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Let R be a (possibly noncommutative) ring and let $\mathcal{C}$ be a class of finitely generated (right) R-modules which is closed under finite direct sums, direct summands, and isomorphisms. Then the set $\mathcal{V}(\mathcal{C})$ of isomorphism classes of modules is a commutative semigroup with operation induced by the direct sum. This semigroup encodes all possible information about direct sum decompositions of modules in $\mathcal{C}$. If the endomorphism ring of each module in $\mathcal{C}$ is semilocal, then $\mathcal{V}(\mathcal{C})$ is a Krull monoid. Although this fact was observed nearly a decade ago, the focus of study thus far has been on ring- and module-theoretic conditions enforcing that $\mathcal{V}(\mathcal{C})$ is Krull. If $\mathcal{V}(\mathcal{C})$ is Krull, its arithmetic depends only on the class group of $\mathcal{V}(\mathcal{C})$ and the set of classes containing prime divisors. In this paper we provide the first systematic treatment to study the direct-sum decompositions of modules using methods from factorization theory of Krull monoids. We do this when $\mathcal{C}$ is the class of finitely generated torsion-free modules over certain one- and two-dimensional commutative Noetherian local rings.

1. Introduction

The study of direct-sum decompositions of finitely generated modules is a classical topic in module theory dating back over a century. In the early 1900s, Wedderburn, Remak, Krull, and Schmidt proved unique direct-sum decomposition results for various classes of groups (see [Maclagan-Wedderburn 1909; Remak 1911; Krull 1925; Schmidt 1929]). A few decades later Azumaya [1950] proved uniqueness of (possibly infinite) direct-sum decomposition of modules provided that each indecomposable module has a local endomorphism ring. In the commutative setting, Evans [1973] gave an example due to Swan illustrating a nonunique direct-sum decomposition of a finitely generated module over a local ring. The past decade has seen a new semigroup-theoretical approach. This approach was first introduced by

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Facchini and Wiegand [2004] and has been used by several authors (for example, see [Baeth 2007; 2009; Baeth and Luckas 2011; Baeth and Saccon 2012; Diracca 2007; Facchini 2002; 2006; 2012; Facchini and Halter-Koch 2003; Facchini et al. 2006; Facchini and Wiegand 2004; Hassler et al. 2007; Herbera and Příhoda 2010; Levy and Odenthal 1996]). Let R be a ring and let $\mathcal{C}$ be a class of right R-modules which is closed under finite direct sums, direct summands, and isomorphisms. For a module M in $\mathcal{C}$, let $[M]$ denote the isomorphism class of M. Let $\mathcal{V}(C)$ denote the set of isomorphism classes of modules in $\mathcal{C}$. (We assume here that $\mathcal{V}(C)$ is indeed a set, and note that this hypothesis holds for all examples we study.) Then $\mathcal{V}(\mathcal{C})$ is a commutative semigroup with operation defined by $[M]+[N]=[M \oplus N]$ and all information about direct-sum decomposition of modules in $\mathcal{C}$ can be studied in terms of factorization of elements in the semigroup $\mathcal{V}(\mathcal{C})$. In particular, the direct-sum decompositions in $\mathcal{C}$ are (essentially) unique (in other words, the Krull–Remak–Schmidt–Azumaya theorem — KRSA — holds) if and only if $\mathcal{V}(C)$ is a free abelian monoid. This semigroup-theoretical point of view was justified by Facchini [2002] who showed that $\mathcal{V}(\mathcal{C})$ is a reduced Krull monoid provided that the endomorphism ring $\operatorname{End}_R(M)$ is semilocal for all modules M in $\mathcal{C}$. This result allows one to describe the direct-sum decomposition of modules in terms of factorization of elements in Krull monoids, a well-studied class of commutative monoids.

However, thus far much of the focus in this direction has been on the study of module-theoretic conditions which guarantee that all endomorphism rings are semilocal, as well as on trying to describe the monoid $\mathcal{V}(\mathcal{C})$ in terms of various ring- and module-theoretic conditions. Although some factorization-theoretic computations have been done in various settings (e.g., the study of elasticity in [Baeth 2009; Baeth and Luckas 2011; Baeth and Saccon 2012] and the study of the ω-invariant in [Diracca 2007]), the general emphasis has not been on the arithmetic of the monoid $\mathcal{V}(\mathcal{C})$. Our intent is to use known module-theoretic results along with factorization-theoretic techniques in order to give detailed descriptions of the arithmetic of direct-sum decompositions of finitely generated torsion-free modules over certain one- and two-dimensional local rings. We hope that this systematic approach will not only serve to inspire others to consider more detailed and abstract factorization-theoretic approaches to the study of direct-sum decompositions, but to provide new and interesting examples for zero-sum theory over torsion-free groups. We refer to [Facchini 2003] and to the opening paragraph in the recent monograph [Leuschke and Wiegand 2012] for broad information on the Krull–Remak–Schmidt–Azumaya theorem, and to the surveys [Facchini 2012; Baeth and Wiegand 2013] promoting this semigroup-theoretical point of view. More details and references will be given in Section 3.

Krull monoids, both their ideal theory and their arithmetic, are well-studied; see [Geroldinger and Halter-Koch 2006] for a thorough treatment. A reduced

Krull monoid is uniquely determined (up to isomorphism) by its class group G, the set of classes $G_{\mathcal{P}} \subset G$ containing prime divisors, and the number of prime divisors in each class. Let $\mathcal{V}(C)$ be a monoid of modules and suppose $\mathcal{V}(\mathcal{C})$ is Krull with class group G and with set of classes containing prime divisors $G_{\mathcal{P}}$. We are interested in determining what this information tells us about direct-sum decompositions of modules. Let M be a module in $\mathcal{C}$ and let $M = M_1 \oplus \cdots \oplus M_\ell$ where $M_1, \ldots, M_\ell$ are indecomposable right R-modules. Then ℓ is called the length of this factorization (decomposition into indecomposables), and the set of lengths $\mathsf{L}(M) \subset \mathbb{N}$ is defined as the set of all possible factorization lengths. Then KRSA holds if and only if $|G| = 1$. Moreover, it is easy to check that $|\mathsf{L}(M)| = 1$ for all M in $\mathcal{C}$ provided that $|G| \le 2$. Clearly, sets of lengths are a measure how badly KRSA fails. Assuming that $\mathcal{V}(\mathcal{C})$ is Krull, M has at least one direct-sum decomposition in terms of indecomposable right R-modules, and, up to isomorphism, only finitely many distinct decompositions. In particular, all sets of lengths are finite and nonempty. Without further information about the class group G and the subset $G_{\mathcal{P}} \subset G$, this is all that can be said. Indeed, there is a standing conjecture that for every infinite abelian group G there is a Krull monoid with class group G and set $G_{\mathcal{P}}$ such that every set of lengths has cardinality one (see [Geroldinger and Göbel 2003]). On the other hand, if the class group of a Krull monoid is infinite and every class contains a prime divisor, then every finite subset of $\mathbb{N}_{\ge 2}$ occurs as a set of lengths (see Proposition 6.2).

Thus an indispensable prerequisite for the study of sets of lengths (and other arithmetical invariants) in Krull monoids is detailed information about not only the class group G, but also on the set $G_{\mathcal{P}} \subset G$ of classes containing prime divisors. For the monoid $\mathcal{V}(\mathcal{C})$, this is of course a module-theoretic task which depends on both the ring R and the class $\mathcal{C}$ of R-modules. Early results gave only extremal sets $G_{\mathcal{P}}$ and thus no further arithmetical investigations were needed. In Sections 4 and 5 we determine, based on deep module-theoretic results, the class group G of $\mathcal{V}(\mathcal{C})$. We then exhibit well-structured sets $G_{\mathcal{P}}$ providing a plethora of arithmetically interesting direct-sum decompositions. In particular, we study the classes of finitely generated modules, finitely generated torsion-free modules, and maximal Cohen–Macaulay modules over one- and two-dimensional commutative Noetherian local rings. We restrict, if necessary, to specific families of rings in order to obtain explicit results for $G_{\mathcal{P}}$, since it is possible that even slightly different sets $G_{\mathcal{P}}$ can induce completely different behavior in terms of the sets of lengths. Given this information, we use transfer homomorphisms, a key tool in factorization theory and introduced in Section 3, which make it possible to study sets of lengths and other arithmetical invariants of general Krull monoids instead in an associated monoid of zero-sum sequences (see Lemma 3.4). These monoids can be studied using methods from additive (group and number) theory (see [Geroldinger 2009]).

Factorization theory describes the nonuniqueness of factorizations of elements in rings and semigroups into irreducible elements by arithmetical invariants such as sets of lengths, catenary, and tame degrees. We will define each of these invariants in Section 2. The goal is to relate the arithmetical invariants with algebraic parameters (such as class groups) of the objects under consideration. The study of sets of lengths in Krull monoids is a central topic in factorization theory. However, since much of this theory was motivated by examples in number theory (such as holomorphy rings in global fields), most of the focus so far has been on Krull monoids with finite class group and with each class containing a prime divisor. This is in contrast to Krull monoids stemming from module theory which often have infinite class group (see Section 4). A key result in Section 6 shows that the arithmetic of these two types of Krull monoids can have drastically different arithmetic.

In combination with the study of various arithmetical invariants of a given Krull monoid, the following dual question has been asked since the beginning of factorization theory: Are arithmetical phenomena characteristic for a given Krull monoid (inside a given class of Krull monoids)? Affirmative answers have been given for the class of Krull monoids with finitely generated class groups where every class contains a prime divisor. Since sets of lengths are the most investigated invariants in factorization theory, the emphasis in the last decade has been on the following question: Within the class of Krull monoids having finite class group and such that every class contains a prime divisor, does the system of sets of lengths of a monoid H characterize the class group of H? A survey of these problems can be found in [Geroldinger and Halter-Koch 2006, Sections 7.1 and 7.2]. For recent progress, see [Schmid 2009b; 2009a; Baginski et al. 2013]. In Theorem 6.8 we exhibit that for many Krull monoids stemming from the module theory of Sections 4 and 5, the system of sets of lengths and the behavior of absolutely irreducible elements characterizes the class group of these monoids.

In Section 2 we introduce some of the main arithmetical invariants studied in factorization theory as well as their relevance to the study of direct-sum decompositions. Our focus is on sets of lengths and on parameters controlling their structure, but we will also need other invariants such as catenary and tame degrees. Section 3 gives a brief introduction to Krull monoids, monoids of modules, and transfer homomorphisms. Sections 4 and 5 provide explicit constructions stemming from module theory of class groups and distribution of prime divisors in the classes. Finally, in Section 6, we present our results on the arithmetic of direct-sum decomposition in the Krull monoids discussed in Sections 4 and 5.

We use standard notation from commutative algebra and module theory (see [Leuschke and Wiegand 2012]) and we follow the notation of [Geroldinger and Halter-Koch 2006] for factorization theory. All monoids of modules $\mathcal{V}(\mathcal{C})$ are written additively, while all abstract Krull monoids are written multiplicatively.

This follows the tradition in factorization theory, and makes sense also here because our crucial tool, the monoid of zero-sum sequences, is written multiplicatively. In particular, our arithmetical results in Section 6 are written in a multiplicative setting but they are derived for the additive monoids of modules discussed in Sections 4 and 5. Since we hope that this article is readable both for those working in ring and module theory as well as those working in additive theory and factorization theory, we often recall concepts of both areas which are well known to the specialists in the respective fields.

2. Arithmetical preliminaries

In this section we gather together the concepts central to describing the arithmetic of nonfactorial monoids. In particular, we exhibit the arithmetical invariants which will be studied in Section 6 and which will give a measure of nonunique direct sum decompositions of classes of modules studied in Sections 4 and 5. When possible, we recall previous work in the area of direct-sum decompositions for which certain invariants have been studied. For more details on nonunique factorization, see [Geroldinger and Halter-Koch 2006]. First we record some preliminary terminology.

Notation. We denote by $\mathbb{N}$ the set of positive integers and set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For every $n \in \mathbb{N}$, C_n denotes a cyclic group of order n. For real numbers $a, b \in \mathbb{R}$ we set $[a, b] = \{x \in \mathbb{Z} : a \le x \le b\}$. We use the convention that $\sup \varnothing = \max \varnothing = \min \varnothing = 0$.

Subsets of the integers. Let $L, L' \subset \mathbb{Z}$. We denote by $L + L' = \{a + b : a \in L, b \in L'\}$ the *sumset* of L and L'. If $\varnothing \ne L \subset \mathbb{N}$, we call

$$\rho(L) = \sup\left\{\frac{m}{n} : m, n \in L\right\} = \frac{\sup L}{\min L} \in \mathbb{Q}_{\ge 1} \cup \{\infty\}$$

the *elasticity* of L. In addition, we set $\rho(\{0\}) = 1$. Distinct elements $k, l \in L$ are called *adjacent* if $L \cap [\min\{k, l\}, \max\{k, l\}] = \{k, l\}$. A positive integer $d \in \mathbb{N}$ is called a *distance* of L if there exist adjacent elements $k, l \in L$ with $d = |k - l|$. We denote by $\Delta(L)$ the *set of distances* of L. Note that $\Delta(L) = \varnothing$ if and only if $|L| \le 1$, and that L is an arithmetical progression with difference $d \in \mathbb{N}$ if and only if $\Delta(L) \subset \{d\}$.

Monoids and rings. By a *monoid* H we always mean a commutative semigroup with identity 1 which satisfies the cancellation law; that is, if a, b, and c are elements of the H with $ab = ac$, then $b = c$.

Let H be a monoid. We denote by $\mathcal{A}(H)$ the set of atoms (irreducible elements) of H, by $\mathsf{q}(H)$ a quotient group of H with $H \subset \mathsf{q}(H) = \{a^{-1}b : a, b \in H\}$, and by $H^\times$ the set of invertible elements of H. We say that H is *reduced* if $H^\times = \{1\}$, and we denote by $H_{\text{red}} = H/H^\times = \{aH^\times : a \in H\}$ the associated reduced monoid.

Let $H' \subset H$ be a subset. We say that H' is *divisor-closed* if $a \in H'$ and $b \in H$ with $b \mid a$ implies that $b \in H'$. Denote by $[H'] \subset H$ the submonoid generated by H'.

A monoid F is called *free abelian with basis* $\mathcal{P} \subset F$ if every $a \in F$ has a unique representation of the form

$$a = \prod_{p \in \mathcal{P}} p^{\mathsf{v}_p(a)} \quad \text{with } \mathsf{v}_p(a) \in \mathbb{N}_0 \text{ and } \mathsf{v}_p(a) = 0 \text{ for almost all } p \in \mathcal{P}.$$

If F is free abelian with basis $\mathcal{P}$, we set $F = \mathcal{F}(\mathcal{P})$ and call

$$|a| = \sum_{p \in \mathcal{P}} \mathsf{v}_p(a)$$

the *length* of a, and

$$\operatorname{supp}(a) = \{p \in \mathcal{P} : \mathsf{v}_p(a) > 0\}$$

the *support* of a. The multiplicative monoid $\mathcal{F}(\mathcal{P})$ is, of course, isomorphic to the additive monoid $(\mathbb{N}_0^{(\mathcal{P})}, +)$.

Throughout this manuscript, all rings have a unit element and, apart from a few motivating remarks in Section 3, all rings are commutative. Let R be a ring. Then we let $R^\bullet = R \setminus \{0\}$ denote the nonzero elements of R and let $R^\times$ denote its group of units. Note that if R is a domain, then $R^\bullet$ is a monoid as defined above. By the dimension of a ring we always mean its Krull dimension.

Abelian groups. Let G be an additive abelian group and let $G_0 \subset G$ a subset. Then $-G_0 = \{-g : g \in G_0\}$, $G_0^\bullet = G_0 \setminus \{0\}$, and $\langle G_0 \rangle \subset G$ denotes the subgroup generated by G_0. A family $(e_i)_{i \in I}$ of elements of G is said to be *independent* if $e_i \neq 0$ for all $i \in I$ and, for every family $(m_i)_{i \in I} \in \mathbb{Z}^{(I)}$,

$$\sum_{i \in I} m_i e_i = 0 \ \text{ implies } \ m_i e_i = 0 \quad \text{for all } i \in I.$$

The family $(e_i)_{i \in I}$ is called a *basis* for G if $G = \bigoplus_{i \in I} \langle e_i \rangle$. The *total rank* $\mathsf{r}^*(G)$ is the supremum of the cardinalities of independent subsets of G. Thus $\mathsf{r}^*(G) = \mathsf{r}_0(G) + \sum_{p \in \mathbb{P}} \mathsf{r}_p(G)$, where $\mathsf{r}_0(G)$ is the torsion-free rank of G and $\mathsf{r}_p(G)$ is the p-rank of G for every prime $p \in \mathbb{P}$.

Factorizations. Let H be a monoid. The free abelian monoid $\mathsf{Z}(H) = \mathcal{F}(\mathcal{A}(H_{\mathrm{red}}))$ is called the *factorization monoid* of H, and the unique homomorphism

$$\pi : \mathsf{Z}(H) \to H_{\mathrm{red}} \text{ satisfying } \pi(u) = u \text{ for each } u \in \mathcal{A}(H_{\mathrm{red}})$$

is called the *factorization homomorphism* of H. For $a \in H$ and $k \in \mathbb{N}$,

- $\mathsf{Z}_H(a) = \mathsf{Z}(a) = \pi^{-1}(aH^\times) \subset \mathsf{Z}(H)$ is the *set of factorizations* of a,
- $\mathsf{Z}_k(a) = \{z \in \mathsf{Z}(a) : |z| = k\}$ is the *set of factorizations* of a of length k,

- $\mathsf{L}_H(a) = \mathsf{L}(a) = \{|z| : z \in \mathsf{Z}(a)\} \subset \mathbb{N}_0$ is the *set of lengths* of a, and
- $\mathcal{L}(H) = \{\mathsf{L}(b) : b \in H\}$ is the *system of sets of lengths* of H.

By definition, we have $\mathsf{Z}(a) = \{1\}$ and $\mathsf{L}(a) = \{0\}$ for all $a \in H^\times$. If H is assumed to be Krull, as is the case in the monoids of modules we study, and $a \in H$, then the set of factorizations $\mathsf{Z}(a)$ is finite and nonempty and hence $\mathsf{L}(a)$ is finite and nonempty. Suppose that there is $a \in H$ with $|\mathsf{L}(a)| > 1$ with distinct $k, l \in \mathsf{L}(a)$. Then for all $N \in \mathbb{N}$, $\mathsf{L}(a^N) \supset \{(N-i)k + il : i \in [0, N]\}$ and hence $|\mathsf{L}(a^N)| > N$. Thus, whenever there is an element $a \in H$ that has at least two factorizations of distinct lengths, there exist elements of H having arbitrarily many factorizations of distinct lengths. This motivates the need for more refined measures of nonunique factorization.

Several invariants such as elasticity and the Δ-set measure nonuniqueness in terms of sets of lengths. Other invariants such as the catenary degree provide an even more subtle measurement in terms of the distinct factorizations of elements. However, these two approaches cannot easily be separated and it is often the case that a factorization-theoretical invariant is closely related to an invariant of the set of lengths. Thus the exposition that follows will introduce invariants as they are needed and so that the relations between these invariants can be made as clear as possible.

The monoid H is called

- *atomic* if $\mathsf{Z}(a) \neq \emptyset$ for all $a \in H$,
- *factorial* if $|\mathsf{Z}(a)| = 1\,k$ for all $a \in H$ (equivalently, H_{red} is free abelian), and
- *half-factorial* if $|\mathsf{L}(a)| = 1$ for all $a \in H$.

Let z, $z' \in \mathsf{Z}(H)$. Then we can write

$$z = u_1 \cdot \ldots \cdot u_l v_1 \cdot \ldots \cdot v_m \quad \text{and} \quad z' = u_1 \cdot \ldots \cdot u_l w_1 \cdot \ldots \cdot w_n,$$

where $l, m, n \in \mathbb{N}_0$ and $u_1, \ldots, u_l,\ v_1, \ldots, v_m,\ w_1, \ldots, w_n \in \mathcal{A}(H_{\text{red}})$ satisfy

$$\{v_1, \ldots, v_m\} \cap \{w_1, \ldots, w_n\} = \emptyset.$$

Then $\gcd(z, z') = u_1 \cdot \ldots \cdot u_l$, and we call

$$\mathsf{d}(z, z') = \max\{m,\ n\} = \max\{|z \gcd(z, z')^{-1}|, |z' \gcd(z, z')^{-1}|\} \in \mathbb{N}_0$$

the *distance* between z and z'. If $\pi(z) = \pi(z')$ and $z \neq z'$, then clearly

$$2 + \big||z| - |z'|\big| \le \mathsf{d}(z, z').$$

For subsets $X, Y \subset \mathsf{Z}(H)$, we set

$$\mathsf{d}(X, Y) = \min\{\mathsf{d}(x, y) : x \in X,\ y \in Y\},$$

and thus $\mathsf{d}(X, Y) = 0$ if and only if ($X \cap Y \neq \emptyset$, $X = \emptyset$, or $Y = \emptyset$).

From this point on, we will assume all monoids to be atomic. Since the monoids described in Sections 4 and 5 are of the form $\mathcal{V}(\mathcal{C})$ for $\mathcal{C}$ a subclass of finitely generated modules over a commutative Noetherian ring, they are Krull and hence atomic.

The set of distances and chains of factorizations. We now recall the Δ-set of a monoid H, an invariant which describes the sets of lengths of elements in H, and illustrate its relationship with distances between factorizations of elements in H. We denote by

$$\Delta(H) = \bigcup_{L \in \mathcal{L}(H)} \Delta(L) \subset \mathbb{N}$$

the *set of distances* of H. By definition, $\Delta(H) = \emptyset$ if and only if H is half-factorial. For a more thorough investigation of factorizations in H, we will need a distinguished subset of the set of distances. Let $\Delta^*(H)$ denote the set of all $d = \min \Delta(S)$ for some divisor-closed submonoid $S \subset H$ with $\Delta(S) \neq \emptyset$. By definition, we have $\Delta^*(H) \subset \Delta(H)$.

Suppose that H is not factorial. Then there exists an element $a \in H$ with $|\mathsf{Z}(a)| > 1$, and so there exist distinct $z, z' \in \mathsf{Z}(a)$. Then, for $N \in \mathbb{N}$, we have $\mathsf{Z}(a^N) \supset \{z^{N-i}(z')^i : i \in [0, N]\}$. Although $\mathsf{d}(z^N, (z')^N) = Nd(z, z') > N$ suggests that the factorizations z^N and $(z')^N$ of a^N are very different,

$$\mathsf{d}(z^{N-i}(z')^i, z^{N-i+1}(z')^{i-1}) = \mathsf{d}(z, z')$$

for each $i \in [1, N]$. This illustrates that the distance alone is too coarse of an invariant, and motivates the study of the catenary degree as a way of measuring how distinct two factorizations are. As will be described below, there is a structure theorem for the set of lengths of a Krull monoid. However, except in very simple situations, there is no known structure theorem for the set of factorizations of an element in a Krull monoid. Thus we use the catenary degree, its many variations, the tame degree, and other invariants help to measure the subtle distinctions between factorizations.

Let $a \in H$ and $N \in \mathbb{N}_0 \cup \{\infty\}$. A finite sequence $z_0, \ldots, z_k \in \mathsf{Z}(a)$ is called a (*monotone*) N*-chain of factorizations* if $\mathsf{d}(z_{i-1}, z_i) \le N$ for all $i \in [1, k]$ and ($|z_0| \le \cdots \le |z_k|$ or $|z_0| \ge \cdots \ge |z_k|$ respectively). We denote by $\mathsf{c}(a)$ (or by $\mathsf{c}_{\text{mon}}(a)$ respectively) the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ such that any two factorizations z, $z' \in \mathsf{Z}(a)$ can be concatenated by an N-chain (or by a monotone N-chain respectively). Then

$$\mathsf{c}(H) = \sup\{\mathsf{c}(b) : b \in H\} \in \mathbb{N}_0 \cup \{\infty\},$$
$$\mathsf{c}_{\text{mon}}(H) = \sup\{\mathsf{c}_{\text{mon}}(b) : b \in H\} \in \mathbb{N}_0 \cup \{\infty\}$$

denote the *catenary degree* and the *monotone catenary degree* of H. The monotone catenary degree is studied by using the two auxiliary notions of the equal and the adjacent catenary degrees. Let $\mathsf{c}_{\mathrm{eq}}(a)$ denote the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ such that any two factorizations $z, z' \in \mathsf{Z}(a)$ with $|z| = |z'|$ can be concatenated by a monotone N-chain. We call

$$\mathsf{c}_{\mathrm{eq}}(H) = \sup\{\mathsf{c}_{\mathrm{eq}}(b) : b \in H\} \in \mathbb{N}_0 \cup \{\infty\}$$

the *equal catenary degree* of H. We set

$$\mathsf{c}_{\mathrm{adj}}(a) = \sup\{\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) : k, l \in \mathsf{L}(a) \text{ are adjacent}\},$$

and the *adjacent catenary degree* of H is defined as

$$\mathsf{c}_{\mathrm{adj}}(H) = \sup\{\mathsf{c}_{\mathrm{adj}}(b) : b \in H\} \in \mathbb{N}_0 \cup \{\infty\}.$$

Obviously, we have

$$\mathsf{c}(a) \le \mathsf{c}_{\mathrm{mon}}(a) = \sup\{\mathsf{c}_{\mathrm{eq}}(a), \mathsf{c}_{\mathrm{adj}}(a)\} \le \sup \mathsf{L}(a) \quad \text{for all } a \in H,$$

and hence

$$\mathsf{c}(H) \le \mathsf{c}_{\mathrm{mon}}(H) = \sup\{\mathsf{c}_{\mathrm{eq}}(H), \mathsf{c}_{\mathrm{adj}}(H)\}.$$

Note that $\mathsf{c}_{\mathrm{adj}}(H) = 0$ if and only if H is half-factorial, and if H is not half-factorial, then $2 + \sup \Delta(H) \le \mathsf{c}(H)$. Moreover, $\mathsf{c}_{\mathrm{eq}}(H) = 0$ if and only if for all $a \in H$ and all $k \in \mathsf{L}(a)$ we have $|\mathsf{Z}_k(a)| = 1$. Corollary 2.12 of [Coykendall and Smith 2011] implies that if D is a domain, we have that $\mathsf{c}_{\mathrm{eq}}(D^\bullet) = 0$ if and only if $D^\bullet$ is factorial.

We call

$$\sim_{H,\mathrm{eq}} = \big\{(x, y) \in \mathsf{Z}(H) \times \mathsf{Z}(H) : \pi(x) = \pi(y) \text{ and } |x| = |y|\big\}$$

the *monoid of equal-length relations* of H. Let $Z \subset \mathsf{Z}(H)$ be a subset. We say that an element $x \in Z$ is *minimal* in Z if for all elements $y \in Z$ with $y \mid x$ it follows that $x = y$. We denote by $\mathrm{Min}(Z)$ the *set of minimal elements* in Z. Let $x \in Z$. Since the number of elements $y \in Z$ with $y \mid x$ is finite, there exists an $x^* \in \mathrm{Min}(Z)$ with $x^* \mid x$.

Lemma 2.1. *Let H be an atomic monoid.*

(1) $\mathsf{c}_{\mathrm{eq}}(H) \le \sup\big\{|x| : (x, y) \in \mathcal{A}(\sim_{H,\mathrm{eq}})$ *for some* $y \in \mathsf{Z}(H) \setminus \{x\}\big\}$.

