^2 + \sum_{i=1}^{2} R^N(X_i, H, X_i, H)$$
$$\leq \sum_{i=1}^{2} R^N(X_i, H, X_i, H) \leq 2|H|^2 K_0,$$

and $|A_H|^2 \leq 2K_0|H|^2$. As M is complete with K^M nonnegative, it is parabolic and $|S_2|^2$, a subharmonic function bounded from above, must be constant (see [Huber 1957]):

$$K^M(|A_H|^2 - 4|H|^4) = 0,$$

while $\nabla A_H = 0$; in particular, $|A_H|^2$ is constant. □

Remark 19. When the dimension of the target is three, we can replace the curvature condition by an upper bound on the Ricci tensor.

The Hopf theorem [1983] shows that a compact simply connected CMC surface immersed in a three-dimensional Euclidean space must be umbilical, hence an embedded round sphere, and the condition of biharmonicity allows us to extend this to any codomain. This result has some strict implications on the set of pseudoumbilical points and hints at the difficulties of working with non-CMC surfaces. An interesting parallel can be drawn with [Fetcu and Pinheiro 2013].

Theorem 20. *Let $\phi : (M^2, g) \to (N^n, h)$ be a proper-biharmonic Riemannian immersion with mean curvature vector field H, with M^2 oriented. Let z be a complex coordinate on M^2; then the function $\langle B(\partial z, \partial z), H\rangle$ is holomorphic if and only if the norm of H is constant.*

Proof. The tangent part of the biharmonic equation is

$$\operatorname{grad}\frac{|H|^2}{2} + \operatorname{trace} A_{\nabla^\perp_{\cdot} H}(\,\cdot\,) + \operatorname{trace}(R^N(d\phi(\,\cdot\,), H)d\phi(\,\cdot\,))^T = 0.$$

Let $g = \lambda^2(dx^2 + dy^2)$ and

$$\tfrac{1}{2}\partial x(|H|^2)\partial x + \tfrac{1}{2}\partial y(|H|^2)\partial y + A_{\nabla^\perp_{\partial x} H}(\partial x) + A_{\nabla^\perp_{\partial y} H}(\partial y)$$
$$+ (R^N(\partial x, H)\partial x + R^N(\partial y, H)\partial y)^T = 0;$$

therefore

$$\frac{\lambda^2}{2}\partial x(|H|^2) + \langle A_{\nabla^\perp_{\partial x} H}(\partial x), \partial x\rangle + \langle A_{\nabla^\perp_{\partial y} H}(\partial y), \partial x\rangle + R^N(\partial y, H, \partial x, \partial y) = 0,$$

and

$$\frac{\lambda^2}{2}\partial y(|H|^2) + \langle A_{\nabla^\perp_{\partial x} H}(\partial x), \partial y\rangle + \langle A_{\nabla^\perp_{\partial y} H}(\partial y), \partial y\rangle + R^N(\partial x, H, \partial y, \partial x) = 0,$$

which is equivalent to

$$\frac{\lambda^2}{2}\partial x(|H|^2)+\langle B(\partial x,\partial x),\nabla^{\perp}_{\partial x}H\rangle+\langle B(\partial x,\partial y),\nabla^{\perp}_{\partial y}H\rangle +R^N(\partial y,H,\partial x,\partial y)=0, \tag{1}$$

and

$$\frac{\lambda^2}{2}\partial y(|H|^2)+\langle B(\partial y,\partial x),\nabla^{\perp}_{\partial x}H\rangle+\langle B(\partial y,\partial y),\nabla^{\perp}_{\partial y}H\rangle +R^N(\partial x,H,\partial y,\partial x)=0. \tag{2}$$

Since $\partial z=(\partial x-i\partial y)/2$ and $\partial\bar{z}=(\partial x+i\partial y)/2$, we see that

$$B(\partial z,\partial z)=\tfrac{1}{2}(\lambda^2H-B(\partial y,\partial y)-iB(\partial x,\partial y))$$

and

$$\langle B(\partial z,\partial z),H\rangle=\tfrac{1}{2}(\lambda^2|H|^2-\langle B(\partial y,\partial y),H\rangle-i\langle B(\partial x,\partial y),H\rangle).$$

Next we compute $\partial\bar{z}\langle B(\partial z,\partial z),H\rangle$:

$$\begin{aligned}
&(\partial x+i\partial y)\big(\lambda^2|H|^2-\langle B(\partial y,\partial y),H\rangle-i\langle B(\partial x,\partial y),H\rangle\big)\\
&\quad=2\lambda\frac{\partial\lambda}{\partial x}|H|^2+\lambda^2\partial x(|H|^2)-\langle\nabla^{\perp}_{\partial x}B(\partial y,\partial y),H\rangle-\langle B(\partial y,\partial y),\nabla^{\perp}_{\partial x}H\rangle\\
&\qquad+\langle\nabla^{\perp}_{\partial y}B(\partial x,\partial y),H\rangle+\langle B(\partial x,\partial y),\nabla^{\perp}_{\partial y}H\rangle\\
&\qquad+i\Big\{2\lambda\frac{\partial\lambda}{\partial y}|H|^2+\lambda^2\partial y(|H|^2)-\langle\nabla^{\perp}_{\partial y}B(\partial y,\partial y),H\rangle-\langle B(\partial y,\partial y),\nabla^{\perp}_{\partial y}H\rangle\\
&\qquad\qquad-\langle\nabla^{\perp}_{\partial x}B(\partial x,\partial y),H\rangle-\langle B(\partial x,\partial y),\nabla^{\perp}_{\partial x}H\rangle\Big\}\\
&\quad=A+iB.
\end{aligned}$$

