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CERTAIN SELF-HOMOTOPY EQUIVALENCES ON WEDGE PRODUCTS OF MOORE SPACES

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For a based 1-connected finite CW-complex X, let $\mathcal{E}(X)$ denote the group of homotopy classes of self-homotopy equivalences on X, and $\mathcal{E}_\sharp^{\dim+r}(X)$ the subgroup of $\mathcal{E}(X)$ of homotopy classes of self-homotopy equivalences on X that induce the identity homomorphism on the homotopy groups of X in dimensions $\le \dim X + r$. For two given Moore spaces $M_1 = M(Z_q, n+1)$ and $M_2 = M(Z_p, n)$ with $n \ge 5$, we investigate the subsets of $[M_1, M_2]$ and $[M_2, M_1]$ consisting of homotopy classes of maps that induce the trivial homomorphism between the homotopy groups of M_1 and those of M_2 in dimensions $\le \dim X + r$. Using the results of this investigation, we completely determine the subgroups $\mathcal{E}_\sharp^{\dim+r}\big(M(Z_q, n+1) \vee M(Z_p, n)\big)$, where p and q are positive integers, for $n \ge 5$ and $r = 0, 1$.

1. Introduction

If X and Y are based topological spaces, let $[X, Y]$ denote the set of homotopy classes of based maps from X to Y, let $\mathcal{E}(X)$ denote the subset of $[X, X]$ that consists of homotopy classes of self-homotopy equivalences of X and let $\mathcal{E}_\sharp^{\dim+r}(X)$ denote the set of homotopy classes of self-homotopy equivalences that induce the identity on the homotopy groups of X in dimensions at most $\dim X + r$. Then, $\mathcal{E}(X)$ is a group with a group operation given by the composition of homotopy classes, and $\mathcal{E}_\sharp^{\dim+r}(X)$ is a subgroup of $\mathcal{E}(X)$. The group $\mathcal{E}(X)$ and certain natural subgroups including $\mathcal{E}_\sharp^{\dim+r}(X)$ are fundamental objects in homotopy theory and have been studied extensively. For a survey of the known results and applications of $\mathcal{E}(X)$, see [Arkowitz 1990].

When G is an abelian group, we let $M(G, n)$ denote the Moore space, that is, the space with G as a single nonvanishing homology group at n-level. Also, in this case, $M(G, n)$ is a simply connected space. We note that if $n \ge 3$, then $M(G, n)$ is characterized by

$$\widetilde{H}_i(M(G, n)) \cong \begin{cases} G & \text{if } i = n, \\ 0 & \text{if } i \ne n. \end{cases}$$

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Let $C(G,n)$ denote the co-Moore space of type (G,n) defined by

$$\widetilde{H}^i(C(G,n)) \cong \begin{cases} G & \text{if } i = n, \\ 0 & \text{if } i \neq n. \end{cases}$$

If G is a finitely generated abelian group and $G = F \oplus T$, where F is a free abelian group of rank r and T is a finite group, then $M(G,n) = M(F,n) \vee M(T,n)$ and $C(G,n) = M(F,n) \vee M(T,n-1)$ for $n \geq 3$.

Arkowitz and Maruyama [1998] showed that $\mathcal{E}_\sharp^{\dim}(M(G,n)) \cong \oplus^{(r+s)s} Z_2$ and $\mathcal{E}_\sharp^{\dim+1}(M(G,n)) = 1$ for $n > 3$, where r is the rank of G and s is the number of 2-torsion summands in G. Moreover, they completely determined $\mathcal{E}_\sharp^{\dim}(C(G,n))$ for $n \geq 3$ by means of 2×2 matrices, where G is a finitely generated abelian group.

