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**NONCONCORDANT LINKS WITH HOMOLOGY COBORDANT
ZERO-FRAMED SURGERY MANIFOLDS**

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We use topological surgery theory to give sufficient conditions for the zero-framed surgery manifold of a 3-component link to be homology cobordant to the zero-framed surgery on the Borromean rings (also known as the 3-torus) via a topological homology cobordism preserving the free homotopy classes of the meridians.

This enables us to give examples of 3-component links with unknotted components and vanishing pairwise linking numbers, such that any two of these links have homology cobordant zero-surgeries in the above sense, but the zero-surgery manifolds are not homeomorphic. Moreover, the links are not concordant to one another, and in fact they can be chosen to be height h but not height $h + 1$ symmetric grope concordant, for each h which is at least three.

1. Introduction

It is well known that the study of homology cobordism of 3-manifolds is essential for understanding the concordance of knots and links: homology cobordism of the *exteriors* of links in S^3 is equivalent to concordance in a homology $S^3 \times I$, and an additional mild normal generation condition for π_1 is equivalent to topological concordance in $S^3 \times I$ (this also holds modulo the 4-dimensional Poincaré conjecture in the smooth case).

We recall the definitions: two m -component links L_0 and L_1 in S^3 are said to be *topologically* (respectively *smoothly*) *concordant* if there exist m locally flat (respectively smoothly embedded) disjoint annuli in $S^3 \times [0, 1]$ cobounded by components of $L_0 \times \{0\}$ and $-L_1 \times \{1\}$. Two 3-manifolds M_0 and M_1 bordered by a 2-manifold Σ , that is, endowed with a marking $\mu_i : \Sigma \xrightarrow{\cong} \partial M_i$, are *topologically* (respectively *smoothly*) *homology cobordant* if there is a topological (respectively smooth) 4-manifold W with

$$\partial W = M_0 \sqcup -M_1 \sqcup \Sigma \times [0, 1] / (\mu_0(x) \sim x \times \{0\}, \mu_1(x) \sim x \times \{1\}, x \in \Sigma),$$

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such that the inclusions $M_i \rightarrow W$ ($i = 0, 1$) induce isomorphisms on integral homology groups. In this paper links are oriented, and link exteriors are always bordered by $\bigsqcup_m S^1 \times S^1$ under the zero framing.

In high dimensions, concordance classification results were obtained by studying homology surgery, with the aim of surgeries being to produce a homology cobordism of the exteriors (for example, see [Cappell and Shaneson 1974; 1980; Le Dimet 1988]). On the other hand, for knots and links in dimension three, the *zero-surgery manifolds* and their 4-dimensional homology cobordisms have been extensively used in the literature in order to understand the structure peculiar to low dimensions, especially in the topological category. Recall that performing zero-framed surgery on a link in S^3 yields a closed 3-manifold, called the zero-surgery manifold.

The classical invariants such as the knot signature and Levine’s algebraic knot concordance class [Levine 1969a; 1969b] are obtained from the zero-surgery manifold of a knot, via the Blanchfield form. Also, higher-order knot concordance obstructions, such as Casson–Gordon invariants [Casson and Gordon 1978; 1986], and Cochran–Orr–Teichner L^2 -signatures [Cochran et al. 2003] are obtained from the zero-surgery manifold (often together with the homology class of the meridian).

A natural interesting question is whether the homology cobordism class of a zero-surgery manifold determines the concordance class of a knot or link or if it determines the homology cobordism class of the exterior.

In this paper we show, in a strong sense involving homotopy of meridians, that the answer is negative for a large class of *links* satisfying a certain nonvanishing condition on Milnor’s $\bar{\mu}$ -invariants, even in the framework of symmetric grope and Whitney tower generalisations of concordance and homology cobordism in the sense of [Cochran et al. 2003; Cha 2014]. Also, we employ topological surgery in dimension 4 to give a new construction of homology cobordisms of zero-surgery manifolds. Next we state our main theorems, after which we will discuss these aspects further.

Theorem 1.1. *Suppose $h \geq 3$. Then there are infinitely many 3-component links L_0, L_1, \dots , with vanishing pairwise linking numbers and with unknotted components, satisfying the following for any $i \neq j$:*

- (1) *The zero-surgery manifolds M_{L_i} and M_{L_j} are not homeomorphic.*
- (2) *There is a topological homology cobordism between M_{L_i} and M_{L_j} in which the k -th meridians of L_i and L_j are freely homotopic for each $k = 1, 2, 3$.*
- (3) *The links L_i and L_j are height h but not height $h + 1$ symmetric grope concordant. In particular, L_i and L_j are not concordant.*

For a definition of height h symmetric grope concordance, see Definition 4.2. Our links are obtained from the Borromean rings by performing a satellite construction

along a curve lying in the kernel of the map $\pi_1(S^3 \setminus L) \rightarrow \pi_1(M_L)$ induced by inclusion.

As a counterpoint to [Theorem 1.1](#), we show that there are infinite families of links with the same nonvanishing Milnor invariants, with homeomorphic zero-surgery manifolds preserving the homotopy classes of the meridians, but which are not concordant.

The Milnor invariant of an m -component link associated to a multi-index $I = i_1 i_2 \cdots i_r$ with $i_j \in \{1, \dots, m\}$, as defined in [[Milnor 1957](#)], will be denoted by $\bar{\mu}_L(I)$. We denote its length by $|I| := r$. Define $k(m) := \lfloor \log_2(m-1) \rfloor$.

Theorem 1.2. *Let I be a multi-index with nonrepeating indices with length $m \geq 2$. For any $h \geq k(m) + 2$ there are infinitely many m -component links L_0, L_1, \dots , with unknotted components, satisfying the following:*

- (1) *The L_i have identical $\bar{\mu}$ -invariants, $\bar{\mu}_{L_i}(I) = 1$, and $\bar{\mu}_{L_i}(J) = 0$ for $|J| < |I|$.*
- (2) *There is a homeomorphism between the zero-surgery manifolds M_{L_i} and M_{L_j} which preserves the homotopy classes of the meridians.*
- (3) *The links L_i and L_j are height h but not height $h + 1$ symmetric grope concordant. In particular, L_i and L_j are not concordant.*

The case when $m \geq 3$ should be compared with [Theorem 1.1](#) since then the links L_i have vanishing pairwise linking numbers. To construct such links we start with certain iterated Bing doubles constructed using T. Cochran's algorithm, which realise the Milnor invariant required. We then perform satellite operations which affect the concordance class of the link but do not change the homeomorphism type of the zero-surgery manifold.

We remark that we could also phrase [Theorems 1.1](#) and [1.2](#) in terms of symmetric Whitney tower concordance instead of grope concordance.

In the three subsections below, we discuss some features of [Theorem 1.1](#), regarding (i) the use of topological surgery in dimension 4, (ii) link concordance versus zero-surgery homology cobordism, and (iii) link exteriors and the homology surgery approach.

1A. Topological surgery for 4-dimensional homology cobordism. An interesting aspect of the proof of [Theorem 1.1](#) is that we employ topological surgery in dimension 4 to give a sufficient condition for certain zero-surgery manifolds of 3-component links to be homology cobordant. It is well known that topological surgery in dimension 4 is useful for obtaining homology cobordisms (and consequently concordances), although the current state of the art in terms of "good" groups, for which the π_1 -null disc lemma is known, is still insufficient for the general case. M. Freedman showed that knots of Alexander polynomial one are concordant to the unknot [[Freedman and Quinn 1990](#), [Theorem 11.7B](#)]. J. Davis

[2006] extended the program to show that 2-component links with Alexander polynomial one are concordant to the Hopf link. These two cases use topological surgery over fundamental groups \mathbb{Z} and \mathbb{Z}^2 , respectively. Due to the rarity of good groups for 4-dimensional topological surgery, there are not many other situations where such positive results on knot and link concordance can currently be proven. As another case, S. Friedl and P. Teichner [2005] found sufficient conditions for a knot to be homotopy ribbon, and in particular slice, with a certain ribbon group $\mathbb{Z} \ltimes \mathbb{Z}[\frac{1}{2}]$.

We give another instance of the utility of topological surgery for constructing homology cobordisms, using the group \mathbb{Z}^3 , which is manageable from the point of view of topological surgery in dimension 4. Indeed, our sufficient condition for zero-surgery manifolds to be homology cobordant focuses on the Borromean rings as a base link. The zero-surgery manifold M_{Bor} of the Borromean rings is the 3-torus $T^3 = S^1 \times S^1 \times S^1$, whose fundamental group is \mathbb{Z}^3 .

To state our result, we with the following notation: Let

$$\Lambda := \mathbb{Z}[\mathbb{Z}^3] = \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}].$$

Denote the zero-surgery manifold of a link L by M_L as before. For a 3-component link L with vanishing pairwise linking numbers, there is a canonical homotopy class of maps $f_L : M_L \rightarrow M_{\text{Bor}} = T^3$ which send the homotopy class of the i -th meridian of L to that of the Borromean rings, namely the i -th circle factor of T^3 . After choosing an identification of $\pi_1(T^3) = \mathbb{Z}^3$, we can use this to define the Λ -coefficient homology $H_1(M_L; \Lambda)$. We say that a map $f : M_L \rightarrow T^3$ is a Λ -homology equivalence if f is homotopic to f_L and f induces isomorphisms on $H_*(-; \Lambda)$.

Theorem 1.3. *Suppose L is a 3-component link whose components have trivial Arf invariants and there exists a Λ -homology equivalence $M_L \rightarrow T^3$. Then there is a homology cobordism W between M_L and $T^3 = M_{\text{Bor}}$ for which the inclusion-induced maps $\pi_1(M_L) \rightarrow \pi_1(W) \xrightarrow{\cong} \pi_1(T^3)$ are such that the composition from left to right takes meridians to meridians.*

1B. Link concordance versus zero-surgery homology cobordism. We review the general question of whether links with homology cobordant zero-surgery manifolds are concordant. The answer to the basic question is easily seen to be no, once one knows of a result of C. Livingston [1983] that there are knots not concordant to their reverses. Note that a knot and its reverse have the same zero-surgery manifold. This leads us to consider some additional conditions on the homology cobordism, involving the meridians. In what follows, meridians are always positively oriented.

First, observe that the exteriors of two links are homology cobordant if and only if the zero-framed meridians cobound framed annuli disjointly embedded in a homology cobordism of the zero-surgery manifolds. (For the if direction, note that the exterior of the framed annuli is a homology cobordism of the link exteriors.)

In particular, it holds if two links (or knots) are concordant.

Regarding the knot case, in [Cochran et al. 2013], T. Cochran, B. Franklin, M. Hedden and P. Horn considered homology cobordisms of zero-surgery manifolds in which the meridians are *homologous*: in the smooth category, they showed that the existence of such a homology cobordism is insufficient for knots to be concordant. In the topological case this is still left unknown.

Concerning a stronger *homotopy* analogue, the following is unknown in both the smooth and topological cases:

Question 1.4. If there is a homology cobordism of zero-surgery manifolds of two knots in which the meridians are homotopic, are the knots concordant? Or concordant in a homology $S^3 \times I$?

For the link case, results in the literature give nonconcordant examples whose zero-surgery manifolds admit a homology cobordism with homotopic meridians. As a generic example in the topological category, consider a 2-component link with linking number one. The zero-surgery manifold is a homology 3-sphere, which bounds a contractible topological 4-manifold by [Freedman and Quinn 1990, Corollary 9.3C]. Taking the connected sum of such 4-manifolds, one obtains the following: *the zero-surgery manifolds of any two linking number one 2-component links cobound a simply connected topological homology cobordism*. Note that in this case the meridians are automatically homotopic. There are many linking number one 2-component links which are not concordant, as can be detected, for example, by the multivariable Alexander polynomial [Kawauchi 1978; Nakagawa 1978]. For related in-depth study, the reader is referred to, for instance, [Cha and Ko 1999; Friedl and Powell 2011; Cha 2014]. With our respective coauthors, we gave nonconcordant linking number one links with two unknotted components, for which abelian invariants such as the multivariable Alexander polynomial are unable to obstruct them from being concordant.