With (1),

$$\begin{aligned}
A&=2\lambda\frac{\partial\lambda}{\partial x}|H|^2+\tfrac{1}{2}\lambda^2\partial x(|H|^2)-\langle\nabla^{\perp}_{\partial x}B(\partial y,\partial y),H\rangle-\langle B(\partial y,\partial y),\nabla^{\perp}_{\partial x}H\rangle\\
&\qquad+\langle\nabla^{\perp}_{\partial y}B(\partial x,\partial y),H\rangle-\langle B(\partial x,\partial x),\nabla^{\perp}_{\partial x}H\rangle-R(\partial y,H,\partial x,\partial y)\\
&=2\lambda\frac{\partial\lambda}{\partial x}|H|^2+\tfrac{1}{2}\lambda^2\partial x(|H|^2)-\langle\nabla^{\perp}_{\partial x}B(\partial y,\partial y),H\rangle+\langle\nabla^{\perp}_{\partial y}B(\partial x,\partial y),H\rangle\\
&\qquad-\langle 2\lambda^2H,\nabla^{\perp}_{\partial x}H\rangle-R(\partial y,H,\partial x,\partial y)\\
&=2\lambda\frac{\partial\lambda}{\partial x}|H|^2-\tfrac{1}{2}\lambda^2\partial x(|H|^2)-\langle\nabla^{\perp}_{\partial x}B(\partial y,\partial y),H\rangle+\langle\nabla^{\perp}_{\partial y}B(\partial x,\partial y),H\rangle\\
&\qquad-R(\partial y,H,\partial x,\partial y).
\end{aligned}$$

From the Codazzi equation,

$$\begin{aligned}\langle\nabla^\perp_{\partial y}B(\partial x,\partial y),H\rangle &= \langle(\nabla^\perp_{\partial y}B)(\partial x,\partial y),H\rangle+\langle B(\nabla_{\partial y}\partial x,\partial y),H\rangle+\langle B(\partial x,\nabla_{\partial y}\partial y),H\rangle\\ &= \langle\nabla^\perp_{\partial x}B(\partial y,\partial y),H\rangle-2\langle B(\nabla_{\partial x}\partial y,\partial y),H\rangle+R(\partial y,\partial x,H,\partial y)\\ &\quad+\langle B(\nabla_{\partial y}\partial x,\partial y),H\rangle+\langle B(\partial x,\nabla_{\partial y}\partial y),H\rangle;\end{aligned}$$

therefore

$$\begin{aligned}A &= 2\lambda\frac{\partial\lambda}{\partial x}|H|^2-\tfrac{1}{2}\lambda^2\partial x(|H|^2)-\langle\nabla^\perp_{\partial x}B(\partial y,\partial y),H\rangle+\langle\nabla^\perp_{\partial x}B(\partial y,\partial y),H\rangle\\ &\quad-\langle B(\nabla_{\partial x}\partial y,\partial y),H\rangle+\langle B(\partial x,\nabla_{\partial y}\partial y),H\rangle+R(\partial y,\partial x,H,\partial y)\\ &\quad-R(\partial y,H,\partial x,\partial y)\\ &= 2\lambda\frac{\partial\lambda}{\partial x}|H|^2-\tfrac{1}{2}\lambda^2\partial x(|H|^2)-\left\langle B\left(\frac{1}{\lambda}\left(\frac{\partial\lambda}{\partial y}\partial x+\frac{\partial\lambda}{\partial x}\partial y\right),\partial y\right),H\right\rangle\\ &\quad+\left\langle B\left(\frac{1}{\lambda}\left(-\frac{\partial\lambda}{\partial x}\partial x+\frac{\partial\lambda}{\partial y}\partial y\right),\partial x\right),H\right\rangle\\ &= -\tfrac{1}{2}\lambda^2\partial x(|H|^2).\end{aligned}$$

Identical arguments for the imaginary part B, using (2), yield

$$B=\tfrac{1}{2}\lambda^2\partial y(|H|^2).$$ □

Remark 21. If $\phi:(M^2,g)\to(N^n,h)$ is a CMC proper-biharmonic Riemannian immersion, with M^2 oriented. Then $\langle B(\partial z,\partial z),H\rangle dz^2$ is globally defined and, if M^2 has no pseudoumbilical point, it is equal to $dz^2/4$ and therefore M^2 is an affine manifold.

Corollary 22. *Let $\phi:(M^2,g)\to(N^n,h)$ be a CMC proper-biharmonic Riemannian immersion, with M^2 oriented. If M^2 is not pseudoumbilical, then its pseudoumbilical points are isolated.*

Theorem 20 yields:

Theorem 23. *Let $\phi:(M^2,g)\to(N^n,h)$ be a CMC proper-biharmonic Riemannian immersion. If M^2 is a topological sphere $\mathbb{S}^2$, then M is pseudoumbilical.*

Proof. Since $\langle B(\partial z,\partial z),H\rangle=0$, we have

$$\langle B(\partial x,\partial x)-B(\partial y,\partial y),H\rangle=0\quad\text{and}\quad\langle B(\partial x,\partial y),H\rangle=0,$$

which is equivalent to

$$\langle A_H(\partial x),\partial x\rangle=\langle A_H(\partial y),\partial y\rangle\quad\text{and}\quad\langle A_H(\partial x),\partial y\rangle=\langle A_H(\partial y),\partial x\rangle=0.$$ □

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