(2) *For $d \in \Delta(H)$ let $A_d = \{x \in \mathsf{Z}(H) : |x| - d \in \mathsf{L}(\pi(x))\}$. Then* $\mathsf{c}_{\mathrm{adj}}(H) \le \sup\{|x| : x \in \mathrm{Min}(A_d),\ d \in \Delta(H)\}$.

Proof. See [Blanco et al. 2011, Proposition 4.4]. □

Unions of sets of lengths and the refined elasticities. We now return to studying sets of lengths. We note that the elasticity of certain monoids of modules were studied in [Baeth and Luckas 2011; Baeth and Saccon 2012], but that in Section 6 we will provide results which generalize these results to larger classes of Krull monoids. In addition, we will fine tune these results by also computing the refined elasticities. Let $k, l \in \mathbb{N}$. If $H \neq H^\times$, then

$$\mathcal{U}_k(H) = \bigcup_{\substack{k \in L \\ L \in \mathcal{L}(H)}} L$$

is the union of all sets of lengths containing k. In other words, $\mathcal{U}_k(H)$ is set of all $m \in \mathbb{N}$ for which there exist $u_1, \ldots, u_k, v_1, \ldots, v_m \in \mathcal{A}(H)$ with $u_1 \cdot \ldots \cdot u_k = v_1 \cdot \ldots \cdot v_m$. When $H^\times = H$, we set $\mathcal{U}_k(H) = \{k\}$. In both cases, we define $\rho_k(H) = \sup \mathcal{U}_k(H) \in \mathbb{N} \cup \{\infty\}$ and $\lambda_k(H) = \min \mathcal{U}_k(H) \in [1, k]$. Clearly, we have $\mathcal{U}_1(H) = \{1\}$, $k \in \mathcal{U}_k(H)$, and since $\mathcal{U}_k(H) + \mathcal{U}_l(H) \subset \mathcal{U}_{k+l}(H)$, it follows that

$$\lambda_{k+l}(H) \le \lambda_k(H) + \lambda_l(H) \le k + l \le \rho_k(H) + \rho_l(H) \le \rho_{k+l}(H).$$

The *elasticity* $\rho(H)$ of H is defined as

$$\rho(H) = \sup\{\rho(L) : L \in \mathcal{L}(H)\} \in \mathbb{R}_{\ge 1} \cup \{\infty\},$$

and it is not difficult to verify that

$$\rho(H) = \sup\left\{\frac{\rho_k(H)}{k} : k \in \mathbb{N}\right\} = \lim_{k \to \infty} \frac{\rho_k(H)}{k}.$$

The structure of sets of lengths. To describe the structure of sets of lengths and of their unions, we need the concept of arithmetical progressions as well as various generalizations. Let $l, M \in \mathbb{N}_0$, $d \in \mathbb{N}$, and $\{0, d\} \subset \mathcal{D} \subset [0, d]$. We set

$$P_l(d) = d\mathbb{Z} \cap [0, ld] = \{0, d, 2d, \ldots, ld\}.$$

Thus a subset $L \subset \mathbb{Z}$ is an arithmetical progression (with *difference* $d \in \mathbb{N}$ and *length* $l \in \mathbb{N}_0$) if $L = \min L + P_l(d)$. A subset $L \subset \mathbb{Z}$ is called an *almost arithmetical multiprogression* (AAMP for short) with *difference* d, *period* $\mathcal{D}$, and *bound* M, if

$$L = y + (L' \cup L^* \cup L'') \subset y + \mathcal{D} + d\mathbb{Z},$$

where

- L^* is finite and nonempty with $\min L^* = 0$ and $L^* = (\mathcal{D} + d\mathbb{Z}) \cap [0, \max L^*]$,
- $L' \subset [-M, -1]$ and $L'' \subset \max L^* + [1, M]$, and
- $y \in \mathbb{Z}$.

Note that an AAMP is finite and nonempty, and that an AAMP with period $\{0, d\}$ and bound $M = 0$ is a (usual) arithmetical progression with difference d.

The ω-invariant and the tame degrees. We now study the ω-invariant as well as local and global tame degrees. We note that these notions have been studied in specific noncommutative module-theoretic situations in terms of the so-called *semiexchange property* (see [Diracca 2007]). Moreover, when describing the sets of lengths of elements within a Krull monoid H in terms of AAMPs (see Proposition 6.2), the bound M (described above) is a tame degree related to the monoid H. We begin with the definition. For an atom $u \in H$, let $\omega(H, u)$ denote the smallest $N \in \mathbb{N} \cup \{\infty\}$ having the following property:

> For any multiple a of u and any factorization $a = v_1 \cdot \ldots \cdot v_n$ of a, there exists a subset $\Omega \subset [1, n]$ such that $|\Omega| \le N$ and
> $$u \text{ divides } \prod_{\nu \in \Omega} v_\nu .$$

Furthermore, we set

$$\omega(H) = \sup\{\omega(H, u) : u \in \mathcal{A}(H)\} \in \mathbb{N} \cup \{\infty\}.$$

An atom $u \in H$ is prime if and only if $\omega(H, u) = 1$, and thus H is factorial if and only if $\omega(H) = 1$. If H satisfies the ascending chain condition on divisorial ideals (in particular, H is a Krull monoid or a Noetherian domain), then $\omega(H, u) < \infty$ for all $u \in \mathcal{A}(H)$ [Geroldinger and Hassler 2008, Theorem 4.2]. Roughly speaking, the tame degree $\mathsf{t}(H, u)$ is the maximum of $\omega(H, u)$ and a factorization length of $u^{-1} \prod_{\nu \in \Omega} v_\nu$ in H. More precisely, for an atom $u \in H$, the local tame degree $\mathsf{t}(H, u)$ is the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ having the following property:

> For any multiple a of u and any factorization $a = v_1 \cdot \ldots \cdot v_n$ of a which does not contain u, there is a short subproduct which is a multiple of u, say $v_1 \cdot \ldots \cdot v_m$, and a refactorization of this subproduct which contains u, say $v_1 \cdot \ldots \cdot v_m = u u_2 \cdot \ldots \cdot u_\ell$, such that $\max\{\ell, m\} \le N$.

Thus the local tame degree $\mathsf{t}(H, u)$ measures the distance between any factorization of a multiple a of u and a factorization of a which contains u. As before, we set

$$\mathsf{t}(H) = \sup\{\mathsf{t}(H, u) : u \in \mathcal{A}(H)\} \in \mathbb{N}_0 \cup \{\infty\}.$$

We conclude this section with the following lemma (see [Geroldinger and Halter-Koch 2006, Chapter 1; Geroldinger and Kainrath 2010]) which illustrates how the primary invariants measure the nonuniqueness of factorizations and show that all of these invariants are trivial if the monoid is factorial.

Lemma 2.2. *Let H be an atomic monoid.*

(1) *H is half-factorial if and only if $\rho(H) = 1$ if and only if $\rho_k(H) = k$ for every $k \in \mathbb{N}$.*

(2) *H is factorial if and only if $\mathsf{c}(H) = \mathsf{t}(H) = 0$ if and only if $\omega(H) = 1$.*

(3) $\mathsf{c}(H) = 0$ *or* $\mathsf{c}(H) \ge 2$, *and if* $\mathsf{c}(H) \le 2$, *then* H *is half-factorial.*

(4) $\mathsf{c}(H) \le \omega(H) \le \mathsf{t}(H) \le \omega(H)^2$, *and if* H *is not factorial*, *then*

$$\max\{2, \rho(H)\} \le \omega(H).$$

(5) *If* $\mathsf{c}(H) = 3$, *every* $L \in \mathcal{L}(H)$ *is an arithmetical progression with difference* 1.

3. Krull monoids, monoids of modules, and transfer homomorphisms

The theory of Krull monoids is presented in detail in the monographs [Halter-Koch 1998; Geroldinger and Halter-Koch 2006]. Here we gather the terminology required for our treatment. We then present an introduction to monoids of modules — the key objects of our study. Finally, we recall important terminology and results about monoids of zero-sum sequences and transfer homomorphisms — the key tools in our arithmetical investigations.

Krull monoids. Let H and D be monoids. A monoid homomorphism $\varphi : H \to D$ is called

- a *divisor homomorphism* if $\varphi(a) \mid \varphi(b)$ implies that $a \mid b$ for all $a, b \in H$.
- *cofinal* if for every $a \in D$ there exists some $u \in H$ such that $a \mid \varphi(u)$.
- a *divisor theory* (for H) if $D = \mathcal{F}(\mathcal{P})$ for some set $\mathcal{P}$, φ is a divisor homomorphism, and for every $a \in \mathcal{F}(\mathcal{P})$, there exists a finite nonempty subset $X \subset H$ satisfying $a = \gcd(\varphi(X))$.

We call $\mathcal{C}(\varphi) = \mathsf{q}(D)/\mathsf{q}(\varphi(H))$ the class group of φ, use additive notation for this group, and for $a \in \mathsf{q}(D)$, we denote by $[a] = a\,\mathsf{q}(\varphi(H)) \in \mathsf{q}(D)/\mathsf{q}(\varphi(H))$ the class containing a. Clearly $D/H = \{[a] : a \in D\} \subset \mathcal{C}(\varphi)$ is a submonoid with quotient group $\mathcal{C}(\varphi)$. The homomorphism φ is cofinal if and only if $\mathcal{C}(\varphi) = D/H$ and, by definition, every divisor theory is cofinal. Let $\varphi : H \to D = \mathcal{F}(\mathcal{P})$ be a divisor homomorphism. Then $\varphi(H) = \{a \in D : [a] = [1]\}$ and

$$G_{\mathcal{P}} = \{[p] = p\mathsf{q}(\varphi(H)) : p \in \mathcal{P}\} \subset \mathcal{C}(\varphi)$$

is called the *set of classes containing prime divisors*. Moreover, $\langle G_{\mathcal{P}} \rangle = \mathcal{C}(\varphi)$ and $[G_{\mathcal{P}}] = \{[a] : a \in D\}$.

The monoid H is called a *Krull monoid* if it satisfies one of the following equivalent conditions:

(a) H is completely integrally closed and satisfies the accending chain condition on divisorial ideals.

(b) H has a divisor theory.

(c) H has a divisor homomorphism into a free abelian monoid.

If H is a Krull monoid, then a divisor theory is unique up to unique isomorphism, and the class group associated to a divisor theory depends only on H. It is called the class group of H and will be denoted by $\mathcal{C}(H)$. Moreover, a reduced Krull monoid H with divisor theory $H \hookrightarrow \mathcal{F}(\mathcal{P})$ is uniquely determined up to isomorphism by its *characteristic* $(G, (m_g)_{g\in G})$ where G is an abelian group together with an isomorphism $\Phi : G \to \mathcal{C}(H)$ and with family $(m_g)_{g\in G}$ of cardinal numbers $m_g = |\mathcal{P} \cap \Phi(g)|$ (see [Geroldinger and Halter-Koch 2006, Theorem 2.5.4], and the forthcoming Lemma 3.4).

It is well known that a domain R is a Krull domain if and only if its multiplicative monoid $R^\bullet$ is a Krull monoid, and we set the class group of R to be $\mathcal{C}(R) = \mathcal{C}(R^\bullet)$. Property (a) shows that a Noetherian domain is Krull if and only if it is integrally closed. In addition, many well-studied classes of commutative monoids such as regular congruence monoids in Krull domains and Diophantine monoids are Krull. The focus of the present paper is on Krull monoids stemming from module theory.

Monoids of modules. Let R be a (not necessarily commutative) ring and $\mathcal{C}$ a class of (right) R-modules. We say that $\mathcal{C}$ is *closed under finite direct sums, direct summands, and isomorphisms* provided the following holds: Whenever M, M_1 and M_2 are R-modules with $M \cong M_1 \oplus M_2$, we have $M \in \mathcal{C}$ if and only if $M_1, M_2 \in \mathcal{C}$. We say that $\mathcal{C}$ satisfies the KRSA theorem if the following holds:

> If $k, l \in \mathbb{N}$ and $M_1, \dots, M_k, N_1, \dots, N_l$ are indecomposable modules in $\mathcal{C}$ with $M_1 \oplus \cdots \oplus M_k \cong N_1 \oplus \cdots \oplus N_l$, then $l = k$ and, after a possible reordering of terms, $M_i \cong N_i$ for all $i \in [1, k]$.

Suppose that $\mathcal{C}$ is closed under finite direct sums, direct summands, and isomorphisms. For a module $M \in \mathcal{C}$, we denote by $[M]$ its isomorphism class, and by $\mathcal{V}(\mathcal{C})$ the set of isomorphism classes. (For our purposes here, we tacitly assume that this is actually a set. For the classes of modules studied in Sections 4 and 5 this is indeed the case. For the involved set-theoretical problems in a more general context, see [Facchini 2012, Section 2].) Then $\mathcal{V}(\mathcal{C})$ is a commutative semigroup with operation $[M] + [N] = [M \oplus N]$ and all information about direct-sum decompositions of modules in $\mathcal{C}$ can be studied in terms of factorizations in the semigroup $\mathcal{V}(\mathcal{C})$. By definition, $\mathcal{C}$ satisfies KRSA if and only if $\mathcal{V}(\mathcal{C})$ is a free abelian monoid, which holds if $\mathrm{End}_R(M)$ is local for each indecomposable M in $\mathcal{C}$ (see [Leuschke and Wiegand 2012, Theorem 1.3]).

If the endomorphism ring $\mathrm{End}_R(M)$ is semilocal for all modules M in $\mathcal{C}$, then $\mathcal{V}(\mathcal{C})$ is a Krull monoid ([Facchini 2002, Theorem 3.4]). There is an abundance of recent work which provides examples of rings and classes of modules over these rings for which all endomorphism rings are semilocal (see [Facchini 2004; 2006; 2012]. For monoids of modules, a characterization of when the class group is a torsion group is given in [Facchini and Halter-Koch 2003]).

Suppose that $\mathcal{V}(\mathcal{C})$ is a Krull monoid. Then to understand the structure of direct-sum decompositions of modules in $\mathcal{C}$ is to understand the arithmetic of the reduced Krull monoid $\mathcal{V}(\mathcal{C})$. Since any reduced Krull monoid H is uniquely determined by its class group and by the distribution of prime divisors (that is, the characteristic of H), one must study these parameters.

In the present paper we will focus on the following classes of modules over a commutative Noetherian local ring S, each closed under finite direct sums, direct summands, and isomorphisms. For a commutative Noetherian local ring S, we denote by

- $\mathcal{M}(S)$ the semigroup of isomorphism classes of finitely generated S-modules,
- $\mathcal{T}(S)$ the semigroup of isomorphism classes of finitely generated torsion-free S-modules, and
- $\mathfrak{C}(S)$ the semigroup of isomorphism classes of maximal Cohen–Macaulay (MCM) S-modules.

Note that in order to make $\mathfrak{C}(S)$ a semigroup, we insist that $[0_S] \in \mathfrak{C}(S)$, even though the zero module is not MCM. We say that a commutative Noetherian local ring S has finite representation type if there are, up to isomorphism, only finitely many indecomposable MCM S-modules. Otherwise we say that S has infinite representation type.

Throughout, let $(R, \mathfrak{m})$ be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$, and let $(\widehat{R}, \widehat{\mathfrak{m}})$ denote its $\mathfrak{m}$-adic completion. Let $\mathcal{V}(R)$ and $\mathcal{V}(\widehat{R})$ be any of the above three semigroups. If M is an R-module such that $[M] \in \mathcal{V}(R)$, then $\widehat{M} \cong M \otimes_R \widehat{R}$ is an $\widehat{R}$-module with $[\widehat{M}] \in \mathcal{V}(\widehat{R})$, and every such $\widehat{R}$-module is called *extended*. Note that R has finite representation type if and only if $\widehat{R}$ has finite representation type (see [Leuschke and Wiegand 2012, Chapter 10]), and that the dimension of R is equal to the dimension of $\widehat{R}$. The following crucial result shows that the monoid $\mathcal{V}(R)$ is Krull.

Lemma 3.1. *Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$, and let $(\widehat{R}, \widehat{\mathfrak{m}})$ denote its $\mathfrak{m}$-adic completion.*

(1) *For each indecomposable finitely generated $\widehat{R}$-module M, $\operatorname{End}_{\widehat{R}}(M)$ is local, and therefore $\mathcal{M}(\widehat{R})$, $\mathcal{T}(\widehat{R})$, and $\mathfrak{C}(\widehat{R})$ are free abelian monoids.*

(2) *The embedding $\mathcal{M}(R) \hookrightarrow \mathcal{M}(\widehat{R})$ is a divisor homomorphism. It is cofinal if and only if every finitely generated $\widehat{R}$-module is a direct summand of an extended module.*

(3) *The embeddings $\mathcal{T}(R) \hookrightarrow \mathcal{T}(\widehat{R})$ and $\mathfrak{C}(R) \hookrightarrow \mathfrak{C}(\widehat{R})$ are divisor homomorphisms.*

In particular, $\mathcal{M}(R), \mathcal{T}(R)$, *and* $\mathfrak{C}(R)$ *are reduced Krull monoids. Moreover*, *the embeddings in* (2) *and* (3) *are injective and map* R*-modules onto the submonoid of extended* $\widehat{R}$*-modules.*

Proof. Property (1) holds by the KRSA theorem (see [Leuschke and Wiegand 2012, Chapter 1]).

Wiegand [2001] proved that the given embedding is a divisor homomorphism (see also [Baeth and Wiegand 2013, Theorem 3.6]). The characterization of cofinality follows from the definition and thus (2) holds.

Let M, N be R-modules such that either $[M], [N] \in \mathcal{V}(R)$ where $\mathcal{V}(R)$ denotes either $\mathcal{T}(R)$ or $\mathfrak{C}(R)$ and suppose that $[\widehat{M}]$ divides $[\widehat{N}]$ in $\mathcal{V}(\widehat{R})$. Then we have divisibility in $\mathcal{M}(\widehat{R})$, and hence in $\mathcal{M}(R)$ by (2). Since $\mathcal{V}(R) \subset \mathcal{M}(R)$ is divisor-closed, it follows that $[M]$ divides $[N]$ in $\mathcal{V}(R)$, proving (3).

Together, (2) and (3) show that $\mathcal{M}(R), \mathcal{T}(R)$, and $\mathfrak{C}(R)$ satisfy Property (c) in the definition of Krull monoids. Since each of these monoids is reduced, the maps induced by $[M] \mapsto [\widehat{M}]$ are injective. □

Note that the embedding $\mathcal{M}(R) \hookrightarrow \mathcal{M}(\widehat{R})$ is not necessarily cofinal, as is shown in [Hassler and Wiegand 2009; Frankild et al. 2008]. In Sections 4 and 5 we will study in detail the class group and the distribution of prime divisors of these Krull monoids, in the case of one-dimensional and two-dimensional commutative Noetherian local rings.

Monoids of zero-sum sequences. We now introduce Krull monoids having a combinatorial flavor which are used to model arbitrary Krull monoids. Let G be an additive abelian group and let $G_0 \subset G$ be a subset. Following the tradition in additive group and number theory, we call the elements of $\mathcal{F}(G_0)$ *sequences* over G_0. Thus a sequence $S \in \mathcal{F}(G_0)$ will be written in the form

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G_0} g^{\mathsf{v}_g(S)}.$$

We will use all notions (such as the length) as in general free abelian monoids. We set $-S = (-g_1) \cdot \ldots \cdot (-g_l)$, and call $\sigma(S) = g_1 + \cdots + g_l \in G$ the *sum* of S. The monoid

$$\mathcal{B}(G_0) = \{S \in \mathcal{F}(G_0) : \sigma(S) = 0\}$$

is called the *monoid of zero-sum sequences* over G_0, and its elements are called *zero-sum sequences* over G_0. Obviously, the inclusion $\mathcal{B}(G_0) \hookrightarrow \mathcal{F}(G_0)$ is a divisor homomorphism, and hence $\mathcal{B}(G_0)$ is a reduced Krull monoid by Property (c) in the definition of Krull monoids. By definition, the inclusion $\mathcal{B}(G_0) \hookrightarrow \mathcal{F}(G_0)$ is cofinal if and only if for every $g \in G_0$ there is an $S \in \mathcal{B}(G_0)$ with $g \mid S$; equivalently, there is no proper subset $G_0' \subsetneq G_0$ such that $\mathcal{B}(G_0') = \mathcal{B}(G_0)$. If $|G| \neq 2$, then $\mathcal{C}(\mathcal{B}(G)) \cong G$, and every class contains precisely one prime divisor.

For every arithmetical invariant $*(H)$, as defined for a monoid H in Section 2, it is usual to write $*(G_0)$ instead of $*(\mathcal{B}(G_0))$ (whenever the meaning is clear from the context). In particular, we set $\mathcal{A}(G_0) = \mathcal{A}(\mathcal{B}(G_0))$, $\mathcal{L}(G_0) = \mathcal{L}(\mathcal{B}(G_0))$, $\mathsf{c}_{\text{mon}}(G_0) = \mathsf{c}_{\text{mon}}(\mathcal{B}(G_0))$, etc.

The study of sequences, subsequence sums, and zero-sums is a flourishing subfield of additive group and number theory (see, for example, [Gao and Geroldinger 2006; Geroldinger and Ruzsa 2009; Grynkiewicz 2013]). The *Davenport constant* $\mathsf{D}(G_0)$, defined as

$$\mathsf{D}(G_0) = \sup\{|U| : U \in \mathcal{A}(G_0)\} \in \mathbb{N}_0 \cup \{\infty\},$$

is among the most studied invariants in additive theory and will play a crucial role in the computations of arithmetical invariants (see the discussion after Lemma 3.4). We will need the following two simple lemmas which we present here so as to not clutter the exposition of Section 6.

Lemma 3.2. *Suppose that the inclusion $\mathcal{B}(G_0) \hookrightarrow \mathcal{F}(G_0)$ is cofinal. The following are equivalent.*

(a) *There exist nontrivial submonoids $H_1, H_2 \subset \mathcal{B}(G_0)$ such that $\mathcal{B}(G_0) = H_1 \times H_2$.*

(b) *There exist nonempty subsets $G_1, G_2 \subset G_0$ such that $G_0 = G_1 \uplus G_2$ and $\mathcal{B}(G_0) = \mathcal{B}(G_1) \times \mathcal{B}(G_2)$.*

(c) *There exist nonempty subsets $G_1, G_2 \subset G_0$ such that $G_0 = G_1 \uplus G_2$ and $\mathcal{A}(G_0) = \mathcal{A}(G_1) \uplus \mathcal{A}(G_2)$.*

Proof. Clearly (b) implies (a). The converse follows from [Geroldinger and Halter-Koch 2006, Proposition 2.5.6]. The implication (b) implies (c) is obvious. We now show that (c) implies (b). Let $B \in \mathcal{B}(G_0)$. Since $\mathcal{B}(G_0)$ is a Krull monoid, it is atomic and hence $B = U_1 \cdot \ldots \cdot U_l$ with $U_1, \ldots, U_l \in \mathcal{A}(G_0)$. After renumbering (if necessary), we can find $k \in [0, l]$ such that $U_1, \ldots, U_k \in \mathcal{A}(G_1)$ and $U_{k+1}, \ldots, U_l \in \mathcal{A}(G_2)$. Thus $\mathcal{B}(G_0) = \mathcal{B}(G_1)\mathcal{B}(G_2)$. If $B \in \mathcal{B}(G_1) \cap \mathcal{B}(G_2)$, then B is a product of atoms from $\mathcal{A}(G_1)$ and a product of atoms from $\mathcal{A}(G_2)$. Since their intersection is empty, both products are empty. Therefore $B = 1$ and hence $\mathcal{B}(G_0) = \mathcal{B}(G_1) \times \mathcal{B}(G_2)$. □

Lemma 3.2.(c) shows that $\mathcal{B}(G^\bullet)$ is not a direct product of submonoids. Suppose that $0 \in G_0$. Then $0 \in \mathcal{B}(G_0)$ is a prime element and $\mathcal{B}(G_0) = \mathcal{B}(\{0\}) \times \mathcal{B}(G_0^\bullet)$. But $\mathcal{B}(\{0\}) = \mathcal{F}(\{0\}) \cong (\mathbb{N}_0, +)$, and thus all the arithmetical invariants measuring the nonuniqueness of factorizations of $\mathcal{B}(G_0)$ and of $\mathcal{B}(G_0^\bullet)$ coincide. Therefore we can assume that $0 \notin G_0$ whenever it is convenient.

Lemma 3.3. *Let G be an abelian group and let $G_0 \subset G$ be a subset such that $1 < \mathsf{D}(G_0) < \infty$.*

(1) *For all $k \in \mathbb{N}$,*

$$\rho(G_0) \le \mathsf{D}(G_0)/2, \quad k \le \rho_k(G_0) \le k\rho(G_0), \quad \rho(G_0)^{-1}k \le \lambda_k(G_0) \le k.$$

(2) *Suppose that $\rho_2(G_0) = \mathsf{D}(G_0)$. Then $\rho(G_0) = \mathsf{D}(G_0)/2$, and for all $k \in \mathbb{N}$,*

$$\rho_{2k}(G_0) = k\mathsf{D}(G_0) \quad \text{and} \quad k\mathsf{D}(G_0) + 1 \le \rho_{2k+1}(G_0) \le k\mathsf{D}(G_0) + \frac{\mathsf{D}(G_0)}{2}.$$

Moreover, if $j, l \in \mathbb{N}_0$ are such that $l\mathsf{D}(G_0) + j \ge 1$, then

$$2l + \frac{2j}{\mathsf{D}(G_0)} \le \lambda_{l\mathsf{D}(G_0)+j}(G_0) \le 2l + j.$$

Proof. By definition, $\lambda_k(G_0) \le k \le \rho_k(G_0)$. Since $\rho(G_0) = \sup\{\rho_k(G_0)/k : k \in \mathbb{N}\}$, it follows that $\rho_k(G_0) \le k\rho(G_0)$ and $k \le \rho(G_0)\lambda_k(G_0)$. Furthermore, $2\rho_k(G_0) \le k\mathsf{D}(G_0)$ for all $k \in \mathbb{N}$ implies that $\rho(G_0) \le \mathsf{D}(G_0)/2$. This gives (1).

We now prove (2). Since $\rho_k(G_0) + \rho_l(G_0) \le \rho_{k+l}(G_0)$ for every $k, l \in \mathbb{N}$, (1) implies that

$$k\mathsf{D}(G_0) = k\rho_2(G_0) \le \rho_{2k}(G_0) \le (2k)\frac{\mathsf{D}(G_0)}{2} = k\mathsf{D}(G_0),$$

and hence

$$\begin{aligned} k\mathsf{D}(G_0) + 1 = \rho_{2k}(G_0) + \rho_1(G_0) \le \rho_{2k+1}(G_0) &\le (2k+1)\rho(G_0) \\ &\le k\mathsf{D}(G_0) + \frac{\mathsf{D}(G_0)}{2}. \end{aligned}$$

Let $j, l \in \mathbb{N}_0$ be such that $l\mathsf{D}(G_0)+j \ge 1$. For convenience, set $\rho_0(G_0) = \lambda_0(G_0) = 0$. Since

$$2l = \frac{2}{\mathsf{D}(G_0)} l\mathsf{D}(G_0) \le \lambda_{l\mathsf{D}(G_0)}(G_0) \quad \text{and} \quad \rho_{2l}(G_0) = l\mathsf{D}(G_0),$$

it follows that $\lambda_{l\mathsf{D}(G_0)}(G_0) = 2l$, and hence

$$\begin{aligned} 2l + \frac{2j}{\mathsf{D}(G_0)} &= \frac{2}{\mathsf{D}(G_0)}\big(l\mathsf{D}(G_0) + j\big) = \rho(G_0)^{-1}\big(l\mathsf{D}(G_0) + j\big) \\ &\le \lambda_{l\mathsf{D}(G_0)+j}(G_0) \le \lambda_{l\mathsf{D}(G_0)}(G_0) + \lambda_j(G_0) \le 2l + j. \qquad \square \end{aligned}$$

Transfer homomorphisms. Transfer homomorphisms are a central tool in factorization theory. In order to study a given monoid H, one constructs a transfer homomorphism $\theta : H \to B$ to a simpler monoid B, studies factorizations in B, and then lifts arithmetical results from B to H. In the case of Krull monoids, transfer homomorphisms allow one to study nearly all of the arithmetical invariants introduced in Section 2 in an associated monoid of zero-sum sequences. We now gather the basic tools necessary for this approach.