There are other examples which have knotted components: in [Cochran et al. 2013, end of Section 1], the authors discuss 2-component linking number zero links with homeomorphic zero-surgery manifolds which have nonconcordant (knotted) components. These links are obviously not concordant, and it can be seen that the homeomorphisms preserve meridians up to homotopy.

By contrast with the above examples, our links have unknotted components and vanishing pairwise linking numbers. Another feature exhibited by the links of Theorems 1.1 and 1.2 is that the entire subtlety of symmetric grope concordance of links can occur, within a single homology cobordism/homeomorphism class of the zero-surgeries, even modulo local knot tying.

We remark that all the links of Theorems 1.1 and 1.2 lie in the same “ k -solv-equivalence class” for all k in the sense of [Cochran and Kim 2008, Definition 2.5].

1C. *Link exteriors and the homology surgery approach.* Our results serve to underline the philosophy that when investigating the relative problem of whether two links are concordant, and neither of them are the unlink, one should consider obstructions to homology cobordism of the link exteriors viewed as bordered manifolds, rather than to homology cobordism of the zero-surgery manifolds, even in low dimensions. This was implemented in, for example, [Kawauchi 1978; Nakagawa 1978; Cha 2014] (see also [Friedl and Powell 2011] for a related approach).

Although we stated our results in terms of grope concordance of links in Theorems 1.1 and 1.2 given above, in fact we show more: the link exteriors are far from being homology cobordant, as measured in terms of Whitney towers. A more detailed discussion is given in Section 5. For the purpose of distinguishing exteriors, we use the amenable Cheeger–Gromov ρ -invariant technology for bordered 3-manifolds (particularly for link exteriors) developed in [Cha 2014], generalising applications of ρ -invariants to concordance and homology cobordism in [Cochran et al. 2003; 2009; Cha and Orr 2012].

We will now discuss our results from the viewpoint of the homology surgery approach to link concordance classification, initiated by S. Cappell and J. Shaneson [1974; 1980] and implemented in high dimensions by J. Le Dimet [1988] using P. Vogel’s homology localisation of spaces [1978]. The strategy consists of two parts. Consider the problem of comparing two given link exteriors. First we decide whether the exteriors have the same “Poincaré type”, which roughly means that they have homotopy-equivalent Vogel homology localisations. If so, there is a common finite target space, into which the exteriors are mapped by homology equivalences rel. boundary. Once this is the case, a surgery problem is defined, and one can try to decide whether homology surgery gives a homology cobordism of the exteriors. The first step is obstructed by homotopy invariants (including Milnor $\bar{\mu}$ -invariants in low dimension). The failure of the second step is measured by surgery obstructions, which are not yet fully formulated in low dimension (even modulo the fact that 4-dimensional surgery might not work), since the fundamental group plays a more sophisticated central rôle; see [Powell 2012] for the beginning of an algebraic surgery approach to this problem in the context of knot slicing.

Our examples illustrate that for many Poincaré types, namely those in Theorems 1.1 and 1.2, we get a rich theory of surgery obstructions within each Poincaré type, which is invisible via zero-surgery manifolds. We remark that for our links L_i in Theorems 1.1 and 1.2, there is a homology equivalence of the exterior of each L_i into that of a fixed one, say L_1 , since we use satellite constructions (see Section 4). It follows that the exteriors have the same Poincaré type in the above sense. In this paper, (parts of the not yet fully formulated) homology surgery obstructions in dimension 4 have their incarnation in Theorem 5.2, the Amenable Signature Theorem.

Organisation of the paper. In [Section 2](#), we explore the implications of the hypothesis that a homology equivalence $f : M_L \rightarrow T^3$ as in [Theorem 1.3](#) exists, and we prove [Theorem 1.3](#) in [Section 3](#). In [Section 4](#), we construct links with a given Milnor invariant with nonrepeating indices, and perform satellite operations on the links to construct the links of [Theorems 1.1](#) and [1.2](#), which are height h symmetric grope concordant. In [Section 5](#), we show that none of these links are height $h + 1$ grope concordant to one another.

2. Homology type of zero-surgery manifolds and the 3-torus

This section discusses the hypotheses of [Theorem 1.3](#). We begin the section by briefly reminding the reader who is familiar with Kirby calculus of a nice way to see the following fact.

Lemma 2.1. *The zero-surgery manifold of the Borromean rings is homeomorphic to the 3-torus.*

Proof. Place dots on two components of the Borromean rings and a zero near the other. Each component of the Borromean rings is a commutator in the meridians of the other two components, so this is a Kirby diagram for $T^2 \times D^2$, whose boundary is T^3 . The 1-handles (dotted circles) can be replaced with zero-framed 2-handles without changing the boundary. \square

In the following proposition we expand on the meaning and implications of the condition in [Theorem 1.3](#). Denote the exterior of a link L by $X_L := S^3 \setminus \nu L$ as before.

Proposition 2.2. *Suppose that L is a 3-component link. Then the following are equivalent:*

- (1) *There is a Λ -homology equivalence $f : M_L \rightarrow T^3$.*
- (2) *The preferred longitudes generate the link module $H_1(X_L; \Lambda)$.*
- (3) *The pairwise linking numbers of L vanish and $H_1(M_L; \Lambda) = 0$.*

Furthermore, (any of) the above conditions imply that L has multivariable Alexander polynomial $\Delta_L = (t_1 - 1)(t_2 - 1)(t_3 - 1)$, and this implies that the Milnor invariant $\bar{\mu}_L(123)$ is equal to ± 1 .

Proof. First we will observe that (2) and (3) are equivalent. Longitudes of L represent elements in $H_1(X_L; \Lambda) \cong \pi_1(X_L)^{(1)}/\pi_1(X_L)^{(2)}$ if and only if they are zero in $H_1(X_L; \mathbb{Z}) \cong \mathbb{Z}^3$; that is, the pairwise linking numbers are zero. If this is the case, $H_1(M_L; \Lambda)$ is isomorphic to $H_1(X_L; \Lambda)/\langle \text{longitudes} \rangle$, since M_L is obtained by attaching three 2-handles to E_L along the longitudes and then attaching three 3-handles along the boundary. It follows that longitudes generate $H_1(X_L; \Lambda)$ if and only if $H_1(M_L; \Lambda) = 0$.

Suppose (1) holds. Denote the meridians of L by μ_i ($i = 1, 2, 3$) and the linking number of the i -th and j -th components by ℓ_{ij} . The i -th longitude λ_i , which is homologous to $\sum_{j \neq i} \ell_{ij} \mu_j$, is zero in $H_1(M_L; \mathbb{Z}) \cong H_1(T^3; \mathbb{Z})$. Since $\{f_*([\mu_i])\}$ forms a basis of $H_1(T^3; \mathbb{Z}) \cong \mathbb{Z}^3$, it follows by linear independence that $\ell_{ij} = 0$ for any i and j . Also, $H_1(M_L; \Lambda) \cong H_1(T^3; \Lambda) = 0$. This shows that (3) holds.

Suppose (3) holds. Start with a map $g : \partial X_L = \bigsqcup_3 S^1 \times S^1 \rightarrow T^3$ that sends μ_i to the i -th S^1 factor and λ_i to a point. Observe that $g_* : H_1(\partial X_L; \mathbb{Z}) \rightarrow H_1(T^3; \mathbb{Z})$ factors through the inclusion-induced map $i_* : H_1(\partial X_L; \mathbb{Z}) \rightarrow H_1(M_L; \mathbb{Z})$ and the identifications $H_1(M_L; \mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}^3 \xleftarrow{\cong} H_1(T^3; \mathbb{Z})$; this follows from the fact that $H_1(\partial X_L; \mathbb{Z}) \cong \mathbb{Z}^6$ is generated by the μ_i and λ_i and that both g_* and i_* are quotient maps, with their kernels generated by the λ_i . Since T^3 is a $K(\mathbb{Z}^3, 1)$, elementary obstruction theory shows that g extends to a map $f : M_L \rightarrow T^3$.

Consider the universal coefficient spectral sequence (see, e.g., [Levine 1977, Theorem 2.3]) $E_{p,q}^2 = \text{Ext}_\Lambda^p(H_q(M_L; \Lambda), \Lambda) \Rightarrow H^n(M_L; \Lambda)$. We have $E_{0,1}^2 = 0$ since $H_1(M_L; \Lambda) = 0$, and $E_{1,0}^2 = \text{Ext}_\Lambda^1(\mathbb{Z}, \Lambda) = H^1(T^3; \Lambda) = 0$. It follows that $H^1(M_L; \Lambda) = 0$. By duality, $H_2(M_L; \Lambda) = 0$. Also, $H_3(M_L; \Lambda) = 0$ since the \mathbb{Z}^3 -cover of M_L is noncompact. Since $H_0(M_L; \Lambda) \cong \mathbb{Z} \cong H_0(T^3; \Lambda)$ and $H_i(T^3; \Lambda) = 0$ for $i > 0$, it follows that f is a Λ -homology equivalence. This completes the proof of the equivalence of (1), (2) and (3).

Suppose (1), (2) and (3) hold. Recall that the scalar multiplication of a loop by t_i in the module $H_1(X_L; \Lambda)$ is defined to be conjugation by μ_i . Since λ_i and μ_i commute, we have $(t_i - 1)\lambda_i = 0$ in $H_1(X_L; \Lambda)$. From this and (2), it follows that there is an epimorphism of $A := \bigoplus_{i=1}^3 \Lambda / \langle t_i - 1 \rangle$ onto $H_1(X_L; \Lambda)$. Since the zeroth elementary ideal of A is the principal ideal generated by $(t_1 - 1)(t_2 - 1)(t_3 - 1)$, it follows that Δ_L is a factor of $(t_1 - 1)(t_2 - 1)(t_3 - 1)$. We now invoke the Torres condition (see, e.g., [Kawauchi 1996, Theorem 7.4.1]):

$$\Delta_L(1, t_2, t_3) = (t_2^{\ell_{12}} t_3^{\ell_{13}} - 1) \Delta_{L'}(t_2, t_3),$$

where L' is the sublink of L with the first component deleted and ℓ_{ij} is the pairwise linking number. Since $\ell_{ij} = 0$ by (3), we have $\Delta_L(1, t_2, t_3) = 0$. It follows that $t_1 - 1$ is a factor of Δ_L . Similarly $t_2 - 1$ and $t_3 - 1$ are factors. Therefore $\Delta_L(t_1, t_2, t_3) = (t_1 - 1)(t_2 - 1)(t_3 - 1)$.

To show the last part, suppose that $\Delta_L(t_1, t_2, t_3) = (t_1 - 1)(t_2 - 1)(t_3 - 1)$. By [Kawauchi 1996, Proposition 7.3.14], the single-variable Alexander polynomial $\Delta_L(t)$ of L is given by

$$\Delta_L(t) = (t - 1) \Delta_L(t, t, t) = (t - 1)^4 \doteq ((\sqrt{t})^{-1} - \sqrt{t})^4.$$

It follows that L has Conway polynomial $\nabla_L(z) = z^4$, by the standard substitution $z = (\sqrt{t})^{-1} - \sqrt{t}$. Cochran [1985, Theorem 5.1] identified the coefficient of z^4

in $\nabla_L(z)$ with $(\mu_L(123))^2$ for 3-component links with pairwise linking number zero. Applying this to our case, it follows that $\bar{\mu}_L(123) = \pm 1$. \square

3. Construction of homology cobordisms using topological surgery

This section gives the proof of [Theorem 1.3](#). The proof will use surgery theory, and will parallel the proof given by Davis [\[2006\]](#) (see also [\[Hillman 2002, Section 7.6\]](#)). We will provide some details in order to fill in where the treatment in [\[Davis 2006\]](#) was terse, and to convince ourselves that the analogous arguments work in the case of interest.

