FOLIATIONS OF A SMOOTH METRIC MEASURE SPACE
BY HYPERSURFACES
WITH CONSTANT $f$-MEAN CURVATURE

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We study smooth codimension-one foliations \(\mathcal{F}\) of a smooth metric measure space whose leaves have the same constant \(f\)-mean curvature. Firstly, we show that all the leaves of \(\mathcal{F}\) are \(f\)-minimal hypersurfaces when either the smooth metric measure space is compact and has nonnegative Bakry–Émery Ricci curvature, or the limit of the ratio of the weighted volume of a geodesic ball \(B\) and the weighted area of a geodesic sphere \(\partial B\) vanishes. Secondly, we prove that every leaf of \(\mathcal{F}\) is strongly \(f\)-stable. Lastly, we show that there is no complete proper foliation of the Gaussian space whose leaves have the same constant \(f\)-mean curvature. In particular, there are no foliations of \(\mathbb{R}^{n+1}\) whose leaves are complete proper self-similar solutions for mean curvature flow.

1. Introduction and the statement of results

The study of smooth codimension-one foliations of manifolds has a long history in mathematics (see [Lawson 1974] and reference therein). In [Barbosa et al. 1987; 1991; Meeks 1988; Oshikiri 1981], there are very interesting results on foliations whose leaves have constant mean curvature. In this paper, we consider foliations of a smooth metric measure space whose leaves are hypersurfaces having the same \(f\)-mean curvatures. The main questions we consider here concern the rigidity and \(f\)-minimality of such foliations of a smooth metric measure space. Extending the classical results (i.e., when \(f\) is constant) to a smooth metric measure space requires \(f\) or \(\nabla f\) to be bounded in many cases; see [Morgan 2005; Wei and Wylie 2009], for example. Our proof follows the one from the case where \(f\) is constant [Barbosa et al. 1987; 1991] but without any further assumption on \(f\). Moreover, for particular weight functions \(f\), we get rigidity results for self-similar surfaces or translating solitons which are models for singularities of mean curvature flow.

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Recall that a smooth metric measure space \((M^{n+1}, g, f)\) is a smooth Riemannian manifold \((M^{n+1}, \tilde{g})\) with a positive density \(e^{-f}\) used to weight the volume of domains and the area of hypersurfaces. Let \(\Sigma\) be an isometrically immersed hypersurface in \((M^{n+1}, \tilde{g})\). Denote by \(dv\) and \(dA\) the Riemannian volume forms on \(M\) and \(\Sigma\) with respect to \(\tilde{g}\) and the induced metric \(g = i^{\ast}\tilde{g}\), respectively. Then the weighted volume and area are given by \(dv_{m} = e^{-f}dv\) and \(dA_{m} = e^{-f}dA\), respectively.

Smooth metric measure spaces naturally arise in various fields. The Gaussian space, i.e., Euclidean space with the Gaussian density \(e^{-\pi|x|^{2}}\), appears in the study of probability and statistics. Many interesting solitons in geometric flows (e.g., self-similar solutions and translating solitons to the mean curvature flow, and Ricci solitons to the Ricci flow) are represented by \(f\)-minimal hypersurfaces in a smooth metric measure space (see [Bakry and Émery 1985; Cheng et al. 2012; Colding and Minicozzi 2012; Huisken and Sinestrari 1999; Morgan 2005; Pyo 2014] and the references therein).

With the upper bar, we denote the geometric quantities on the ambient space \((M^{n+1}, \tilde{g})\). For example, \(\bar{\nabla}, \bar{d}, \bar{\nabla}^{2}, \bar{\Delta}, \bar{\text{div}}\) and \(\bar{\text{Ric}}\), denote the Levi-Civita connection, exterior differentiation, Hessian, Laplacian, divergence and Ricci tensor of \((M^{n+1}, \tilde{g})\), respectively. For a smooth metric measure space, we naturally consider the Bakry–Émery Ricci tensor \(\text{Ric}_{f}\), which is defined by

\[
\text{Ric}_{f} = \text{Ric} + \nabla^{2}f,
\]

and the \(f\)-Laplacian \(\bar{\Delta}_{f} = \bar{\Delta} - \tilde{g}(\bar{\nabla}f, \bar{\nabla})\) on \(M\), which is a selfadjoint operator with respect to the weighted measure \(dv_{m}\). For a smooth vector field \(\xi\), the \(f\)-divergence of \(\xi\) is defined by

\[
\bar{\text{div}}_{f}\xi = e^{f}\bar{\text{div}}(e^{-f}\xi).
\]