Jeong [2010] computed the groups $\mathcal{E}_\sharp^{\dim}(Y)$ for $Y = M(Z_p, n+1) \vee M(Z_p, n)$, $n \geq 5$ as follows:

$$\mathcal{E}_\sharp^{\dim}(Y) \cong \begin{cases} Z_p & \text{if } p \text{ is odd}, \\ Z_2 \oplus Z_2 & \text{if } p \equiv 2 \pmod 4, \\ Z_2 \oplus Z_2 \oplus Z_2 & \text{if } p \equiv 0 \pmod 4. \end{cases}$$

In this paper we study the self-homotopy equivalences on the wedge product $X = M(Z_q, n+1) \vee M(Z_p, n)$ for $n \geq 5$, where p and q are positive integers. For two given Moore spaces $M_1 = M(Z_q, n+1)$ and $M_2 = M(Z_p, n)$, we compute $[M_1, M_2]$ and $[M_2, M_1]$ and find their generators. Moreover, we investigate the subset of $[M_1, M_2]$ or $[M_2, M_1]$ that consists of elements whose induced homomorphisms are trivial between the homotopy groups of M_1 and those of M_2 in dimensions at most $\dim X + r$ with $r = 0, 1$. Using these results, we completely determine the groups $\mathcal{E}_\sharp^{\dim+r}(X)$ for $r = 0, 1$. As a result, we obtain Table 1 and the following:

$$\mathcal{E}_\sharp^{\dim+1}(X) \cong \begin{cases} 1 & \text{if } q \text{ is odd or } p \text{ is odd } (d = 1), \\ Z_d & \text{if } q \text{ is odd or } p \text{ is odd } (d \neq 1), \\ Z_{d/2} \oplus Z_2 & \text{if } p \equiv 0 \pmod 4 \text{ and } (p, 24) = 4 \text{ or } 12\ (d \neq 1), \\ Z_{d/2} & \text{if } p \equiv 0 \pmod 4 \text{ and } (p, 24) = 8 \text{ or } 24\ (d \neq 1), \\ Z_{d/2} & \text{if } q \equiv 2,\ p \equiv 2 \pmod 4, \\ Z_{d/2} \oplus Z_2 & \text{if } q \equiv 0,\ p \equiv 2 \pmod 4, \end{cases}$$

where d is the greatest common divisor of p and q.

The space X is neither a Moore space nor a co-Moore space but is characterized by finite homology groups and cohomology groups. That is,

$$\widetilde{H}_i(X) \cong \begin{cases} Z_p & \text{if } i = n, \\ Z_q & \text{if } i = n+1, \\ 0 & \text{otherwise}, \end{cases}$$

	q is odd $d=1$	q is odd $d\neq 1$	$q\equiv 2 \pmod 4$	$q\equiv 0 \pmod 4$
p is odd ($d=1$)	1	$\cdot$	Z_2	Z_2
p is odd ($d\neq 1$)	$\cdot$	Z_d	$Z_2\oplus Z_d$	$Z_2\oplus Z_d$
$p\equiv 2 \pmod 4$	1	Z_d	$Z_2\oplus Z_{d/2}\oplus Z_2$	$Z_2\oplus Z_{d/2}\oplus Z_4$
$p\equiv 0 \pmod 4$	1	Z_d	$Z_2\oplus Z_{d/2}\oplus Z_2\oplus Z_2$	$Z_2\oplus Z_{d/2}\oplus Z_2\oplus Z_2$

Table 1. Isomorphism class of the groups $\mathcal{E}_\sharp^{\dim}(X)$.

and

$$\widetilde{H}^i(X,\pi)\cong\begin{cases}\mathrm{Hom}(Z_p,\pi) & \text{if } i=n,\\ \mathrm{Ext}(Z_p,\pi)\oplus\mathrm{Hom}(Z_q,\pi) & \text{if } i=n+1,\\ \mathrm{Ext}(Z_q,\pi) & \text{if } i=n+2,\\ 0 & \text{otherwise.}\end{cases}$$

From this perspective, X is an interesting space for studying self-homotopy equivalences.