For the convenience of the reader we restate [Theorem 1.3](#) here.

Theorem 1.3. *Suppose L is a 3-component link whose components have trivial Arf invariants and there exists a Δ -homology equivalence $M_L \rightarrow T^3$. Then there is a homology cobordism W between M_L and $T^3 = M_{\text{Bor}}$ for which the inclusion-induced maps $\pi_1(M_L) \rightarrow \pi_1(W) \xrightarrow{\cong} \pi_1(T^3)$ are such that the composition from left to right takes meridians to meridians.*

Remark 3.2. It is an interesting question to determine whether there are extra conditions which can be imposed in order to see that the Arf invariants of the components are forced to vanish by the homological assumptions. In the cases of knots and 2-component links with Alexander polynomial one, the Arf invariants of the components are automatically trivial. For knots, $\Delta_K(-1)$ computes the Arf invariant, by [\[Levine 1966\]](#). For 2-component links one observes that $\Delta_L(t, 1)$ and $\Delta_L(1, t)$ give the Alexander polynomials of the components, by the Torres condition, and then applies Levine's theorem. These arguments do not seem to extend to the 3-component case of current interest.

The proof of [Theorem 1.3](#) will occupy the rest of this section. In order to produce a homology cobordism, we will first show that there is a normal cobordism between normal maps $f : M_L \rightarrow T^3$ and $\text{Id} : T^3 \rightarrow T^3$. Interestingly, we can work with smooth manifolds in order to establish the existence of a normal cobordism. This will make arguments which invoke tangent bundles and transversality easier to digest. Only at the end of the proof of [Theorem 1.3](#), where we take connected sums with the E_8 -manifold, and where we claim that the vanishing of a surgery obstruction implies that surgery can be done, do we need to leave the realm of smooth manifolds.

Definition 3.3. Let X be an n -dimensional manifold with a vector bundle $\nu \rightarrow X$. A degree-one normal map (F, b) over X is an n -manifold M with a map $F : M \rightarrow X$ which induces an isomorphism $F_* : H_n(M; \mathbb{Z}) \xrightarrow{\cong} H_n(X; \mathbb{Z})$, together with a stable trivialisation $b : TM \oplus F^*\nu \oplus \varepsilon^l \cong \varepsilon^k$.

A degree-one normal cobordism (J, e) between normal maps $(F : M \rightarrow X, b)$ and $(G : N \rightarrow X, c)$ is an $(n + 1)$ -dimensional cobordism between M and N with a map $J : W \rightarrow X \times I$ extending $F : M \rightarrow X \times \{0\}$ and $G : N \rightarrow X \times \{1\}$, which induces an isomorphism

$$J_* : H_{n+1}(W, \partial W; \mathbb{Z}) \xrightarrow{\cong} H_{n+1}(X \times I, X \times \{0, 1\}; \mathbb{Z}),$$

together with a stable trivialisation $e : TW \oplus J^*(\nu \times I) \oplus \varepsilon^{l'} \cong \varepsilon^{k'}$.

For us, let $X = T^3$, and let ν be its tangent bundle. We fix a framing on the stable tangent bundle of the target torus T^3 once and for all. Note that this canonically determines a trivialisation of the tangent bundle of $F^*\nu$, for any map $F : M \rightarrow X$, by the following diagram, in which the bottom composition is the constant map, denoted $*$, and the top composition is the pull back $F^*\nu$. The middle composition is the induced framing.

$$\begin{array}{ccccc} M \times \{0\} & \xrightarrow{F \times \text{Id}} & T^3 \times \{0\} & \xrightarrow{\nu} & \text{BO}(n) \\ \downarrow & & \downarrow & & \downarrow \\ M \times I & \xrightarrow{F \times \text{Id}} & T^3 \times I & \longrightarrow & \text{BO} \\ \uparrow & & \uparrow & & \uparrow \\ M \times \{1\} & \xrightarrow{F \times \text{Id}} & T^3 \times \{1\} & \xrightarrow{*} & \text{BO}(n) \end{array}$$

A framing of the tangent bundle of the domain therefore determines a normal map.

Lemma 3.4. *Let L be a link whose components have trivial Arf invariants, and let $f : M_L \rightarrow T^3$ be a degree-one normal map which induces a \mathbb{Z} -homology isomorphism and which maps a chosen meridian μ_i to the i -th S^1 factor of T^3 for $i = 1, 2, 3$. We can make a homotopy of f and choose a framing on M_L so that $f : M_L \rightarrow T^3$ and $\text{Id} : T^3 \rightarrow T^3$ are degree-one normal cobordant.*

Proof. We need to show that we can choose a framing on M_L such that the disjoint union $M_L \sqcup -T^3$ represent the trivial element of $\Omega_3^{\text{fr}}(T^3)$. We compute this bordism group:

$$\tilde{\Omega}_3^{\text{fr}}(T^3) \cong \tilde{\Omega}_4^{\text{fr}}(\Sigma T^3) \cong \tilde{\Omega}_4^{\text{fr}}(S^2 \vee S^2 \vee S^2 \vee S^3 \vee S^3 \vee S^3 \vee S^4),$$

with this last isomorphism induced by a homotopy equivalence of spaces. There is a copy of S^{i+1} for each i -cell of T^3 , for $i = 1, 2, 3$. To see this homotopy equivalence, we need to see that the attaching maps of the cells are null-homotopic. The suspension of the 1-skeleton of T^3 is $S^2 \vee S^2 \vee S^2$. The Hilton–Milnor theorem [Hilton 1955, Theorem A] computes the homotopy groups of a wedge of spheres.

The attaching maps for the 2-cells of T^3 become the attaching maps for the 3-cells of ΣT^3 , namely maps in

$$\pi_2(S^2 \vee S^2 \vee S^2) \cong \bigoplus_3 \pi_2(S^2) \cong \bigoplus_3 \mathbb{Z},$$

where the first isomorphism is by the Hilton–Milnor theorem. The commutator attaching maps become trivial in the abelian $\pi_2(S^2)$. Therefore the 3-skeleton of ΣT^3 is a wedge $S^2 \vee S^2 \vee S^2 \vee S^3 \vee S^3 \vee S^3$. The attaching map for the 3-cell of T^3 becomes the attaching map for the 4-cell of ΣT^3 , a map in

$$\pi_3(S^2 \vee S^2 \vee S^2 \vee S^3 \vee S^3 \vee S^3) \cong \bigoplus_{1 \leq i \leq 3} \pi_3(S^3) \oplus \bigoplus_3 \pi_3(S^2) \oplus \bigoplus_{1 \leq i < j \leq 3} \pi_3(S^3),$$

again by the Hilton–Milnor theorem, where the last three $\pi_3(S^3)$ summands include into $\pi_3(S^2 \vee S^2 \vee S^2 \vee S^3 \vee S^3 \vee S^3)$ via composition with the Whitehead product: let $f_i : S^2 \rightarrow S_i^2$ be a generator of $\pi_2(S_i^2)$, where S_i^2 is the i -th S^2 component in the wedge. Then the Whitehead product is the homotopy class of the map $[f_i, f_j] \in \pi_3(S_i^2 \vee S_j^2)$, which is the attaching map for the 4-cell in a standard cellular decomposition of $S^2 \times S^2$. Since $\pi_2(S^1) \cong \pi_2(S^1 \vee S^1) \cong 0$, the summands associated to the S^2 components of the wedge do not arise from a suspension. The summands associated to the S^3 components are null-homotopic since the 3-cell of T^3 is attached to each 2-cell twice, once on either side. This completes the explanation of the claimed homotopy equivalence:

$$\Sigma T^3 \simeq S^2 \vee S^2 \vee S^2 \vee S^3 \vee S^3 \vee S^3 \vee S^4.$$

By Mayer–Vietoris, the bordism group $\tilde{\Omega}_4^{\text{fr}}(S^2 \vee S^2 \vee S^2 \vee S^3 \vee S^3 \vee S^3 \vee S^4)$ is a direct sum

$$\begin{aligned} \bigoplus_3 \tilde{\Omega}_4^{\text{fr}}(S^2) \oplus \bigoplus_3 \tilde{\Omega}_4^{\text{fr}}(S^3) \oplus \tilde{\Omega}_4^{\text{fr}}(S^4) &\cong \bigoplus_3 \tilde{\Omega}_2^{\text{fr}}(S^0) \oplus \bigoplus_3 \tilde{\Omega}_1^{\text{fr}}(S^0) \oplus \tilde{\Omega}_0^{\text{fr}}(S^0) \\ &\cong \bigoplus_3 \Omega_2^{\text{fr}} \oplus \bigoplus_3 \Omega_1^{\text{fr}} \oplus \Omega_0^{\text{fr}} \\ &\cong \bigoplus_3 \mathbb{Z}_2 \oplus \bigoplus_3 \mathbb{Z}_2 \oplus \mathbb{Z}. \end{aligned}$$

Therefore

$$\Omega_3^{\text{fr}}(T^3) \cong \Omega_3^{\text{fr}} \oplus \bigoplus_3 \mathbb{Z}_2 \oplus \bigoplus_3 \mathbb{Z}_2 \oplus \mathbb{Z} \cong \mathbb{Z}_{24} \oplus \bigoplus_3 \mathbb{Z}_2 \oplus \bigoplus_3 \mathbb{Z}_2 \oplus \mathbb{Z}.$$

The isomorphism is given as follows. Let

$$\text{pr}_i : T^3 = S^1 \times S^1 \times S^1 \longrightarrow S^1$$

be given by projection onto the i -th factor. Similarly, let

$$\text{qr}_i : T^3 = S^1 \times S^1 \times S^1 \longrightarrow S^1 \times S^1$$

be given by forgetting the i -th factor. Let $F : M \rightarrow T^3$ be an element of $\Omega_3^{\text{fr}}(T^3)$. Making all maps transverse to a point, we obtain an 8-tuple

$$\begin{aligned} & ([M], (\text{pr}_1 \circ F)^{-1}(*), (\text{pr}_2 \circ F)^{-1}(*), (\text{pr}_3 \circ F)^{-1}(*), \\ & (\text{qr}_1 \circ F)^{-1}(*), (\text{qr}_2 \circ F)^{-1}(*), (\text{qr}_3 \circ F)^{-1}(*), F^{-1}(*)) \\ & \in \Omega_3^{\text{fr}} \oplus \bigoplus_3 \Omega_2^{\text{fr}} \oplus \bigoplus_3 \Omega_1^{\text{fr}} \oplus \Omega_0^{\text{fr}} \cong \mathbb{Z}_{24} \oplus \bigoplus_3 \mathbb{Z}_2 \oplus \bigoplus_3 \mathbb{Z}_2 \oplus \mathbb{Z}. \end{aligned}$$

We consider each of the summands in turn.

By choosing the appropriate orientation on M_L and making the degree-one normal maps transverse to a point, one can arrange for the disjoint union $f^{-1}(*) \sqcup -\text{Id}^{-1}(*)$ to be equal to $\{\text{pt}\} \sqcup -\{\text{pt}\} = 0 \in \Omega_0^{\text{fr}}$.

As observed in [Davis 2006, proof of the lemma], we can change the framing so that the elements of Ω_1^{fr} agree. First, we change the framing on each of three chosen meridians μ_i to the link components L_i .

Orientable k -plane vector bundles over S^1 are classified by homotopy classes of maps $[S^1, \text{BSO}(k)]$. Consider the exact sequence

$$\pi_2(\text{BSO}) \longrightarrow \pi_2(\text{BSO}, \text{BSO}(k)) \longrightarrow \pi_1(\text{BSO}(k)) \xrightarrow{\gamma} \pi_1(\text{BSO}).$$

A stably trivial vector bundle over S^1 gives us an element of $\ker(\gamma)$. A choice of trivialisaton of the vector bundle gives us a null homotopy and therefore an element of $\pi_2(\text{BSO}, \text{BSO}(k))$. The possible choices of stable trivialisations, or framings, are indexed by $\pi_2(\text{BSO}) \cong \pi_1(\text{SO}) \cong \mathbb{Z}_2$.