Let \(\nu\) be a unit normal vector field to \(\Sigma\) in \(M\). With the induced metric \(g = i^{\ast}\tilde{g}\) on \(\Sigma\), the second fundamental form of \((\Sigma, g)\) is given by \(A(X, Y) = g(\bar{\nabla}_{X}Y, \nu)\) for any two tangent vectors \(X\) and \(Y\) on \(\Sigma\), and the mean curvature by \(H = \text{tr}(A)\). For the hypersurface \(\Sigma\) in \((M, \tilde{g}, f)\), we define the \(f\)-mean curvature \(H_{f}\) with respect to \(\nu\) as follows:

\[
H_{f} = H + \tilde{g}(\bar{\nabla}f, \nu),
\]

which is obtained by the first variation formula of the weighted area. For \((\Sigma, g), \nabla, d, \Delta\) and \(\text{div}\) denote the Levi-Civita connection, exterior differentiation, Laplacian and divergence on \(\Sigma\), respectively.

The following is proved for foliations of a compact smooth metric measure space with nonnegative Bakry–Émery Ricci curvature:
Theorem 2. Let \((M^{n+1}, \bar{g}, f)\) be a compact smooth metric measure space with nonnegative Bakry–Émery Ricci curvature and \(\mathcal{F}\) a codimension-one smooth foliation of \(M\) whose leaves have the same constant \(f\)-mean curvature. Then every leaf of \(\mathcal{F}\) is a totally geodesic and \(f\)-minimal hypersurface with vanishing Bakry–Émery Ricci curvature in the normal direction.

In a smooth metric measure space \((M^{n+1}, \bar{g}, f)\), we define the ratio

\[
\Lambda_f (R, p) = \frac{\text{vol}_f (\partial B_p (R))}{\text{vol}_f (B_p (R))},
\]

where \(\text{vol}_f (B_p (R))\) and \(\text{vol}_f (\partial B_p (R))\) are the weighted volume of the geodesic ball \(B_p (R)\) and the geodesic sphere \(\partial B_p (R)\) for a point \(p\), respectively. For smooth metric measure spaces of vanishing \(\Lambda_f (R, p)\) as \(R \to \infty\), we show:

Theorem 6. Let \(\mathcal{F}\) be an orientable codimension-one foliation of \((M^{n+1}, \bar{g}, f)\) such that every orientable leaf \(L\) of \(\mathcal{F}\) has the same constant \(f\)-mean curvature. If \(\lim_{R \to \infty} \Lambda_f (R, p) = 0\) for some \(p \in M\), then leaves of \(\mathcal{F}\) are \(f\)-minimal hypersurfaces of \((M^{n+1}, \bar{g}, f)\).

We remark that the Gaussian space and \((\mathbb{R}^{n+1}, d\bar{s}_0, f = x^{n+1})\) enjoy the property that, for any point \(p\), the ratio \(\Lambda_f (R, p)\) vanishes as \(R \to \infty\).

In Section 3, we prove:

Theorem 11. Let \((M^{n+1}, \bar{g}, f)\) be an orientable smooth metric measure space and \(\mathcal{F}\) a smooth codimension-one foliation of \(M\) by orientable leaves. If each leaf of \(\mathcal{F}\) has the same constant \(f\)-mean curvature, then each leaf of \(\mathcal{F}\) is strongly \(f\)-stable.

Theorem 13. There are no complete proper foliations in the Gaussian space \((\mathbb{R}^{n+1}, d\bar{s}_0, f = |X|^2 / 2)\) whose leaves have the same constant \(f\)-mean curvature. In particular, there are no foliations of \(\mathbb{R}^{n+1}\) whose leaves are complete proper self-similar solutions for mean curvature flow.