A monoid homomorphism $\theta : H \to B$ is called a *transfer homomorphism* if it has the following properties:

(T1) $B = \theta(H)B^\times$ and $\theta^{-1}(B^\times) = H^\times$.

(T2) If $u \in H$, $b, c \in B$ and $\theta(u) = bc$, then there exist $v, w \in H$ such that $u = vw$, $\theta(v) \simeq b$ and $\theta(w) \simeq c$.

The next result provides the link between the arithmetic of Krull monoids and additive group and number theory. This interplay is highlighted in the survey [Geroldinger 2009].

Lemma 3.4. *Let H be a Krull monoid, $\varphi : H \to D = \mathcal{F}(\mathcal{P})$ a cofinal divisor homomorphism, $G = \mathcal{C}(\varphi)$ its class group, and $G_\mathcal{P} \subset G$ the set of classes containing prime divisors. Let $\widetilde{\boldsymbol{\beta}} : D \to \mathcal{F}(G_\mathcal{P})$ denote the unique homomorphism defined by $\widetilde{\boldsymbol{\beta}}(p) = [p]$ for all $p \in \mathcal{P}$.*

(1) *The inclusion $\mathcal{B}(G_\mathcal{P}) \hookrightarrow \mathcal{F}(G_\mathcal{P})$ is cofinal, and the homomorphism*

$$\boldsymbol{\beta} = \widetilde{\boldsymbol{\beta}} \circ \varphi : H \to \mathcal{B}(G_\mathcal{P})$$

is a transfer homomorphism.

(2) *For all $a \in H$, $\mathsf{L}_H(a) = \mathsf{L}_{\mathcal{B}(G_\mathcal{P})}(\boldsymbol{\beta}(a))$. In particular, $\mathcal{L}(H) = \mathcal{L}(G_\mathcal{P})$, $\Delta(H) = \Delta(G_\mathcal{P})$, $\mathcal{U}_k(H) = \mathcal{U}_k(G_\mathcal{P})$, $\rho_k(H) = \rho_k(G_\mathcal{P})$, and $\lambda_k(H) = \lambda_k(G_\mathcal{P})$ for each $k \in \mathbb{N}$.*

(3) *Suppose that H is not factorial. Then $\mathsf{c}(H) = \mathsf{c}(G_\mathcal{P})$, $\mathsf{c}_{\text{adj}}(H) = \mathsf{c}_{\text{adj}}(G_\mathcal{P})$, $\mathsf{c}_{\text{mon}}(H) = \mathsf{c}_{\text{mon}}(G_\mathcal{P})$, $\Delta^*(H) = \Delta^*(G_\mathcal{P})$, and $\omega(H) \le \mathsf{D}(G_\mathcal{P})$.*

Proof. See [Geroldinger and Halter-Koch 2006, Section 3.4] for details pertaining to most of the invariants. For the statements on the monotone catenary degree, see [Geroldinger et al. 2010]. Roughly speaking, all of the statements in (2) are straightforward, but the statements in (3) are more subtle. Note that a statement corresponding to (3) does not hold true for the tame degree (see [Gao et al. 2015]). □

In summary, if the monoid of modules $\mathcal{V}(R)$ is Krull with class group G and set $G_\mathcal{P}$ of classes containing prime divisors, then the arithmetic of direct-sum decompositions can be studied in the monoid $\mathcal{B}(G_\mathcal{P})$ of zero-sum sequences over $G_\mathcal{P}$. In particular, if $H = \mathcal{M}(R)$ and $D = \mathcal{M}(\widehat{R})$ as in Lemma 3.1 and all notation is as in Lemma 3.4, then

$$\mathsf{D}(G_\mathcal{P}) = \sup\bigl\{l : \widehat{M} \cong N_1 \oplus \cdots \oplus N_l \text{ with } [M] \in \mathcal{A}(H) \text{ and } [N_i] \in \mathcal{A}(D)\ \forall i \in [1, l]\bigr\}.$$

4. Monoids of modules: class groups and distribution of prime divisors, I

Throughout this section we use the following setup:

(S) $(R, \mathfrak{m})$ *denotes a one-dimensional analytically unramified commutative Noetherian local ring with unique maximal ideal* $\mathfrak{m}$, $k = R/\mathfrak{m}$ *its residue field,* $\widehat{R}$ *its* $\mathfrak{m}$*-adic completion, and* $\mathrm{spl}(R) = |\mathrm{spec}(\widehat{R})| - |\mathrm{spec}(R)|$ *the splitting number of R.*

In this section we investigate the characteristic of the Krull monoids $\mathcal{M}(R)$ and $\mathcal{T}(R)$ for certain one-dimensional local rings. This study is based on deep module-theoretic work achieved over the past several decades. We gather together module-theoretic information and proceed using a recent construction (see 4.4) to obtain results on the class group and on the set $G_{\mathcal{P}}$ of classes containing prime divisors. The literature does not yet contain a systematic treatment along these lines. Indeed, early results (see Theorem 4.2 below) indicated only the existence of extremal sets $G_{\mathcal{P}}$ which imply either trivial direct-sum decompositions or that all arithmetical invariants describing the direct-sum decompositions are infinite. In either case there was no need for further arithmetical study. Here we reveal that finite and well-structured sets $G_{\mathcal{P}}$ occur in abundance. Thus, as we will see in Section 6, the arithmetical behavior of direct-sum decompositions is well-structured.

We first gather basic ring and module-theoretic properties. By definition, $\widehat{R}$ and R are both reduced and the integral closure of R is a finitely generated R-module. Moreover, we have $\mathfrak{C}(R) = \mathcal{T}(R)$. Let M be a finitely generated R-module. If $\mathfrak{p}$ is a minimal prime ideal of R, then $R_{\mathfrak{p}}$ is a field, $M_{\mathfrak{p}}$ is a finite-dimensional $R_{\mathfrak{p}}$-vector space, and we set $\mathrm{rank}_{\mathfrak{p}}(M) = \dim_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$. If $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ are the minimal prime ideals of R, then $\mathrm{rank}(M) = (r_1, \ldots, r_s)$ where $r_i = \mathrm{rank}_{\mathfrak{p}_i}(M)$ for all $i \in [1, s]$. The module M is said to have constant rank if $r_1 = \cdots = r_s$.

We start with a beautiful result of Levy and Odenthall, which gives us a tool to determine which finitely generated $\widehat{R}$-modules are extended from R-modules.

Proposition 4.1 [Levy and Odenthal 1996, Theorem 6.2]. *Let M be a finitely generated torsion-free $\widehat{R}$-module. Then M is extended if and only if* $\mathrm{rank}_{\mathfrak{p}}(M) = \mathrm{rank}_{\mathfrak{q}}(M)$ *whenever* $\mathfrak{p}$ *and* $\mathfrak{q}$ *are minimal prime ideals of* $\widehat{R}$ *with* $\mathfrak{p} \cap R = \mathfrak{q} \cap R$. *In particular, if R is a domain, then M is extended if and only if its rank is constant.*

We start our discussion with a result which completely determines the characteristic of the Krull monoid $\mathcal{M}(R)$. The arithmetic of this monoid is studied in Proposition 6.2.2.

Theorem 4.2 [Hassler et al. 2007, Theorem 6.3]. *Let G denote the class group of $\mathcal{M}(R)$ and let $G_{\mathcal{P}} \subset G$ denote the set of classes containing prime divisors.*

(1) *If R is not Dedekind-like, then G is free abelian of rank* $\mathrm{spl}(R)$ *and each class contains* $|k|\aleph_0$ *prime divisors.*

(2) *If R is a DVR, then $G = 0$.*
(3) *If R is Dedekind-like but not a DVR, then either*
   (a) $\mathrm{spl}(R) = 0$ *and* $G = 0$, *or*
   (b) $\mathrm{spl}(R) = 1$, *G is infinite cyclic with $G = \langle e \rangle$ and $G_{\mathcal{P}} = \{-e, 0, e\}$. Each of the classes e and $-e$ contain $\aleph_0$ prime divisors and the class 0 contains $|k|\aleph_0$ prime divisors.*

Thus, for the rest of this section, we focus our attention on $\mathcal{T}(R)$. To determine if the divisor homomorphism $\mathcal{T}(R) \hookrightarrow \mathcal{T}(\widehat{R})$ is a divisor theory, we will require additional information. For now, we easily show that it is always cofinal.

Proposition 4.3. *The embedding $\mathcal{T}(R) \hookrightarrow \mathcal{T}(\widehat{R})$ is a cofinal divisor homomorphism.*

Proof. By Lemma 3.1 the embedding is a divisor homomorphism. If M is a finitely generated torsion-free $\widehat{R}$-module, we can consider its rank, $\mathrm{rank}(M) = (r_1, \dots, r_t)$, where t is the number of minimal primes of $\widehat{R}$. If $r_1 = \cdots = r_t$, then by Proposition 4.1 M is extended, say $M = \widehat{N}$ for some finitely generated torsion-free R-module N and the result is trivial. If the rank of M is not constant, set $r = \max\{r_1, \dots, r_t\}$ and consider the $\widehat{R}$-module

$$L = (\widehat{R}/\mathfrak{q}_1)^{r-r_1} \oplus \cdots \oplus (\widehat{R}/\mathfrak{q}_t)^{r-r_t},$$

where $\mathfrak{q}_1, \dots, \mathfrak{q}_t$ denote the minimal primes of $\widehat{R}$. Then $\mathrm{rank}(N \oplus L) = (r, \dots, r)$ is constant and hence $N \oplus L$ is extended, say $N \oplus L \cong \widehat{P}$ for some finitely generated torsion-free R-module P. Clearly M is isomorphic to a direct summand of $\widehat{P}$ and the result follows. □

Since $\mathcal{T}(\widehat{R})$ is free abelian, we can identify it with the free abelian monoid $\mathbb{N}_0^{(\mathcal{P})}$, where $\mathcal{P}$ is an index set for the isomorphism classes of indecomposable finitely generated torsion-free $\widehat{R}$-modules. We then use Proposition 4.1 to describe $\mathcal{T}(R)$ in detail. The following construction has been used numerous times (see, for example, [Baeth and Luckas 2011; Baeth and Saccon 2012; Facchini et al. 2006]).

Construction 4.4. • Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the distinct minimal prime ideals of R. For each $i \in [1, s]$, let $\mathfrak{q}_{i,1}, \dots, \mathfrak{q}_{i,t_i}$ be the minimal primes of $\widehat{R}$ lying over $\mathfrak{p}_i$. Note that $\mathrm{spl}(R) = \sum_{i=1}^{s}(t_i - 1)$.

• Let $\mathcal{P}$ be the set of isomorphism classes of indecomposable finitely generated torsion-free $\widehat{R}$-modules.

• Let $\mathsf{A}(R)$ be the $\mathrm{spl}(R) \times |\mathcal{P}|$ matrix whose column indexed by the isomorphism class $[M] \in \mathcal{P}$ is

$$\begin{bmatrix} r_{1,1} - r_{1,2} & \cdots & r_{1,1} - r_{1,t_1} & \cdots & r_{s,1} - r_{s,2} & \cdots & r_{s,1} - r_{s,t_s} \end{bmatrix}^T,$$

where $r_{i,j} = \mathrm{rank}_{\mathfrak{q}_{i,j}}(M)$.

Then $\mathcal{T}(R) \cong \ker(\mathsf{A}(R)) \cap \mathbb{N}^{(\mathcal{P})} \subset \mathbb{N}_0^{(\mathcal{P})}$ is a Diophantine monoid.

If one has a complete description of how the minimal prime ideals of $\widehat{R}$ lie over the minimal prime ideals of R together with the ranks of all indecomposable finitely generated torsion-free $\widehat{R}$-modules, then Construction 4.4 completely describes the monoid $\mathcal{T}(R)$. In certain cases (e.g., Section 4A) we are able to obtain all of this information. Other times we know only some of the ranks that occur for indecomposable $\widehat{R}$-modules and thus have only a partial description for $\mathcal{T}(R)$. However, as was shown in [Baeth and Saccon 2012], the ranks of indecomposable cyclic $\widehat{R}$-modules gives enough information about the columns of $\mathsf{A}(R)$ to prove that $\mathcal{T}(R) \hookrightarrow \mathcal{T}(\widehat{R})$ is nearly always a divisor theory. First we recall that if $\mathfrak{q}_1, \dots, \mathfrak{q}_t$ are the minimal primes of $\widehat{R}$, and $E \subset [1, t]$. Then

$$\operatorname{rank}\left(\frac{\widehat{R}}{\bigcap_{i \in E} \mathfrak{q}_i}\right) = (r_1, \dots, r_t), \qquad \text{where } r_i = \begin{cases} 1 & \text{if } i \in E, \\ 0 & \text{if } i \notin E. \end{cases}$$

Thus *every* nontrivial t-tuple of zeros and ones can be realized as the rank of a nonzero (necessarily indecomposable) cyclic $\widehat{R}$-module. Thus we have the following:

Construction 4.5. Let all notation be as in Construction 4.4. After renumbering if necessary, there is $p \in [0, s]$ such that $t_1, \dots, t_p \geq 2$ and such that $t_i = 1$ for each $i \in [p+1, s]$. Then $\operatorname{spl}(R) = \sum_{j=1}^{p} t_j - p$. For each $i \in [1, p]$, let A_i be the set of $(t_i - 1) \times 1$ column vectors all of whose entries are either 0 or 1, and let B_i be the set of $(t_i - 1) \times 1$ column vectors all of whose entries are either 0 or -1.

We now define $\mathcal{T}$ to be the $\operatorname{spl}(R) \times \prod_{i=1}^{p}(2^{t_i} - 1)$ matrix, each of whose columns has the form

$$\begin{bmatrix} T_1 \\ \hline \vdots \\ \hline T_p \end{bmatrix}, \qquad \text{where } T_i \in A_i \cup B_i \text{ for each } i \in [1, p].$$

With the notation as in Constructions 4.4 and 4.5, we give a realization result which shows that the matrix $\mathcal{T}$ occurs as a submatrix of $\mathsf{A}(R)$.

Proposition 4.6 [Baeth and Saccon 2012, Proposition 3.7]. *For each column $\boldsymbol{\alpha}$ of $\mathcal{T}$, there exist nonnegative integers $r_{i,j}$ and an indecomposable torsion-free $\widehat{R}$-module $M_{\boldsymbol{\alpha}}$ of rank*

$$(r_{1,1}, \dots, r_{1,t_1}, \dots, r_{p,1}, \dots, r_{p,t_p}, r_{p+1,1}, \dots, r_{s,1})$$

such that

$$\boldsymbol{\alpha} = \begin{bmatrix} r_{1,1} - r_{1,2} & \cdots & r_{1,1} - r_{1,t_1} & \cdots & r_{p,1} - r_{p,2} & \cdots & r_{p,1} - r_{p,t_p} \end{bmatrix}^T.$$

In particular, the matrix $\mathsf{A}(R)$ nearly always satisfies the hypotheses of the following lemma.

Lemma 4.7 [Baeth and Saccon 2012, Lemma 4.1]. *Fix an integer $q \geq 1$, and let I_q denote the $q \times q$ identity matrix. Let $\mathcal{P}$ be an index set, and let $\mathcal{D}$ be a $q \times |\mathcal{P}|$ integer matrix whose columns are indexed by $\mathcal{P}$. Assume $\mathcal{D} = \big[D_1 \mid D_2\big]$, where D_1 is the $q \times (2q+2)$ integer matrix*

$$\left[\begin{array}{c|c|cc} & & 1 & -1 \\ I_q & -I_q & \vdots & \vdots \\ & & 1 & -1 \end{array}\right],$$

and D_2 is an arbitrary integer matrix with q rows (and possibly infinitely many columns). Let $H = \ker(\mathcal{D}) \cap \mathbb{N}_0^{(\mathcal{P})}$.

(1) *The map $\mathcal{D} : \mathbb{Z}^{(\mathcal{P})} \to \mathbb{Z}^{(q)}$ is surjective.*

(2) *The natural inclusion $H \hookrightarrow \mathbb{N}_0^{(\mathcal{P})}$ is a divisor theory.*

(3) $\ker(\mathcal{D}) = \mathsf{q}(H)$.

(4) *$\mathcal{C}(H) \cong \mathbb{Z}^{(q)}$, and this isomorphism maps the set of classes containing prime divisors onto the set of distinct columns of $\mathcal{D}$.*

In particular, we observe the following: Given a fixed column α of $\mathcal{D}$, the cardinality of $\{\beta : \beta$ is a column of $\mathcal{D}$ and $\beta = \alpha\}$ is equal to the cardinality of prime divisors in the class corresponding to α. Therefore, the characteristic of the Krull monoid H is completely given by the matrix $\mathcal{D}$.

Based on the previous results, one easily obtains the following theorem which provides the framework for our study of the characteristic of $\mathcal{T}(R)$.

Theorem 4.8. (1) *If $\operatorname{spl}(R) = 0$, then $\mathcal{T}(R) \cong \mathcal{T}(\widehat{R})$ is free abelian.*

(2) *If $\operatorname{spl}(R) \geq 2$ then the embedding $\mathcal{T}(R) \hookrightarrow \mathcal{T}(\widehat{R})$ is a divisor theory. Moreover,*

   (a) $\mathcal{T}(R) \cong \ker(\mathsf{A}(R)) \cap \mathbb{N}_0^{(\mathcal{P})}$,

   (b) *$\mathcal{C}(\mathcal{T}(R)) \cong \mathbb{Z}^{(\operatorname{spl}(R))}$, and this isomorphism maps the set of classes containing prime divisors onto the set of distinct columns of $\mathsf{A}(R)$.*

Suppose that $\operatorname{spl}(R) = 1$. The embedding $\mathcal{T}(R) \hookrightarrow \mathcal{T}(\widehat{R})$ is a divisor theory if and only if the defining matrix $\mathsf{A}(R)$ contains at least two positive and at least two negative entries (see Proposition 6.1.2).

In many cases, computing the ranks of indecomposable $\widehat{R}$-modules and hence the columns of the defining matrix $\mathsf{A}(R)$ is difficult. However, an additional hypotheses on R implies that the set of classes containing prime divisors satisfies $G_{\mathcal{P}} = -G_{\mathcal{P}}$, a crucial property for all arithmetical investigations (see Proposition 6.2 and the subsequent remarks).

Corollary 4.9. *Suppose in addition that $\widehat{R} \cong S/(f)$ where $(S, \mathfrak{n})$ is a hypersurface, that is, a regular Noetherian local ring of dimension two and where $0 \neq f \in \mathfrak{n}$. If G is the class group of $\mathcal{T}(R) \hookrightarrow \mathcal{T}(\widehat{R})$ and $G_{\mathcal{P}}$ is the set of classes containing prime divisors, then $G_{\mathcal{P}} = -G_{\mathcal{P}}$.*

Proof. With the hypotheses given, we can apply [Baeth and Saccon 2012, Proposition 6.2] to see that if M is any indecomposable $\widehat{R}$-module with rank $(r_1, \dots, r_t)$, then there is an indecomposable $\widehat{R}$-module N with rank $(m-r_1, m-r_2, \dots, m-r_t)$ for some $m \geq \max\{r_1, \dots, r_t\}$. Using Construction 4.4 we see that if $\alpha = \begin{bmatrix} a_1 \cdots a_q \end{bmatrix}$ is the column of $\mathsf{A}(R)$ indexed by M, then $-\alpha$ is the column indexed by N. Therefore, since $G_{\mathcal{P}}$ corresponds to the distinct columns of $\mathsf{A}(R)$, $G_{\mathcal{P}} = -G_{\mathcal{P}}$. □

Remark 4.10. Although the system of equations developed in Construction 4.4 is somehow natural, it is not the only system of equations which can be used to define $\mathcal{T}(R)$. Indeed, the matrix $\mathsf{A}(R)$ can be adjusted by performing any set of elementary row operations. If J is an elementary matrix corresponding to such a set of row operations, then $\mathcal{T}(R) \cong \ker(\mathsf{A}(R)) \cap \mathbb{N}_0^{(\mathcal{P})} \cong \ker(J\mathsf{A}(R)) \cap \mathbb{N}_0^{(\mathcal{P})}$. Moreover, this isomorphism gives rise to an automorphism of $\mathcal{C}(\mathcal{T}(R))$ mapping the set of classes containing prime divisors to another set of classes containing prime divisors. Example 4.20 illustrates the usefulness of considering an alternate defining matrix for $\mathcal{T}(R)$.

4A. ***Finite representation type.*** *Throughout this subsection, let R be as in Setup* (S)*, and suppose in addition that R has finite representation type.*

Decades of work, going back to [Green and Reiner 1978], and including [Wiegand and Wiegand 1994; Cimen 1998; Arnavut et al. 2007; Baeth 2007], culminated in a precise classification of tuples that can occur as the ranks of indecomposable torsion-free R-modules [Baeth and Luckas 2009]. We note that since R has finite representation type, both R and $\widehat{R}$ have at most three minimal primes (see [Cimen et al. 1995, Theorem 0.5]).

Proposition 4.11 [Baeth and Luckas 2009, Main Theorem 1.2]. (1) *If $\widehat{R}$ is a domain, then every indecomposable finitely generated torsion-free $\widehat{R}$-module has rank* 1, 2 *or* 3.

(2) *If $\widehat{R}$ has exactly two minimal prime ideals, then every indecomposable finitely generated torsion-free $\widehat{R}$-module has rank* $(0, 1)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, $(2, 1)$ *or* $(2, 2)$.

(3) *If $\widehat{R}$ has exactly two minimal prime ideals, then every indecomposable finitely generated torsion-free $\widehat{R}$-module has rank* $(0, 0, 1)$, $(0, 1, 0)$, $(1, 0, 0)$, $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$, $(1, 1, 1)$ *or* $(2, 1, 1)$.

Note the lack of symmetry in case (3): With a predetermined order on the minimal prime ideals of $\widehat{R}$, there is an indecomposable module of rank $(2, 1, 1)$, but not of rank $(1,2,1)$ or $(1,1,2)$. As is stated in [Baeth and Luckas 2009, Remark 5.2], even for a fixed number of minimal primes, not each of these tuples will occur as the rank of an indecomposable module for each ring. However, since when applying Construction 4.4 we cannot distinguish between an indecomposable of rank $(2,1)$ and one of rank $(1,0)$, and since all nontrivial tuples of zeros and ones occur as ranks of indecomposable cyclic modules, we have [Baeth and Luckas 2011, Proposition 3.3]:

(1) If $\operatorname{spl}(R)=1$, then $\mathsf{A}(R)=\begin{bmatrix}1 & \cdots & 1 & -1 & \cdots & -1 & 0 & \cdots & 0\end{bmatrix}$.

(2) If $\operatorname{spl}(R)=2$, then $\mathsf{A}(R)=\begin{bmatrix}0 & -1 & 1 & -1 & 1 & 0 & 0 & 1 & \cdots \\ -1 & 0 & 1 & -1 & 0 & 1 & 0 & 1 & \cdots\end{bmatrix}$.

When $\operatorname{spl}(R)=1$, we are guaranteed at least one entry for each of 1, -1, and 0, coming from the ranks of indecomposable cyclic $\widehat{R}$-modules. If we have at most one 1 or at most one -1 in the defining matrix $\mathsf{A}(R)$, then it must be the case that R is a domain, $\widehat{R}$ has exactly two minimal primes $\mathfrak{p}$ and $\mathfrak{q}$, and up to isomorphism either $\widehat{R}/\mathfrak{p}$ is the only indecomposable torsion-free $\widehat{R}$-module of rank (r,s) with $r-s=1$ or the $\widehat{R}/\mathfrak{q}$ is the only indecomposable torsion-free $\widehat{R}$-module of rank (r,s) with $r-s=-1$. If this is the situation, we say that R satisfies condition (†). In case $\operatorname{spl}(R)=2$, we are guaranteed that each column listed appears at least once as a column of $\mathsf{A}(R)$.

We then have the following refinement of Theorem 4.8 when R has finite representation type. The arithmetic of this monoid is studied in Proposition 6.2.2, Theorem 6.4, and Corollary 6.10.

Theorem 4.12 [Baeth and Luckas 2011, Proposition 3.3]. (1) *If* $\operatorname{spl}(R)=1$ *and* R *satisfies condition* (†) *then* $\mathcal{T}(R)\hookrightarrow\mathcal{T}(\widehat{R})$ *is not a divisor theory but* $\mathcal{T}(R)$ *is free abelian.*

(2) *If* $\operatorname{spl}(R)=1$ *and* R *does not satisfy condition* (†), *then* $\mathcal{T}(R)\hookrightarrow\mathcal{T}(\widehat{R})$ *is a divisor theory with infinite cyclic class group* $G=\langle e\rangle$, *and* $G_{\mathcal{P}}=\{-e,0,e\}$.

(3) *If* $\operatorname{spl}(R)=2$, *then* $\mathcal{T}(R)\hookrightarrow\mathcal{T}(\widehat{R})$ *is a divisor theory and* $\mathcal{C}(\mathcal{T}(R))\cong\mathbb{Z}^{(2)}$. *Moreover, this isomorphism maps the set of classes containing prime divisors onto*

$$\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}-1\\-1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}-1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}0\\-1\end{bmatrix}\right\}.$$

4B. ***Infinite representation type.*** *Throughout this subsection, let* R *be as in Setup* (S), *and suppose in addition that* R *has infinite representation type.*

Unfortunately, in this case, there is no known complete list of the tuples that can occur as ranks of indecomposable finitely generated torsion-free R-modules. Thus we cannot give a full description of $\mathcal{T}(R)$ using Construction 4.4. However, with the additional assumption that $\widehat{R}/\mathfrak{q}$ has infinite representation type for some minimal prime ideal $\mathfrak{q}$ of $\widehat{R}$, we can produce a wide variety of interesting ranks and can provide a partial description of $\mathcal{T}(R)$. This information is enough to show that, very much unlike the finite representation type case of Section 4A, all of the arithmetical invariants we study are infinite.

Proposition 4.13 [Saccon 2010, Theorem 3.4.1]. *Let S be a one-dimensional analytically unramified commutative Noetherian local ring with residue field K, and with t minimal prime ideals $\mathfrak{q}_1, \dots, \mathfrak{q}_t$ such that $S/\mathfrak{q}_1$ has infinite representation type. Let $(r_1, \dots, r_t)$ be a nonzero t-tuple of nonnegative integers with $r_i \le 2r_1$ for all $i \in [2, t]$.*

(1) *There exists an indecomposable torsion-free S-module of rank $(r_1, \dots, r_t)$.*

(2) *If the residue field K is infinite, then the set of isomorphism classes of indecomposable torsion-free S-modules of rank $(r_1, \dots, r_t)$ has cardinality $|K|$.*

By Proposition 4.13 the conditions of Lemma 4.7 are satisfied. Therefore the map $\mathcal{T}(R) \hookrightarrow \mathcal{T}(\widehat{R})$ is a divisor theory and the class group $\mathcal{C}(\mathcal{T}(R))$ is free abelian of rank $\operatorname{spl}(R)$. Our main result of this subsection is a refinement of Theorem 4.8. Its arithmetical consequences are given in Proposition 6.2.1, strongly improving the arithmetical characterizations given in [Baeth and Saccon 2012].