We can therefore, if necessary, change the framing on each μ_i to be the bounding framing using an element of $\pi_1(\text{SO}(2))$ which maps to the nontrivial element of $\pi_1(\text{SO})$. Use the element of $\pi_1(\text{SO}(2))$ to change the framing on the normal bundle of μ_i in M_L . We claim that these changes in the framing can be extended to the whole of M_L . To see this, we argue as follows. The dual of the inclusion map $H^1(M_L; \mathbb{Z}) \rightarrow H^1(\mu_i; \mathbb{Z})$ is surjective, since each $[\mu_i]$ is a generator of $H_1(M_L; \mathbb{Z})$. The change of framing map $\mu_i \rightarrow \text{SO}(2)$ represents a homotopy class of maps in $[\mu_i, S^1]$ and therefore an element of $H^1(\mu_i; \mathbb{Z})$. Since this pulls back to an element of $H^1(M_L; \mathbb{Z})$, which can in turn produce a map $M_L \rightarrow \text{SO}(2)$, the change of framing map can be extended as claimed.

Let $N_i \subset M_L$ be the submanifolds given by $(\text{qr}_i \circ f)^{-1}(*)$, after perturbing f to make $\text{qr}_i \circ f$ transverse to a point. As the inverse image of the i -th S^1 factor of T^3 (e.g., $f^{-1}(S^1 \times \{*\} \times \{*\})$), N_i is a collection of circles. After a homotopy of f , it can be arranged, by the assumption on f , that N_i is a single meridian μ_i , which has

the bounding framing and therefore represents the zero element in Ω_1^{fr} . To make this arrangement, it suffices to be able to remove circles N_i whose image in T^3 is null-homologous. But in T^3 , a null-homologous curve is also null-homotopic. Therefore we can make a homotopy of f so that N_i misses $S^1 \times \{*\} \times \{*\}$.

After another homotopy, the inverse image $(\text{pr}_i \circ f)^{-1}(*)$ can be arranged to be a capped-off Seifert surface $F_i \cup D^2$, where F_i is a Seifert surface for L_i (possibly with closed connected components). To see this, we again use our assumption that f sends the i -th meridian μ_i to the i -th circle. This assumption enables us to homotope f so that $\text{pr}_i \circ f$ sends a regular neighbourhood $\mu_i \times D^2$ to S^1 by projection onto the first factor. Then the inverse image is as desired. A homotopy of f preserves the framed bordism class of $(\text{pr}_i \circ f)^{-1}(*)$, and the class $[F_i \cup D^2] \in \Omega_2^{\text{fr}}$ is determined by the Arf invariant of L_i . By hypothesis, this vanishes.

Finally, again following [Davis 2006] (see also [Freedman and Quinn 1990, proof of Lemma 11.6B]), the framing can be altered in the neighbourhood of a point to change the element $[M] \in \Omega_3^{\text{fr}}$ to the trivial element. We recall the definition of the J -homomorphism $J : \pi_3(\text{SO}) \rightarrow \pi_3^S \cong \Omega_3^{\text{fr}}$, for the convenience of the reader, where π_k^S is the k -th stable homotopy group of spheres. (Incidentally, $\pi_3(\text{SO}) \cong \mathbb{Z}$ and $\pi_3^S \cong \mathbb{Z}_{24}$.) Given $\theta : S^3 \rightarrow \text{SO}$, choose a k sufficiently large so that we can represent θ by a map $\theta : S^3 \rightarrow \text{SO}(k)$. We proceed to construct a map $(J(\theta) : S^{k+3} \rightarrow S^k) \in \pi_3^S$. So:

$$S^{k+3} = S^3 \times D^k \cup_{S^3 \times D^{k-1}} D^4 \times S^{k-1}.$$

Define a map

$$\begin{aligned} j(\theta) : S^3 \times D^k &\rightarrow S^3 \times D^k \\ (x, y) &\mapsto (x, \theta(x)(y)), \end{aligned}$$

since $\theta(x) \in \text{SO}(k)$ acts on D^k by identifying D^k with the unit ball in \mathbb{R}^k . This map extends to a homeomorphism $j(\theta)$ of $S^3 \times D^k \cup_{S^3 \times D^{k-1}} D^4 \times S^{k-1}$. Form the composition

$$\begin{aligned} S^{k+3} = S^3 \times D^k \cup_{S^3 \times D^{k-1}} D^4 \times S^{k-1} &\xrightarrow{j(\theta)} S^3 \times D^k \cup_{S^3 \times D^{k-1}} D^4 \times S^{k-1} \\ &\xrightarrow{\text{col}} S^3 \times S^k \xrightarrow{\text{proj}_1} S^k, \end{aligned}$$

where col is the collapse map which squashes $D^4 \times S^{k-1}$ and proj_1 is the projection onto the first factor. This gives an element of π_3^S , which is the image of θ under $J : \pi_3(\text{SO}) \rightarrow \pi_3^S \cong \Omega_3^{\text{fr}}$.

This J -homomorphism is onto [Adams 1966, Example 7.17], so that composing the framing in a neighbourhood D^3 of a point with the choice of map $\theta \in \pi_3(\text{SO}) = [(D^3, \partial D^3), (\text{SO}, *)]$ such that $-J(\theta) = [M] \in \Omega_3^{\text{fr}}$ changes the class in Ω_3^{fr} as desired.

This shows the existence of a normal cobordism W' . To see that this is of degree one, note that the map to T^3 which extends over W' can be used to define a map to $T^3 \times I$, by defining a map $g : W' \rightarrow I$ such that $g(M_L) = \{0\}$ and $g(T^3) = \{1\}$. Now consider the commutative diagram

$$\begin{array}{ccc} H_4(W', \partial W'; \mathbb{Z}) & \longrightarrow & H_3(\partial W'; \mathbb{Z}) \\ \downarrow & & \downarrow \\ H_4(T^3 \times I, T^3 \times \{0, 1\}; \mathbb{Z}) & \longrightarrow & H_3(T^3 \times \{0, 1\}; \mathbb{Z}) \end{array}$$

Going right, then down, the fundamental class $[W', \partial W']$ maps to

$$(1, -1) \in H_3(T^3 \times \{0, 1\}; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

By commutativity, the relative fundamental class $[W', \partial W']$ must map to a generator of $H_4(T^3 \times I, T^3 \times \{0, 1\}; \mathbb{Z})$. \square

A Λ -homology equivalence is also an integral homology equivalence, by the following argument. By definition (see above the statement of [Theorem 1.3](#)), a Λ -homology equivalence induces an isomorphism on $H_1(-; \mathbb{Z})$. By duality, we also have an isomorphism on $H_2(-; \mathbb{Z})$. It remains to see that $f : M_L \rightarrow T^3$ is a degree-one map. The assumption that

$$f_* : H_*(M_L; \Lambda) \xrightarrow{\cong} H_*(T^3; \Lambda)$$

is an isomorphism implies that the relative homology vanishes: $H_*(T^3, M_L; \Lambda) \cong 0$. The universal coefficient spectral sequence then implies that $H_*(T^3, M_L; \mathbb{Z}) \cong 0$ since all the E^2 terms $\text{Tor}_p^\Lambda(H_q(T^3, M_L; \Lambda), \mathbb{Z})$ vanish. Therefore a Λ -homology equivalence as in [Theorem 1.3](#) is a degree-one map.

[Lemma 3.4](#) then establishes the existence of a choice of stable framing b on M_L such that there is a degree-one normal cobordism

$$(F' : W' \longrightarrow T^3 \times I, e')$$

between $(f : M_L \rightarrow T^3, b)$ and $(\text{Id} : T^3 \rightarrow T^3, c)$. Choosing such a framing, we proceed to apply surgery theory to alter W' into a homology cobordism. Davis' observation [[2006](#)] was that the framing on W' is not an intrinsic part of the concordance problem, but rather necessary additional data which is required in order to be able to apply surgery theory. Without the information provided by the self-intersection form, it is not possible to obtain algebraic sufficient conditions which ensure that surgery can be performed. Nevertheless, as we shall see, there is a certain amount of freedom in the choice of framing data.

Before giving the proof of [Theorem 1.3](#), we first give the definition of the Wall even-dimensional surgery obstruction groups, which we will use in the proof.

Definition 3.5 [Wall 1999, Chapter 5]. Let A be a ring with involution. A $(-1)^k$ -Hermitian sesquilinear quadratic form on a free based A -module M is a $(-1)^k$ -Hermitian sesquilinear form $\lambda : M \times M \rightarrow A$ together with a quadratic enhancement. A quadratic enhancement of a form $\lambda : M \times M \rightarrow A$ is a function $\mu : M \rightarrow A/\{a - (-1)^k \bar{a} \mid a \in A\}$ such that

- (1) $\lambda(x, x) = \mu(x) + \overline{\mu(x)}$,
- (2) $\mu(x + y) - \mu(x) - \mu(y) = \lambda(x, y)$,
- (3) $\mu(ax) = a\mu(x)\bar{a}$,

for all $x, y \in M$ and for all $a \in A$.

A hyperbolic quadratic form is a direct sum of standard hyperbolic forms, where the standard hyperbolic form (H, χ, ν) is given by

$$\left(A \oplus A, \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}, \nu((1, 0)^T) = 0 = \nu((0, 1)^T) \right).$$

The even-dimensional surgery obstruction group $L_{2k}(A)$ is defined to be the Witt group of nonsingular $(-1)^k$ -Hermitian sesquilinear quadratic forms on free based A -modules, where addition in the Witt group is by direct sum, and the equivalence class of the hyperbolic forms is the identity element, where the equivalence relation is as follows. Quadratic forms (M, λ, μ) and (M', λ', μ') are said to be equivalent if there are hyperbolic forms (H, χ, ν) and (H', χ', ν') such that there is an isomorphism of forms $(M \oplus H, \lambda \oplus \chi, \mu \oplus \nu) \cong (M' \oplus H', \lambda' \oplus \chi', \mu' \oplus \nu')$. This completes the definition of $L_{2k}(A)$.

For us, A will be the group ring $\mathbb{Z}[\pi]$ of some group π ; initially π will be \mathbb{Z}^3 , so that we take $A = \mathbb{Z}[\mathbb{Z}^3] = \Lambda$. We omit the definition of the odd-dimensional L -groups since they will only play a peripheral rôle in the proof of [Theorem 1.3](#).

Proof of Theorem 1.3. First, do surgery below the middle dimension [[ibid.](#), Chapter 1] on (W', F', e') to create a normal cobordism $(F : W \rightarrow T^3 \times I, e)$ which is 2-connected, i.e., W is connected and $\pi_1(W) \cong \pi_1(T^3) \cong \mathbb{Z}^3$. The induced map $F_* : \pi_2(W) \rightarrow \pi_2(T^3 \times I)$ is automatically surjective since T^3 is aspherical.

The Wall surgery obstruction [[ibid.](#), Chapter 5] of the normal cobordism $(F : W \rightarrow T^3 \times I, e)$ is now defined in $L_4(\mathbb{Z}[\mathbb{Z}^3])$ to be given by the intersection form

$$\lambda_{W'} : H_2(W'; \Lambda) \times H_2(W'; \Lambda) \longrightarrow \Lambda,$$

together with the quadratic enhancement

$$\mu : H_2(W; \Lambda) \longrightarrow \mathbb{Z}[\mathbb{Z}^3]/\{a = \bar{a} \mid a \in \mathbb{Z}[\mathbb{Z}^3]\}$$

defined by counting the self-intersections of an immersion of a sphere $S^2 \looparrowright W$ representing an element of $H_2(W; \Lambda) \cong \pi_2(W)$, where the regular homotopy class

of the immersion is fixed by the framing e to be the unique class of immersions for which the induced trivialisation of TS^2 extends over the null-homotopy of S^2 in T^3 .