2. Foliation whose leaves are \(f\)-minimal hypersurfaces

Let us start with the key lemma about the \(f\)-divergence of \(\nabla_v v\). The proof is analogous to that of Proposition 2.14 in [Barbosa et al. 1991], but we include its proof in the Appendix for the sake of completeness.

Lemma 1. Let \(\mathcal{F}\) be a smooth codimension-one foliation of a smooth metric measure space \((M^{n+1}, \bar{g}, f)\) and \(v\) a unit normal vector field to the leaves of \(\mathcal{F}\) in some open subset \(U\) of \(M\). Define a tangent vector field \(\xi = \nabla_v v\). Then on \(U\), we have:

\[
\begin{align*}
\text{(a)} \quad & \overline{\text{div}}_f v = -H_f; \\
\text{(b)} \quad & \overline{\text{div}}_f \xi = \text{div}_f \xi - \frac{\|\xi\|^2}{\bar{g}};
\end{align*}
\]
Theorem 2. Let \((M^{n+1}, \bar{g}, f)\) be a compact smooth metric measure space with nonnegative Bakry–Émery Ricci curvature and \(\mathcal{F}\) a codimension-one smooth foliation of \(M\) whose leaves have the same constant \(f\)-mean curvature. Then every leaf of \(\mathcal{F}\) is a totally geodesic and \(f\)-minimal hypersurface with vanishing Bakry–Émery Ricci curvature in the normal direction.

Proof. Since \(H_f\) is constant in \(M\), \(\nu(H_f) \equiv 0\). Then Lemma 1(c) implies that
\[
\text{div}_f \xi = |\xi|_{\bar{g}}^2 + |A|^2 + \overline{\text{Ric}}_f(v, v) - \nu H_f.
\]
on any leaf of \(\mathcal{F}\), and therefore Lemma 1(b) implies that
\[
\overline{\text{div}}_f \xi = |A|^2 + \overline{\text{Ric}}_f(v, v).
\]
Recall that \(d\nu_m = e^{-f} dv\). Integrating both sides and applying Stokes’ theorem on \(M\), we get
\[
0 = \int_M \overline{\text{div}}_f \xi \, d\nu_m = \int_M |A|^2 + \overline{\text{Ric}}_f(v, v) \, d\nu_m,
\]
that is, \(|A|^2 = 0\) and \(\overline{\text{Ric}}_f(v, v) = 0\) on \(M\). Therefore, every leaf is a totally geodesic hypersurface with vanishing Bakry–Émery Ricci curvature in the normal direction.

Since \(M\) is compact, there exists a point \(m \in M\) such that \(f(m) = \max_M f\). At \(m\), we have \(\nabla f(m) = 0\). Therefore \(H_f(L) = -\bar{g}(\nabla f(m), v) = 0\), where \(L\) is the leaf which contains the point \(m\). So, \(H_f \equiv 0\) on any leaf of \(\mathcal{F}\). This completes the proof. \(\square\)

Remark 3. (1) The compactness condition in Theorem 2 is necessary. The smooth metric measure space \((\mathbb{R}^{n+1}, ds_0, f = x_{n+1})\) has vanishing Bakry–Émery Ricci curvature and is noncompact. Translating solitons under the mean curvature flow do not change shape and are just translated in a direction with a constant speed. Up to rotating and scaling, they are represented by \(x_{n+1}\)-minimal hypersurfaces in the smooth metric measure space \((\mathbb{R}^{n+1}, ds_0, f = x_{n+1})\) (see [Huisken and Sinestrari 1999]). By [Altschuler and Wu 1994] for \(n = 2\), and [Gui, Jian and Ju 2010] for \(n \geq 3\), there exists an entire rotationally symmetric strictly convex graphical hypersurface \(U\), which gives a foliation by \(x_{n+1}\)-minimal hypersurfaces. But clearly \(U\) is not a totally geodesic hypersurface.