Theorem 4.14. *Suppose that* $\operatorname{spl}(R) \ge 1$ *and that there is at least one minimal prime ideal $\mathfrak{q}$ of $\widehat{R}$ such that $\widehat{R}/\mathfrak{q}$ has infinite representation type. Then $\mathcal{C}(\mathcal{T}(R))$ is free abelian of rank* $\operatorname{spl}(R)$ *and the set of classes containing prime divisors contains an infinite cyclic subgroup.*

Proof. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ denote the minimal primes of R and, for each $i \in [1, s]$, let $\mathfrak{q}_{i,1}, \dots, \mathfrak{q}_{i,t_i}$ denote the set of minimal primes of $\widehat{R}$ lying over $\mathfrak{p}_i$. Without loss of generality, assume that $\widehat{R}/\mathfrak{q}_{1,1}$ has infinite representation type. If $t_1 = 1$, then without loss of generality, $t_2 > 1$. From Proposition 4.13 there is, for each pair (r, s) of nonnegative integers (not both zero), an indecomposable $\widehat{R}$-module M with $\operatorname{rank}_{\mathfrak{q}_{2,1}}(M) = r$, $\operatorname{rank}_{\mathfrak{q}_{2,2}}(M) = s$, and $\operatorname{rank}_{\mathfrak{q}_{i,j}}(M) = 0$ for all $(i, j) \notin \{(1, 1), (2, 1), (2, 2)\}$. Now suppose that $t_1 > 1$. Then we have, from Proposition 4.13, for each pair (r, s) of nonnegative integers (not both zero) satisfying $r - s \ge -s$, an indecomposable $\widehat{R}$-module M with $\operatorname{rank}_{\mathfrak{q}_{1,1}}(M) = r$, $\operatorname{rank}_{\mathfrak{q}_{1,2}}(M) = s$, and $\operatorname{rank}_{\mathfrak{q}_{i,j}}(M) = 0$ for all $(i, j) \notin \{(1, 1), (1, 2)\}$. In either case, using Construction 4.4 we see that the set

$$\left\{\begin{bmatrix} x & 0 & \cdots & 0 \end{bmatrix}^T : x \in \mathbb{Z}\right\}$$

occurs as a set of columns for $\mathsf{A}(R)$ and hence occurs as a subset of the set of classes containing prime divisors. □

4C. ***Divisor-closed submonoids of $\mathcal{T}(R)$.*** Suppose that R has infinite representation type but, in contrast to Theorem 4.14, suppose that $\widehat{R}/\mathfrak{q}$ has finite representation type for each minimal prime $\mathfrak{q}$ of $\widehat{R}$. Then there is no known classification of all ranks of indecomposable finitely generated torsion-free R-modules. Specific rings have been studied in the literature, but even in these settings, a complete solution has been unattainable. We now give such an example which we will return to in Section 4D.

Example 4.15. Let K be an algebraically closed field of characteristic zero. Consider the ring $S = K[\![x, y]\!]/(x^4 - xy^7)$ which has exactly two minimal primes xS and $(x^3 - y^7)S$. Detailed constructions in [Karr and Wiegand 2011; Saccon 2010] show that S has indecomposable modules of ranks (m, m), $(m + 1, m)$, and $(m + 2, m)$ for each positive integer m. Moreover, [Baeth and Saccon 2012, Proposition 6.2] guarantees indecomposable modules of ranks $(s - (m + 1), s - m)$ and $(t - (m + 2), t - m)$, where $s \geq m + 1$ and $t \geq m + 2$ are positive integers. Determining what other tuples occur as ranks of indecomposable torsion-free S-modules appears to be quite difficult.

Thus, since studying $\mathcal{T}(R)$ as a whole is out of reach at the present state of knowledge, we pick finitely many R-modules $M_1, \ldots, M_n$, and study the direct-sum relations among them. In more technical terms, instead of studying the full Krull monoid $\mathcal{T}(R)$, we focus on divisor-closed submonoids. Suppose that H is a Krull monoid and $H \hookrightarrow \mathcal{F}(\mathcal{P})$ a cofinal divisor homomorphism. If $H' \subset H$ is a divisor-closed submonoid, then $H' \hookrightarrow H \hookrightarrow \mathcal{F}(\mathcal{P})$ is a divisor homomorphism. For each of the arithmetical invariants $*(\cdot)$ introduced in Section 2, we have $*(H') \leq *(H)$ or $*(H') \subset *(H)$; for example we have $\mathsf{c}(H') \leq \mathsf{c}(H)$, $\mathcal{L}(H') \subset \mathcal{L}(H)$, and so on. Moreover, if H' is the smallest divisor-closed submonoid containing finitely many elements $a_1, \ldots, a_k \in H$, it is also the smallest divisor-closed submonoid containing $a_1 \cdot \ldots \cdot a_k$.

For the rest of Section 4, we study divisor-closed submonoids of $\mathcal{T}(R)$ generated by a single R-module M, regardless of whether R has finite or infinite representation type. We denote this monoid by $\operatorname{add}(M)$. Before discussing specific examples in Section 4D, we carefully recall the consequences of our main Construction 4.4 for such submonoids.

Construction 4.16. Let R and $\widehat{R}$ be as in Construction 4.4. Let M be a finitely generated torsion-free R-module. Then $\operatorname{add}(M)$ consists of all isomorphism classes $[N] \in \mathcal{T}(R)$ such that N is isomorphic to a direct summand of $M^{(n)}$ for some finite positive integer n.

Write $\widehat{M} = L_1^{(n_1)} \oplus \cdots \oplus L_k^{(n_k)}$, where the L_i are pairwise nonisomorphic indecomposable finitely generated torsion-free $\widehat{R}$-modules and the n_i are positive integers. If $[N] \in \operatorname{add}(M)$, then $[\widehat{N}] \in \operatorname{add}(\widehat{M})$ and thus, since direct-sum decomposition is essentially unique over $\widehat{R}$,

$$\widehat{N} \cong L_1^{(a_1)} \oplus \cdots \oplus L_k^{(a_k)},$$

with each a_i a nonnegative integer at most n_i. Thus there is a divisor homomorphism $\Psi : \mathrm{add}(M) \to \mathbb{N}_0^{(k)}$ given by $[N] \mapsto (a_1, \dots, a_k)$. We identify $\mathrm{add}(M)$ with the saturated submonoid $\Gamma(M) = \Psi(\mathrm{add}(M))$ of $\mathbb{N}_0^{(k)}$.

Moreover, if $\mathsf{A}(M)$ is the $\mathrm{spl}(R) \times k$ integer-valued matrix for which the l-th column is the transpose of the row vector

$$\begin{bmatrix} r_{1,1} - r_{1,2} & \cdots & r_{1,1} - r_{1,t_1} & \cdots & r_{s,1} - r_{s,2} & \cdots & r_{s,1} - r_{s,t_s} \end{bmatrix},$$

where $r_{i,j} = \mathrm{rank}_{\mathfrak{q}_{i,j}}(V_l)$, then $\mathrm{add}(M) \cong \Gamma(M) = \ker(\mathsf{A}(M)) \cap \mathbb{N}_0^{(k)}$.

We now state a corollary of Theorem 4.8 for $\mathrm{add}(M)$.

Corollary 4.17. *Let M be a finitely generated torsion-free R-module as in Construction 4.16.*

(1) *If* $\mathrm{spl}(R) = 0$, *then* $\mathrm{add}(M) \cong \mathrm{add}(\widehat{M})$ *is free abelian.*

(2) *If* $\mathrm{spl}(R) \geq 1$ *and* $\mathsf{A}(M)$ *satisfies the conditions of Lemma 4.7, then the inclusion* $\Gamma(M) \subset \mathbb{N}_0^{(k)}$ *is a divisor theory. Moreover*:

   (a) $\mathrm{add}(M) \cong \ker(\mathsf{A}(M)) \cap \mathbb{N}_0^{(k)}$.

   (b) $\mathcal{C}(\mathrm{add}(M)) \cong \mathbb{Z}^{(\mathrm{spl}(R))}$, *and this isomorphism maps the set of classes containing prime divisors onto the set of distinct columns of* $\mathsf{A}(M)$.

Before considering explicit examples, we give a realization result (see also [Leuschke and Wiegand 2012, Chapter 1]).

Proposition 4.18. *Let H be a reduced Krull monoid with free abelian class group G of rank q and let $G_{\mathcal{P}} \subset G$ denote the set of classes containing prime divisors. Suppose that $G_{\mathcal{P}}$ is finite and that G has a basis $(e_1, \dots, e_q)$ such that*

$$G_0 = \{e_0 = e_1 + \cdots + e_q, e_1, \dots, e_q, -e_0, \dots, -e_q\} \subset G_{\mathcal{P}}.$$

Then there exists an analytically unramified commutative Noetherian local domain S and a finitely generated torsion-free S-module M such that $\mathrm{add}(M) \cong H$.

Proof. Let $\Phi : G \to \mathbb{Z}^{(q)}$ denote the isomorphism which maps $(e_1, \dots, e_q)$ onto the standard basis of $\mathbb{Z}^{(q)}$. Let S be an analytically unramified Noetherian local domain with completion $\widehat{S}$ having $q+1$ minimal primes $Q_0, \dots, Q_q$ such that $\widehat{S}/Q_0$ has infinite representation type. For $\boldsymbol{s} = [s_1 \ \cdots \ s_q] \in \Phi(G_{\mathcal{P}})$, set $r_0 = s_1 + \cdots + s_q$ and $r_i = \sum_{j \neq i} s_j$ for each $i \in [1, q]$. By Proposition 4.13 there exists an indecomposable finitely generated torsion-free $\widehat{S}$-module $N_{\boldsymbol{s}}$ such that $\mathrm{rank}(N_{\boldsymbol{s}}) = (r_0, \dots, r_q)$. Set

$$N = \bigoplus_{\boldsymbol{s} \in \Phi(G_{\mathcal{P}})} N_{\boldsymbol{s}},$$

and write $\operatorname{rank}(N) = (a_0, \dots, a_q)$. Set $a = \max\{a_0, \dots, a_q\}$ and

$$L = \bigoplus_{i=0}^{q} (\widehat{S}/Q_i)^{(a-a_i)}.$$

Then $N \oplus L$ is a finitely generated torsion-free $\widehat{S}$-module with constant rank and is thus extended from a finitely generated torsion-free S-module M. By Construction 4.16 and Corollary 4.17 we see that $\operatorname{add}(M)$ has class group isomorphic to $\mathbb{Z}^{(q)}$ and this isomorphism maps the set of prime divisors onto the elements of the set $\Phi(G_{\mathcal{P}})$. □

4D. *Examples.* In this section we provide the constructions of naturally occurring monoids $\operatorname{add}(M)$ where M is a finitely generated torsion-free R-module. In particular, we construct specific modules M, whose completion $\widehat{M}$ is often a direct sum of indecomposable cyclic $\widehat{R}$-modules and we determine the class group G of $\operatorname{add}(M)$ and the set of classes $G_{\mathcal{P}} \subset G$ containing prime divisors. Note that the Krull monoids $\mathcal{M}(R)$ of *all* finitely generated R-modules and $\mathcal{T}(R)$ of *all* finitely generated torsion-free R-modules have class groups $G' \supset G$ and a set $G'_{\mathcal{P}}$ of classes containing prime divisors such that $G'_{\mathcal{P}} \supset G_{\mathcal{P}}$. Since $\operatorname{add}(M)$ is a divisor-closed submonoid of both $\mathcal{M}(R)$ and of $\mathcal{T}(R)$, a study of the arithmetic of $\operatorname{add}(M)$ provides a partial description of $\mathcal{M}(R)$ and $\mathcal{T}(R)$. Moreover, the values of arithmetical invariants of $\operatorname{add}(M)$ give lower bounds on the same arithmetical invariants of $\mathcal{M}(R)$ and $\mathcal{T}(R)$.

In each of the following examples we construct an $\widehat{R}$-module $L = L_1^{n_1} \oplus \cdots \oplus L_k^{n_k}$ of constant rank, where $L_1, \dots, L_k$ are pairwise nonisomorphic indecomposable $\widehat{R}$-modules. Then, by Corollary 4.17 with $\widehat{M} \cong L$ for some R-module M,

$$\operatorname{add}(M) \cong \ker\big(\mathsf{A}(M)\big) \cap \mathbb{N}_0^{(k)} \subset \mathbb{N}_0^{(k)} \cong \operatorname{add}(L).$$

In particular, we do so in such a way that the natural map $\operatorname{add}(M) \hookrightarrow \operatorname{add}(L)$ is a divisor theory with class group isomorphic to $\mathbb{Z}^{(\operatorname{spl}(R))}$ and where the set of classes containing prime divisors maps onto the distinct columns of $\mathsf{A}(M)$.

Example 4.19. We now construct a monoid of modules whose arithmetic will be studied in Proposition 6.12. Let S be as in Example 4.15. Then there are indecomposable torsion-free S-modules M_1, M_{-1}, M_2, M_{-2}, N_1, N_{-1}, N_2, and N_{-2} with ranks (respectively) $(2,1)$, $(1,2)$, $(3,1)$, $(1,3)$, $(3,2)$, $(3,2)$, $(2,3)$, $(4,2)$, and $(2,4)$. Set L to be the direct sum of these eight indecomposable S-modules. By Lech's theorem [1986], there exists a Noetherian local domain $(R, \mathfrak{m})$ with $\mathfrak{m}$-adic completion $\widehat{R} \cong S$. Since L has constant rank, L is extended from some R-module M, and $\operatorname{add}(M) \hookrightarrow \operatorname{add}(L) \cong \mathbb{N}_0^{(8)}$ is a divisor theory with infinite cyclic class group G and with $G_{\mathcal{P}} = \{-2e, -e, e, 2e\}$ where $G = \langle e \rangle$.

Example 4.20. We now provide an example that illustrates the convenience of choosing an alternate defining matrix for $\operatorname{add}(M)$, as is described in Remark 4.10. Its arithmetic is given in Theorem 6.4. Suppose that R has two minimal prime ideals $\mathfrak{p}_1, \mathfrak{p}_2$ and that $\widehat{R}$ has five minimal prime ideals $\mathfrak{q}_{(1,1)}, \mathfrak{q}_{(1,2)}, \mathfrak{q}_{(1,3)}, \mathfrak{q}_{(2,1)}$, and $\mathfrak{q}_{(2,2)}$, with $\mathfrak{q}_{(i,j)}$ lying over $\mathfrak{p}_i$ for each $i \in [1,2]$ and for each j. Set

$$L = \frac{\widehat{R}}{\mathfrak{q}_{(1,1)} \cap \mathfrak{q}_{(1,3)} \cap \mathfrak{q}_{(2,2)}} \oplus \frac{\widehat{R}}{\mathfrak{q}_{(1,1)} \cap \mathfrak{q}_{(1,2)} \cap \mathfrak{q}_{(2,1)}} \oplus \frac{\widehat{R}}{\mathfrak{q}_{(1,2)} \cap \mathfrak{q}_{(2,1)}} \oplus \frac{\widehat{R}}{\mathfrak{q}_{(1,1)}} \oplus \frac{\widehat{R}}{\mathfrak{q}_{(1,3)} \cap \mathfrak{q}_{(2,2)}} \oplus \frac{\widehat{R}}{\mathfrak{q}_{(1,2)} \cap \mathfrak{q}_{(1,3)} \cap \mathfrak{q}_{(2,1)} \cap \mathfrak{q}_{(2,2)}}.$$

Since L has constant rank 3, there is an R-module M such that $\widehat{M} \cong L$. Then $\operatorname{add}(M) \cong \ker(\mathsf{A}(M)) \cap \mathbb{N}_0^{(6)}$ where

$$\mathsf{A}(M) = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ -1 & 1 & 1 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & -1 \\ -1 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = J\mathsf{A}(M).$$

Thus

$$\operatorname{add}(M) \cong \ker(J\mathsf{A}(M)) \cap \mathbb{N}_0^{(6)} \cong \ker \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & -1 \end{bmatrix} \cap \mathbb{N}_0^{(6)}.$$

Since the matrix $\mathsf{A}(M)$ has rank two, the representation of $\operatorname{add}(M)$ as a Diophantine matrix defined by two equations more clearly describes this monoid. Moreover, since the map from $\mathbb{Z}^{(6)}$ to $\mathbb{Z}^{(2)}$ is surjective (the map $\mathsf{A}(M) : \mathbb{Z}^{(6)} \to \mathbb{Z}^{(3)}$ is not surjective), we immediately see that $\mathcal{C}(\operatorname{add}(M)) \cong \mathbb{Z}^{(2)}$, and this isomorphism maps the set of classes containing prime divisors onto

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}.$$

Example 4.21. We now consider a monoid $\operatorname{add}(M)$ which generalizes the monoid $\mathcal{T}(R)$ when R has finite representation type, and its arithmetic is studied in Theorem 6.7 and Corollary 6.10. Suppose that $\widehat{R}$ has $q+1$ minimal primes $\mathfrak{q}_1, \dots, \mathfrak{q}_{q+1}$, and set

$$L = \bigoplus_{\substack{I \subset [1,q+1] \\ I \neq \varnothing}} \frac{\widehat{R}}{\bigcap_{i \in I} \mathfrak{q}_i}.$$

From the symmetry of the set of ranks of the indecomposable cyclic $\widehat{R}$-modules

$$\frac{\widehat{R}}{\bigcap_{i\in I} \mathfrak{q}_i}$$

we immediately see that L has constant rank $(2^q, \dots, 2^q)$ and is therefore extended from some R-module M. Then

$$\mathrm{add}(M) \cong \ker(\mathsf{A}(M)) \cap \mathbb{N}_0^{(q)},$$

where $\mathsf{A}(M)$ is an $q \times 2^{q+1} - 1$ integer-valued matrix with columns $[\epsilon_1 \ \cdots \ \epsilon_q]^T$ where either $\epsilon_i \in \{0, 1\}$ for all $i \in [1, q]$ or $\epsilon_i \in \{0, -1\}$ for all $i \in [1, q]$.

Since the columns of $\mathsf{A}(M)$ contain a basis for $\mathbb{Z}^{(q)}$, $\mathrm{add}(M) \hookrightarrow \mathrm{add}(L) \cong \mathbb{N}_0^{(q)}$ is a divisor theory with class group $\mathcal{C}(\mathrm{add}(M)) \cong \mathbb{Z}^{(q)}$, and this isomorphism maps the set of classes containing prime divisors onto

$$\Big(\big\{[\epsilon_1 \ \cdots \ \epsilon_q]^T : \epsilon_i \in \{0,1\}\big\} \cup \big\{[\epsilon_1 \ \cdots \ \epsilon_q]^T : \epsilon_i \in \{0,-1\}\big\}\Big) \setminus \big\{[0 \ \cdots \ 0]\big\}.$$

Example 4.22. In this example we construct a monoid $\mathrm{add}(M)$ which generalizes the monoid of Example 4.21 by including all vectors having entries in $\{-1, 0, 1\}$ in the set $G_{\mathcal{P}}$. This larger set of classes containing prime divisors adds much complexity to the arithmetic. Suppose that R has q minimal primes and that $\widehat{R}$ has $2q$ minimal primes

$$\mathfrak{q}_{(1,1)}, \mathfrak{q}_{(1,2)}, \mathfrak{q}_{(2,1)}, \dots, \mathfrak{q}_{(q,2)},$$

where $\mathfrak{q}_{(i,j)} \cap R = \mathfrak{q}_{(i',j')} \cap R$ if and only if $i = i'$. As in the previous example, let

$$L = \bigoplus_{\substack{I \subset \{(1,1),\dots,(q,2)\} \\ I \neq \varnothing}} \frac{\widehat{R}}{\bigcap_{(i,j)\in I} \mathfrak{q}_{(i,j)}}.$$

From the symmetry of the set of ranks of the indecomposable cyclic $\widehat{R}$-modules $\widehat{R}/\bigcap_{\{i,j\}\in I} \mathfrak{q}_{i,j}$, we immediately see that L has constant rank $(2^{2q-1}, \dots, 2^{2q-1})$ and is therefore extended from some R-module M. Then

$$\mathrm{add}(M) \cong \ker(\mathsf{A}(M)) \cap \mathbb{N}_0^{(q)},$$

where $\mathsf{A}(M)$ is an $q \times 2^{2q} - 1$ integer-valued matrix with columns of the form

$$\big[r_{(1,1)} - r_{(1,2)} \quad r_{(2,1)} - r_{(2,2)} \quad \cdots \quad r_{(q,1)} - r_{(q,2)}\big]^T,$$

where $(r_{(1,1)}, r_{(1,2)}, \dots, r_{(q,2)})$ is the rank of one of the $2^{2q} - 1$ indecomposable cyclic $\widehat{R}$-modules — that is, any one of the q-tuples of 1s and 0s (not all 0). In other words, the columns of $\mathsf{A}(M)$ are exactly the 3^q columns $[\epsilon_1 \ \cdots \ \epsilon_q]^T$, where

$\epsilon_i \in \{-1, 0, 1\}$ for all $i \in [1, q]$, repeated with some multiplicity. For example, the column of all zeros occurs for each of the indecomposable cyclic $\widehat{R}$-modules

$$\frac{\widehat{R}}{\bigcap_{(i,j)\in I} \mathfrak{q}_{(i,j)}},$$

where $(i, 1) \in I$ if and only if $(i, 2) \in I$.

Since the columns of $\mathsf{A}(M)$ contain a basis for $\mathbb{Z}^{(q)}$,

$$\operatorname{add}(M) \hookrightarrow \operatorname{add}(L) \cong \mathbb{N}_0^{(2^{2q}-1)}$$

is a divisor theory whose class group $G \cong \mathbb{Z}^{(q)}$, and this isomorphism maps the set of classes containing prime divisors onto

$$\bigl\{[\epsilon_1 \ \cdots \ \epsilon_q]^T : \epsilon_i \in \{-1, 0, 1\}\bigr\}.$$

Example 4.23. In this example we consider $\operatorname{add}(M)$ when the completion of M is isomorphic to a direct sum of some (but not all) of the indecomposable cyclic $\widehat{R}$-modules. In this case, the example is constructed in such a way that $\mathcal{B}(G_{\mathcal{P}})$ is a direct product of nontrivial submonoids (see Lemma 3.2). Suppose that R has q minimal primes and that $\widehat{R}$ has $3q$ minimal primes

$$\bigl\{\mathfrak{q}_{(i,j)} : i \in [1, q], \ j \in [1, 3]\bigr\},$$

where $\mathfrak{q}_{(i,j)} \cap R = \mathfrak{q}_{(i',j')} \cap R$ if and only if $i = i'$. Let L be the $\widehat{R}$-module

$$\bigoplus_{i=1}^{q} \Bigl(\widehat{R}/\mathfrak{q}_{(i,1)} \oplus \widehat{R}/\mathfrak{q}_{(i,2)} \oplus \widehat{R}/\mathfrak{q}_{(i,3)} \oplus \widehat{R}/(\mathfrak{q}_{(i,1)} \cap \mathfrak{q}_{(i,2)}) \oplus \widehat{R}/(\mathfrak{q}_{(i,1)} \cap \mathfrak{q}_{(i,3)}) \oplus \widehat{R}/(\mathfrak{q}_{(i,2)} \cap \mathfrak{q}_{(i,3)}) \Bigr).$$

We see immediately that L has constant rank $(3, \ldots, 3)$ and thus L is extended from some R-module M. Then $\operatorname{add}(M) \cong \ker(\mathsf{A}(M)) \cap \mathbb{N}_0^{(2q)}$ where $\mathsf{A}(M)$ is an $2q \times 6q$ integer-valued matrix with columns

$$\bigl\{e_{2k-1}, e_{2k}, e_{2k-1} + e_{2k}, -e_{2k-1}, -e_{2k}, -e_{2k-1} - e_{2k} : k \in [1, q]\bigr\},$$

where $(e_1, \ldots, e_{2q})$ denotes the canonical basis of $\mathbb{Z}^{(2q)}$.

For $k \in [1, q]$, we set

$$G_k = \bigl\{e_{2k-1}, e_{2k}, e_{2k-1} + e_{2k}, -e_{2k-1}, -e_{2k}, -e_{2k-1} - e_{2k}\bigr\}.$$

Then $G_{\mathcal{P}} = \biguplus_{k\in[1,q]} G_k$ is the set of classes containing prime divisors and $\mathcal{B}(G_{\mathcal{P}}) = \mathcal{B}(G_1) \times \cdots \times \mathcal{B}(G_q)$. From Proposition 6.1 we will see that $\mathcal{B}(G_k) \hookrightarrow \mathcal{F}(G_k)$

is a divisor theory, whence $\mathcal{B}(G_{\mathcal{P}}) \hookrightarrow \mathcal{F}(G_{\mathcal{P}})$ and $\mathrm{add}(M) \hookrightarrow \mathrm{add}(L)$ are divisor theories. The arithmetic of this monoid is studied in Proposition 6.12 and Corollary 6.15.

Example 4.24. As in Example 4.23, suppose that R has q minimal primes and suppose that the completion $\widehat{R}$ of R has $3q$ minimal primes

$$\{\mathfrak{q}_{(i,j)} : i \in [1,q],\, j \in [1,3]\},$$

where $\mathfrak{q}_{(i,j)} \cap R = \mathfrak{q}_{(i',j')} \cap R$ if and only if $i = i'$. Further suppose that $\widehat{R} = S/(f)$ where $(S, \mathfrak{n})$ is a regular Noetherian local ring of dimension two and where $0 \neq f \in \mathfrak{n}$ and that $\widehat{R}/\mathfrak{q}_{(i,j)}$ has infinite representation type for all pairs (i,j). By Proposition 4.13, for each $k \in [1,q]$ there are indecomposable finitely generated torsion-free $\widehat{R}$-modules M_k and N_k of ranks $(r_{1,1}, \ldots, r_{q,3})$ and $(s_{1,1}, \ldots, s_{q,3})$ where

$$r_{i,j} = \begin{cases} 0 & \text{if } i \neq k, \\ 2 & \text{if } i = k,\, j \in [1,2], \\ 0 & \text{if } i = k,\, j = 3, \end{cases} \quad \text{and} \quad s_{i,j} = \begin{cases} 0 & \text{if } i \neq k, \\ 3 & \text{if } i = k,\, j = 1, \\ 2 & \text{if } i = k,\, j = 2, \\ 0 & \text{if } i = k,\, j = 3. \end{cases}$$

Moreover, by Corollary 4.9, for each $k \in [1,q]$ there are constant $t_k \geq 2$ and $t'_k \geq 3$ and indecomposable finitely generated torsion-free $\widehat{R}$-modules M'_k and N'_k having ranks $(r'_{1,1}, \ldots, r'_{q,3})$ and $(s'_{1,1}, \ldots, s'_{q,3})$ where

$$r'_{i,j} = \begin{cases} t_k & \text{if } i \neq k, \\ t_k - 2 & \text{if } i = k,\, j \in [1,2], \\ t_k & \text{if } i = k,\, j = 3, \end{cases} \quad \text{and} \quad s'_{i,j} = \begin{cases} t'_k & \text{if } i \neq k, \\ t'_k - 3 & \text{if } i = k,\, j = 1, \\ t'_k - 2 & \text{if } i = k,\, j = 2, \\ t'_k & \text{if } i = k,\, j = 3. \end{cases}$$

Let

$$L = \widehat{R} \oplus \Big(\bigoplus_{k=1}^{q} (M_k \oplus N_k \oplus M'_k \oplus N'_k) \oplus \widehat{R}/(\mathfrak{q}_{(i,1)} \cap \mathfrak{q}_{(i,3)}) \oplus \widehat{R}/(\mathfrak{q}_{(i,1)} \cap \mathfrak{q}_{(i,2)}) \oplus \widehat{R}/\mathfrak{q}_{(i,2)} \oplus \widehat{R}/\mathfrak{q}_{(i,3)} \Big).$$

Since L has constant rank, L is extended from an R-module M. Then $\mathrm{add}(M) \cong \ker(\mathsf{A}(M)) \cap \mathbb{N}_0^{(2q)}$ where $\mathsf{A}(M)$ is an $2q \times (8q+1)$ integer-valued matrix with columns 0 and

$$\{e_{2k-1}, e_{2k}, 2e_{2k}, e_{2k-1} + 2e_{2k}, -e_{2k-1}, -e_{2k}, -2e_{2k}, -e_{2k-1} - 2e_{2k} : k \in [1,q]\},$$

where $(e_1, \ldots, e_{2q})$ denotes the canonical basis of $\mathbb{Z}^{(2q)}$.