The fact that the homology of the boundary $H_j(M_L; \Lambda) \cong H_j(T^3; \Lambda)$ vanishes for $j = 1, 2$, is used crucially here to see that the intersection form λ_W is nonsingular, as observed by the surgeon in the “dialogue” of [Davis 2006].

By [Wall 1999, Proposition 13B.8], which is based on Shaneson’s formula $L_n(\mathbb{Z}[\pi \times \mathbb{Z}]) \cong L_n(\mathbb{Z}[\pi]) \oplus L_{n-1}(\mathbb{Z}[\pi])$, when π has trivial Whitehead group [Shaneson 1969] we have that

$$\begin{aligned} L_4(\mathbb{Z}[\mathbb{Z}^3]) &\cong \bigoplus_{i=0}^3 \binom{3}{i} L_{4-i}(\mathbb{Z}) \cong L_4(\mathbb{Z}) \oplus \bigoplus_3 L_3(\mathbb{Z}) \oplus \bigoplus_3 L_2(\mathbb{Z}) \oplus L_1(\mathbb{Z}) \\ &\cong L_0(\mathbb{Z}) \oplus \bigoplus_3 L_2(\mathbb{Z}), \end{aligned}$$

where the last isomorphism is by periodicity of the L -groups and the fact that the odd-dimensional simply connected L -groups vanish. The even-dimensional simply connected L -groups $L_{2k}(\mathbb{Z})$ are computed [Kervaire and Milnor 1963], when $k = 0 \pmod{2}$, as

$$\begin{aligned} L_0(\mathbb{Z}) &\xrightarrow{\cong} \mathbb{Z} \\ (M, \lambda, \mu) &\mapsto \sigma(\mathbb{R} \otimes_{\mathbb{Z}} M, \text{Id} \otimes \lambda) / 8, \end{aligned}$$

while for the dimensions where $k = 1 \pmod{2}$ they are computed via

$$\begin{aligned} L_2(\mathbb{Z}) &\xrightarrow{\cong} \mathbb{Z}_2 \\ (M, \lambda, \mu) &\mapsto \text{Arf}(\mathbb{Z}_2 \otimes_{\mathbb{Z}} M, \text{Id} \otimes \lambda, \text{Id} \otimes \mu). \end{aligned}$$

We need to see that we can make further alterations to W in order to make the surgery obstruction vanish.

First, we take the connected sum with $-\sigma(W)/8$ copies of the E_8 manifold, namely the simply connected 4-manifold which is constructed by plumbing disc bundles $D^2 \times D^2$ according to the E_8 lattice. It turns out that the boundary of the resulting 4-manifold is the Poincaré homology sphere. One then caps off with the contractible topological 4-manifold whose boundary is the Poincaré homology sphere [Freedman and Quinn 1990, Corollary 9.3C]. This produces the E_8 manifold, a closed topological 4-manifold. It has a nonsingular intersection form, with a quadratic enhancement induced from a normal map to S^4 , and its signature is 8. By a negative copy of this 4-manifold we of course mean the same manifold but with the opposite choice of orientation. By making such a modification to W , we obtain a new normal map, which by abuse of notation we again denote by (W, F, e) , for which the obstruction in $L_0(\mathbb{Z})$ is trivial. Note that W still has fundamental group \mathbb{Z}^3 since $\pi_1(E_8 \text{ manifold}) \cong \{1\}$, and moreover ∂W is unchanged.

Next, we may need to alter W again, so that the three Arf invariant obstructions in $L_2(\mathbb{Z})$ vanish. For $i = 1, 2, 3$, define maps

$$\text{qr}_i : T^3 \times I = S^1 \times S^1 \times S^1 \times I \longrightarrow S^1 \times S^1$$

which forget the i -th S^1 factor and the I factor. Perform a homotopy of F to ensure that $\text{qr}_i \circ F$ is transverse to $* \in S^1 \times S^1$, and such that

$$F^{-1}(S^1 \times \{*\} \times \{*\} \times \{\partial I\}) \longrightarrow S^1 \times \{*\} \times \{*\} \times \{\partial I\}$$

is a homotopy equivalence (and similarly with the $*$ terms moved appropriately for $i = 2, 3$). This homotopy equivalence was already arranged in the proof of [Lemma 3.4](#), when we saw that the elements of Ω_1^{fr} can be removed. Let S_i be the surfaces $(\text{qr}_i \circ F)^{-1}(*)$; each surface has boundary ∂S_i given by the meridian μ_i and the corresponding S^1 factor of T^3 .

Let $\text{pr}_i : T^3 \times I = S^1 \times S^1 \times S^1 \times I \rightarrow S^1 \times I$ be the map which remembers the i -th S^1 factor and the I factor. Making F transverse to a point, $(\text{pr}_i \circ F)^{-1}(*)$ is a surface $\Sigma_i \subset W$. Since $F(S_i \cap \Sigma_i)$ is a single point and F is of degree one, we can assume that S_i and Σ_i intersect in a single point. By choosing different points in the I factor, we can ensure that the Σ_i are all distinct.

Now, as in [\[Davis 2006\]](#), for each i with nonzero-surgery obstruction in the corresponding $L_2(\mathbb{Z})$ summand of $L_4(\mathbb{Z}[\mathbb{Z}^3])$, remove a neighbourhood $\Sigma_i \times D^2$ of Σ_i and replace it with $\Sigma_i \times \text{cl}(S^1 \times S^1 \setminus D^2)$. That is, replace the D^2 factor with a punctured torus, but define the framing on the torus to be the framing which yields Arf invariant one, that is, the Lie framing on both S^1 factors. Since Σ_i is dual to S_i , this adds one to the Arf invariant of the element of $L_2(\mathbb{Z})$ represented by S_i , and so changes the Arf invariant one summands to having Arf invariant zero.

After these alterations we have a normal map $(G' : V' \rightarrow T^3 \times I, k')$, with vanishing surgery obstruction. Since the fundamental group \mathbb{Z}^3 is *good* in the sense of Freedman (polycyclic groups are good [\[Freedman and Quinn 1990, Theorem 5.1A\]](#)), the surgery sequence is exact in the topological category — see [\[ibid., Theorem 11.3A\]](#). We can therefore find embedded two-spheres representing a half-basis for $\pi_2(G')$, perform surgery, and obtain a topological 4-manifold V which is homotopy equivalent to $T^3 \times I$; in particular, V is a homology cobordism between M_L and T^3 .

Moreover, the following diagram commutes:

$$\begin{array}{ccccc} \pi_1(M_L) & \longrightarrow & \pi_1(V) & \longleftarrow & \pi_1(T^3) \\ f_* \downarrow & & \downarrow \cong & & \text{Id} \downarrow \cong \\ \pi_1(T^3) & \xrightarrow{\cong} & \pi_1(T^3 \times I) & \xleftarrow{\cong} & \pi_1(T^3) \end{array}$$

Since the meridians μ_i of L are mapped to standard generators of $\pi_1(T^3)$, an easy diagram chase shows that the homotopy classes of the meridians are preserved in the homology cobordism V . \square

4. Construction of links and grope concordance

In this section we give constructions of certain links with a given Milnor invariant, and construct grope concordances, using the methods of [Cochran 1990] and [Cha 2014].

4A. Iterated Bing doubles with a prescribed Milnor invariant. Let I be a multi-index with nonrepeating indices with length $m := |I| \geq 2$. We describe a rooted binary tree $T(m)$ associated to $m \geq 2$, which has m leaves: the right subtree of the root just consists of a single vertex, and the left subtree $T^\dagger(m)$ is the complete binary tree of height $h(m) := \lceil \log_2(m-1) \rceil$ with the rightmost $2(m-h(m)-1)$ pairs of leaves (and edges ending at these) removed. (By convention, a binary tree is always embedded in a plane with the root on the top.) That is, $T^\dagger(m)$ is a minimal height binary tree with $m-1$ leaves. For example, $T(m)$ for $m=7$ is shown in Figure 1.

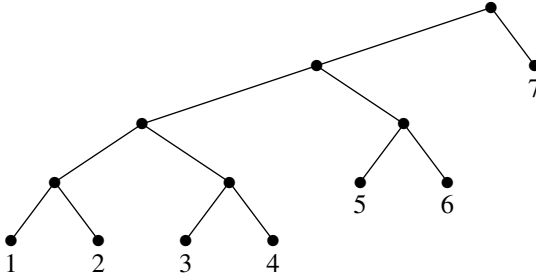


Figure 1. The tree $T(m)$ for $m=7$, labelled with $I=1234567$.

Following the proof of [Cochran 1990, Theorem 7.2], a rooted binary tree T describes a link with components corresponding to the leaves of T . First, a complete binary tree of height one is associated to a Hopf link. If T is obtained from T' by attaching two new leaves to a leaf v of T' , then the link associated to T is obtained from that of T' by Bing doubling the component corresponding to v .

Consider the link described by the tree $T(m)$. Labelling the leaves of $T(m)$ from left to right with the multi-index I (see Figure 1 for $I=1234567$), the components of the link are ordered. We denote this ordered link by L_I . Then, by [Cochran 1990, Theorem 8.1], the link L_I has $\bar{\mu}_{L_I}(I) = \pm 1$ and $\bar{\mu}_{L_I}(I') = 0$ for $|I'| < |I|$.

4B. Satellite construction and grope concordance of links. To construct links which are grope concordant, we employ the method of [Cha 2014, Section 4]. We

begin by giving the definition of grope concordance. The use of gropes in this context first appeared in [Cochran et al. 2003].

Definition 4.1 [Freedman and Teichner 1995]. A *grope* is a pair (2-complex, base circle) of a certain type described below. A grope has a height $h \in \mathbb{N}$. For $h = 1$ a grope is precisely a compact oriented surface Σ with a single boundary component which is the base circle. A grope of height $h + 1$ is defined inductively as follows: let $\{\alpha_i \mid i = 1, \dots, 2 \cdot \text{genus}\}$ be a standard symplectic basis of circles for Σ . Then a grope of height $h + 1$ is formed by attaching gropes of height h to each α_i along the base circles.

An annular grope is defined by replacing the bottom stage surface by a surface with two boundary components.

Definition 4.2 [Cha 2014, Definition 2.16]. Two m -component links L and L' in S^3 are *height n grope concordant* if there are m framed annular gropes G_i of height n , $i = 1, \dots, m$, disjointly embedded in $S^3 \times [0, 1]$, with the boundary of G_i the zero-framed i -th component of $L_i \subset S^3 \times \{0\}$ and $-L'_i \subset S^3 \times \{1\}$.

As mentioned in the introduction, we could also phrase our theorems in terms of Whitney towers, but for simplicity of exposition we stick to gropes. See [ibid., Section 2] for an exposition on gropes, Whitney towers, and n -solvable cobordisms (our Section 5 also contains a limited discussion of n -solvable cobordisms).

We recall that a *capped grope of height k* is a grope of height k together with 2-discs attached along each of the standard symplectic basis curves of the top-layer surfaces. The attached 2-discs are called *caps*, and the grope itself is called the *body*. We always assume that a capped grope embedded in a 4-manifold is framed.

We denote the exterior of a link L by X_L . If L is a link in S^3 , η is an unknotted circle in S^3 disjoint from L , and K is a knot, then we denote the satellite link of L with axis η and companion K by $L(\eta, K)$; this is the image of L under the homeomorphism $X_\eta \cup_\partial X_K \xrightarrow{\cong} S^3$, where the gluing identifies the longitude of η with the meridian of K , and vice versa.