(2) The theorem of [Bonnet and Myers 1941] says that a complete Riemannian manifold \(M\) is compact when \(M\) has Ricci curvature bounded from below by a positive constant. But this does not hold in general for a smooth metric measure space. One such example is the Gaussian space \((\mathbb{R}^{n+1}, ds_0, f = |X|^2/2)\). There are some generalizations of the Bonnet–Myers theorem with conditions on \(f\) [Morgan 2005; Wei and Wylie 2009].
Theorem 4. Let \((M^{n+1}, \bar{g}, f)\) be a smooth metric measure space with positive Bakry–Émery Ricci curvature. Any smooth codimension-one foliation of \(M\) whose leaves have the same constant \(f\)-mean curvature cannot have a compact leaf.

Proof. Suppose that, on the contrary, there exists a compact leaf \(L\) in the foliation \(\mathcal{F}\). Lemma 1(c) implies that
\[
\text{div}_f \xi = |\xi|^2_{\bar{g}} + |A|^2 + \operatorname{Ric}_f(v, v)
\]
on \(L\). Weighting both sides by \(dA_m = e^{-f} dA\), integrating, and applying Stokes’ theorem on \(L\), we get a contradiction. □

Let \(\mathcal{F}\) be a smooth orientable codimension-one foliation and \(L\) a leaf of \(\mathcal{F}\). The weighted volume element \(dA_m = \varphi_f\) of \(L\) is defined as follows:
\[
\varphi_f(X_1, \ldots, X_n) = e^{-f} g(X_1 \wedge \cdots \wedge X_n, v),
\]
where the \(X_i\) are tangent vector fields \((i = 1, \ldots, n)\).

With a positively oriented frame field \(\{e_1, \ldots, e_n, e_{n+1} = v\}\), and its dual coframe \(\{\omega_1, \ldots, \omega_n, \omega_{n+1}\}\), the weighted volume elements \(dA_m = \varphi_f\) and \(dv_m = \Phi_m\) are expressed by
\[
\varphi_f = e^{-f} \omega_1 \wedge \cdots \wedge \omega_n,
\]
\[
\Phi_f = e^{-f} \omega_1 \wedge \cdots \wedge \omega_{n+1}.
\]
Both these weighted volume elements are related by the Rummler-type identity [Rummler 1979] as follows:

Lemma 5. Let \((M^{n+1}, \bar{g}, f)\) be an orientable smooth metric measure space and \(\mathcal{F}\) a smooth codimension-one foliation of \(M\) by orientable leaves. Then
\[
\bar{d}\varphi_f = (-1)^{n+1} H_f \Phi_f,
\]
where \(\varphi_f\) is a weighted volume element of leaves of \(\mathcal{F}\).

Proof. Taking exterior differentiation on \(\varphi_f\), we have
\[
\bar{d}\varphi_f = -e^{-f} \bar{d}f \wedge \omega_1 \wedge \cdots \wedge \omega_n + e^{-f} \bar{d}(\omega_1 \wedge \cdots \wedge \omega_n).
\]
Since
\[
\bar{d}f = e_1 f \omega_1 + \cdots + e_{n+1} f \omega_{n+1}
\]
and
\[
\bar{d}(\omega_1 \wedge \cdots \wedge \omega_n) = (-1)^{n+1} H \omega_1 \wedge \cdots \wedge \omega_{n+1},
\]
we have
\[
\bar{d}\varphi_f = (-1)^{n+1} e^{-f} (e_{n+1} f) \omega_1 \wedge \cdots \wedge \omega_{n+1} + (-1)^{n+1} e^{-f} H \omega_1 \wedge \cdots \wedge \omega_{n+1}
\]
\[
= (-1)^{n+1} H_f \Phi_f.
\]
□
Let $p$ be a point in $M$, and $B_p(R)$ a geodesic ball in $(M, \bar{g})$ of radius $R$ centered at $p$. The boundary of $B_p(R)$ is denoted by $\partial B_p(R)$. Define the ratio of the weighted volume of $B_p(R)$ and $\partial B_p(R)$ as follows:

$$\Lambda_f(R, p) = \frac{\text{vol}_f(\partial B_p(R))}{\text{vol}_f(B_p(R))},$$

where $\text{vol}_f(B_p(R))$ and $\text{vol}_f(\partial B_p(R))$ are the weighted volumes of $B_p(R)$ and $\partial B_p(R)$, respectively.