Throughout this paper, all topological spaces are based and have the based homotopy type of a finite l-connected CW-complex. All maps and homotopies will preserve base points. For the spaces X and Y, we denote by $[X, Y]$ the set of homotopy classes of maps from X to Y. We do not distinguish between the notation of a map $X\to Y$ and that of its homotopy class in $[X, Y]$. If a group G is generated by a set $\{a_1,\dots,a_n\}$, then we denote the group by $G\{a_1,\dots,a_2\}$ or $G=\langle a_1,\dots,a_n\rangle$.

2. Preliminaries

Let X be a space. Then, we denote by SX the suspension of X and by S^nX the iterated suspension defined by $S^nX=S(S^{n-1}X)$. Let $f: A\to B$ be a map and let $C_f=B\cup_f CA$ be the mapping cone of f. Then, we have a Puppe sequence [1958] for f,

$$A\xrightarrow{f}B\xrightarrow{i}C_f\xrightarrow{\pi}SA\xrightarrow{Sf}SB\xrightarrow{Si}SC_f\xrightarrow{S\pi}S^2A\xrightarrow{S^2f}S^2B\longrightarrow\cdots,$$

such that the following sequence is exact for any space X:

$$\cdots\longrightarrow[SC_f,X]\xrightarrow{S\pi^*}[SB,X]\xrightarrow{Sf^*}[SA,X]\xrightarrow{\pi^*}[C_f,X]\xrightarrow{i^*}[B,X]\xrightarrow{f^*}[A,X],$$

where S^nf is a suspension map induced by f.

If A is m-connected and B is n-connected, then we have the following exact sequence for any CW-complex Y with dimension at most $m+n$ as a dual sequence of the above sequence [Blakers and Massey 1952]:

$$[Y, A] \xrightarrow{f_*} [Y, B] \xrightarrow{i_*} [Y, C_f] \xrightarrow{\pi_*} [Y, SA] \xrightarrow{Sf_*} [Y, SB] \longrightarrow \cdots .$$

Both sequences will be called *the exact sequences associated with the cofibration*

$$B \to C_f \to SA.$$

Proposition 2.1 [Arkowitz and Maruyama 1998]. *If* X *is* $(k-1)$*-connected*, Y *is* $(l-1)$*-connected*, $k, l \geq 2$ *and* $\dim P \leq k+l-1$, *then the projections* $X \vee Y \to X$ *and* $X \vee Y \to Y$ *induce a bijection*

$$[P, X \vee Y] \to [P, X] \oplus [P, Y].$$

Proposition 2.1 is a consequence of [Spanier 1966, p. 405] since the inclusion $X \vee Y \to X \times Y$ is a $(k+l-1)$-equivalence.

Next, we consider abelian groups G_1 and G_2 and Moore spaces $M_1 = M(G_1, n_1)$ and $M_2 = M(G_2, n_2)$. Let $X = M_1 \vee M_2$. We denote by $i_j : M_j \to X$ the inclusion and by $p_j : X \to M_j$ the projection, where $j = 1, 2$. If $f : X \to X$, then we define $f_{jk} : M_k \to M_j$ by $f_{jk} = p_j f i_k$ for $j, k = 1, 2$.

If $f : X \to Y$ is a map, then $f_{\sharp n} : \pi_n(X) \to \pi_n(Y)$ denotes the induced homomorphism in dimension n.

Proposition 2.2 [Arkowitz and Maruyama 1998]. *The function* θ *that assigns to each* $f \in [X, X]$ *the* 2×2 *matrix*

$$\theta(f) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

where $f_{jk} \in [M_k, M_j]$, *is a bijection. In addition*:

(1) $\theta(f+g) = \theta(f) + \theta(g)$, *so* θ *is an isomorphism* $[X, X] \to \bigoplus_{j,k=1,2}[M_k, M_j]$.