Following [ibid., Definition 4.2], we call (L, η) a *satellite configuration of height k* if L is a link in S^3 , η is an unknotted circle in S^3 disjoint from L , and the 0-linking parallel of η in $X_\eta = X_\eta \times \{0\}$ bounds a capped grope of height k embedded in $X_\eta \times [0, 1]$ with body disjoint from $L \times [0, 1]$. The caps should be embedded in $X_\eta \times [0, 1]$ but may intersect $L \times [0, 1]$.

Lemma 4.3, stated below, describes how iterated satellite constructions using satellite configurations give us grope concordant links. The setup is as follows. Fix n . (To obtain Theorems 1.1 and 1.2, set $h = n + 2$.) Suppose that (L_0, η) is a satellite configuration of height $k \leq n$. (Later we will use the link L_1 described above as L_0 .) Suppose that (K_i, α_i) is a satellite configuration of height one, with K_i a slice knot, for $i = 0, \dots, n - k - 1$. Let J_0^j be the connected sum

of N_j copies of the knot described in [Cochran and Teichner 2007, Figure 3.6], where $\{N_j\}$ is an increasing sequence of integers which will be specified later. (Indeed, these will be given in terms of the Cheeger–Gromov bound on the ρ -invariants and, for the links of Theorem 1.1, in terms of the Kneser–Haken bound on the number of disjoint nonparallel incompressible surfaces. See Section 5, just before the proof of Theorem 5.3, and Section 4D, just before Lemma 4.7.) Define $J_i^j := K_{i-1}(\alpha_{i-1}, J_{i-1}^j)$ inductively for $i = 1, \dots, n-k$. Finally define $L_j := L_0(\eta, J_{n-k}^j)$.

Lemma 4.3 [Cha 2014, Proposition 4.7]. *The link L_j is height $n+2$ grope concordant to L_0 for all j .*

Proof. The same as the proof of [loc. cit.], except that L_0 replaces the Hopf link in the last sentence. \square

The following observation on the satellite construction is useful.

Lemma 4.4. *If $L' = L(\eta, K)$ is obtained from L by a satellite construction, then L and L' have the same Milnor $\bar{\mu}$ -invariants.*

Proof. It is well known that a satellite construction $L' = L(\eta, K)$ comes with an integral homology equivalence $f : (X_{L'}, \partial X_{L'}) \rightarrow (X_L, \partial X_L)$ which restricts to a homeomorphism on the boundary preserving longitudes and meridians (see, e.g., [Cha 2010, proof of Proposition 4.8; Cha and Orr 2013, Lemma 5.3]). As in [Cha et al. 2012, Lemma 2.1], by [Stallings 1965] it follows that f induces an isomorphism $\pi_1(X_L)/\pi_1(X_L)_q \cong \pi_1(X_{L'})/\pi_1(X_{L'})_q$ that preserves the classes of meridians and longitudes for any q , and consequently L and L' have identical $\bar{\mu}$ -invariants. \square

4C. Satellite configuration of iterated Bing doubles. Now we consider again the link L_I described in Section 4A. Recall that $k(m) := \lfloor \log_2(m-1) \rfloor$, where $m = |I|$. Let η be the zero-framed longitude of the component of L_I labelled with m , namely the component of the original Hopf link that is never Bing doubled in the construction of L_I .

- Lemma 4.5.** (1) *The pair (L_I, η) is a satellite configuration of height $k(m)$.*
 (2) *The curve η is nonzero in $\pi_1(L_I)/\pi_1(L_I)_m$.*
 (3) *For any knot K , the link $L_I(\eta, K)$ has zero-surgery manifold homeomorphic to the zero-surgery manifold of L_I .*

We remark that Lemma 4.5(2) will be used in Section 5.

Proof. Denote $L := L_I$ for this proof.

(1) We go back to the construction of L , and construct the grope as we construct L . We begin with the Hopf link (i.e., $m = 2$), and the curve η as a longitude of L_2 .

We also begin with a thickened cap $D^2 \times [-1, 1]$, such that $\partial D^2 \times \{0\} = \eta$. This intersects the other component of the Hopf link in a single point.

Every time a component K is Bing doubled in the construction of L , we arrange that one of the clasps lies in $D^2 \times [-1, 1]$, and then replace the thickened cap that intersected K with a genus-one capped surface with a single boundary component, whose body surface misses the new Bing doubled components, and such that each cap intersects one of the two new components. See [Figure 2](#), which is somewhat reminiscent of a figure in [\[Freedman and Quinn 1990, Chapter 2.1\]](#).

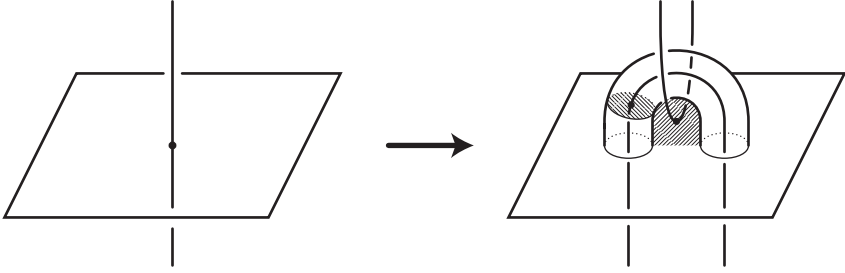


Figure 2. Replacing a cap with a capped surface.

Since a complete binary tree of height $k(m)$ can be embedded in $T(m)$, we obtain a symmetric embedded capped grope of the required height, with the body lying in the link exterior X_L and the caps intersecting the link transversely.

(2) The nonvanishing of the Milnor invariant $\bar{\mu}_L(I)$ implies that all of the longitudes of L are nontrivial in $\pi_1(X_L)/\pi_1(X_L)_{|I|}$.

(3) A Kirby diagram for the 3-manifold M_L given by zero-framed surgery on L can be produced by putting a 0 next to every component of L . If we perform a satellite construction with pattern K and with η as axis, this is equivalent to tying all the strands of L which intersect a disc D , whose boundary is η , in the knot K , with framing zero. In other words, replace the trivial string link in $D \times [0, 1]$ with the string link obtained by taking suitably many parallel copies of K .

But we can make a crossing change of these parallel copies of K at will, by performing handle slides, sliding the parallel strands over the zero-framed 2-handle attached along the component parallel to η . This gives a Kirby presentation of a homeomorphic 3-manifold.

By making sufficiently many such crossing changes/handle slides, all the parallel strands which the satellite construction ties in the knot K can be unknotted, recovering the link L . Thus the zero-surgery manifolds of the satellite link and the original link are homeomorphic. It is easy to see that the homotopy classes of the meridians of L are preserved under such homeomorphisms. \square

Now, let $n \geq k(|I|) = k(m)$. Let L_j be the links obtained by the construction just before [Lemma 4.3](#), using our (L_I, η) as (L_0, η) , and using the Stevedore satellite configuration described in [[Cha 2014](#), Figure 6], which for the reader's convenience is shown in [Figure 3](#), as the (K_i, α_i) . Then by [Lemma 4.3](#) and [Lemma 4.5\(1\)](#), the links L_j are height $n + 2$ grope concordant to the link $L_0 = L_I$.

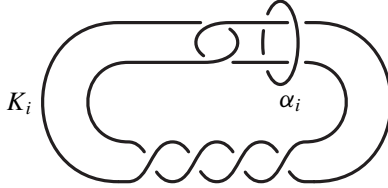


Figure 3. Stevedore satellite configuration (K_i, α_i) .

[Lemma 4.4](#) shows that the links L_j satisfy [Theorem 1.2\(1\)](#). They also satisfy [Theorem 1.2\(2\)](#) by [Lemma 4.5\(3\)](#). We have also proved, in [Lemma 4.3](#), the first part of [Theorem 1.2\(3\)](#): the links L_j are mutually height $n + 2$ grope concordant. The second part of [Theorem 1.2\(3\)](#), namely the failure of the links to be pairwise height $n + 3$ grope concordant, will be shown in [Section 5](#).

4D. Examples with nonhomeomorphic zero-surgery manifolds. In order to produce examples satisfying [Theorem 1.1\(1\)](#), we alter the construction of [Sections 4B](#) and [4C](#) to give examples with nonhomeomorphic zero-surgery manifolds. We consider the case of $m = 3$ and $I = 123$ only. Then the link $L := L_I$ described in [Section 4A](#) is the Borromean rings. Let η be the simple closed curve in $S^3 \setminus L$ shown in [Figure 4](#); x , y , and z denote the components of L .

The pair (L, η) also has two of the properties stated in [Lemma 4.5](#), for $m = 3$:

Lemma 4.6. (1) *The pair (L, η) is a satellite configuration of height one.*

(2) *In $\pi_1(X_L)$, $\eta = [x, y][[x, y], x]$, where x , y , and z are the Wirtinger generators corresponding to the dotted arcs in [Figure 4](#). Also, η is nontrivial in $\pi_1(X_L)/\pi_1(X_L)_3$.*

Here $[a, b]$ denotes the commutator $aba^{-1}b^{-1}$.

Proof. (1) Tubing the obvious disc bounded by η along the components of L that intersect it, we obtain a genus-two surface V with boundary η which is shown in [Figure 5](#). This is the body of the desired capped grope. The whole capped grope is the body taken together with the four caps shown in [Figure 5](#) as shaded discs.

(2) The claim that $\eta = [x, y][[x, y], x]$ follows from a straightforward computation in terms of the Wirtinger generators, reading undercrossings of η starting from the dot on η in [Figure 4](#). Since L has vanishing linking number, due to Milnor [[1957](#)]

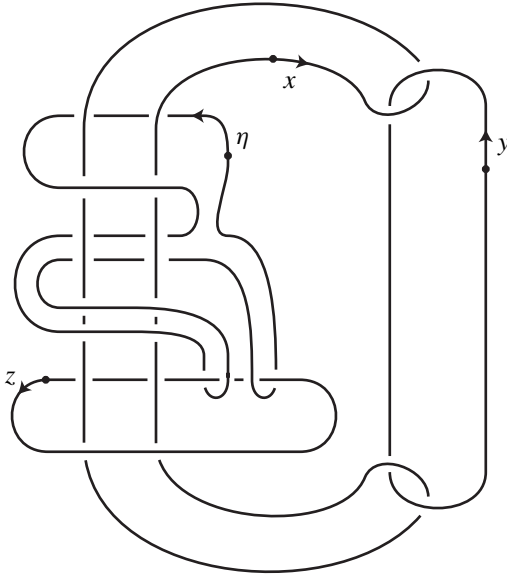


Figure 4. A satellite configuration on the Borromean rings.

(see also [Stallings 1965]), $\pi_1(X_L)/\pi_1(X_L)_3$ is isomorphic to F/F_3 , where F is the free group generated by x , y , and z . Consequently, $[[x, y], x] \in \pi_1(X_L)_3$ and $[x, y] \notin \pi_1(X_L)_3$. From this the second conclusion follows. \square

As in Section 4C, we apply the construction described just before Lemma 4.3, using our (L, η) as the seed link (L_0, η) and using the Stevedore satellite configuration described in [Cha 2014, Figure 6] (see our Figure 3) as (K_i, α_i) for $i = 0, \dots, n-2$ as above. Let the resulting links be the L_j . Then by Lemma 4.3, the L_j are height

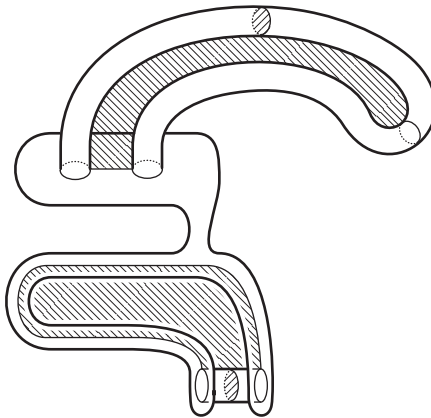


Figure 5. The capped grope bounded by η .

$n + 2$ grope concordant to the Borromean rings L , so these satisfy the first part of [Theorem 1.1\(3\)](#). The second part of [Theorem 1.1\(3\)](#), on the failure of the links to be pairwise height $n + 3$ grope concordant, will be shown in [Section 5](#).