For $k \in [1, q]$, we set

$$G_k = \{e_{2k-1}, e_{2k}, 2e_{2k}, e_{2k-1} + 2e_{2k}, -e_{2k-1}, -e_{2k}, -2e_{2k}, -e_{2k-1} - 2e_{2k}\}.$$

Then $G_{\mathcal{P}} = \biguplus_{k\in[1,q]} G_k$ is the set of classes containing prime divisors and $\mathcal{B}(G_{\mathcal{P}}) = \mathcal{B}(G_1) \times \cdots \times \mathcal{B}(G_q)$. From Proposition 6.1 we will see that $\mathcal{B}(G_k) \hookrightarrow \mathcal{F}(G_k)$ is a divisor theory, whence $\mathcal{B}(G_{\mathcal{P}}) \hookrightarrow \mathcal{F}(G_{\mathcal{P}})$ and $\text{add}(M) \hookrightarrow \text{add}(N)$ are divisor theories. The arithmetic of this monoid is studied in Proposition 6.13 and Corollary 6.15.

Example 4.25. In our final example we construct a tuple $(G, G_{\mathcal{P}})$ which generalizes the monoid $\mathcal{T}(R)$ when R has finite representation type (see Theorem 4.12). The arithmetic of such Krull monoids is studied in Theorem 6.4 and Corollary 6.10. Suppose that $\widehat{R}$ has $q + 1$ minimal primes $\mathfrak{q}_1, \dots, \mathfrak{q}_{q+1}$, and set

$$L = \bigoplus_{j=1}^{q+1} \Big(\big(\widehat{R} / \bigcap_{i \neq j} \mathfrak{q}_i\big) \oplus (\widehat{R}/\mathfrak{q}_j) \Big).$$

Note that L has constant rank $(q, \dots, q)$ and is hence extended from some R-module M. Then $\text{add}(M) \cong \ker(\mathsf{A}(M)) \cap \mathbb{N}_0^{(q)}$ where $\mathsf{A}(M)$ is an $q \times 2q$ integer-valued matrix with columns

$$e_1, \dots, e_q, e_0 = e_1 + e_2 + \cdots + e_q, -e_1, \dots, -e_q, -e_0.$$

By Proposition 6.1, $\text{add}(M) \hookrightarrow \text{add}(L) \cong \mathbb{N}_0^{(2q)}$ is a divisor theory with class group $\mathcal{C}(\text{add}(M)) \cong \mathbb{Z}^{(q)}$, and this isomorphism maps the set of classes containing prime divisors onto

$$\{e_0 = e_1 + \cdots + e_q, e_1, \dots, e_q, -e_0, \dots, -e_q\}.$$

5. Monoids of modules: class groups and distribution of prime divisors, II

In this section we investigate the characteristic of the Krull monoids $\mathcal{T}(R)$ and $\mathfrak{C}(R)$ for two-dimensional Noetherian local Krull domains (see Theorem 5.4). We will show that, apart from a well-described exceptional case, their class groups are both isomorphic to the factor group $\mathcal{C}(\widehat{R})/\iota(\mathcal{C}(R))$, where $\iota : \mathcal{C}(R) \to \mathcal{C}(\widehat{R})$ is the natural homomorphism between the class groups of R and $\widehat{R}$. In a well-studied special case where R is factorial and $\widehat{R}$ is a hypersurface with finite representation type, this factor group is a finite cyclic group (see Theorem 5.5). This is in strong contrast to the results on one-dimensional rings in the previous section where all class groups are torsion-free.

Let S be a Krull domain and let $\mathcal{I}_v^*(S)$ denote the monoid of nonzero divisorial ideals. Then $\varphi : S \to \mathcal{I}_v^*(S)$, defined by $a \mapsto aS$, is a divisor theory. In this section we view $\mathcal{C}(S)$ as the class group of this specific divisor theory. First we give a classical result (see [Bourbaki 1988, Chapter VII, Section 4.7]).

Lemma 5.1. *Let S be a Noetherian Krull domain. One can associate to each finitely generated S-module M a class $\boldsymbol{c}(M) \in \mathcal{C}(S)$ in such a way that*

(1) *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of finitely generated S-modules, then $\boldsymbol{c}(M) = \boldsymbol{c}(M') + \boldsymbol{c}(M'')$.*

(2) *If I is a fractional ideal of S and I_v the divisorial ideal generated by I, then $\boldsymbol{c}(I) = \boldsymbol{c}(I_v)$.*

Note that if S is any Noetherian domain, every ideal of S is obviously an indecomposable finitely generated torsion-free S-module. If, in addition, the ring has dimension two, then we have the following stronger result.

Lemma 5.2. *Let S be a Noetherian local Krull domain of dimension two.*

(1) *Every divisorial ideal of S is an indecomposable MCM S-module* [Evans and Griffith 1985, Lemma 1.1 and Theorem 3.6].

(2) *In addition, assume that the $\mathfrak{m}$-adic completion $\widehat{S}$ of S is a Krull domain. Then a finitely generated torsion-free $\widehat{S}$-module N is extended from an S-module if and only if $\boldsymbol{c}(N)$ is in the image of the natural homomorphism $\iota : \mathcal{C}(S) \to \mathcal{C}(\widehat{S})$* [Rotthaus et al. 1999, Proposition 3].

We now give a result on abstract Krull monoids which encapsulates the structure of the monoids of modules described in Theorem 5.4.

Lemma 5.3. *Let $D = \mathcal{F}(\mathcal{P})$ be a free abelian monoid, G an additive abelian group, $\psi : D \to G$ a homomorphism, and $H = \psi^{-1}(0) \subset D$.*

(1) *If $H \subset D$ is cofinal, then the inclusion $H \hookrightarrow D$ is a divisor homomorphism and $\overline{\psi} : D/H \to \psi(D) \subset G$ given by $\overline{\psi}([a]) = \psi(a)$ is an isomorphism.*

(2) *The inclusion $H \hookrightarrow D$ is a divisor theory if and only if $\langle \psi(\mathcal{P}) \rangle = [\psi(\mathcal{P} \setminus \{q\})]$ for every $q \in \mathcal{P}$. If this is the case, then $\overline{\psi} : D/H \to \psi(D)$ is an isomorphism and, for every $g \in \psi(D)$, the set $\mathcal{P} \cap \psi^{-1}(g)$ is the set of prime divisors in the class $\overline{\psi}^{-1}(g)$.*

(3) *If the restriction $\psi \mid_{\mathcal{P}} : \mathcal{P} \to G$ of ψ to $\mathcal{P}$ is an epimorphism, then $H \hookrightarrow D$ is cofinal. Moreover, it is a divisor theory apart from the following exception:*

$$G = \{0, g\} \quad \text{and} \quad |\mathcal{P} \cap \psi^{-1}(g)| = 1.$$

If $H \hookrightarrow D$ is not a divisor theory, then H is factorial.

Proof. For the proofs of (1) and (2), see [Geroldinger and Halter-Koch 2006, Proposition 2.5.1]. We now consider the proof of (3).

Let $a \in D$. Since $\psi \mid_{\mathcal{P}} : \mathcal{P} \to G$ is an epimorphism, there exists $p \in \mathcal{P} \subset D$ such that $\psi(p) = -\psi(a)$. Therefore $ap \in H$ and the inclusion $H \hookrightarrow D$ is cofinal. In order to show that $H \hookrightarrow D$ is a divisor theory we distinguish three cases. First suppose that $|G| = 1$. Then $|D/H| = 1$, and hence $H = D$. Next suppose that $|G| > 2$. By (2) we must verify that

$$\psi(q) \in [\psi(\mathcal{P} \setminus \{q\})] \quad \text{for every } q \in \mathcal{P}.$$

Let $q \in \mathcal{P}$. Since $|G| > 2$, there exist $g_1, g_2 \in G \setminus \{0, \psi(q)\}$ with $\psi(q) = g_1 + g_2$. Since the restriction $\psi \mid_{\mathcal{P}} : \mathcal{P} \to G$ is an epimorphism, there exist $p_1, p_2 \in \mathcal{P} \setminus \{q\}$ with $\psi(p_i) = g_i$ for $i \in [1, 2]$. Therefore

$$\psi(q) = g = g_1 + g_2 = \psi(p_1) + \psi(p_2) \in [\psi(\mathcal{P} \setminus \{q\})].$$

Finally, suppose that $|G| = 2$. Then $H \hookrightarrow D$ is a divisor theory if and only if $\langle \psi(\mathcal{P}) \rangle = [\psi(\mathcal{P} \setminus \{q\})]$ for every $q \in \mathcal{P}$ if and only if there exist distinct $q_1, q_2 \in \mathcal{P}$ such that $\psi(q_i) \neq 0$ for $i \in [1, 2]$. Clearly, if $q \in \mathcal{P}$ is the unique element of $\mathcal{P}$ with $\psi(q) \neq 0$, then H is free abelian with basis $\mathcal{P} \setminus \{q\} \cup \{q^2\}$. □

We are now able to determine both the class group and the set of classes containing prime divisors for the monoids $\mathcal{T}(R)$ and $\mathfrak{C}(R)$. This generalizes and refines the results of [Baeth 2009]. Since each divisorial ideal over a two-dimensional local ring is MCM and thus finitely generated and torsion-free, Theorem 5.4 can be stated in parallel both for $\mathcal{T}(R)$ and $\mathfrak{C}(R)$.

Theorem 5.4. *Let $(R, \mathfrak{m})$ be a Noetherian local Krull domain of dimension two whose $\mathfrak{m}$-adic completion $\widehat{R}$ is also a Krull domain. Let $\mathcal{V}(R)$ (respectively $\mathcal{V}(\widehat{R})$) denote either $\mathcal{T}(R)$ (respectively $\mathcal{T}(\widehat{R})$) or $\mathfrak{C}(R)$ (respectively $\mathfrak{C}(\widehat{R})$), and let $\iota : \mathcal{C}(R) \to \mathcal{C}(\widehat{R})$ be the natural map.*

(1) *The embedding $\mathcal{V}(R) \hookrightarrow \mathcal{V}(\widehat{R})$ is a cofinal divisor homomorphism. The class group of this divisor homomorphism is isomorphic to $G = \mathcal{C}(\widehat{R})/\iota(\mathcal{C}(R))$ and every class contains a prime divisor. Moreover the embedding is a divisor theory except if $\widehat{R}$ satisfies the following condition*:

   (E) *$|G| = 2$ and, up to isomorphism, there is precisely one nonextended indecomposable $\widehat{R}$-module M with $[M] \in \mathcal{V}(\widehat{R})$.*

   In particular, $\mathcal{V}(R)$ satisfies KRSA if and only if either $|G| = 1$ or $\widehat{R}$ satisfies (E).

(2) *Suppose that the embeddings $\mathcal{T}(R) \hookrightarrow \mathcal{T}(\widehat{R})$ and $\mathfrak{C}(R) \hookrightarrow \mathfrak{C}(\widehat{R})$ are both divisor theories. Then their class groups are isomorphic. If $(G, (m_g)_{g \in G})$ is the characteristic of $\mathcal{T}(R)$ and $(G, (n_g)_{g \in G})$ is the characteristic of $\mathfrak{C}(R)$, then $m_g \geq n_g$ for all $g \in G$. Moreover, $\sum_{g \in G} m_g$ infinite.*

(A_n)	$k[\![x, y, z]\!]/(x^2 + y^2 + z^{n+1})$,	$n \geq 1$
(D_n)	$k[\![x, y, z]\!]/(x^2 z + y^2 + z^{n-1})$,	$n \geq 4$
(E_6)	$k[\![x, y, z]\!]/(x^3 + y^2 + z^4)$	
(E_7)	$k[\![x, y, z]\!]/(x^3 + xz^3 + y^2)$	
(E_8)	$k[\![x, y, z]\!]/(x^2 + y^3 + z^5)$	

Table 1. Two-dimensional rings with finite representation type.

Proof. We set $D = \mathcal{F}(\mathcal{P}) = \mathcal{V}(\widehat{R})$. By (1) of Lemma 5.1 there is a homomorphism

$$\psi : \mathcal{V}(\widehat{R}) \to \mathcal{C}(\widehat{R})/\iota(\mathcal{C}(R)) = G \quad \text{given by } [M] \mapsto \boldsymbol{c}(M) + \iota(\mathcal{C}(R)).$$

By (1) of Lemma 5.2, every divisorial ideal of $\widehat{R}$ is an indecomposable MCM $\widehat{R}$-module. That is, the class of each divisorial ideal of $\widehat{R}$ in $\mathcal{C}(\widehat{R})$ is the image of some $[M] \in \mathcal{V}(\widehat{R})$, where M is an indecomposable MCM $\widehat{R}$-module. In other words, ψ restricted to $\mathcal{P} = \mathcal{A}(D)$ is an epimorphism and $\psi(\mathcal{A}(D)) = \mathcal{C}(\widehat{R})/\iota(\mathcal{C}(R))$.

By Lemma 5.3, the inclusion $H = \psi^{-1}(0) \subset D$ is a cofinal divisor homomorphism. By (2) of Lemma 5.2, H is the image of the embedding $\mathcal{V}(R) \hookrightarrow \mathcal{V}(\widehat{R})$, and thus the embedding $\mathcal{V}(R) \hookrightarrow \mathcal{V}(\widehat{R})$ is a divisor theory if and only if the inclusion $H \hookrightarrow D$ is a divisor theory. By Lemma 5.3 this always holds apart from the described Exception (E). A Krull monoid is factorial if and only if its class group is trivial. Thus, if $\mathcal{V}(R) \hookrightarrow \mathcal{V}(\widehat{R})$ is a divisor theory then KRSA holds for $\mathcal{V}(R)$ if and only if $|G| = 0$. If $\mathcal{V}(R) \hookrightarrow \mathcal{V}(\widehat{R})$ is not a divisor theory, then the inclusion $H \hookrightarrow D$ is not a divisor theory. By Lemma 5.3, H is factorial, whence $\mathcal{V}(R)$ is factorial.

Since each MCM R-module is finitely generated and torsion-free, it is clear that $m_g \geq n_g$ for each $g \in G$. From [Bass 1962] we know that there infinitely many nonisomorphic indecomposable finitely generated torsion-free $\widehat{R}$-modules, and therefore $\sum_{g \in G} m_g$ infinite. □

Let R be as in the above theorem and assume in addition that

- R contains a field and $k = R/\mathfrak{m}$ is algebraically closed with characteristic zero;
- $\widehat{R}$ is a hypersurface, that is, $\widehat{R}$ is isomorphic to a three-dimensional regular Noetherian local ring modulo a regular element;
- R has finite representation type.

Such rings were classified in [Buchweitz et al. 1987; Knörrer 1987] and are given, up to isomorphism, in Table 1. Note that since $\widehat{R}$ has finite representation type and each divisorial ideal of $\widehat{R}$ is and indecomposable MCM $\widehat{R}$-module, $\mathcal{C}(\widehat{R})$ and hence $\mathfrak{C}(R)$ is finite.

An amazing theorem of Heitmann [1993] gives the existence of a local factorial domain whose completion is a ring as in Table 1. In this situation we can determine the characteristic of $\mathfrak{C}(R)$.

Theorem 5.5. *Let* $(R, \mathfrak{m})$ *be a Noetherian local factorial domain with* $\mathfrak{m}$*-adic completion* $\widehat{R}$ *isomorphic to a ring in Table 1.*

(1) *If* $\widehat{R}$ *is a ring of type* (A_n), *then* $\mathcal{C}(\mathfrak{C}(R))$ *is cyclic of order* $n+1$ *and each class contains exactly one prime divisor.*

(2) *Suppose* $\widehat{R}$ *is a ring of type* (D_n).

   (a) *If* n *is even*, *then* $\mathcal{C}(\mathfrak{C}(R)) \cong C_2 \oplus C_2$. *The trivial class contains* $n/2$ *prime divisors. Two nontrivial classes each contain a single prime divisor and their sum contains* $(n-2)/2$ *prime divisors.*

   (b) *If* n *is odd*, *then* $\mathcal{C}(\mathfrak{C}(R))$ *is cyclic of order four. The classes of order four each contain a single prime. The remaining classes each contain* $(n-1)/2$ *prime divisors.*

(3) *If* $\widehat{R}$ *is a ring of type* (E_6), *then* $\mathcal{C}(\mathfrak{C}(R))$ *is cyclic of order three. The trivial class contains three prime divisors*, *while each remaining class contains two prime divisors.*

(4) *If* $\widehat{R}$ *is a ring of type* (E_7), *then* $\mathcal{C}(\mathfrak{C}(R))$ *is cyclic of order two. The trivial class contains five prime divisors and the nontrivial class contains three prime divisors.*

(5) *If* $\widehat{R}$ *is a ring of type* (E_8), *then* $\mathcal{C}(\mathfrak{C}(R))$ *is trivial*, *with the trivial class containing all nine prime divisors.*

Proof. The class groups $\mathcal{C}(\widehat{R})$ for $\widehat{R}$ a ring listed in Table 1 were given in [Brieskorn 1967–1968]. Since R is factorial, $\mathcal{C}(R) = 0$ and by Theorem 5.4, $\mathcal{C}(\mathfrak{C}(R)) \cong \mathcal{C}(\widehat{R})$. Following the proof of [Baeth 2009, Theorem 4.3] one can compute the class of each indecomposable $\widehat{R}$-module in $\mathcal{C}(\widehat{R})$ by using the Auslander–Reiten sequence for $\widehat{R}$. The result follows by considering the map $\overline{\psi}$ defined in the proof of Theorem 5.4. □

The above theorem completely determines the characteristic of the monoid $\mathfrak{C}(R)$. With this information, in addition to being able to completely describe the arithmetic of $\mathfrak{C}(R)$ as we do in Theorem 6.8, we can easily enumerate the atoms of $\mathfrak{C}(R)$ (the nonisomorphic indecomposable MCM modules). We now illustrate this ability with an example. If $\boldsymbol{\beta} : \mathfrak{C}(R) \to \mathcal{B}(G_{\mathcal{P}})$ is the transfer homomorphism of Lemma 3.4, then $\mathcal{A}(\mathfrak{C}(R)) = \boldsymbol{\beta}^{-1}\big(\mathcal{A}(\mathcal{B}(G_{\mathcal{P}}))\big)$. Suppose that $\widehat{R}$ is a ring of type (D_n) with n even. Then $\widehat{R}$ has exactly $n+1$ nonisomorphic indecomposable MCM modules. If $C_2 \oplus C_2 = \{0, e_1, e_2, e_1 + e_2\}$, then

$$\mathcal{A}(C_2 \oplus C_2) = \{0, e_1^2, e_2^2, (e_1+e_2)^2, e_1e_2(e_1+e_2)\},$$

and hence R has exactly

$$|\mathcal{A}(\mathfrak{C}(R))| = \frac{n}{2} + 1\cdot 1 + 1\cdot 1 + \frac{n-2}{2}\cdot\frac{n-2}{2} + 1\cdot 1\cdot\frac{n-2}{2} = \frac{n^2+8}{4}$$

nonisomorphic indecomposable MCM modules.

We conclude this section by noting that a two-dimensional local Krull domain $(R, \mathfrak{m})$ having completion isomorphic to a ring in Table 1 may not be factorial. However, Theorem 5.4 implies that $\mathcal{C}(\mathfrak{C}(R))$ is a factor group of a group given in Theorem 5.5. In particular, $\mathcal{C}(\mathfrak{C}(R))$ is a finite cyclic group such that every class contains a prime divisor, and thus the arithmetic of $\mathfrak{C}(R)$ is described in Theorem 6.8.

6. The arithmetic of monoids of modules

In this section we study the arithmetic of the Krull monoids that have been discussed in Sections 4 and 5. Thus, using the transfer properties presented in Section 2, we describe the arithmetic of direct-sum decompositions of modules. Suppose that H is a Krull monoid having a divisor homomorphism $\varphi : H \to \mathcal{F}(\mathcal{P})$ and let $G_{\mathcal{P}} \subset \mathcal{C}(\varphi)$ be the set of classes containing prime divisors. The first subsection deals with quite general sets $G_{\mathcal{P}}$ and provides results on the finiteness or nonfiniteness of various arithmetical parameters. The second subsection studies three specific sets $G_{\mathcal{P}}$, provides explicit results on arithmetical parameters, and establishes a characterization result (Theorems 6.4, 6.7, 6.8, and Corollary 6.10). The third subsection completely determines the system of sets of lengths in case of small subsets $G_{\mathcal{P}}$. It shows that small subsets in torsion groups and in torsion-free groups can have the same systems of sets of lengths, and it reveals natural limits for arithmetical characterization results (Corollary 6.15).

6A. ***General sets $G_{\mathcal{P}}$ of classes containing prime divisors.*** In this subsection we consider the algebraic and arithmetic structure of Krull monoids with respect to $G_{\mathcal{P}}$. We will often assume that $G_{\mathcal{P}} = -G_{\mathcal{P}}$, a property which has a strong influence on the arithmetic of H. Recall that $G_{\mathcal{P}} = -G_{\mathcal{P}}$ holds in many of the (finite and infinite representation type) module-theoretic contexts described in Sections 4 and 5. More generally, all configurations $(G, G_{\mathcal{P}})$ occur for certain monoids of modules (see Proposition 4.18, [Herbera and Příhoda 2010] and [Leuschke and Wiegand 2012, Chapter 1]) and, by Claborn's realization theorem, all configurations $(G, G_{\mathcal{P}})$ occur for Dedekind domains (see [Geroldinger and Halter-Koch 2006, Theorem 3.7.8]). In addition, every abelian group can be realized as the class group of a Dedekind domain which is a quadratic extension of a principal ideal domain, and in this case we have $G_{\mathcal{P}} = -G_{\mathcal{P}}$ (see [Leedham-Green 1972]).

Proposition 6.1. *Let H be a Krull monoid, $\varphi : H \to \mathcal{F}(\mathcal{P})$ a divisor homomorphism with class group $G = \mathcal{C}(\varphi)$, and let $G_{\mathcal{P}} \subset G$ denote the set of classes containing prime divisors.*

(1) *If $G_{\mathcal{P}}$ is finite, then $\mathcal{A}(G_{\mathcal{P}})$ is finite and hence $\mathsf{D}(G_{\mathcal{P}}) < \infty$. If G has finite total rank, then $G_{\mathcal{P}}$ is finite if and only if $\mathcal{A}(G_{\mathcal{P}})$ is finite if and only if $\mathsf{D}(G_{\mathcal{P}}) < \infty$.*

(2) *If $G_{\mathcal{P}} = -G_{\mathcal{P}}$, then $[G_{\mathcal{P}}] = G$. Moreover, the map $\varphi : H \to D$ and the inclusion $\mathcal{B}(G_{\mathcal{P}}) \hookrightarrow \mathcal{F}(G_{\mathcal{P}})$ are both cofinal.*

(3) *Suppose that G is infinite cyclic, say $G = \langle e \rangle$, and that $\{-e, e\} \subset G_{\mathcal{P}}$. Then $\mathcal{B}(G_{\mathcal{P}}) \hookrightarrow \mathcal{F}(G_{\mathcal{P}})$ is a divisor theory if and only if there exist $k, l \in \mathbb{N}_{\geq 2}$ such that $-ke, le \in G_{\mathcal{P}}$.*

(4) *Let $r, \alpha \in \mathbb{N}$ with $r + \alpha > 2$. Let $(e_1, \ldots, e_r) \in G_{\mathcal{P}}^r$ be independent and let $e_0 \in G_{\mathcal{P}}$ such that $\alpha e_0 = e_1 + \cdots + e_r$, $\{-e_0, \ldots, -e_r\} \subset G_{\mathcal{P}}$, and $\langle e_0, \ldots, e_r \rangle = G$.*
   - (a) *The map $\varphi : H \to \mathcal{F}(\mathcal{P})$ and the inclusion $\mathcal{B}(G_{\mathcal{P}}) \hookrightarrow \mathcal{F}(G_{\mathcal{P}})$ are divisor theories with class group isomorphic to G.*
   - (b) *If $0 \notin G_{\mathcal{P}}$, then $\mathcal{B}(G_{\mathcal{P}})$ is not a direct product of nontrivial submonoids.*

Proof. (1) follows from [Geroldinger and Halter-Koch 2006, Theorem 3.4.2].

If $G_{\mathcal{P}} = -G_{\mathcal{P}}$, then $[G_{\mathcal{P}}] = \langle G_{\mathcal{P}} \rangle = G$. By Lemma 3.4, (2) follows once we verify that φ is cofinal. If $p \in \mathcal{P}$, then there is a $q \in \mathcal{P}$ with $q \in -[p]$, whence there is an $a \in H$ with $\varphi(a) = pq$, and so φ is cofinal.

If $\{-ke : k \in \mathbb{N}\} \cap G_{\mathcal{P}} = \{-e\}$ or $\{ke : k \in \mathbb{N}\} \cap G_{\mathcal{P}} = \{e\}$, then $\mathcal{B}(G_{\mathcal{P}})$ is factorial. Since $\mathcal{F}(G_{\mathcal{P}}) \neq B(G_{\mathcal{P}})$, the inclusion $\mathcal{B}(G_{\mathcal{P}}) \hookrightarrow F(G_{\mathcal{P}})$ is not a divisor theory. Conversely, suppose that there exist $k, l \in \mathbb{N}_{\geq 2}$ such that $-ke, le \in G_{\mathcal{P}}$. Let $m \in \mathbb{N}$. If $me \in G_{\mathcal{P}}$, then $me = \gcd\big((me)(-e)^m, (me)^k(-ke)^m\big)$, and if $-me \in G_{\mathcal{P}}$, then $-me = \gcd\big((-me)e^m, (-me)^l(le)^m\big)$. Thus every element of $G_{\mathcal{P}}$ is a greatest common divisor of a finite set of elements from $\mathcal{B}(G_{\mathcal{P}})$ and hence $\mathcal{B}(G_{\mathcal{P}}) \hookrightarrow F(G_{\mathcal{P}})$ is a divisor theory.