(2) $\theta(fg) = \theta(f)\theta(g)$, *where* fg *denotes composition in* $[X, X]$ *and* $\theta(f)\theta(g)$ *denotes matrix multiplication.*

(3) *If* $\alpha_r : \pi_r(M_1) \oplus \pi_r(M_2) \to \pi_r(M_1 \vee M_2)$ *is the homomorphism induced by the inclusions and* $\beta_r : \pi_r(M_1 \vee M_2) \to \pi_r(M_1) \oplus \pi_r(M_2)$ *the homomorphism induced by the projections respectively*, *then*

$$\beta_r f_{\sharp r} \alpha_r(x, y) = \big(f_{11\,\sharp r}(x) + f_{12\,\sharp r}(y),\, f_{21\,\sharp r}(x) + f_{22\,\sharp r}(y)\big)$$

for $x \in \pi_r(M_1)$ *and* $y \in \pi_r(M_2)$.

Proposition 2.3 [Araki and Toda 1965]. (1) $\pi_n(M(Z_q, n)) \cong Z_q$ *for all* q.

(2) $\pi_{n+1}(M(Z_q, n)) \cong \begin{cases} 0 & \text{if } q \text{ is odd}, \\ Z_2 & \text{if } q \text{ is even}. \end{cases}$

(3) $\pi_{n+2}(M(Z_q,n)) \cong \begin{cases} 0 & \text{if } q \text{ is odd,} \\ Z_4 & \text{if } q \equiv 2 \pmod 4, \\ Z_2 \oplus Z_2 & \text{if } q \equiv 0 \pmod 4. \end{cases}$

(4) $\pi_{n+3}(M(Z_q,n)) \cong \begin{cases} Z_{(q,24)} & \text{if } q \text{ is odd,} \\ Z_{(q,24)} \oplus Z_2 & \text{if } q \text{ is even.} \end{cases}$

The generators of $[S^{n+i}, S^n]$ can be summarized thus [Toda 1962]:

	$i<0$	$i=0$	$i=1$	$i=2$	$i=3$	$i=4,5$
$[S^{n+i}, S^n]$	0	Z	Z_2	Z_2	Z_{24}	0
Generator		ι	η	η^2	ν	0

Proposition 2.4 [Araki and Toda 1965].

(1) $[M(Z_q,n), S^n] \cong \begin{cases} 0 & \text{if } q \text{ is odd,} \\ Z_2 & \text{if } q \text{ is even.} \end{cases}$

(2) $[M(Z_q,n+1), S^n] \cong \begin{cases} 0 & \text{if } q \text{ is odd,} \\ Z_4 & \text{if } q \equiv 2 \pmod 4, \\ Z_2 \oplus Z_2 & \text{if } q \equiv 0 \pmod 4. \end{cases}$

Proposition 2.5 [Arkowitz and Maruyama 1998]. *For the Moore space $X = M(G,n)$:*

(1) $\mathcal{E}_{\sharp}^{\dim}(X) \cong \oplus^{(r+s)s} Z_2$, *where r is the rank of G and s is the number of* 2-*torsion summands in G.*

(2) $\mathcal{E}_{\sharp}^{\dim+1}(X) \cong 1$ *if $n > 3$.*

Proposition 2.6 (universal coefficient theorem for homotopy groups with coefficients [Hilton 1965]). *There is an exact sequence*

$$0 \to \operatorname{Ext}(G, \pi_{n+1}(X)) \to \pi_n(G; X) \to \operatorname{Hom}(G, \pi_n(X)) \to 0,$$

where $\pi_n(G;X)$, the n-th homotopy group of X with coefficients in G, is given by $\pi_n(G;X) = [M(G,n), X]$, where $M(G,n)$ is a Moore space.