**Theorem 6.** Let $\mathcal{F}$ be an orientable codimension-one foliation of $(M^{n+1}, \bar{g}, f)$ such that every orientable leaf $L$ of $\mathcal{F}$ has the same constant $f$-mean curvature. If $\lim_{R \to \infty} \Lambda_f(R, p) = 0$ for some $p \in M$, then leaves of $\mathcal{F}$ are $f$-minimal hypersurfaces of $(M^{n+1}, \bar{g}, f)$.

**Proof.** Suppose not. Then, choosing a normal vector field, we may assume that

$$(-1)^{n+1} H_f > 0.$$

Let $\sigma_f$ be a weighted volume element of $\partial B_p(R)$. That is, for a local orthonormal frame field $\{X_1, \ldots, X_n\}$ which is tangent to $\partial B_p(R)$,

$$\sigma_f(X_1, \ldots, X_n) = e^{-f}.$$

On $\partial B_p(R)$, we have $\varphi_f \leq \sigma_f$.

By Lemma 5, we have

$$\text{vol}_f(B_p(R)) = \int_{B_p(R)} \Phi_f = \int_{B_p(R)} \frac{(-1)^{n+1}}{H_f} \varphi_f$$

$$\leq \frac{(-1)^{n+1}}{H_f} \int_{\partial B_p(R)} \sigma_f$$

$$= \frac{(-1)^{n+1}}{H_f} \text{vol}_f(\partial B_p(R)).$$

Therefore

$$0 < (-1)^{n+1} H_f \leq \frac{\text{vol}_f(\partial B_p(R))}{\text{vol}_f(B_p(R))} = \Lambda_f(R, p).$$

As $R$ goes to $\infty$, we get a contradiction, and this completes the proof. \qed

Let $X = (x_1, \ldots, x_{n+1})$ be the position vector in $\mathbb{R}^{n+1}$ and $|X|^2 = x_1^2 + \cdots + x_{n+1}^2$. Self-shrinkers under the mean curvature flow in $\mathbb{R}^{n+1}$ are represented by $|X|^2/2$-minimal hypersurfaces in the Gaussian space $(\mathbb{R}^{n+1}, ds_0, f = |X|^2/2)$ (see [Colding and Minicozzi 2012]).
By direct computation,
\[ \lim_{R \to \infty} \frac{\text{vol}_f(\partial B_p(R))}{\text{vol}_f(B_p(R))} = 0 \]
in the Gaussian space, and therefore the following corollary is obtained:

**Corollary 7.** Let \( \mathcal{F} \) be an orientable codimension-one foliation of the Gaussian space such that every orientable leaf \( L \) of \( \mathcal{F} \) has the same constant \( f \)-mean curvature. Then leaves of \( \mathcal{F} \) are self-shrinkers.

By direct computation,
\[ \lim_{R \to \infty} \frac{\text{vol}_f(\partial B_p(R))}{\text{vol}_f(B_p(R))} = 0 \]
also holds in \((\mathbb{R}^{n+1}, ds_0, f = x_{n+1})\), and therefore the following corollary is obtained:

**Corollary 8.** Let \( \mathcal{F} \) be an orientable codimension-one foliation of \((\mathbb{R}^{n+1}, ds_0, f = x_{n+1})\) such that every orientable leaf \( L \) of \( \mathcal{F} \) has the same constant \( f \)-mean curvature. Then leaves of \( \mathcal{F} \) are translating solitons.

Let \( \Sigma \) be a hypersurface in \((\mathbb{R}^{n+1}, ds_0, f = x_{n+1})\) and \( H_f \) its \( f \)-mean curvature. Translating \( \Sigma \) in the direction of \( e_{n+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1} \), the \( f \)-mean curvature does not change. Using this property we get a Bernstein-type theorem for constant \( f \)-mean curvature surfaces.

**Corollary 9.** Let \( x_{n+1} = F(x_1, \ldots, x_n) \) be a hypersurface of constant \( f \)-mean curvature defined on \( \{x_{n+1} = 0\} \) in \((\mathbb{R}^{n+1}, ds_0, f = x_{n+1})\). Then the \( f \)-mean curvature must be zero.