We now suppose that G and $G_{\mathcal{P}}$ are as in (4). To prove (a) it is sufficient to show that $\mathcal{B}(G_{\mathcal{P}}) \hookrightarrow \mathcal{F}(G_{\mathcal{P}})$ is a divisor theory. By [Geroldinger and Halter-Koch 2006, Proposition 2.5.6] we need only verify that $\langle G_{\mathcal{P}} \rangle = [G_{\mathcal{P}} \setminus \{g\}]$ for every $g \in G_{\mathcal{P}}$. Let $g \in G$. We will show that

$$[G_{\mathcal{P}} \setminus \{g\}] = [G_{\mathcal{P}}] = [e_0, \ldots, e_r, -e_0, \ldots, -e_r].$$

If $g \notin \{e_0, \ldots, e_r, -e_0, \ldots, -e_r\}$, the assertion is clear. By symmetry it suffices to consider the case where $g \in \{e_0, \ldots, e_r\}$. If $g = e_i$ for some $i \in [1, r]$, then

$$e_i = \alpha e_0 + (-e_1) + \cdots + (-e_{i-1}) + (-e_{i+1}) + \cdots + (-e_r) \in [\{\pm e_\nu : \nu \in [0, r] \setminus \{i\}],$$

and hence

$$[G_{\mathcal{P}} \setminus \{g\}] \supset [\{\pm e_\nu : \nu \in [0, r] \setminus \{i\}\}] = [\pm e_0, \ldots, \pm e_r] = G\,.$$

If $g = e_0$, then

$$e_0 = e_1 + \cdots + e_r + (\alpha - 1)(-e_0) \in [G_{\mathcal{P}} \setminus \{e_0\}],$$

and hence $[G_{\mathcal{P}} \setminus \{g\}] = [G_{\mathcal{P}}] = G$.

To prove (b) we use Lemma 3.2 and suppose that $0 \notin G_{\mathcal{P}}$ with $G_{\mathcal{P}} = G_1 \uplus G_2$ such that $\mathcal{A}(G_{\mathcal{P}}) = \mathcal{A}(G_1) \uplus \mathcal{A}(G_2)$. We must show that either G_1 or G_2 is empty. Suppose that $V = e_1 \cdot \ldots \cdot e_r(-e_0)^\alpha \in \mathcal{A}(G_1)$. Since $(-e_0)e_0, \ldots, (-e_r)e_r \in \mathcal{A}(G_{\mathcal{P}})$, it follows that $\{\pm e_0, \ldots, \pm e_r\} \subset G_1$. Let $g \in G_{\mathcal{P}}$. Since $G_{\mathcal{P}} \subset G = [\pm e_0, \ldots, \pm e_r]$, there exists $U \in \mathcal{A}(G_{\mathcal{P}})$ such that $g \in \operatorname{supp}(U) \subset \{g, \pm e_0, \ldots, \pm e_r\}$, and hence $g \in G_1$. Thus $G_1 = G_{\mathcal{P}}$ and $G_2 = \varnothing$. □

For our characterization results, we need to recall the concept of an absolutely irreducible element, a classical notion in algebraic number theory. An element u in an atomic monoid H is called *absolutely irreducible* if $u \in \mathcal{A}(H)$ and $|\mathsf{Z}(u^n)| = 1$ for all $n \in \mathbb{N}$; equivalently, the divisor-closed submonoid of H generated by u is factorial. Suppose that $H \hookrightarrow \mathcal{F}(\mathcal{P})$ is a divisor theory with class group G and that $u = p_1^{k_1} \cdot \ldots \cdot p_m^{k_m}$ where $m, k_1, \ldots, k_m \in \mathbb{N}$ and where $p_1, \ldots, p_m \in \mathcal{P}$ are pairwise distinct. Then u is absolutely irreducible. if and only if $(k_1, \ldots, k_m)$ is a minimal element of the set

$$\Gamma = \{(s_1, \ldots, s_m) \in \mathbb{N}_0^l : p_1^{s_1} \cdot \ldots \cdot p_m^{s_m} \in H\} \setminus \{\mathbf{0}\}$$

relative to the usual product ordering, and the torsion-free rank of $\langle [p_1], \ldots, [p_m] \rangle$ in G is $m-1$ (see [Geroldinger and Halter-Koch 2006, Proposition 7.1.4]). In particular, if $[p_1] \in G$ has finite order, then $p_1^{\operatorname{ord}([p_1])}$ is absolutely irreducible, and if $[p_1] \in G$ has infinite order, then $p_1 q_1$ is absolutely irreducible for all $q_1 \in \mathcal{P} \cap (-[p_1])$.

Proposition 6.2. *Let H be a Krull monoid, $\varphi : H \to \mathcal{F}(\mathcal{P})$ a cofinal divisor homomorphism with class group G, and let $G_{\mathcal{P}} \subset G$ denote the set of classes containing prime divisors.*

(1) *Suppose that $G_{\mathcal{P}}$ is infinite.*

   (a) *If $G_{\mathcal{P}}$ has an infinite subset G_0 such that $G_0 \cup (-G_0) \subset G_{\mathcal{P}}$ and $\langle G_0 \rangle$ has finite total rank, then $\mathcal{U}_k(H)$ is infinite for each $k \ge 2$. Moreover, $\mathsf{D}(G_{\mathcal{P}}) = \rho_k(H) = \omega(H) = \mathsf{t}(H) = \infty$.*

   (b) *If there exists $e \in G$ such that $G_{\mathcal{P}} \cap \{ke : k \in \mathbb{N}\}$ and $G_{\mathcal{P}} \cap \{-ke : k \in \mathbb{N}\}$ are both infinite, then $\Delta(H)$ is infinite and $\mathsf{c}(H) = \mathsf{c}_{\mathrm{mon}}(H) = \infty$.*

   (c) *If $G_{\mathcal{P}}$ contains an infinite group, then every finite subset $L \subset \mathbb{N}_{\ge 2}$ occurs as a set of lengths.*

(2) *Suppose that $G_{\mathcal{P}}$ is finite and that H is not factorial.*

   (a) *The set $\Delta(H)$ is finite and there is a constant $M_1 \in \mathbb{N}$ such that every set of lengths is an AAMP with difference $d \in \Delta^*(H)$ and bound M_1.*

(b) *There is a constant* $M_2 \in \mathbb{N}$ *such that, for every* $k \geq 2$, *the set* $\mathcal{U}_k(H)$ *is an AAMP with period* $\{0, \min \Delta(H)\}$ *and bound* M_2.

(c) $\mathsf{c}(H) \leq \omega(H) \leq \mathsf{t}(H) \leq 1 + \frac{1}{2}\mathsf{D}(G_{\mathcal{P}})(\mathsf{D}(G_{\mathcal{P}}) - 1)$ *and*

$$\mathsf{c}_{\mathrm{mon}}(H) < \frac{|G^{\bullet}_{\mathcal{P}}| + 2}{2}\big((2|G^{\bullet}_{\mathcal{P}}| + 2)(|G^{\bullet}_{\mathcal{P}}| + 2)(\mathsf{D}(G_{\mathcal{P}}) + 1)\big)^{|G^{\bullet}_{\mathcal{P}}|+1}.$$

(d) *Suppose that* $G_{\mathcal{P}} = -G_{\mathcal{P}}$. *Then* $\omega(H) = \mathsf{D}(G_{\mathcal{P}})$, $\rho(H) = \mathsf{D}(G_{\mathcal{P}})/2$, $\rho_{2k}(H) = k\mathsf{D}(G_{\mathcal{P}}) < \infty$, *and* $\lambda_{k\mathsf{D}(G_{\mathcal{P}})+j}(H) = 2k + j$ *for all* $k \in \mathbb{N}$ *and* $j \in [0, 1]$. *If G is torsion-free, then* $\mathsf{D}(G_{\mathcal{P}})$ *is the maximal number s of absolutely irreducible atoms* $u_1, \dots, u_s$ *such that* $2 \in \mathsf{L}(u_1 \cdot \ldots \cdot u_s)$.

(e) *If, in particular,* $G_{\mathcal{P}} = -G_{\mathcal{P}}$ *and* $\mathsf{D}(G_{\mathcal{P}}) = 2$, *then* $\mathcal{U}_k(H) = \{k\}$ *for all* $k \in \mathbb{N}$ *and* $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}(H) = \omega(H) = \mathsf{t}(H) = 2$.

Proof. Throughout the proof we implicitly assume the results of Lemma 2.2 and of Lemma 3.4. In particular, we have $\rho(H) \leq \omega(H)$ and $\mathsf{c}(H) \leq \omega(H) \leq \mathsf{D}(G_{\mathcal{P}})$.

For (1), suppose that $G_{\mathcal{P}}$ is infinite. We first prove (a). Theorem 3.4.2 in [Geroldinger and Halter-Koch 2006] implies that $\mathcal{A}(G_0)$ and $\mathsf{D}(G_0)$ are infinite. Thus, for every $k \in \mathbb{N}$, there is $U_k \in \mathcal{A}(G_0)$ with $|U_k| \geq k$ and hence $\mathsf{L}\big(U_k(-U_k)\big) \supset \{2, |U_k|\}$. This implies that $\mathcal{U}_2(G_0)$ is infinite and thus $\mathcal{U}_k(G_0)$ is infinite for all $k \geq 2$. Therefore $\rho_k(H) = \rho_k(G_{\mathcal{P}}) = \infty$ for all $k \geq 2$ and, since $\rho(H) \leq \omega(H) \leq \mathsf{t}(H)$, each of these invariants is infinite.

Item 1(b) follows from [Geroldinger et al. 2010, Theorem 4.2].

Item 1(c) is a realization result is due to Kainrath. See [Kainrath 1999] or [Geroldinger and Halter-Koch 2006, Theorem 7.4.1].

Now, in order to prove (2), we suppose that $G_{\mathcal{P}}$ is finite and that H is not factorial. Then $\mathsf{D}(G_{\mathcal{P}}) > 1$ and $2 \leq \mathsf{c}(H) \leq \omega(H) \leq \mathsf{t}(H)$. By Proposition 6.1, $\mathcal{B}(G_{\mathcal{P}})$ is finitely generated and $\mathsf{D}(G_{\mathcal{P}}) < \infty$. The respective upper bounds given in (c) for $\mathsf{c}_{\mathrm{mon}}(H)$ and $\mathsf{t}(H)$ can be found in [Geroldinger and Yuan 2013, Theorem 3.4] and [Geroldinger and Halter-Koch 2006, Theorem 3.4.10].

We now consider (a). Since $2 + \sup \Delta(H) \leq \mathsf{c}(H) < \infty$, (c) implies that $\Delta(H)$ is finite. Since $\mathcal{B}(G_{\mathcal{P}})$ is finitely generated, the assertion on the structure of sets of lengths follows from [Geroldinger and Halter-Koch 2006, Theorem 4.4.11].

Since $\mathsf{t}(H) < \infty$ and $\Delta(H)$ is finite, [Gao and Geroldinger 2009, Theorems 3.5 and 4.2] imply the assertion in (b) on the structure of the unions of sets of lengths.

In order to prove (d), we suppose that $G_{\mathcal{P}} = -G_{\mathcal{P}}$. The statements about $\rho_{2k}(H)$, $\rho(H)$, and $\lambda_{k\mathsf{D}(G_{\mathcal{P}})+j}(H)$ follow from Lemma 3.3, and it remains to show that $\omega(H) = \mathsf{D}(G_{\mathcal{P}})$. We have $\omega(H) \leq \mathsf{D}(G_{\mathcal{P}}) < \infty$. If $\mathsf{D}(G_{\mathcal{P}}) = 2$, then $\omega(H) = \mathsf{D}(G_{\mathcal{P}})$. Suppose that $\mathsf{D}(G_{\mathcal{P}}) \geq 3$. If $V = g_1 \cdot \ldots \cdot g_l \in \mathcal{A}(G_{\mathcal{P}})$ with $|V| = l = \mathsf{D}(G_{\mathcal{P}})$ and $U_i = (-g_i)g_i$ for all $i \in [1, l]$, then $V \mid U_1 \cdot \ldots \cdot U_l$ but yet V divides no proper subproduct of $U_1 \cdot \ldots \cdot U_l$. Thus $\mathsf{D}(G_{\mathcal{P}}) \leq \omega(G_{\mathcal{P}}) \leq \omega(H)$.

Let t denote the maximal number of absolutely irreducible atoms with the required property. Since $\rho(H) = \mathsf{D}(G_{\mathcal{P}})/2$, it follows that $t \le \mathsf{D}(G_{\mathcal{P}})$. Let $V = g_1 \cdot \ldots \cdot g_l \in \mathcal{A}(G_{\mathcal{P}})$ with $|V| = l = \mathsf{D}(G_{\mathcal{P}})$. For $i \in [1, l]$ choose an element $p_i \in \mathcal{P} \cap g_i$ and an element $q_i \in \mathcal{P} \cap (-g_i)$. Since G is torsion-free, the element $u_i = p_i q_i \in H$ is absolutely irreducible for each $i \in [1, l]$ and, by construction, we have $2 \in \mathsf{L}(u_1 \cdot \ldots \cdot u_l)$.

The statement in (e) follows immediately from (c) and (d). □

Let all notation be as in Proposition 6.2. We note that if $G_{\mathcal{P}}$ is infinite but without a subset G_0 as in (1a), then none of the conclusions of (1a) need hold. A careful analysis of the case where G is an infinite cyclic groups is handled in [Geroldinger et al. 2010]. We also note that the description of the structure of sets of lengths given in (2a) is best possible (see [Schmid 2009c]).

By Lemma 3.4, many arithmetical phenomena of a Krull monoid H are determined by the tuple $(G, G_{\mathcal{P}})$. We now provide a first result indicating that conversely arithmetical phenomena give us back information on the class group. Indeed, our next corollary characterizes arithmetically whether the class group of a Krull monoid is torsion-free or not. To do so we must study the arithmetical behavior of elements similar to absolutely irreducible elements. Note that such a result cannot be accomplished via sets of lengths alone (see Propositions 6.12 and 6.13 and (1c) of Proposition 6.2; in fact, there is an open conjecture that every abelian group is the class group of a half-factorial Krull monoid [Geroldinger and Göbel 2003]).

Proposition 6.3. *Let H be a Krull monoid with class group G. Then G has an element of infinite order if and only if there exists an irreducible element $u \in H$ having the following two arithmetical properties.*

(a) *Whenever there are $v \in H \setminus H^\times$ and $m \in \mathbb{N}$ with $v \mid u^m$, then $u \mid v^n$ for some $n \in \mathbb{N}$.*

(b) *There exist $l \ge 2$ and $a_1, \ldots, a_l \in H$ such that $u \mid a_1 \cdot \ldots \cdot a_l$ but yet*

$$u \nmid a_\nu^{-1} (a_1 \cdot \ldots \cdot a_l)^N$$

for each $\nu \in [1, l]$ and for every $N \in \mathbb{N}$.

Proof. We may assume that H is reduced. Consider a divisor theory $H \hookrightarrow \mathcal{F}(\mathcal{P})$ and denote by $G_{\mathcal{P}} \subset G$ the set of classes containing prime divisors.

First suppose that G is a torsion group and let $u \in \mathcal{A}(H)$ have Property (a). Then $u = p_1^{k_1} \cdot \ldots \cdot p_m^{k_m}$ for some $m, k_1, \ldots, k_m \in \mathbb{N}$ and pairwise distinct elements $p_1, \ldots, p_m \in \mathcal{P}$. Then (a) implies that $k = 1$ and hence u is absolutely irreducible. Thus Property (b) cannot hold for any $l \ge 2$.

Conversely, suppose that G is not a torsion group. Since $[G_{\mathcal{P}}] = G$ there exists a $p \in \mathcal{P}$ such that $[p] \in G$ has infinite order, and there is an element $u' \in \mathcal{A}(H)$ with

$p \mid u'$. Suppose that $u' = p_1 \cdot \ldots \cdot p_n \cdot q_1 \cdot \ldots \cdot q_r$, where

$$p = p_1, p_2, \ldots, p_n, q_1, \ldots, q_r \in \mathcal{P},$$

$[p_1], \ldots, [p_n]$ have infinite order, and $[q_1], \ldots, [q_r]$ have finite order, each of which divides some integer N. Then $(q_1 \cdot \ldots \cdot q_r)^N \in H$, whence $(p_1 \cdot \ldots \cdot p_n)^N \in H$. After a possible reordering there is an atom $u = p_1^{k_1} \cdot \ldots \cdot p_m^{k_m} \in \mathcal{A}(H)$ dividing a power of $(p_1 \cdot \ldots \cdot p_n)^N$ such that there is no atom $v \in \mathcal{A}(H)$ with $\operatorname{supp}_{\mathcal{P}}(v) \subsetneq \{p_1, \ldots, p_m\}$. Thus u satisfies Property (a). Since $H \hookrightarrow \mathcal{F}(\mathcal{P})$ is a divisor theory, there exist $b_1, \ldots, b_s \in H$ such that

$$p_1^{k_1} \cdot \ldots \cdot p_{m-1}^{k_{m-1}} = \gcd(b_1, \ldots, b_s).$$

Hence there is an $i \in [1, s]$, say $i = 1$, such that $p_m \nmid b_1$. Similarly, there are $c_1, \ldots, c_t \in H$ such that $p_m^{k_m} = \gcd(c_1, \ldots, c_t)$. Without loss of generality, there exists $i \in [1, m-1]$ such that $p_i \nmid c_1$. Therefore $u \mid b_1 c_1$, but yet $u \nmid b_1^N$ and $u \nmid c_1^N$ for any $N \in \mathbb{N}$, and so Property (b) is satisfied. □

Propositions 6.1, 6.2, and 6.3 provide abstract finiteness and nonfiniteness results. To obtain more precise information on the arithmetical invariants, we require specific information on $G_{\mathcal{P}}$. In the next subsection we will use such specific information to give more concrete results.

6B. ***Specific sets $G_{\mathcal{P}}$ of classes containing prime divisors and arithmetical characterizations.*** We now provide an in-depth study of the arithmetic of three classes of Krull monoids studied in Sections 4 and 5. Theorem 6.4 describes the arithmetic of the monoids discussed in Examples 4.12, 4.20, 4.25 and in Theorem 4.12. Its arithmetic is simple enough that we can more or less give a complete description.

Theorem 6.4. *Let H be a Krull monoid with class group G and suppose that*

$$G_{\mathcal{P}} = \{e_0, \ldots, e_r, -e_0, \ldots, -e_r\} \subset G$$

is the set of classes containing prime divisors, where $r, \alpha \in \mathbb{N}$ with $r + \alpha > 2$ and $(e_1, \ldots, e_r)$ is an independent family of elements each having infinite order such that $e_1 + \cdots + e_r = \alpha e_0$. Then:

(1) *$\mathcal{A}(G_{\mathcal{P}}) = \{V, -V, U_\nu : \nu \in [0, r]\}$, where $V = (-e_0)^\alpha e_1 \cdot \ldots \cdot e_r$ and $U_\nu = (-e_\nu) e_\nu$ for all $\nu \in [0, r]$. In particular, $\mathsf{D}(G_{\mathcal{P}}) = r + \alpha$.*

(2) *Suppose that*

$$S = \prod_{i=0}^{r} e_i^{k_i} (-e_i)^{l_i} \in \mathcal{F}(G_{\mathcal{P}}),$$

where $k_0, l_0, \ldots, k_r, l_r \in \mathbb{N}_0$. Then $S \in \mathcal{B}(G_{\mathcal{P}})$ if and only if

$$l_i = \alpha^{-1}(k_0 - l_0) + k_i$$

for all $i \in [1, r]$. *If* $S \in \mathcal{B}(G_{\mathcal{P}})$ *with* $k_0 \ge l_0$ *and* $k^* = \min\{k_1, \dots, k_r\}$, *then*

$$\mathsf{Z}(S) = \left\{ V^{\nu}(-V)^{\alpha^{-1}(k_0 - l_0) + \nu} U_0^{l_0 - \alpha\nu} \prod_{i=1}^{r} U_i^{k_i - \nu} : \nu \in [0, \min\{\alpha^{-1} l_0, k^*\}] \right\}$$

and

$$\mathsf{L}(S) = \left\{ \alpha^{-1}(k_0 - l_0) + l_0 + k_1 + \dots + k_r - (r + \alpha - 2)\nu : \nu \in [0, \min\{\alpha^{-1} l_0, k^*\}] \right\}.$$

(3) *The system of sets of lengths of* H *can be described as follows.*

(a) $\Delta(H) = \{r + \alpha - 2\}$.

(b) $\rho(H) = \mathsf{D}(G_{\mathcal{P}})/2$.

(c) *For each* $k \in \mathbb{N}$, *the set* $\mathcal{U}_k(H)$ *is an arithmetical progression with difference* $r + \alpha - 2$.

(d) *For each* $k \in \mathbb{N}$ *and each* $j \in [0, 1]$, $\rho_{2k+j}(H) = k\mathsf{D}(G_{\mathcal{P}}) + j$.

(e) *For each* $l \in \mathbb{N}_0$, $\lambda_{l\mathsf{D}(G_{\mathcal{P}})+j}(H) = 2l + j$ *whenever* $j \in [0, \mathsf{D}(G_{\mathcal{P}}) - 1]$ *and* $l\mathsf{D}(G_{\mathcal{P}}) + j \ge 1$.

(f) *Finally*,

$$\mathcal{L}(H) = \left\{ m + \{2k^* + (r + \alpha - 2)\lambda : \lambda \in [0, k^*]\} : m, k^* \in \mathbb{N}_0 \right\}.$$

(4) $\mathsf{c}(H) = \mathsf{c}_{\mathrm{mon}}(H) = \omega(H) = \mathsf{t}(H) = \mathsf{D}(G_{\mathcal{P}}) = r + \alpha$.

Proof. By Lemma 3.4, all assertions on lengths of factorizations and on catenary degrees can be proved working in $\mathcal{B}(G_{\mathcal{P}})$ instead of H.

Obviously, $\{U_\nu : \nu \in [0, r]\} \subset \mathcal{A}(G_{\mathcal{P}})$ and to prove (1) it remains to verify that if $W \in \mathcal{A}(G_{\mathcal{P}})$ with $W \ne U_\nu$, then $W = V$. Note that $e_0 \in \langle e_1, \dots, e_r \rangle$ but that $\langle e_0 \rangle \cap \langle e_i : i \in I \rangle = \{0\}$ for any proper subset $I \subsetneq [1, r]$. Thus, if

$$W = \prod_{i \in I} e_i^{k_i} (-e_0)^{k_0} \in \mathcal{A}(G_{\mathcal{P}}) \setminus \{U_\nu : \nu \in [0, r]\},$$

where $\emptyset \ne I \subset [1, r]$ and $k_0, k_i \in \mathbb{N}$ for all $i \in I$, then $k_0 e_0 = \sum_{i \in I} k_i e_i \in \langle e_i : i \in I \rangle$ and hence $I = [1, r]$. Assume to the contrary that there is $i \in [1, r]$ such that $k_i > 1$. Since $V \in \mathcal{B}(G_{\mathcal{P}})$ and $W \in \mathcal{A}(G_{\mathcal{P}})$, it follows that $k_0 \in [1, \alpha - 1]$. Then

$$0 \ne (k_1 - 1)e_1 + \dots + (k_r - 1)e_r = (k_0 - \alpha)e_0 \in [e_1, \dots, e_r] \cap [-e_1, \dots, -e_r] = \{0\},$$

a contradiction. Thus $k_1 = \dots = k_r = 1$ and we obtain that $k_0 = \alpha$, whence $W = V \in \mathcal{A}(G_{\mathcal{P}})$.

To prove (2), suppose that $S \in \mathcal{B}(G_{\mathcal{P}})$ and that $l_0 \ge k_0$. Then

$$S' = (-e_0)^{l_0 - k_0} \prod_{i=1}^{r} e_i^{k_i} (-e_i)^{l_i} \in \mathcal{B}(G_{\mathcal{P}}),$$

whence $l_0 - k_0 = \alpha m_0 \in \alpha\mathbb{N}_0$, $S'' = \prod_{i=1}^{r} e_i^{k_i - m_0} (-e_i)^{l_i} \in \mathcal{B}(G_{\mathcal{P}})$, and

$$l_i = k_i - m_0 = k_i - \frac{l_0 - k_0}{\alpha} = \frac{k_0 - l_0}{\alpha} + k_i \quad \text{for each } i \in [1, r].$$

The same holds true if $l_0 \le k_0$. Conversely, if $l_1, \dots, l_r$ satisfy the asserted equations, then obviously $\sigma(S) = 0$.

Suppose that $S \in \mathcal{B}(G_{\mathcal{P}})$ and that $k_0 \ge l_0$. Then

$$\begin{aligned}
S &= e_0^{k_0}(-e_0)^{l_0} \prod_{i=1}^{r} e_i^{k_i}(-e_i)^{k_i+\alpha^{-1}(k_0-l_0)} \\
&= \big((-e_0)e_0\big)^{l_0}\big(e_0{}^{\alpha}(-e_1)\cdot\ldots\cdot(-e_r)\big)^{\alpha^{-1}(k_0-l_0)} \prod_{i=1}^{r} \big((-e_i)e_i\big)^{k_i} \\
&= \big((-e_0)e_0\big)^{l_0-\alpha\nu}\big(e_0{}^{\alpha}(-e_1)\cdot\ldots\cdot(-e_r)\big)^{\alpha^{-1}(k_0-l_0)+\nu}\big((-e_0)^{\alpha}e_1\cdot\ldots\cdot e_r\big)^{\nu} \\
&\qquad \times \prod_{i=1}^{r} \big((-e_i)e_i\big)^{k_i-\nu} \\
&= U_0^{l_0-\alpha\nu}(-V)^{\alpha^{-1}(k_0-l_0)+\nu} V^{\nu} \prod_{i=1}^{r} U_i^{k_i-\nu}
\end{aligned}$$

for each $\nu \in [0, \min\{\alpha^{-1} l_0, k^*\}]$. Therefore $\mathsf{Z}(S)$ and hence $\mathsf{L}(S)$ have the given forms.

We now consider the statements of (3). The assertion on $\Delta(G_{\mathcal{P}})$ follows immediately from (2). Since $\Delta(G_{\mathcal{P}}) = \{r + \alpha - 2\}$, all sets $\mathcal{U}_k(G_{\mathcal{P}})$ are arithmetical progressions with difference $r + \alpha - 2$. The assertion on each $\rho_{2k}(G_{\mathcal{P}})$ and each $\rho(G_{\mathcal{P}})$ follow from Proposition 6.2.

In order to determine $\mathcal{L}(G_{\mathcal{P}})$, let $S \in \mathcal{B}(G_{\mathcal{P}})$ be given with all parameters as in (2). First suppose that $l_0 \ge \alpha k^*$. Then

$$\begin{aligned}
\mathsf{L}(S) &= (\alpha^{-1}(k_0 - l_0) + l_0 + k_1 + \dots + k_r) \\
&\qquad -(r+\alpha)k^* + \{(r+\alpha)k^* - (r+\alpha-2)\nu : \nu \in [0, k^*]\} \\
&= (\alpha^{-1}(k_0 - l_0) + l_0 + k_1 + \dots + k_r) \\
&\qquad -(r+\alpha)k^* + \{2k^* + (r+\alpha-2)\lambda : \lambda \in [0, k^*]\}.
\end{aligned}$$

Thus $\mathsf{L}(S)$ has the form

$$\mathsf{L}(S) = m + \{2k^* + (r+\alpha-2)\lambda : \lambda \in [0, k^*]\}$$

for some $m, k^* \in \mathbb{N}_0$. Conversely, for every choice of $m, k^* \in \mathbb{N}_0$, there is an $S \in \mathcal{B}(G_{\mathcal{P}})$ such that $\mathsf{L}(S)$ has the given form.