3. Generators of the sets of homotopy classes on Moore spaces

In this section, we find generators of homotopy groups of Moore spaces and the sets of homotopy classes between two Moore spaces. Let

$$M_1 = M(Z_q, n+1) = S^{n+1} \cup_q e^{n+2} \quad \text{and} \quad M_2 = M(Z_p, n) = S^n \cup_p e^{n+1},$$

with $p, q \geq 1$. Then, there are two mapping cone sequences

$$S^{n+1} \xrightarrow{q\iota_1} S^{n+1} \xrightarrow{i_1} S^{n+1} \cup_q e^{n+2} \xrightarrow{\pi_1} S^{n+2} \xrightarrow{q\iota_1} S^{n+2}$$

and

$$S^n \xrightarrow{p_{\iota_2}} S^n \xrightarrow{i_2} S^n \cup_p e^{n+1} \xrightarrow{\pi_2} S^{n+1} \xrightarrow{p_{\iota_2}} S^{n+1},$$

where p_{ι_2} and q_{ι_1} are maps with degree p and q respectively.

Remark 3.1. We find generators of $\pi_m(M(Z_r, n))$, for $n \leq m \leq n+2$.

Recall that $\pi_n(M(Z_r, n)) \cong Z_r$. From the mapping cone sequence

$$S^n \xrightarrow{r_\iota} S^n \xrightarrow{i} M(Z_r, n) \xrightarrow{\pi} S^{n+1} \xrightarrow{r_\iota} S^{n+1},$$

we obtain the long exact sequence

$$\pi_n(S^n) \xrightarrow{r_{\iota\sharp}} \pi_n(S^n) \xrightarrow{i_\sharp} \pi_n(M(Z_r, n)) \xrightarrow{\pi_\sharp} \pi_n(S^{n+1}) \xrightarrow{r_{\iota\sharp}} \pi_n(S^{n+1}).$$

By the results in [Toda 1962], we have the sequence

$$Z\{\iota\} \xrightarrow{r_{\iota\sharp}} Z\{\iota\} \xrightarrow{i_\sharp} \pi_n(M(Z_r, n)) \longrightarrow 0,$$

so $i_\sharp$ is surjective. Thus, $\pi_n(M(Z_r, n)) \cong Z\{\iota\}/\operatorname{Im}(r_{\iota\sharp})$. Let $i_\sharp(\iota) = i$. Then, we can take i as a generator of $\pi_n(M(Z_r, n))$.

Next, we find a generator of $\pi_{n+1}(M(Z_r, n))$. There are two cases according to the parity of the positive integer r. If r is odd, then $\pi_{n+1}(M(Z_r, n))$ is trivial. If r is even, then we can take $i_\sharp(\eta)$ as a generator of $\pi_{n+1}(M(Z_r, n))$, where η is the generator of $\pi_{n+1}(S^n)$.

Finally, we find a generator of $\pi_{n+2}(M(Z_r, n))$. Consider the exact sequence

$$\pi_{n+2}(S^n) \xrightarrow{r_{\iota\sharp}} \pi_{n+2}(S^n) \xrightarrow{i_\sharp} \pi_{n+2}(M(Z_r, n)) \xrightarrow{\pi_\sharp} \pi_{n+2}(S^{n+1}) \xrightarrow{r_{\iota\sharp}} \pi_{n+2}(S^{n+1}).$$

Then by the results in [Toda 1962], we have the exact sequence

$$Z_2\{\eta^2\} \xrightarrow{r_{\iota\sharp}} Z_2\{\eta^2\} \xrightarrow{i_\sharp} \pi_{n+2}(M(Z_r, n)) \xrightarrow{\pi_\sharp} Z_2\{\eta\} \xrightarrow{q_{\iota\sharp}} Z_2\{\eta\}.$$

Since r is an even number, we obtain the exact sequence

$$0 \longrightarrow Z_2\{\eta^2\} \xrightarrow{i_\sharp} \pi_{n+2}(M(Z_r, n)) \xrightarrow{\pi_\sharp} Z_2\{\eta\} \longrightarrow 0.$$

If $r \equiv 2 \pmod 4$, then $\pi_{n+2}(M(Z_r, n)) \cong Z_4\{\bar{\eta}\}$ such that $i_\sharp(\eta^2) = 2\bar{\eta}$ and $\pi_\sharp(\bar{\eta}) = \eta$. On the other hand, if $r \equiv 0 \pmod 4$, then $\pi_{n+2}(M(Z_r, n)) \cong Z_2 \oplus Z_2\{\eta_1, \eta_2\}$ such that $i_\sharp(\eta^2) = \eta_1$ and $\pi_\sharp(\eta_2) = \eta$.