Furthermore, the links L_j satisfies the hypothesis of [Theorem 1.3](#). First note that since our satellite operation does not change the knot type of the components, L_j has unknotted components. In particular, the Arf invariants of the components vanish. Recall from the proof of [Lemma 4.4](#) that there is a homology equivalence $f : X_{L_j} \rightarrow X_{L_0}$ obtained from the satellite construction $L_j = L_0(\eta, J_{n-1}^j)$; indeed, f is obtained by gluing the identity map of $X_{L_0 \sqcup \eta}$ with the standard homology equivalence

$$(X_{J_{n-1}^j}, \partial X_{J_{n-1}^j}) \longrightarrow (S^1 \times D^2, S^1 \times S^1)$$

along $S^1 \times S^1$. Since our curve $\eta \subset S^3 - L_0$ lies in the commutator subgroup of $\pi_1(S^3 - L_0)$, f is indeed a Λ -homology equivalence $X_{L_j} \rightarrow X_{L_0}$, by a Mayer–Vietoris argument. Filling it in with 3 solid tori, we obtain a Λ -homology equivalence $M_{L_j} \rightarrow T^3 = M_{L_0}$ as desired. Therefore, by applying [Theorem 1.3](#), it follows that the links L_j satisfy [Theorem 1.1\(2\)](#). We need to confirm that the L_j satisfy [Theorem 1.1\(1\)](#), namely, that the M_{L_j} are not homeomorphic. The underlying idea is as follows. Recall that L_j is defined by a satellite construction, starting with a knot J_0^j . In many cases, the JSJ pieces of the exterior of J_0^j become parts of the JSJ decomposition of M_{L_j} , so that the M_{L_j} have distinct JSJ decompositions. Since a complete proof of this seems to require complicated arguments (a technical issue is that an essential torus might not be parallel to a JSJ torus, because of Seifert fibred pieces), we will present a simpler argument using only the number of incompressible tori; this is enough for our purpose.

We need the following. The Kneser–Haken finiteness theorem [[Haken 1961](#)] states that for each 3-manifold M , there is a bound, say $C_{KH}(M)$, on the number of disjoint pairwise nonparallel incompressible surfaces that can be embedded in M . Recall that the knot J_0^j used in the construction of the link L_j is a connected sum of N_j knots, where $\{N_j\}$ was an increasing sequence to be specified (see the paragraph before [Lemma 4.3](#)). Here is the first requirement on the N_j : we choose the N_j inductively in such a way that $N_j > \max\{C_{KH}(M_{L_k}) \mid k = 0, 1, \dots, j - 1\}$.

Lemma 4.7. *The zero-surgery manifolds M_{L_i} and M_{L_j} are not homeomorphic for $i \neq j$.*

Proof. Recall that $M_{L_0} = M_L$ is the 3-torus T^3 . Consider $Y := M_L \setminus \nu(\eta)$, where $\nu(\eta)$ is an open tubular neighbourhood of η . For notational convenience, denote the exterior of J_{n-1}^j by $X := X_{J_{n-1}^j}$. The 3-manifold M_{L_j} is obtained by gluing Y and X along their boundaries. Let $T = \partial Y = \partial X$ be the common boundary torus. Note that M_{L_0} can also be described in the same way, using $J_{n-1}^0 := \text{unknot}$; in this case, the torus T is compressible in M_{L_0} since X is a solid torus.

Claim. For $j \geq 1$, the torus T is incompressible in Y .

Using the claim, we will show that the 3-manifolds M_{L_j} are not pairwise homeomorphic. Suppose $j \geq 1$. Since the knot J_{n-1}^j is obtained from an iterated satellite construction with the first-stage knot J_0^j a connected sum of N_j nontrivial knots, the exterior X of J_{n-1}^j has at least N_j incompressible tori, including the boundary T . Since $M_{L_j} = Y \cup_T X$ and T is incompressible in Y , it follows that there are N_j nonparallel incompressible tori in M_{L_j} . For any $k < j$, since $N_j > C_{KH}(M_{L_k})$, it follows that M_{L_j} is not homeomorphic to M_{L_k} .

Now, to complete the proof, we will verify the claim. If there is an essential curve on T which bounds a disc in Y , then it must be a zero-linking longitude, say η' , of η , since the meridian of η is a generator of $H_1(Y \setminus \eta) = \mathbb{Z}^4$. By the following lemma, we have a contradiction. \square

Lemma 4.8. *The class of η' is nontrivial in the fundamental group $\pi_1(Y \setminus \eta)$.*

Proof. We consider a Wirtinger presentation of $\pi_1(Y \setminus \eta)$ given as follows: it has 24 generators, denoted by x_1, \dots, x_{24} , associated to arcs in Figure 4. Here (x_1, \dots, x_{10}) , (x_{11}, x_{12}) , (x_{13}, \dots, x_{16}) , and (x_{17}, \dots, x_{24}) are those associated to the arcs of the components x , y , z , and η , respectively. In each component, the arc with a dot on it is the first one, and other arcs are ordered along the orientation. There are 27 relators:

$$\begin{aligned} & x_1 x_{11} \bar{x}_1 \bar{x}_{12}, \quad x_{11} x_1 \bar{x}_{11} \bar{x}_2, \quad x_2 x_{18} \bar{x}_2 \bar{x}_{19}, \quad x_{19} x_2 \bar{x}_{19} \bar{x}_3, \quad x_3 x_{20} \bar{x}_3 \bar{x}_{21}, \\ & x_3 x_{23} \bar{x}_3 \bar{x}_{22}, \quad x_{22} x_4 \bar{x}_{22} \bar{x}_3, \quad x_{21} x_4 \bar{x}_{21} \bar{x}_5, \quad x_5 x_{16} \bar{x}_5 \bar{x}_{13}, \quad x_{13} x_5 \bar{x}_{13} \bar{x}_6, \\ & x_{11} x_7 \bar{x}_{11} \bar{x}_6, \quad x_7 x_{11} \bar{x}_7 \bar{x}_{12}, \quad x_{13} x_8 \bar{x}_{13} \bar{x}_7, \quad x_8 x_{16} \bar{x}_8 \bar{x}_{15}, \quad x_{21} x_9 \bar{x}_{21} \bar{x}_8, \\ & x_{22} x_9 \bar{x}_{22} \bar{x}_{10}, \quad x_{10} x_{23} \bar{x}_{10} \bar{x}_{24}, \quad x_{10} x_{20} \bar{x}_{10} \bar{x}_{19}, \quad x_{19} x_1 \bar{x}_{19} \bar{x}_{10}, \quad x_1 x_{18} \bar{x}_1 \bar{x}_{17}, \\ & x_{15} x_{22} \bar{x}_{15} \bar{x}_{21}, \quad x_{22} x_{15} \bar{x}_{22} \bar{x}_{14}, \quad x_{24} x_{13} \bar{x}_{24} \bar{x}_{14}, \quad x_{13} x_{24} \bar{x}_{13} \bar{x}_{17}, \\ & \bar{x}_{11} \bar{x}_{19} x_{22} \bar{x}_{21} \bar{x}_{13} x_{11} x_{13} x_{21} \bar{x}_{22} x_{19}, \quad \bar{x}_1 x_7, \quad \bar{x}_{24} x_{22} x_8 \bar{x}_5. \end{aligned}$$

Indeed, the first 24 are the standard Wirtinger relators for the 4-component link $L \sqcup \eta$ (thus one of these may be omitted), and the last 3 relators arise from the zero-surgery performed along L . It is straightforward to read off the curve η' :

$$\eta' = x_1 \bar{x}_2 x_{10} \bar{x}_3 x_{15} x_3 \bar{x}_{10} \bar{x}_{13}.$$

We define a representation $\rho : \pi_1(Y \setminus \eta) \rightarrow \mathrm{SL}(2, \mathbb{Z}_5)$ by mapping the above 24 generators, respectively, to:

$$\begin{aligned} & \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 4 & 4 \end{bmatrix}, \\ & \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 3 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}, \\ & \begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 4 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

It can be verified that all the relators are sent to the identity, by a straightforward computation. (We found the representation ρ using a computer program.) Also, we have that

$$\rho(\eta') = \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}$$

is not the identity. This completes the proof. \square

5. Grope concordance and amenable signatures

In this section we show that the links described in Sections 4C and 4D are not height $n + 3$ grope concordant by using amenable signature obstructions from [Cha 2014]. In fact, the amenable signatures we use are obstructions to being n -solvably cobordant, which is a relative analogue for manifolds with boundary, or bordered manifolds, of the notion of n -solvability of [Cochran et al. 2003]. For our purpose it suffices to consider the case of link exteriors; an n -solvable cobordism between the exteriors X and X' of two links with the same number of components is a 4-manifold W with $\partial W = X \cup_{\partial} -X'$ satisfying the conditions described in [Cha 2014, Definition 2.8], where the boundary tori of X and X' are identified along the zero framing. Since we do not use the defining condition right now, instead of spelling it out here, we begin with its relationship to grope concordance. The following theorem originates from [Cochran et al. 2003, Theorem 8.11], and was given in our context in [Cha 2014].

Theorem 5.1 [*ibid.*, Theorems 2.16 and 2.13, and Remark 2.11]. *If two links are height $n + 2$ grope concordant, then their exteriors are n -solvably cobordant as bordered 3-manifolds.*

As our key ingredient to detect nonsolvably cobordant 3-manifolds and therefore non-grope-concordant links, we will use the Amenable Signature Theorem, which was first introduced in [Cha and Orr 2012] for homology cobordism of closed 3-manifolds and then generalised to n -solvable cobordisms of bordered 3-manifolds in [Cha 2014]. We state a special case which will be sufficient for our purpose. For a closed 3-manifold M and a homomorphism $\phi : \pi_1(M) \rightarrow G$, denote the von Neumann–Cheeger–Gromov ρ -invariant by $\rho^{(2)}(M, \phi) \in \mathbb{R}$. See, e.g., [Cochran et al. 2003, Section 5] as well as [Chang and Weinberger 2003; Harvey 2008; Cha 2008; Cha and Orr 2012] for definitions and useful properties of $\rho^{(2)}(M, \phi)$. Precise references for the properties that we need will be recalled as we go along.

Theorem 5.2 (A special case of [Cha 2014, Amenable Signature Theorem 3.2]). *Suppose W is an $(n + 1)$ -solvable cobordism between two bordered 3-manifolds X and X' , and G admits a subnormal series*

$$G = G_0 \supset G_1 \supset \cdots \supset G_n \supset G_{n+1} = \{e\}$$

with each quotient G_i/G_{i+1} torsion-free abelian. Then $\rho^{(2)}(X \cup_{\partial} -X', \phi) = 0$ for any $\phi : \pi_1(X \cup_{\partial} -X') \rightarrow G$ which factors through $\pi_1(W)$.

Recall that in our construction of the links L_j , the knot J_0^j was the connected sum of N_j copies of Cochran and Teichner's knot, say J . Now we proceed to specify the integers N_j . Denote by $\rho^{(2)}(K) := \int_{S^1} \sigma_K(\omega) d\omega$ the integral of the Levine–Tristram signature function over the circle normalised to length one. We have $\rho^{(2)}(J_0^j) = N_j \rho^{(2)}(J) = 4N_j/3$ by additivity under connected sum and [Cochran and Teichner 2007, Lemma 4.5]. Due to Cheeger and Gromov [1985], for any closed 3-manifold Y there is a constant $C_Y > 0$ such that $|\rho^{(2)}(Y, \psi)| < C_Y$ for any ψ . From now on we abbreviate $\ell := n - k(m)$. Define

$$R := C_{X_{L_0 \cup_{\partial} -X_{L_0}}} + 2 \sum_{i=0}^{\ell-1} C_{M_{K_i}}.$$

We choose the large integers N_j inductively in such a way that

$$N_j > 3R/4 + \max\{N_k \mid k < j\}.$$

Then we have

$$\rho^{(2)}(J_0^j) > R + \rho^{(2)}(J_0^k)$$

whenever $j > k$. For Theorem 1.1, we make these choices so that the condition in the preamble to Lemma 4.7 relating to the Kneser–Haken bound is simultaneously satisfied.