Now suppose that $l_0 \le \alpha k^* - 1$ and set $m_0 = \lfloor l_0/\alpha \rfloor$. Then

$$\begin{aligned}
\mathsf{L}(S) &= (\alpha^{-1}(k_0 - l_0) + l_0 + k_1 + \dots + k_r) \\
&\qquad -(r+\alpha)m_0 + \{(r+\alpha)m_0 - (r+\alpha-2)\nu : \nu \in [0, m_0]\}
\end{aligned}$$

$$= (\alpha^{-1}(k_0 - l_0) + l_0 + k_1 + \cdots + k_r) \\ -(r+\alpha)m_0 + \{2m_0 + (r+\alpha-2)\lambda : \lambda \in [0, m_0]\},$$

and hence $\mathsf{L}(S)$ has the form

$$\mathsf{L}(S) = m + \{2m_0 + (r+\alpha-2)\lambda : \lambda \in [0, m_0]\}$$

for some $m \in \mathbb{N}$ and $m_0 \in \mathbb{N}_0$.

Next we verify that, for every $k \in \mathbb{N}$, $\rho_{2k+1}(G_{\mathcal{P}}) \le k\mathsf{D}(G_{\mathcal{P}}) + 1$. By [Geroldinger and Halter-Koch 2006, Proposition 1.4.2], for all $k \in \mathbb{N}$,

$$\rho_k(G_{\mathcal{P}}) = \sup\{\sup L : L \in \mathcal{L}(G_{\mathcal{P}}), k = \min L\}.$$

Thus we may choose $k \in \mathbb{N}$ and $L \in \mathcal{L}(G_{\mathcal{P}})$ with $\min L = 2k+1$. Then, there exist $l, m, k^* \in \mathbb{N}_0$ such that

$$L = m + \{2k^* + (r+\alpha-2)\lambda : \lambda \in [0, k^*]\}$$

with $m = 2l+1$ and $2k+1 = \min L = 2(k^*+l)+1$. Now

$$\begin{aligned} \max L &= m + (r+\alpha)k^* = 2l + 1 + (r+\alpha)(k-l) \\ &= (r+\alpha)k + 1 - (r+\alpha-2)l \le k\mathsf{D}(G_{\mathcal{P}}) + 1, \end{aligned}$$

and thus $\rho_{2k+1}(G_{\mathcal{P}}) \le k\mathsf{D}(G_{\mathcal{P}}) + 1$.

It remains to verify the assertions on the $\lambda_{l\mathsf{D}(G_{\mathcal{P}})+j}(G_{\mathcal{P}})$. Let $l \in \mathbb{N}_0$ and $j \in [0, \mathsf{D}(G_{\mathcal{P}}) - 1]$. Then Lemma 3.3 implies $\lambda_{l\mathsf{D}(G_p)+j}(G_{\mathcal{P}}) \le 2l + j$, and that equality holds if $j \in [0, 1]$. It remains to verify that $\lambda_{l\mathsf{D}(G_p)+j}(G_{\mathcal{P}}) \ge 2l + j$ when $j \in [2, \mathsf{D}(G_{\mathcal{P}}) - 1]$. Let $L \in \mathcal{L}(G_{\mathcal{P}})$ with $l\mathsf{D}(G_{\mathcal{P}}) + j \in L$. Then there exist $m, k^* \in \mathbb{N}_0$ such that

$$\begin{aligned} L &= m + \{2k^* + (r+\alpha-2)\lambda : \lambda \in [0, k^*]\} \\ &= m + k^*\mathsf{D}(G_{\mathcal{P}}) - \{(\mathsf{D}(G_{\mathcal{P}}) - 2)\nu : \nu \in [0, k^*]\}. \end{aligned}$$

Suppose $l\mathsf{D}(G_{\mathcal{P}}) + j = \max L - \nu(\mathsf{D}(G_{\mathcal{P}}) - 2) = m + k^*\mathsf{D}(G_{\mathcal{P}}) - \nu(\mathsf{D}(G_{\mathcal{P}}) - 2)$ for some $\nu \in [0, k^*]$. Then $j \equiv m + 2\nu \mod \mathsf{D}(G_{\mathcal{P}})$ and hence $m + 2\nu \ge j$. This implies

$$(k^* - \nu)\mathsf{D}(G_{\mathcal{P}}) + j \le m + k^*\mathsf{D}(G_{\mathcal{P}}) - \nu(\mathsf{D}(G_{\mathcal{P}}) - 2) = l\mathsf{D}(G_{\mathcal{P}}) + j,$$

and hence $l \ge k^* - \nu$. Therefore we obtain

$$\begin{aligned} \min L &= l\mathsf{D}(G_{\mathcal{P}}) + j - (k^* - \nu)(\mathsf{D}(G_{\mathcal{P}}) - 2) \\ &= (l - k^* + \nu)\mathsf{D}(G_{\mathcal{P}}) + j + 2(k^* - \nu) \ge 2l + j, \end{aligned}$$

and thus $\lambda_{l\mathsf{D}(G_p)+j}(G_{\mathcal{P}}) \ge 2l + j$.

Finally we consider the catenary degrees of H and prove the statements given in (4). Using Proposition 6.2 we infer

$$\mathsf{D}(G_{\mathcal{P}}) = r + \alpha = 2 + \max \Delta(G_{\mathcal{P}}) \le \mathsf{c}(G_{\mathcal{P}}) = \mathsf{c}(H) \le \omega(H) \le \mathsf{t}(H).$$

Since $\mathsf{c}(G_{\mathcal{P}}) \le \mathsf{c}_{\text{mon}}(G_{\mathcal{P}})$, it remains to show that $\mathsf{c}_{\text{mon}}(G_{\mathcal{P}}) \le \mathsf{D}(G_{\mathcal{P}})$ and that $\mathsf{t}(H) \le \mathsf{D}(G_{\mathcal{P}})$.

We proceed in two steps. First we verify that

$$\mathsf{c}_{\text{mon}}(G_{\mathcal{P}}) = \max\{\mathsf{c}_{\text{eq}}(G_{\mathcal{P}}), \quad \mathsf{c}_{\text{adj}}(G_{\mathcal{P}})\} \le r + \alpha.$$

Since

$$\mathcal{A}(\sim_{\mathcal{B}(G_{\mathcal{P}}),\text{eq}}) = \{(-V, -V), (V, V), (U_\nu, U_\nu), (-U_\nu, -U_\nu) : \nu \in [0, r]\},$$

it follows that $\mathsf{c}_{\text{eq}}(G_{\mathcal{P}}) = 0$. If $A_{r+\alpha-2} = \{x \in \mathsf{Z}(G_{\mathcal{P}}) : |x| - (r+\alpha-2) \in \mathsf{L}(\pi(x))\}$, then $\text{Min}(A_{r+\alpha-2}) = \{U_0^\alpha U_1 \cdot \ldots \cdot U_r\}$, and hence $\mathsf{c}_{\text{adj}}(G_{\mathcal{P}}) \le r + \alpha$ by Lemma 2.1.

In order to show that $\mathsf{t}(H) \le \mathsf{D}(G_{\mathcal{P}})$, we must verify the following assertion (see [Geroldinger and Hassler 2008, Theorem 3.6]).

(A) *Let $j \in \mathbb{N}$ and $w, w_1, \ldots, w_j \in \mathcal{A}(H)$ be such that w divides the product $w_1 \cdot \ldots \cdot w_j$ yet w divides no proper subproduct of $w_1 \cdot \ldots \cdot w_j$. Then*

$$\min \mathsf{L}(w^{-1} w_1 \cdot \ldots \cdot w_j) \le \mathsf{D}(G_{\mathcal{P}}) - 1.$$

Proof of (A). We use the transfer homomorphism $\boldsymbol{\beta} : H \to \mathcal{B}(G_{\mathcal{P}})$ as defined in Lemma 3.4. Set $W = \boldsymbol{\beta}(w)$ and $W_i = \boldsymbol{\beta}(w_i)$ for each $i \in [1, j]$. Then $j \le |W|$ and $W, W_1, \ldots, W_j \in \mathcal{A}(G_{\mathcal{P}})$. Clearly

$$\min \mathsf{L}(w^{-1} w_1 \cdot \ldots \cdot w_j) \le \max \mathsf{L}(W^{-1} W_1 \cdot \ldots \cdot W_j) \le \frac{|W_1 \cdot \ldots \cdot W_j| - |W|}{2}.$$

Thus, if $|W| = 2$, then

$$\min \mathsf{L}(w^{-1} w_1 \cdot \ldots \cdot w_j) \le \frac{|W_1| + |W_2| - |W|}{2} \le \mathsf{D}(G_{\mathcal{P}}) - 1.$$

It remains to consider the case $W \in \{-V, V\}$, and by symmetry we may suppose that $W = V$. If $|W_1| = \cdots = |W_j| = 2$, then $j = |V|$ and $w^{-1} w_1 \cdot \ldots \cdot w_j \in \mathcal{A}(H)$. Suppose there is $\nu \in [1, j]$, say $\nu = 1$, such that $\boldsymbol{\beta}(w_1) \in \{-V, V\}$. Since w does not divide a subproduct of $w_1 \cdot \ldots \cdot w_j$ and $\gcd(V, -V) = 1$, it follows that $\boldsymbol{\beta}(w_1) = V$. Then $\mathsf{L}(w^{-1} w_1 \cdot \ldots \cdot w_j) = \mathsf{L}(W^{-1} W_1 \cdot \ldots \cdot W_j) = \mathsf{L}(W_2 \cdot \ldots \cdot W_j)$ and hence

$$\min \mathsf{L}(w^{-1} w_1 \cdot \ldots \cdot w_j) \le j - 1 \le |V| - 1 = \mathsf{D}(G_{\mathcal{P}}) - 1. \qquad \square$$

The next corollary again reveals that certain arithmetical phenomena characterize certain algebraic properties of the class group.

Corollary 6.5. *Let H be a Krull monoid as in Theorem 6.4 with class group G and set $G_{\mathcal{P}}$ of classes containing prime divisors. Then $r+1$ is the minimum of all $s \in \mathbb{N}$ having the following property*:

(P) *There are absolutely irreducible elements $w_1, \ldots, w_s \in \mathcal{A}(H)$ such that $2, \mathsf{D}(G_{\mathcal{P}}) \in \mathsf{L}(w_1^{k_1} \cdot \ldots \cdot w_s^{k_s})$ for some $(k_1, \ldots, k_s) \in \mathbb{N}_0^s$.*

Proof. First we verify that $r+1$ satisfies property (P). For $i \in [0, s]$, let $p_i \in \mathcal{P} \cap e_i$ and $q_i \in \mathcal{P} \cap (-e_i)$ and set $w_i = p_i q_i$. Then $w_0, \ldots, w_s$ are absolutely irreducible elements and, by Theorem 6.4, it follows that $2, \mathsf{D}(G_{\mathcal{P}}) \in \mathsf{L}(w_0^{\alpha} w_1 \cdot \ldots \cdot w_r)$.

Conversely, let $s \in \mathbb{N}$, and let $w_1, \ldots, w_s$ and $k_1, \ldots, k_s$ be as above. For $i \in [1, s]$, we set $W_i = \boldsymbol{\beta}(w_i)$. Since $\rho(G_{\mathcal{P}}) = \mathsf{D}(G_{\mathcal{P}})/2$ and $2, \mathsf{D}(G_{\mathcal{P}}) \in \mathsf{L}(W_1^{k_1} \cdot \ldots \cdot W_s^{k_s})$, it follows that $\sum_{i=1}^{s} k_i |W_i| = \mathsf{D}(G_{\mathcal{P}})$, $|W_1| = \cdots = |W_s| = 2$ and $W_i = (-g_i) g_i$ for $i \in [1, s]$, and that $S = g_1^{k_1} \cdot \ldots \cdot g_s^{k_s} \in \mathcal{A}(G_{\mathcal{P}})$. Now Theorem 6.4 implies $S = (-e_0)^{\alpha} e_1 \cdot \ldots \cdot e_r$, whence $\{W_1, \ldots, W_s\} = \{(-e_0)e_0, \ldots, (-e_r)e_r\}$. Thus

$$|\{w_1, \ldots, w_s\}| \geq |\{W_1, \ldots, W_s\}| = r+1,$$

and so $r+1$ is minimal with Property (P). □

We now begin collecting information in order to study the arithmetic of the Krull monoid presented in Example 4.21. In spite of the simple geometric structure of $G_{\mathcal{P}}$ (the set consists of the vertices of the unit cube and their negatives), the arithmetic of this Krull monoid is highly complex. We get only very limited information. Nevertheless, this will be sufficient to give an arithmetical characterization.

Lemma 6.6. *Let G be an abelian group and let $(e_n)_{n \geq 1}$ be a family of independent elements each having infinite order. For $r \in \mathbb{N}$, set*

$$G_r^+ = \{a_1 e_1 + \cdots + a_r e_r : a_1, \ldots, a_r \in [0, 1]\},$$
$$G_r^- = -G_r^+, \quad G_r = G_r^+ \cup G_r^-.$$

(1) *Let $s \in [2, r]$, $f_0 = e_1 + \cdots + e_s$, and $f_i = f_0 - e_i$ for all $i \in [1, s]$. Then $(f_1, \ldots, f_s)$ is independent, $f_1 + \cdots + f_s = (s-1) f_0$, and*

$$\Delta(\{f_0, \ldots, f_s, -f_0, \ldots, -f_s\}) = \{2s-3\}.$$

(2) *Let $s \in [3, r]$, $f_0 = e_1 + \cdots + e_s$, $f_i = f_0 - e_i$ for each $i \in [1, s-1]$, and set $f_s' = -e_s$. Then $(f_1, \ldots, f_{s-1}, f_s')$ is independent,*

$$f_1 + \cdots + f_{s-1} + f_s' = (s-2) f_0,$$

and

$$\Delta(\{f_0, \ldots, f_{s-1}, f_s', -f_0, \ldots, -f_{s-1}, -f_s'\}) = \{2s-4\}.$$

(3) *If $s \leq [1, r-1]$, then $\mathsf{D}(G_r) \geq \mathsf{D}(G_s) + \mathsf{D}(G_{r-s}) - 1$. In particular, $\mathsf{D}(G_1) = 2$ and $\mathsf{D}(G_r) > \mathsf{D}(G_{r-1})$ for $r \geq 2$.*

Proof. Since $(e_1, \dots, e_s)$ is a basis, there is a matrix A_s with $(f_1, \dots, f_s) = (e_1, \dots, e_s)A_s$. Since $\det(A_s) \neq 0$, it follows that $(f_1, \dots, f_s)$ is independent. By definition, we have $f_1 + \cdots + f_s = (s-1)f_0$. The assertion on the set of distances then follows from Theorem 6.4 and we have proved (1).

We now consider (2). Note that $f_s' = f_s - f_0$. Using (1) we infer that

$$0 = (f_1 + \cdots + f_s) - (s-1)f_0 = (f_1 + \cdots + f_{s-1}) + (f_s - f_0) - (s-2)f_0,$$

and hence $f_1 + \cdots + f_{s-1} + f_s' = (s-2)f_0$. Since $(f_1, \dots, f_{s-1}, -e_s) = (e_1, \dots, e_s)B_s$ for some matrix B_s with $\det(B_s) = (-1)^{2s}\det(A_{s-1}) \neq 0$, it follows that $(f_1, \dots, f_{s-1}, f_s')$ is independent. The assertion on the set of distances follows from Theorem 6.4.

It is clear that $\mathsf{D}(G_1) = 2$ and that $\mathsf{D}(G_r) > \mathsf{D}(G_{r-1})$ whenever $r \geq 2$. To prove the remaining statements of (3), suppose that $s \in [1, r-1]$. After a change of notation, we may suppose that $G_{r-s} \subset \langle e_{s+1}, \dots, e_r \rangle$ such that $\langle G_r \rangle = \langle G_s \rangle \oplus \langle G_{r-s} \rangle$. If $U = a_1 \cdot \ldots \cdot a_k \in \mathcal{A}(G_s)$ with $k = \mathsf{D}(G_s)$ and $V = b_1 \cdot \ldots \cdot b_l \in \mathcal{A}(G_{r-s})$ with $l = \mathsf{D}(G_{r-s})$, then $W = (a_1 + b_1) \cdot a_2 \cdot \ldots \cdot a_k b_2 \cdot \ldots \cdot b_l \in \mathcal{A}(G_r)$, and hence $\mathsf{D}(G_r) \geq |W| = k + l - 1 = \mathsf{D}(G_s) + \mathsf{D}(G_{r-s}) - 1$. □

In Theorem 6.7 we restrict to class groups of rank $r \geq 3$ because when $r \leq 2$ we are in the setting of Theorem 6.4 where we have precise information about arithmetical invariants. For $r \in \mathbb{N}_0$, we denote by F_r the r-th *Fibonacci number*. That is, $\mathsf{F}_0 = 0$, $\mathsf{F}_1 = 1$, and $\mathsf{F}_r = \mathsf{F}_{r-1} + \mathsf{F}_{r-2}$ for all $r \geq 2$.

Theorem 6.7. *Let H be a Krull monoid with free abelian class group G of rank $r \geq 3$ and let $G_{\mathcal{P}} \subset G$ denote the set of classes containing prime divisors. Suppose that there is a basis $(e_1, \dots, e_r)$ of G such that $G_{\mathcal{P}}^{\bullet} = G_{\mathcal{P}}^{+} \cup G_{\mathcal{P}}^{-}$, where*

$$G_{\mathcal{P}}^{+} = \{\epsilon_1 e_1 + \cdots + \epsilon_r e_r : \epsilon_1, \dots, \epsilon_r \in [0,1]\} \quad \textit{and} \quad G_{\mathcal{P}}^{-} = -G_{\mathcal{P}}^{+}.$$

(1) $\mathsf{F}_{r+2} \leq \mathsf{D}(G_{\mathcal{P}})$.

(2) $\mathsf{c}(H) \leq \omega(H) = \mathsf{D}(G_{\mathcal{P}})$, $\rho(H) = \mathsf{D}(G_{\mathcal{P}})/2$, *and* $\rho_{2k}(H) = k\mathsf{D}(G_{\mathcal{P}})$ *for each* $k \in \mathbb{N}$.

(3) $[1, 2r-3] \subset \Delta^*(H) \subset \Delta(H) \subset [1, \mathsf{c}(H) - 2]$.

Proof. See [Baeth et al. 2014] for the proof of assertion (1). Assertion (2) follows from Proposition 6.2.

Note that for every $s \in [2, r]$ we have $2s - 3 \in \Delta^*(H)$ and, by Lemma 6.6, for all $s \in [3, r]$ we have $2s - 4 \in \Delta^*(H)$. This implies that the interval $[1, 2r-3]$ is contained in $\Delta^*(H)$, giving (3). □

The third class of Krull monoids studied in this subsection are Krull monoids with finite cyclic class group having prime divisors in each class. Thus Theorem 6.8 describes the arithmetic of the monoids constructed in Theorem 5.5. Holomorphy rings in global fields are Krull monoids with finite class group and prime divisors in all classes. For this reason this class of Krull monoids has received a great deal of attention.

Theorem 6.8. *Let H be a Krull monoid with finite cyclic class group G of order $|G| = n \geq 3$, and suppose that every class contains a prime divisor. Then*:

(1) $\mathsf{c}(H) = \omega(H) = \mathsf{D}(G) = n$ *and* $\Delta(H) = [1, n-2]$.

(2) *For every $k \in \mathbb{N}$ the set $\mathcal{U}_k(H)$ is a finite interval*, *whence*

$$\mathcal{U}_k(H) = [\lambda_k(H), \rho_k(H)].$$

Moreover, *for all $l \in \mathbb{N}_0$ with $ln + j \geq 1$*,

$$\rho_{2k+j}(H) = kn + j \quad \text{for } j \in [0, 1],$$
$$\lambda_{ln+j}(H) = \begin{cases} 2l + j & \text{for } j \in [0, 1], \\ 2l + 2 & \text{for } j \in [2, n-1]. \end{cases}$$

(3) $\max \Delta^*(H) = n - 2 \quad$ *and* $\quad \max(\Delta^*(H) \setminus \{n-2\}) = \lfloor \frac{n}{2} \rfloor - 1$.

Proof. The proof of (1) can be found in [Geroldinger and Halter-Koch 2006, Theorem 6.7.1] and the proof of (3) can be found in [Geroldinger and Halter-Koch 2006, Theorem 6.8.12]. For (2) see [Geroldinger 2009, Corollary 5.3.2]. □

Much recent research is devoted to the arithmetic of Krull monoids discussed in Theorem 6.8. We briefly address some open questions. Let H be as above and suppose that $n \geq 5$. The precise values of $\mathsf{t}(H)$ and of $\mathsf{c}_{\text{mon}}(H)$ are unknown. It is easy to check that $\mathsf{D}(G) = n < \mathsf{t}(H)$ (in contrast to what we have in Theorem 6.4). For recent results on lower and upper bounds of the tame degree, see [Gao et al. 2015]. We remark that there is a standing conjecture that the monotone catenary degree is that $n = \mathsf{c}(H) = \mathsf{c}_{\text{mon}}(H)$ (this coincides what we have in Theorem 6.4; see [Geroldinger and Yuan 2013]). For recent progress on $\Delta^*(H)$ we refer to [Plagne and Schmid 2013].

Having at least a partial description of the arithmetic of the three monoids described in Theorems 6.4, 6.7, and 6.8, we now work to show that except for in a small number of exceptions, these monoids have vastly different arithmetic. After some preliminary work this distinction is made clear in Corollary 6.10.

Lemma 6.9. *Let G be an abelian group with finite total rank and let $G_0 \subset G$ be a subset with $G_0 = -G_0$. Suppose that $\mathcal{L}(G_0) = \mathcal{L}(C_n)$ for some $n \geq 5$. Then there exists an absolutely irreducible element $U \in \mathcal{A}(G_0)$ with $|U| = \mathsf{D}(G_0)$.*

Proof. First observe that $\mathsf{D}(G_0) = \rho_2(G_0) = \rho_2(C_n) = \mathsf{D}(C_n) = n$ and, by [Geroldinger and Halter-Koch 2006, Theorem 3.4.2], $\mathcal{A}(G_0)$ is finite, say $\mathcal{A}(G_0) =$

$\{U_1, -U_1, \ldots, U_q, -U_q\}$. If $g \in C_n$ with $\mathrm{ord}(g) = n$, then for all $k \in \mathbb{N}$ we have

$$L_k = \{2k + \nu(n-2) : \nu \in [0,k]\} = \mathsf{L}\big(g^{nk}(-g)^{nk}\big) \in \mathcal{L}(C_n) = \mathcal{L}(G_0).$$

Since $\rho(L_k) = \rho(G_0) = \mathsf{D}(G_0)/2$, there exists, for every $k \in \mathbb{N}$, a tuple

$$(k_1, \ldots, k_q) \in \mathbb{N}_0^{(q)}$$

such that $k_1 + \cdots + k_q = k$ and

$$L_k = \mathsf{L}\big((-U_1)^{k_1} U_1^{k_1} \cdot \ldots \cdot (-U_q)^{k_q} U_q^{k_q}\big).$$

Therefore there exists $\lambda \in [1, q]$ such that $\mathsf{L}\big((-U_\lambda)^k U_\lambda^k\big) = L_k$ for every $k \in \mathbb{N}$. Set $U = U_\lambda$ and note that for every $V \in \mathcal{A}(G_0)$ with $V \mid (-U)^k U^k$ for some $k \in \mathbb{N}$, it follows that $|V| \in \{2, n\}$. After changing notation if necessary, we may suppose that there is no $V \in \mathcal{A}(G_0)$ such that $|V| = n$, $\mathrm{supp}(V) \subsetneq \mathrm{supp}(U)$, and $V \mid U^k(-U)^k$ for some $k \in \mathbb{N}$.

In order to show that U is absolutely irreducible, it remains to verify that the torsion-free rank of $\langle \mathrm{supp}(U) \rangle$ is $|\mathrm{supp}(U)| - 1$. Assume to the contrary that there exist $t \in [2, |\mathrm{supp}(U)| - 1]$ and $g_1, \ldots, g_t \in \mathrm{supp}(U)$ which are linearly dependent. Then there are $s \in [1, t]$, $m_1, \ldots, m_s \in \mathbb{N}$, and $m_{s+1}, \ldots, m_t \in -\mathbb{N}$ such that

$$m_1 g_1 + \cdots + m_s g_s + (-m_{s+1})(-g_{s+1}) + \cdots + (-m_t)(-g_t) = 0.$$

Then

$$V = g_1^{m_1} \cdot \ldots \cdot g_s^{m_s} (-g_{s+1})^{-m_{s+1}} \cdot \ldots \cdot (-g_t)^{-m_t} \in \mathcal{B}(G_0).$$

Without restriction we may suppose that the above equation is minimal and that $V \in \mathcal{A}(G_0)$. Since $V \mid U^k(-U)^k$ for some $k \in \mathbb{N}$ and $|V| > 2$, we obtain a contradiction to the minimality of $\mathrm{supp}(U)$. □

The following corollary highlights that the observed arithmetical phenomena in our case studies — Theorems 6.4, 6.7, and 6.8 — are characteristic for the respective Krull monoids. In particular, this illustrates that the structure of direct-sum decompositions over the one-dimensional Noetherian local rings with finite representation type studied in Section 4 can be quite different from the structure of direct-sum decompositions over the two-dimensional Noetherian local Krull domains with finite representation type studied in Section 5. As characterizing tools we use the system of sets of lengths along with the behavior of absolutely irreducible elements.

Corollary 6.10. *For $i \in [1, 3]$, let H_i and H_i' be Krull monoids with class groups G_i and G_i'. Further suppose that*

- *G_1 and G_1' are finitely generated and torsion-free of rank r_1 and r_1' with sets of classes containing prime divisors as in Theorem 6.4 (with parameters $\alpha, \alpha' \in \mathbb{N}$ such that $\alpha + r_1 \geq \alpha' + r_1' > 2$).*

- *G_2 and G_2' are finitely generated and torsion-free of rank $r_2 \geq r_2' \geq 3$ with sets of classes containing prime divisors as in Theorem 6.7.*
- *G_3 and G_3' are finite cyclic of order $|G_3| \geq |G_3'| \geq 5$ such that every class contains a prime divisor.*

Then:

(1) *$\mathcal{L}(H_1) = \mathcal{L}(H_1')$ if and only if $r_1 + \alpha = r_1' + \alpha'$. If this holds, then the arithmetic behavior of the absolutely irreducible elements of H_1 and H_1' coincide in the sense of Corollary 6.5 if and only if $r_1 = r_1'$.*

(2) *$\mathcal{L}(H_2) = \mathcal{L}(H_2')$ if and only if $r_2 = r_2'$.*

(3) *$\mathcal{L}(H_3) = \mathcal{L}(H_3')$ if and only if $|G_3| = |G_3'|$.*

(4) *$\mathcal{L}(H_1) \neq \mathcal{L}(H_2)$ and $\mathcal{L}(H_1) \neq \mathcal{L}(H_3)$.*

(5) *For $i \in [2, 3]$, let s_i denote the maximal number of absolutely irreducible elements $u_1, \ldots, u_{s_i} \in H_i$ such that $2 \in \mathsf{L}(u_1 \cdot \ldots \cdot u_{s_i})$. Then either $\mathcal{L}(H_2) \neq \mathcal{L}(H_3)$ or $s_2 \neq s_3$.*

Proof. The if and only if statement in (1) follows immediately from Theorem 6.4. Suppose that $\mathcal{L}(H_1) = \mathcal{L}(H_1')$. Then the assertion in (1) on the arithmetic behavior of absolutely irreducible elements follows from Corollary 6.5.