**Proof.** Consider the graph \( \text{graph}(F) \) of the function \( F \). The family
\[ \{\text{graph}(F) + te_{n+1}\}_{t \in \mathbb{R}} \]
gives a foliation whose leaves are hypersurfaces in \((\mathbb{R}^{n+1}, ds_0, f = x_{n+1})\) with the same constant \( f \)-mean curvature. From Corollary 8, \( H_f \) vanishes. \( \square \)

When \( f \) is a constant, then Corollary 9 becomes the corollary from p. 82 of [Chern 1965].

3. Stability of foliations whose leaves have the same constant \( f \)-mean curvature

Let \( \Sigma^n \) be a constant \( f \)-mean curvature hypersurface in \((M^{n+1}, \tilde{g}, f)\). The \( f \)-stability operator \( L_f \) is defined as
\[ L_f := \Delta_f + |A|^2 + \overline{\text{Ric}}_f(v, v), \]
where \( \nu \) is a unit normal vector field of \( \Sigma \) (see [Cheng et al. 2012; Colding and Minicozzi 2012; Espinar 2012]).

**Definition 10.** A two-sided hypersurface \( \Sigma \) in \((M^{n+1}, \bar{g}, f)\) with constant \( f \)-mean curvature is said to be strongly \( f \)-stable if for any compactly supported smooth function \( u \in C^\infty_c(\Sigma) \), it satisfies

\[
- \int_\Sigma u \bar{L}_f u \, dA_m = \int_\Sigma |\nabla u|^2 - (|A|^2 + \bar{\text{Ric}}_f(\nu, \nu))u^2 \, dv_m \geq 0.
\]

If \( \Sigma \) is an \( f \)-minimal hypersurface, then strong \( f \)-stability is equivalent to usual \( f \)-stability.

**Theorem 11.** Let \((M^{n+1}, \bar{g}, f)\) be an orientable smooth metric measure space and \( \mathcal{F} \) a smooth codimension-one foliation of \( M \) by orientable leaves. If each leaf of \( \mathcal{F} \) has the same constant \( f \)-mean curvature, then each leaf of \( \mathcal{F} \) is strongly \( f \)-stable.

**Proof.** Let \( L \) be a leaf of \( \mathcal{F} \) and \( u \) a smooth real-valued function which is compactly supported on a domain \( D \) in \( L \) (therefore, \( u \) is zero on \( \partial D \)). Then

\[
- u \bar{L}_f u = - u \triangle_f u - (|A|^2 + \bar{\text{Ric}}_f(\nu, \nu))u^2 = - u \triangle_f u + u^2 \div_f \xi + u^2 |\xi|^2_g.
\]

Here we apply equation (c) in Lemma 1.

Since \( \div_f(u^2 \xi) = 2ug(\nabla u, \xi) + u^2 \div_f \xi \), we get

\[
- u \bar{L}_f u = - u \triangle_f u - 2ug(\nabla u, \xi) + u^2 |\xi|^2_g + \div_f(u^2 \xi).
\]

Weighting both sides by \( dv_m \), integrating over \( D \), and applying Stokes’ theorem twice for the first and the last terms, we have

\[
\int_D - u \bar{L}_f u \, dv_m = \int_D - u \triangle_f u - 2ug(\nabla u, \xi) + u^2 |\xi|^2_g + \div_f(u^2 \xi) \, dv_m
\]

\[
= \int_D |\nabla u|^2_g - 2ug(\nabla u, \xi) + u^2 |\xi|^2_g \, dv_m
\]

\[
= \int_D |\nabla u - u \xi|^2_g \, dv_m \geq 0.
\]

Since \( u \) is an arbitrary function, we conclude that \( L \) is \( f \)-stable. \( \square \)