Now we start the proof that our links L_j are not height $n + 3$ grope concordant to one another. Let X and X' be the exteriors of L_j and L_k , respectively. To distinguish them in the notation, we denote the axis curve η in X by η_j , and we denote the corresponding axis curve in X' by η_k .

Recall that $m = |I|$ and that $k(m) = \lfloor \log_2(m - 1) \rfloor$. Also note that $k(m) + 1 = \lceil \log_2(m) \rceil$. By Theorem 5.1, it suffices to show the following:

Theorem 5.3. *For $n \geq k(m)$, the bordered 3-manifolds X and X' are not $(n + 1)$ -solvably cobordant when $j \neq k$.*

By Theorem 5.1, it then follows that our links L_j and L_k are not height $n + 3$ grope concordant when $j \neq k$.

Proof. The proof proceeds almost identically to that of [Cha 2014, Theorem 4.8], which combines the Amenable Signature Theorem of that reference with a higher-order Blanchfield duality argument for a certain 4-dimensional cobordism introduced in [Cochran et al. 2009] (see our W_0 below). So we will give an outline for our case and discuss differences from [Cha 2014, Theorem 4.8].

Suppose W is an $(n + 1)$ -solvable cobordism with $\partial W = X \cup_{\partial} -X'$. Similarly to [ibid., Section 4.3] (see the paragraph entitled ‘‘Cobordism associated to an iterated

satellite construction”), we consider a cobordism V with

$$\partial V = M_{J_0^j} \sqcup -M_{J_0^k} \sqcup M_{K_0} \sqcup -M'_{K_0} \sqcup \cdots \sqcup M_{K_{\ell-1}} \sqcup -M'_{K_{\ell-1}} \\ \sqcup (X_{L_0} \cup_{\partial} -X_{L_0}) \sqcup -(X \cup_{\partial} -X')$$

which is built by stacking cobordisms associated to satellite constructions [Cochran et al. 2009, p. 1429], where M'_{K_i} is a copy of M_{K_i} , and then construct a cobordism W_0 with

$$\partial W_0 = M_{J_0^j} \sqcup -M_{J_0^k} \sqcup M_{K_0} \sqcup -M'_{K_0} \sqcup \cdots \sqcup M_{K_{\ell-1}} \sqcup -M'_{K_{\ell-1}} \sqcup (X_{L_0} \cup_{\partial} -X_{L_0})$$

by attaching V to W along $X \cup_{\partial} -X'$. We omit the detailed construction of V and W_0 but state a couple of useful facts which can be verified as in [Cha 2014, Section 4.3]. Let $\{\mathcal{P}^r G\}$ be the rational derived series of a group G , i.e., $\mathcal{P}^0 G := G$ and $\mathcal{P}^{r+1} G$ is the kernel of $\mathcal{P}^r G \rightarrow H_1(\mathcal{P}^r G; \mathbb{Q})$. Let ϕ_0 be the quotient map $\pi_1(W_0) \rightarrow G := \pi_1(W_0)/\mathcal{P}^{n+1}\pi_1(W_0)$. Also we denote by ϕ_0 the restrictions of ϕ_0 to the components of ∂W_0 and to $W \subset W_0$, as an abuse of notation. Then we have the following facts:

- (1) $\rho^{(2)}(M_{J_0^j}, \phi_0) - \rho^{(2)}(M_{J_0^k}, \phi_0) + \rho^{(2)}(X_{L_0} \cup_{\partial} -X_{L_0}, \phi_0) \\ + \sum_{i=0}^{\ell-1} \rho^{(2)}(M_{K_i}, \phi_0) - \sum_{i=0}^{\ell-1} \rho^{(2)}(M'_{K_i}, \phi_0) = \rho^{(2)}(X \cup_{\partial} -X', \phi_0)$.
- (2) The image of the meridian of J_0^j in $M_{J_0^j} \subset \partial W_0$ under ϕ_0 is a nontrivial element in the torsion-free abelian subgroup $\mathcal{P}^n \pi_1(W)/\mathcal{P}^{n+1}\pi_1(W)$ of G . Similarly for k instead of j .

The proof of (1) is completely identical to that given in [Cha 2014, Section 4.3] (see the paragraphs entitled “Cobordism associated to an iterated satellite construction” and “Applications of Amenable Signature Theorem”): briefly, the $\rho^{(2)}$ -invariant of ∂W_0 , which is the left-hand side of (1), is equal to the L^2 -signature defect of $W_0 = V \cup_{X \cup_{\partial} -X'} W$ (this is a standard fact from index theory, or can be taken as the definition of $\rho^{(2)}$). It turns out that V has no contribution to the L^2 -signature defect, by [Cochran et al. 2009, Lemma 2.4]. So the left-hand side of (1) is equal to the L^2 -signature defect of W , which is the $\rho^{(2)}$ -invariant of ∂W , namely the right-hand side of (1).

The proof of (2) is almost identical to that given in [Cha 2014, Theorem 4.10]. Only the following change is required: in the initial step of the inductive argument in that result, it was shown that the image of (a parallel copy of) $\eta \subset X \subset \partial W$ is nontrivial under the quotient map $\pi_1(W) \rightarrow \pi_1(W)/\mathcal{P}^2\pi_1(W)$ (see the fourth paragraph of the proof) using a Blanchfield duality argument.

In our case, instead we use [Lemma 5.4](#) below, which is a generalisation of [[Cha et al. 2012](#), Lemma 3.5], to show that the image of η is nontrivial in the quotient $\pi_1(W)/\mathcal{P}^{(k(m)+1)}\pi_1(W)$. The argument used in [Lemma 5.4](#) is essentially an application of Dwyer's theorem.

Lemma 5.4. *If W is an n -solvable cobordism between two link exteriors (or, more generally, bordered 3-manifolds) X and X' , then the inclusions induce isomorphisms*

$$\pi_1(X)/\pi_1(X)_q \cong \pi_1(W)/\pi_1(W)_q \cong \pi_1(X')/\pi_1(X')_q$$

for $q \leq 2^n + 1$.

Proof. Recall Dwyer's theorem [[1975](#)]: if $f : X \rightarrow Y$ induces an isomorphism $H_1(X; \mathbb{Z}) \cong H_1(Y; \mathbb{Z})$ and an epimorphism

$$H_2(X; \mathbb{Z}) \longrightarrow H_2(Y; \mathbb{Z}) / \text{Im}\{H_2(Y; \mathbb{Z}[\pi_1(W)/\pi_1(W)_q]) \rightarrow H_2(Y; \mathbb{Z})\},$$

then f induces an isomorphism $\pi_1(X)_q/\pi_1(X)_{q+1} \cong \pi_1(Y)_q/\pi_1(Y)_{q+1}$.

In our case, by the definition of an n -solvable cobordism [[Cha 2014](#), Definition 2.8], we have $H_1(X; \mathbb{Z}) \cong H_1(W; \mathbb{Z}) \cong H_1(X'; \mathbb{Z})$. Also, by the same definition, there are elements $\ell_1, \dots, \ell_r, d_1, \dots, d_r$ lying in $H_2(W; \mathbb{Z}[\pi_1(W)/\pi_1(W)^{(n)})]$ such that the images of ℓ_i and d_j generate $H_2(W; \mathbb{Z})$. Since $\pi_1(W)^{(n)}$ is contained in $\pi_1(W)_{2^n}$, the H_2 condition of Dwyer's theorem is satisfied. Therefore, it follows that

$$\pi_1(X)_q/\pi_1(X)_{q+1} \cong \pi_1(W)_q/\pi_1(W)_{q+1} \cong \pi_1(X')_q/\pi_1(X')_{q+1}$$

for $q \leq 2^n$ by Dwyer's theorem. From this the desired conclusion follows by the five lemma. \square

Recall that [Lemma 4.5\(2\)](#) implies that $\eta \subset X$ represents a nontrivial element in $\pi_1(X)/\pi_1(X)_m$. Since the above isomorphisms preserve longitudes (and meridians), $\eta_j \subset X$ represents a nontrivial element in $\pi_1(W)/\pi_1(W)_m$. Since L_j has vanishing Milnor invariants of length less than $|I| = m$, we have $\pi_1(X)/\pi_1(X)_m \cong F/F_m$, where F is the free group with rank m , by [[Milnor 1957](#), Theorem 4]. Consequently $\pi_1(W)/\pi_1(W)_m$ is torsion-free.

We note that for any group π , we have $\pi^{(k(q)+1)} = \pi^{(\lceil \log_2(q) \rceil)} \subseteq \pi_q$. Therefore there is a quotient map $\pi_1(W)/\pi_1(W)^{(k(m)+1)} \rightarrow \pi_1(W)/\pi_1(W)_m$, and this map factors through $\pi_1(W)/\mathcal{P}^{(k(m)+1)}\pi_1(W)$ by the definition of $\mathcal{P}^{(k(m)+1)}$ and the fact that the codomain is torsion-free. Since η_j is nontrivial in $\pi_1(W)/\pi_1(W)_m$, η_j is also nontrivial in $\pi_1(W)/\mathcal{P}^{(k(m)+1)}\pi_1(W)$. By replacing j with k and X with X' we obtain the corresponding fact for η_k in X' .

To complete the proof of [Theorem 5.3](#), we proceed as in [[Cha 2014](#), Section 4.3]. Observe that for the normal subgroups $G_i := \mathcal{P}^i\pi_1(W_0)/\mathcal{P}^{n+1}\pi_1(W_0)$ of our G , the quotient G_i/G_{i+1} is torsion-free abelian. So by Amenable Signature [Theorem 5.2](#) we have $\rho^{(2)}(X \cup_{\partial} -X') = 0$. Since the curve η_j represents a nontrivial element

in a torsion-free abelian normal subgroup of G , the image of $\pi_1(M_{J_0^j})$ in G under ϕ_0 is the infinite cyclic group. By L^2 -induction (see, e.g., [Cheeger and Gromov 1985, page 8(2.3); Cochran et al. 2003, Proposition 5.13]) and [Cochran et al. 2004, Proposition 5.1], we have $\rho^{(2)}(M_{J_0^j}, \phi_0) = \rho^{(2)}(J_0^j)$, and similarly for J_0^k . Now, combining these two facts with (1), we obtain

$$(3) \quad \rho^{(2)}(J_0^j) - \rho^{(2)}(J_0^k) + \rho^{(2)}(X_{L_0} \cup_{\partial} -X_{L_0}, \phi_0) \\ + \sum_{i=0}^{\ell-1} \rho^{(2)}(M_{K_i}, \phi_0) - \sum_{i=0}^{\ell-1} \rho^{(2)}(M'_{K_i}, \phi_0) = 0.$$

Recall that

$$\left| \rho^{(2)}(X_{L_0} \cup_{\partial} -X_{L_0}, \phi_0) + \sum_{i=0}^{\ell-1} \rho^{(2)}(M_{K_i}, \phi_0) - \sum_{i=0}^{\ell-1} \rho^{(2)}(M'_{K_i}, \phi_0) \right| \\ < R := C_{X_{L_0} \cup_{\partial} -X_{L_0}} + 2 \sum_{i=0}^{\ell-1} C_{M_{K_i}},$$

and in the preamble to [Theorem 5.3](#), we chose N_j so that $|\rho^{(2)}(J_0^k) - \rho^{(2)}(J_0^j)| > R$ whenever $k \neq j$. Therefore (3) implies that $j = k$. Thus the existence of the $(n+1)$ -solvable cobordism W implies that $j = k$, which is the contrapositive of the desired statement. \square

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