By Remark 3.1, it follows that

$$\pi_{n+1}(M_1) \cong Z_q\{i_1\}, \qquad \pi_n(M_2) \cong Z_p\{i_2\},$$
$$\pi_{n+2}(M_1) \cong Z_2\{i_{1\sharp}(\eta)\}, \qquad \pi_{n+1}(M_2) \cong Z_2\{i_{2\sharp}(\eta)\}.$$

Moreover, $\pi_{n+2}(M_2) \cong Z_4\{\bar{\eta}\}$ or $\pi_{n+2}(M_2) \cong Z_2 \oplus Z_2\{\eta_1, \eta_2\}$.

Lemma 3.2. *Let p and q be positive integers and (p, q) be the greatest common divisor of p and q. Consequently, if $(p, q) = d \neq 1$, then $[M_2, M_1] \cong Z_d\{\pi_2^*(i_1)\}$ and if $(p, q) = 1$, then $[M_2, M_1] \cong 0$.*

Proof. Consider the mapping cone sequence of M_2,

$$S^n \xrightarrow{p_{\iota_2}} S^n \xrightarrow{i_2} S^n \cup_p e^{n+1} \xrightarrow{\pi_2} S^{n+1} \xrightarrow{p_{\iota_2}} S^{n+1}.$$

This sequence induces the following exact sequence:

$$\pi_{n+1}(M_1) \xrightarrow{p_{\iota_2}*} \pi_{n+1}(M_1) \xrightarrow{\pi_2 *} [M_2, M_1] \xrightarrow{i_2 *} \pi_n(M_1) \xrightarrow{p_{\iota_2}*} \pi_n(M_1).$$

Since $\pi_{n+1}(M_1) \cong Z_q\{i_1\}$ and $\pi_n(M_1) \cong 0$, the exact sequence above becomes

$$Z_q\{i_1\} \xrightarrow{p_{\iota_2}^*} Z_q\{i_1\} \xrightarrow{\pi_2^*} [M_2, M_1] \longrightarrow 0.$$

If $(p, q) = 1$, the first $p_{\iota_2}^*$ is an isomorphism, so $[M_2, M_1] \cong 0$. Let $(p, q) = d \neq 1$. Then, since π_2^* is surjective and $p_{\iota_2}^*(i_1) = pi_1$, we have

$$[M_2, M_1] = \operatorname{im} \pi_2^* \cong Z_q\{i_1\}/\operatorname{im} p_{\iota_2}^* \cong Z_d\{\pi_2^*(i_1)\}.$$ □

Lemma 3.3. *If p or q is odd, then $[M_1, M_2] \cong 0$.*

Proof. Consider the mapping cone sequence of M_1,

$$S^{n+1} \xrightarrow{q_{\iota_1}} S^{n+1} \xrightarrow{i_1} S^{n+1} \cup_q e^{n+2} \xrightarrow{\pi_1} S^{n+2} \xrightarrow{q_{\iota_1}} S^{n+2}.$$

Then, we have the exact sequence

$$\pi_{n+2}(M_2) \xrightarrow{q_{\iota_1}^*} \pi_{n+2}(M_2) \xrightarrow{\pi_1^*} [M_1, M_2] \xrightarrow{i_1^*} \pi_{n+1}(M_2) \xrightarrow{q_{\iota_1}^*} \pi_{n+1}(M_2).$$

Let $p \equiv 2 \pmod 4$ and let q be odd. Then, since $\pi_{n+1}(M_2) \cong Z_2$ and $\pi_{n+2}(M_2) \cong Z_4$, we have the sequence

$$Z_4 \xrightarrow{q_{\iota_1}^*} Z_4 \xrightarrow{\pi_1^*} [M_1, M_2] \xrightarrow{i_1^*} Z_2 \xrightarrow{q_{\iota_1}^*} Z_2.$$

Furthermore, since $(q, 4) = 1$ and $(q, 2) = 1$, each $q_{\iota_1}^*$ is an isomorphism. Thus we have the exact sequence

$$0 \to [M_1, M_2] \to 0.$$

Therefore, $[M_1, M_2] \cong 0$.