**Remark 12.** Let \( \Sigma \) be a graph over a domain \( \Omega \subset \{x_{n+1} = 0\} \) in \((\mathbb{R}^{n+1}, ds_0, f = x_{n+1})\) having constant \( f \)-mean curvature. Denote \( \Sigma_t = \Sigma + te_{n+1}, t \in \mathbb{R} \). Then, by Theorem 11, every \( \Sigma_t \) is strongly \( f \)-stable. For example, the family of “grim reapers” \( \Sigma_t = \{(x_1, \ldots, x_n, t - \ln \cos x_1 : |x_1| < \pi/2)\} \) is a foliation in the open manifold \( \{(x_1, \ldots, x_{n+1}) : |x_1| < \pi/2\} \) in \((\mathbb{R}^{n+1}, ds_0, f = x_{n+1})\). So, every grim reaper is strongly \( f \)-stable.
Let $\mathcal{F}$ be a foliation of the Gaussian space $(\mathbb{R}^{n+1}, ds_0, f = |X|^2/2)$. If every leaf of $\mathcal{F}$ is proper (respectively, complete), then $\mathcal{F}$ is said to be proper (respectively, complete).

**Theorem 13.** There are no complete proper foliations in the Gaussian space $(\mathbb{R}^{n+1}, ds_0, f = |X|^2/2)$ whose leaves have the same constant $f$-mean curvature. In particular, there are no foliations of $\mathbb{R}^{n+1}$ whose leaves are complete proper self-similar solutions for mean curvature flow.

Recall Colding and Minicozzi’s result for self-shrinkers in the Gaussian space:

**Theorem 14** [Colding and Minicozzi 2012]. There are no $f$-stable complete self-shrinkers without boundary and with polynomial volume growth in the Gaussian space.

**Proof of Theorem 13.** Suppose, on the contrary, that $\mathcal{F}$ is a complete, proper foliation whose leaves have the same $f$-mean curvature. By Corollary 7 and foliated structure, every leaf $L$ of $\mathcal{F}$ is a self-shrinker without boundary. By Theorem 11, $L$ is $f$-stable. Cheng and Zhou [2013] proved that for self-shrinkers, properness is equivalent to polynomial volume growth. Therefore, $L$ is an $f$-stable complete self-shrinker without boundary and with polynomial volume growth in the Gaussian space. This contradicts Theorem 14. □

**Appendix**

Let $\mathcal{F}$ be a smooth codimension-one foliation of a smooth metric measure space $(M^{n+1}, \bar{g}, f)$. On a leaf of $\mathcal{F}$, the induced metric is denoted by $g = i^*\bar{g}$, where $i$ is the inclusion map. Let $\{e_1, \ldots, e_{n+1}\}$ be a locally defined orthonormal frame field of the tangent bundle of $M$ such that $e_{n+1}$ is normal to the leaves of $\mathcal{F}$. Let us denote the dual coframe field by $\{\omega_1, \ldots, \omega_{n+1}\}$, that is, $\omega_A(e_B) = \delta_{AB}$.

The connection one-forms $\omega_{AB}$ are given by exterior differentiation $\tilde{d}$ of the $\omega_A$, and are uniquely defined by Cartan’s first structure equations:

$$\tilde{d}\omega_A = \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0.$$  

Cartan’s second structure equations yield the curvature tensor

$$(1) \quad \tilde{d}\omega_{AB} = \omega_{AC} \wedge \omega_{CB} + \Omega_{AB},$$

where

$$\Omega_{AB} = \frac{1}{2} R_{ABCD} \omega_D \wedge \omega_C.$$  

Throughout, we adopt Einstein’s convention and the following indexing convention:

$$1 \leq i, j, k, l \leq n, \quad 1 \leq A, B, C, D \leq n + 1.$$
The second fundamental form $A$ of the leaves of $\mathcal{F}$ is given by

$$\omega_{n+1i} = -h_{ij} \omega_j,$$

where

$$h_{ij} = \tilde{g}(A(e_i, e_j), e_{n+1}) = \tilde{g}(\nabla_{e_i} e_j, e_{n+1}).$$

The mean curvature is $H = \sum_i h_{ii}$.

**Proof of Lemma 1.** Consider an adapted orthonormal frame field $\{e_1, \ldots, e_{n+1}\}$ on $U$ such that $e_{n+1} = \nu$. We have

$$\text{div}_f \nu = e^f \text{div} (e^{-f} \nu) = e^f \tilde{g}(\nabla_{e_i} e^{-f} \nu, e_A)$$

$$= -\tilde{g}(e_A f \nu, e_A) + \tilde{g}(\nabla_{e_i} e^{-f} \nu, e_A)$$

$$= -\tilde{g}(\nabla f, \nu) - H = -H_f.$$

Therefore, the equation (a) holds.