To prove (2), first note that one implication is clear, both for H_2 and H_3. Suppose that $\mathcal{L}(H_2) = \mathcal{L}(H_2')$, and let $G_{\mathcal{P}} \subset G_2$ and $G_{\mathcal{P}}' \subset G_2'$ denote the set of classes containing prime divisors. Theorem 6.7 implies that

$$\mathsf{D}(G_{\mathcal{P}}) = \rho_2(H) = \rho_2(H') = \mathsf{D}(G_{\mathcal{P}}'),$$

and thus Lemma 6.6 implies $r_2 = r_2'$. Now consider (3). If $\mathcal{L}(H_3) = \mathcal{L}(H_3')$, then Theorem 6.8 implies that

$$|G_3| - 2 = \max \Delta(H_3) = \max \Delta(H_3') = |G_3'| - 2.$$

For (4), note that $\mathcal{L}(H_1)$ is distinct from both $\mathcal{L}(H_2)$ and $\mathcal{L}(H_3)$ since

$$|\Delta(H_1)| = 1, \quad |\Delta(H_2)| > 1, \quad |\Delta(H_3)| > 1.$$

For (5) we assume that $\mathcal{L}(H_2) = \mathcal{L}(H_3)$ and let $G_{\mathcal{P}} \subset G_2$ denote the set of classes containing prime divisors. Theorems 6.7 and 6.8 imply that

$$\mathsf{D}(G_{\mathcal{P}}) = \rho_2(H_2) = \rho_2(H_3) = |G_3|.$$

By Proposition 6.3 we obtain that $\mathsf{D}(G_{\mathcal{P}}) = s_2$. Now assume to the contrary that $s_2 = s_3$. If $|G_3| = n$, then there are absolutely irreducible elements $u_1, \ldots, u_n$ and atoms $v_1, v_2 \in \mathcal{A}(H_3)$ such that $v_1 v_2 = u_1 \cdot \ldots \cdot u_n$. Without restriction, we suppose H_3 is reduced and we consider a divisor theory $H \hookrightarrow \mathcal{F}(\mathcal{P})$. Since a minimal zero-sum sequence of length n over G_3 consists of one element of order n repeated

n times, the factorization of the atoms $v_1, v_2, u_1, \dots, u_n$ in $\mathcal{F}(\mathcal{P})$ must have the following form: $v_1 = p_1 \cdot \ldots \cdot p_n$, $v_2 = q_1 \cdot \ldots \cdot q_n$, and $u_i = p_i q_i$ for all $i \in [1, n]$, where $p_1, \dots, p_n, q_1, \dots, q_n \in \mathcal{P}$, $[p_1] = \dots = [p_n] \in G_3$, and $[q_1] = \dots = [q_n] = [-p_1]$. But [Geroldinger and Halter-Koch 2006, Proposition 7.1.5] implies that the elements $u_1, \dots, u_n$ are not absolutely irreducible, a contradiction. □

Remark 6.11. Let H_2 and H_3 be as in Corollary 6.10. We set $n = |G_3|$, $r = r_2$, and let $G_{\mathcal{P},r} \subset G_2$ denote the set of classes containing prime divisors. Assume that $\mathcal{L}(H_2) = \mathcal{L}(H_3)$. Then

$$\mathsf{F}_{r+2} \le \mathsf{D}(G_{\mathcal{P},r}) = \rho_2(H_2) = \rho_2(H_3) = n.$$

That is, the orders of the cyclic groups for which $\mathcal{L}(H_2) = \mathcal{L}(H_3)$ grow faster than the sequence of Fibonacci numbers. We conjecture that $\mathcal{L}(H_2)$ and $\mathcal{L}(H_3)$ are always distinct but have not further investigated this (rather delicate combinatorial) problem which would require a more detailed investigation of $\mathsf{D}(G_{\mathcal{P},r})$.

Now suppose that H is a Krull monoid with class group G such that every class contains a prime divisor. If $\mathcal{L}(H) = \mathcal{L}(H_3)$, then following Theorem 6.8, one can show that G is isomorphic to the finite cyclic group G_3 (see [Geroldinger 2009, Corollary 5.3.3]). Therefore sets of lengths characterize Krull monoids with finite cyclic class group having the property that every class contains a prime divisor.

6C. ***Small sets $G_{\mathcal{P}}$ of classes containing prime divisors and limits of arithmetical characterizations.*** In this final subsection we study the arithmetic of Krull monoids having small sets of classes containing prime divisors. This study pertains to the monoids of Theorem 4.12, Example 4.19, Example 4.20, and Theorem 5.5. The most striking phenomenon here is that these systems of sets of lengths are additively closed (see Proposition 6.14). As a consequence, if $\mathcal{L}(H)$ is such a system and H' is a monoid with $\mathcal{L}(H') \subset \mathcal{L}(H)$, then $\mathcal{L}(H \times H') = \mathcal{L}(H)$ (see Example 4.23, Example 4.24, and Corollary 6.15). These phenomena are in strong contrast to the results in the previous subsection, and they show up natural limits for obtaining arithmetical characterization results. Recall that, for $l \in \mathbb{N}_0$ and $d \in \mathbb{N}$, $P_l(d) = \{0, d, \dots, ld\}$.

Proposition 6.12. *Let H be a Krull monoid with infinite cyclic class group G and suppose that*

$$G_{\mathcal{P}} = \{-2e, -e, 0, e, 2e\} \subset G = \langle e \rangle$$

is the set of classes containing prime divisors. Then there is a transfer homomorphism $\theta : H \to \mathcal{B}(C_3)$, and hence

$$\mathcal{L}(H) = \mathcal{L}(C_3) = \mathcal{L}(C_2 \oplus C_2) = \{y + 2k + P_k(1) : y,\ k \in \mathbb{N}_0\}.$$

Moreover, $\mathcal{L}(H)$ *coincides with the system of sets of lengths of the Krull monoid studied in Theorem 6.4 with parameters* $r = 2$ *and* $\alpha = 1$.

Proof. By Lemma 3.4 there is a transfer homomorphism $\boldsymbol{\beta} : H \to \mathcal{B}(G_{\mathcal{P}})$. Since the composition of two transfer homomorphisms is a transfer homomorphism, it is sufficient to show that there is a transfer homomorphism $\theta' : \mathcal{B}(G_{\mathcal{P}}) \to \mathcal{B}(C_3)$. Write $C_3 = \{0, g, -g\}$. Since

$$\mathcal{B}(G_{\mathcal{P}}) = \mathcal{F}(\{0\}) \times \mathcal{B}(G_{\mathcal{P}}^{\bullet}) \quad \text{and} \quad \mathcal{B}(C_3) = \mathcal{F}(\{0\}) \times \mathcal{B}(\{-g, g\}),$$

it suffices to show that there is a transfer homomorphism $\theta : \mathcal{B}(G_{\mathcal{P}}^{\bullet}) \to \mathcal{B}(\{-g, g\})$. In this case, $\mathcal{L}(H) = \mathcal{L}(G_{\mathcal{P}}) = \mathcal{L}(C_3)$. Moreover, $\mathcal{L}(C_3) = \mathcal{L}(C_2 \oplus C_2)$ has the form given in [Geroldinger and Halter-Koch 2006, Theorem 7.3.2] and this coincides with the system of sets of lengths in Theorem 6.4, provided $(r, \alpha) = (2, 1)$.

Note that $\mathcal{A}(G_{\mathcal{P}}^{\bullet}) = \{V, -V, U_1, U_2\}$, where $V = e^2(-2e)$, $U_1 = (-e)e$, and $U_2 = (-2e)(2e)$, and $\mathcal{A}(\{-g, g\}) = \{\overline{V}, -\overline{V}, \overline{U}\}$, where $\overline{V} = g^3$ and $\overline{U} = (-g)g$. Then there is a monoid epimorphism

$$\widetilde{\theta} : \mathcal{F}(G_{\mathcal{P}}^{\bullet}) \to \mathcal{F}(\{-g, g\})$$

satisfying $\widetilde{\theta}(e) = \widetilde{\theta}(-2e) = g$ and $\widetilde{\theta}(-e) = \widetilde{\theta}(2e) = -g$. If

$$A = e^{k_1}(-e)^{k_1'}(2e)^{k_2}(-2e)^{k_2'} \in \mathcal{F}(G_{\mathcal{P}}^{\bullet}) \quad \text{with} \quad k_1, k_1', k_2, k_2' \in \mathbb{N}_0,$$

then $A \in \mathcal{B}(G_{\mathcal{P}}^{\bullet})$ if and only if $k_1 - k_1' + 2(k_2 - k_2') = 0$. If this holds, then $k_1 + k_2' - (k_1' + k_2) \equiv 0 \mod 3$ and hence

$$\widetilde{\theta}(A) = g^{k_1 + k_2'}(-g)^{k_1' + k_2} \in \mathcal{B}(\{-g, g\}).$$

Thus $\theta = \widetilde{\theta} \mid_{\mathcal{B}(G_{\mathcal{P}}^{\bullet})} : \mathcal{B}(G_{\mathcal{P}}^{\bullet}) \to \mathcal{B}(\{-g, g\})$ is a monoid epimorphism satisfying $\theta(V) = \overline{V}$, $\theta(-V) = -\overline{V}$, $\theta(U_1) = \theta(U_2) = \overline{U}$ and $\theta^{-1}(1) = \{1\} = \mathcal{B}(G_{\mathcal{P}}^{\bullet})^{\times}$.

Thus in order to show that θ is a transfer homomorphism, it remains to verify Property (T2). Let $A \in \mathcal{B}(G_{\mathcal{P}}^{\bullet})$ be as above and suppose that

$$\theta(A) = \widetilde{B}\widetilde{C}$$

with $\widetilde{B}, \widetilde{C} \in \mathcal{B}(\{-g, g\})$ and $\widetilde{B} = g^m(-g)^{m'}$ such that $m \in [0, k_1 + k_2']$, $m' \in [0, k_1' + k_2]$ and $m \equiv m' \mod 3$. Our goal is to find $B, C \in \mathcal{B}(G_{\mathcal{P}}^{\bullet})$ such that $A = BC$, $\theta(B) = \widetilde{B}$, and $\theta(C) = \widetilde{C}$. Clearly it is sufficient to find $B \in \mathcal{B}(G_{\mathcal{P}}^{\bullet})$ with $B \mid A$ and $\theta(B) = \widetilde{B}$, that is, to find parameters

$$m_1 \in [0, k_1], \quad m_1' \in [0, k_1'], \quad m_2 \in [0, k_2], \quad m_2' \in [0, k_2'],$$

such that

(C1) $\qquad m_1 + m_2' = m, \quad m_1' + m_2 = m', \quad m_1 - m_1' + 2(m_2 - m_2') = 0.$

To do so we proceed by induction on $|\widetilde{B}|$. If $|\widetilde{B}| = |A|$, then

$$k_1 + k_1' + k_2 + k_2' = |A| = |\widetilde{B}| = m + m',$$

and hence $m = k_1 + k_1'$ and $m' = k_1' + k_2$. Thus we set

$$m_1 = k_1, \quad m_1' = k_1', \quad m_2 = k_2, \quad m_2' = k_2',$$

and the assertion is satisfied with $B = A$. Suppose now that the quadruple (m_1, m_1', m_2, m_2') satisfies (C1) with respect to the pair (m, m'). Dividing $\widetilde{B}$ by an atom of $\mathcal{B}(\{-g, g\})$ (if possible) shows that we must verify that there are solutions to (C1) with respect to each of the pairs $(m-1, m'-1)$, $(m-3, m')$, and $(m, m'-3)$ in $N_0^{(2)}$. One checks respectively that at least one of the following quadruples satisfy (C1).

- $(m_1 - 1, m_1' - 1, m_2, m_2')$ or $(m_1, m_1', m_2 - 1, m_2' - 1)$;
- $(m_1 - 2, m_1', m_2, m_2' - 1)$ or $(m_1 - 3, m_1' - 1, m_2 + 1, m_2')$;
- $(m_1, m_1' - 2, m_2 - 1, m_2')$ or $(m_1 - 1, m_1' - 3, m_2, m_2' + 1)$.

Now the assertion follows by the induction hypothesis. □

Proposition 6.13. *Let H be a Krull monoid with free abelian class group G of rank 2. Let (e_1, e_2) be a basis of G and suppose that*

$$G_{\mathcal{P}} = \{0, e_1, e_2, 2e_2, e_1 + 2e_2, -e_1, -e_2, -2e_2, -e_1 - 2e_2\}$$

is the set of classes containing prime divisors. Then there is a transfer homomorphism $\theta : H \to \mathcal{B}(C_4)$ and hence

$$\begin{aligned}\mathcal{L}(H) &= \mathcal{L}(C_4)\\ &= \bigl\{y + k + 1 + P_k(1) : y, k \in \mathbb{N}_0\bigr\} \cup \bigl\{y + 2k + P_k(2) : y, k \in \mathbb{N}_0\bigr\}\\ &\supset \mathcal{L}(C_3).\end{aligned}$$

Proof. As in Proposition 6.12, it suffices to show that there is a transfer homomorphism $\theta : \mathcal{B}(G_{\mathcal{P}}^\bullet) \to \mathcal{B}(C_4^\bullet)$. Then $\mathcal{L}(H) = \mathcal{L}(G_{\mathcal{P}}) = \mathcal{L}(C_4)$ and $\mathcal{L}(C_4)$ has the form given in [Geroldinger and Halter-Koch 2006, Theorem 7.3.2]. Proposition 6.12 shows that $\mathcal{L}(C_3) \subset \mathcal{L}(C_4)$.

We note that $\mathcal{A}(G_{\mathcal{P}}^\bullet) = \{W, -W, V_1, -V_1, V_2, -V_2, U_1, U_2, U_3, U_4\}$, where

$$\begin{aligned} W &= e_1 e_2 e_2 (-e_1 - 2e_2), & U_3 &= (-e_1 - 2e_2)(e_1 + 2e_2),\\ U_1 &= (-e_1)e_1, & V_1 &= e_1(2e_2)(-e_1 - 2e_2),\\ U_2 &= (-e_2)e_2, & V_2 &= e_2 e_2(-2e_2),\end{aligned}$$

$$U_4 = (-2e_2)(2e_2).$$

We set $C_4 = \{0, g, 2g, -g\}$ and observe that $\mathcal{A}(C_4^\bullet) = \{\overline{W}, -\overline{W}, \overline{V}, -\overline{V}, \overline{U}_1, \overline{U}_2\}$, where

$$\overline{W} = g^4, \quad \overline{V} = g^2(2g), \quad \overline{U}_1 = (-g)g \quad \text{and} \quad \overline{U}_2 = (2g)(2g).$$

There is a monoid epimorphism $\widetilde{\theta} : \mathcal{F}(G_{\mathcal{P}}^\bullet) \to \mathcal{F}(C_4^\bullet)$ satisfying

$$\begin{aligned} \widetilde{\theta}(e_1) = \widetilde{\theta}(e_2) = \widetilde{\theta}(-e_1 - 2e_2) &= g, \\ \widetilde{\theta}(-e_1) = \widetilde{\theta}(-e_2) = \widetilde{\theta}(e_1 + 2e_2) &= -g, \\ \widetilde{\theta}(2e_2) = \widetilde{\theta}(-2e_2) &= 2g. \end{aligned}$$

If

$$A = e_1^{k_1}(-e_1)^{k_1'} e_2^{k_2}(-e_2)^{k_2'}(2e_2)^{k_3}(-2e_2)^{k_3'}(e_1+2e_2)^{k_4}(-e_1-2e_2)^{k_4'} \in \mathcal{F}(G_{\mathcal{P}}^\bullet),$$

with $k_1, k_1', \ldots, k_4, k_4' \in \mathbb{N}_0$, then $A \in \mathcal{B}(G_{\mathcal{P}}^\bullet)$ if and only if

$$k_1 - k_1' + k_4 - k_4' = 0 \quad \text{and} \quad k_2 - k_2' + 2k_3 - 2k_3' + 2k_4 - 2k_4' = 0.$$

If this holds, then

$$k_1 - k_1' + k_2 - k_2' - (k_4 - k_4') + 2k_3 + 2k_3' \equiv 0 \mod 4,$$

and hence

$$\widetilde{\theta}(A) = g^{k_1+k_2+k_4'}(-g)^{k_1'+k_2'+k_4}(2g)^{k_3+k_3'} \in \mathcal{B}(C_4^\bullet).$$

Thus $\theta = \widetilde{\theta}\,|_{\mathcal{B}(G_{\mathcal{P}}^\bullet)} : \mathcal{B}(G_{\mathcal{P}}^\bullet) \to \mathcal{B}(C_4^\bullet)$ is a monoid epimorphism satisfying

$$\begin{aligned} \theta(W) &= \overline{W}, & \theta(-V_1) &= \theta(-V_2) = -\overline{V}, \\ \theta(-W) &= -\overline{W}, & \theta(U_1) &= \theta(U_2) = \theta(U_3) = \overline{U}_1, \\ \theta(V_1) &= \theta(V_2) = \overline{V}, & \theta(U_4) &= \overline{U}_2, \end{aligned}$$
$$\theta^{-1}(1) = \{1\} = \mathcal{B}(G_{\mathcal{P}}^\bullet)^\times.$$

Thus in order to show that θ is a transfer homomorphism, it remains to verify Property (T2). Let $A \in \mathcal{B}(G_{\mathcal{P}}^\bullet)$ be as above and suppose that

$$\theta(A) = \widetilde{B}\widetilde{C}$$

with $\widetilde{B}, \widetilde{C} \in \mathcal{B}(C_4^\bullet)$ and $\widetilde{B} = g^m(-g)^{m'}(2g)^{m''}$ such that

$$m \in [0, k_1 + k_2 + k_4'], \quad m' \in [0, k_1' + k_2' + k_4], \quad m'' \in [0, k_3 + k_3'],$$

and

$$m - m' + 2m'' \equiv 0 \mod 4.$$

Our goal is to find $B, C \in \mathcal{B}(G_{\mathcal{P}}^{\bullet})$ such that $A = BC$, $\theta(B) = \widetilde{B}$, and $\theta(C) = \widetilde{C}$. It will suffice to find $B \in \mathcal{B}(G_{\mathcal{P}}^{\bullet})$ with $B \mid A$ and $\theta(B) = \widetilde{B}$. Thus we must find parameters

$$m_\nu \in [0, k_\nu] \quad \text{and} \quad m'_\nu \in [0, k'_\nu] \qquad \text{for } \nu \in [1, 4],$$

such that

$$\text{(C2)} \qquad \begin{aligned} m_1 + m_2 + m'_4 = m, \quad m'_1 + m'_2 + m_4 = m', \quad m_3 + m'_3 = m'', \\ m_1 - m'_1 + m_4 - m'_4 = 0, \quad m_2 - m'_2 + 2m_3 - 2m'_3 + 2m_4 - 2m'_4 = 0. \end{aligned}$$

We proceed by induction on $|\widetilde{B}| = m + m' + m''$. If $|\widetilde{B}| = |A|$, then we set $m_\nu = k_\nu$ and $m'_\nu = k'_\nu$ for all $\nu \in [1, 4]$, and the assertion is satisfied with $B = A$. Suppose now that the octuplet $(m_1, m'_1, \ldots, m_4, m'_4)$ satisfies (C2) with respect to the triple (m, m', m''). Dividing $\widetilde{B}$ by an element of $\mathcal{A}(C_4^{\bullet})$ (if possible) shows that we must verify that there are solutions to (C2) with respect to each of the triples

$$(m-1, m'-1, m''), \quad (m-2, m', m''-1), \quad (m, m'-2, m''-1),$$
$$(m, m', m''-2), \quad (m-4, m', m''), \quad (m, m'-4, m''),$$

provided that they lie in $\mathbb{N}_0^{(8)}$. As in proof of the previous proposition, one finds the required solutions and hence the assertion follows by the induction hypothesis. □

Let $\mathcal{L}$ be a family of subsets of $\mathbb{Z}$. We say that $\mathcal{L}$ is *additively closed* if the sumset $L + L' \in \mathcal{L}$ for all $L, L' \in \mathcal{L}$.

Proposition 6.14. *Let G be a finite cyclic group. Then $\mathcal{L}(G)$ is additively closed if and only if $|G| \le 4$.*

Proof. We suppose that $|G| = n$ and distinguish four cases.

First assume that $n \le 2$. Since $\mathcal{B}(G)$ is factorial, it follows that

$$\mathcal{L}(G) = \{\{m\} : m \in \mathbb{N}_0\},$$

which is obviously additively closed.

Next assume that $n = 3$. By Proposition 6.12 we have

$$\mathcal{L}(C_3) = \{y + 2k + P_k(1) : y, k \in \mathbb{N}_0\}.$$

If $y_1, y_2, k_1, k_2 \in \mathbb{N}_0$. Then

$$\begin{aligned}(y_1 + 2k_1 + P_{k_1}(1)) + (y_2 + 2k_2 + P_{k_2}(1)) \\ = (y_1 + y_2) + 2(k_1 + k_2) + P_{k_1+k_2}(1) \in \mathcal{L}(C_3),\end{aligned}$$

and hence $\mathcal{L}(C_3)$ is additively closed.

Now assume that $n = 4$. By Proposition 6.13 we have

$$\mathcal{L}(C_4) = \{y + k + 1 + P_k(1) : y,\ k \in \mathbb{N}_0\} \cup \{y + 2k + P_k(2) : y, k \in \mathbb{N}_0\}.$$

Clearly, the sumset of two sets of the first form is of the first form again, and the sumset of two sets of the second form again the second form. Thus it remains to consider the sumset $L_1 + L_2$ where L_1 has the first form, L_2 has the second form, and both L_1 and L_2 have more than one element. If $y_1, y_2 \in \mathbb{N}_0$ and $k_1, k_2 \in \mathbb{N}$, then

$$(y_1 + k_1 + 1 + P_{k_1}(1)) + (y_2 + 2k_2 + P_{k_2}(2)) \\ = (y_1 + y_2) + (k_1 + 2k_2) + 1 + P_{k_1+2k_2}(1) \in \mathcal{L}(C_4).$$

Finally, assume that $n \geq 5$ and assume to the contrary that $\mathcal{L}(G)$ is additively closed. Let $d \in [1, n-2]$. Then $\{2, d+2\} \in \mathcal{L}(G)$ by [Geroldinger and Halter-Koch 2006, Theorem 6.6.2], and hence the k-fold sumset

$$\{2, d+2\} + \cdots + \{2, d+2\} = 2k + P_k(d)$$

lies in $\mathcal{L}(G)$ for all $k \in \mathbb{N}$. Then [Geroldinger and Halter-Koch 2006, Corollary 4.3.16] implies that $n - 3$ divides some $d \in \Delta^*(G)$. By Theorem 6.8 we have

$$\max \Delta^*(G) = n - 2 \quad \text{and} \quad \max(\Delta^*(G) \setminus \{n-2\}) = \left\lfloor \frac{n}{2} \right\rfloor - 1,$$

a contradiction to $n \geq 5$. □

Corollary 6.15. (1) *Let H be an atomic monoid such that $\mathcal{L}(H)$ is additively closed, and let H' be an atomic monoid with $\mathcal{L}(H') \subset \mathcal{L}(H)$. Then*

$$\mathcal{L}(H \times H') = \mathcal{L}(H).$$

(2) *Let H be an atomic monoid with $\mathcal{L}(H) = \mathcal{L}(C_n)$ for $n \in [3, 4]$. For $k \in \mathbb{N}$ and $i \in [1, k]$, let H_i be an atomic monoid with $\mathcal{L}(H_i) \subset \mathcal{L}(C_n)$. Then $\mathcal{L}(H \times H_1 \times \cdots \times H_k) = \mathcal{L}(C_n)$.*

Proof. Since $\mathcal{L}(H \times H') = \{L + L' : L \in \mathcal{L}(H), L' \in \mathcal{L}(H')\}$, (1) follows.

For (2), we set $H' = H_1 \times \cdots \times H_k$. Since $\mathcal{L}(C_n)$ is additively closed by Proposition 6.14, it follows that

$$\mathcal{L}(H') = \{L_1 + \cdots + L_k : L_i \in \mathcal{L}(H_i), i \in [1, k]\} \subset \mathcal{L}(C_n).$$

Finally (1) implies that $\mathcal{L}(H \times H') = \mathcal{L}(H)$. □

We conclude this manuscript by suggesting a rich program for further study. Any progress in these directions will lead to a better understanding of direct-sum decompositions of classes of modules where each module has a semilocal endomorphism ring. Moreover, this program could stimulate new studies in combinatorial factorization theory where much of the focus has been on Krull monoids having finite class group.

Program for further study

A. Module-theoretic aspect. Let R be a ring and let $\mathcal{C}$ be a class of right R-modules which is closed under finite direct sums, direct summands, and isomorphisms, and such that the endomorphism ring $\mathrm{End}_R(M)$ is semilocal for each module M in $\mathcal{C}$ (such classes of modules are presented in a systematic way in [Facchini 2004]). Then $\mathcal{V}(\mathcal{C})$, the monoid of isomorphism classes of modules in $\mathcal{C}$ is a reduced Krull monoid with class group G and set $G_{\mathcal{P}} \subset G$ of classes containing prime divisors.

Since the long-term goal — to determine the characteristic of $\mathcal{V}(\mathcal{C})$ — is out of reach in most cases, the focus of study should be on those properties of $G_{\mathcal{P}}$ which have most crucial influence on the arithmetic of direct-sum decompositions. In particular,

- Is $G_{\mathcal{P}}$ finite or infinite?
- Is $G_{\mathcal{P}}$ well-structured in the sense of Proposition 6.2?

B. Arithmetical aspect of direct-sum decompositions. Let H be a Krull monoid with finitely generated class group G and let $G_{\mathcal{P}} \subset G$ denote the set of classes containing prime divisors.

1. *Finiteness conditions.*

   (a) Characterize the finiteness of arithmetical invariants (introduced in Section 2) and the validity of structural finiteness results (as given in Proposition 6.2, items (2a) and (2b)).

   For infinite cyclic groups much work in this direction can be found done in [Geroldinger et al. 2010].

   (b) If $G_{\mathcal{P}}$ contains an infinite group, then every finite subset $L \subset \mathbb{N}_{\geq 2}$ occurs as a set of lengths in H (see Proposition 6.2) and hence $\Delta(H) = \mathbb{N}$, and $\mathcal{U}_k(H) = \mathbb{N}_{\geq 2}$ for all $k \geq 2$. Weaken the assumption on $G_{\mathcal{P}}$ to obtain similar results.

   A weak condition on $G_{\mathcal{P}}$ enforcing that $\Delta(H) = \mathbb{N}$ can be found in [Hassler 2002].

2. *Upper bounds and precise formulas.* Suppose that G is torsion-free, say $G_{\mathcal{P}} \subset G = \mathbb{Z}^{(q)} \subset (\mathbb{R}^{(q)}, |\cdot|)$, where $|\cdot| : \mathbb{R}^{(q)} \to \mathbb{R}_{\geq 0}$ is a Euclidean norm.

   (a) If $G_{\mathcal{P}} \subset \{x \in \mathbb{R} : |x| \leq M\}$ for some $M \in \mathbb{N}$, then derive upper bounds for the arithmetical invariants in terms of M.

   (b) If $G_{\mathcal{P}}$ has a simple geometric structure (e.g., the set of vertices in a cube; see Examples 4.21 and 4.22), derive precise formulas for the arithmetical invariants, starting with the Davenport constant.

   A first result in this direction can be found in [Baeth et al. 2014].

(c) Determine the extent to which the arithmetic of a Krull monoid with $G_{\mathcal{P}}$ as in (b) is characteristic for $G_{\mathcal{P}}$. In particular, determine how this compares with the arithmetic of a Krull monoid H' where $G'_{\mathcal{P}}$ has the same geometric structure as $G_{\mathcal{P}}$ with different parameters and how this compares with the arithmetic of a Krull monoid having finite class group and prime divisors in all classes.

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