Furthermore,

$$\text{div}_f \xi = e^f \text{div} (e^{-f} \xi)$$

$$= e^f g(\nabla_{e_i} e^{-f} \xi, e_i) + \tilde{g}(\nabla_{e_{n+1}} e^{-f} \xi, e_{n+1})$$

$$= \text{div}_f \xi - \tilde{g}(\xi, \nabla_{e_{n+1}} e_{n+1}) = \text{div}_f \xi - |\xi|^2_{\tilde{g}}.$$

Therefore, the equation (b) holds.

Since $\tilde{d}u = du + e_{n+1}(u) \omega_{n+1}$ for any smooth function $u$ in $U$, from (2) we get

$$\omega_{n+1i} = -h_{ij} \omega_j + g(\xi, e_i) \omega_{n+1}.$$

On the one hand, from (1), we have

$$\tilde{d} \omega_{n+1i} = \omega_{n+1j} \wedge \omega_{ji} + R_{n+1i n+1 k} \omega_k \wedge \omega_{n+1} + \frac{1}{2} R_{n+1i j k} \omega_k \wedge \omega_j$$

$$= (h_{jk} \omega_{ji}(e_{n+1}) - g(\xi, e_i) \omega_{ji}(e_k) + R_{n+1 i n+1 k}) \omega_k \wedge \omega_{n+1}$$

$$+ \text{terms with } \omega_k \wedge \omega_l.$$

On the other hand, from (3),

$$\tilde{d} \omega_{n+1i} = -(dh_{ij} + e_{n+1} h_{ij} \omega_{n+1}) \wedge \omega_j - h_{ij} \omega_{jk} \wedge \omega_k - h_{ij} \omega_{jn+1} \wedge \omega_{n+1}$$

$$+ \tilde{d} g(\xi, e_i) \omega_{n+1} + g(\xi, e_i) \omega_{n+1j} \wedge \omega_j$$

$$= (e_{n+1} h_{ik} + h_{ij} \omega_{jk}(e_{n+1}) - h_{ij} h_{jk} + \tilde{d} g(\xi, e_i)(e_k)$$

$$- g(\xi, e_i) g(\xi, e_j)) \omega_k \wedge \omega_{n+1} + \text{terms with } \omega_k \wedge \omega_l.$$

By investigating both of the coefficients of $\omega_k \wedge \omega_{n+1}$ in $\tilde{d} \omega_{n+1i}$, we have
\( g(\xi, e_i)g(\xi, e_k) + h_{ij} h_{jk} + R_{n+1 n+1} - (dh_{ik} + h_{ij} \omega_{jk} + h_{jk} \omega_{ji})(e_{n+1}) \)
\( = (dg(\xi, e_i) + g(\xi, e_i) \omega_{ji})(e_k). \)

Since \( dg(\xi, e_i)(e_k) = dg(\xi, e_j)(e_k) \) and \( g(\nabla e_i \xi, e_i) = dg(\xi, e_i) + g(\xi, e_j) \omega_{ji}(e_i), \)
\( \text{div}_f \xi = e^f \text{div}(e^{-f} \xi) = \text{div} \xi - g(\nabla f, \xi) \)
\( = \sum_i (dg(\xi, e_i) + g(\xi, e_j) \omega_{ji})(e_i) - g(\nabla f, \xi) \)
\( = \sum_i g(\xi, e_i)^2 + |A|^2 + \overline{Ric}(v, v) - vH - g(\nabla f, \xi) \quad \text{(by (4))} \)
\( = |\xi|^2_g + |A|^2 + \overline{Ric}(v, v) - vH - v g(\nabla f, v) + v g(\nabla f, v) - g(\nabla f, \xi) \)
\( = |\xi|^2_g + |A|^2 - vH + \overline{Ric}(v, v) + \nabla^2 f(v, v) \)
\( = |\xi|^2_g + |A|^2 - vH + \overline{Ric}_f(v, v). \)

This completes the proof.

References


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