In the case where $p \equiv 0 \pmod 4$ and q is odd, we can give a similar proof.

Next, let p be odd. Since $\pi_{n+1}(M_2)$ and $\pi_{n+2}(M_2)$ are trivial groups, so is $[M_1, M_2]$ by exactness. □

Let p and q be even. From the exact sequences associated with the cofibrations $S^{n+1} \to M_1 \to S^{n+2}$ and $S^n \to M_2 \to S^{n+1}$, we have the following commutative diagram with exact rows and columns:

$$
\begin{tikzcd}
{[S^{n+2}, S^n]} \arrow[r, "q_{\iota_1}^*"] \arrow[d, "p_{\iota_2 *}"] & {[S^{n+2}, S^n]} \arrow[r, "\pi_1^*"] \arrow[d, "p_{\iota_2 *}"] & {[M_1, S^n]} \arrow[r, "i_1^*"] \arrow[d, "p_{\iota_2 *}"] & {[S^{n+1}, S^n]} \arrow[r, "q_{\iota_1}^*"] \arrow[d, "p_{\iota_2 *}"] & {[S^{n+1}, S^n]} \arrow[d, "p_{\iota_2 *}"] \\
{[S^{n+2}, S^n]} \arrow[r, "q_{\iota_1}^*"] \arrow[d, "i_{2*}"] & {[S^{n+2}, S^n]} \arrow[r, "\pi_1^*"] \arrow[d, "i_{2*}"] & {[M_1, S^n]} \arrow[r, "i_1^*"] \arrow[d, "i_{2*}"] & {[S^{n+1}, S^n]} \arrow[r, "q_{\iota_1}^*"] \arrow[d, "i_{2*}"] & {[S^{n+1}, S^n]} \arrow[d, "i_{2*}"] \\
{[S^{n+2}, M_2]} \arrow[r, "q_{\iota_1}^*"] \arrow[d, "\pi_{2*}"] & {[S^{n+2}, M_2]} \arrow[r, "\pi_1^*"] \arrow[d, "\pi_{2*}"] & {[M_1, M_2]} \arrow[r, "i_1^*"] \arrow[d, "\pi_{2*}"] & {[S^{n+1}, M_2]} \arrow[r, "q_{\iota_1}^*"] \arrow[d, "\pi_{2*}"] & {[S^{n+1}, M_2]} \arrow[d, "\pi_{2*}"] \\
{[S^{n+2}, S^{n+1}]} \arrow[r, "q_{\iota_1}^*"] \arrow[d, "p_{\iota_2 *}"] & {[S^{n+2}, S^{n+1}]} \arrow[r, "\pi_1^*"] \arrow[d, "p_{\iota_2 *}"] & {[M_1, S^{n+1}]} \arrow[r, "i_1^*"] \arrow[d, "p_{\iota_2 *}"] & {[S^{n+1}, S^{n+1}]} \arrow[r, "q_{\iota_1}^*"] \arrow[d, "p_{\iota_2 *}"] & {[S^{n+1}, S^{n+1}]} \arrow[d, "p_{\iota_2 *}"] \\
{[S^{n+2}, S^{n+1}]} \arrow[r, "q_{\iota_1}^*"] & {[S^{n+2}, S^{n+1}]} \arrow[r, "\pi_1^*"] & {[M_1, S^{n+1}]} \arrow[r, "i_1^*"] & {[S^{n+1}, S^{n+1}]} \arrow[r, "q_{\iota_1}^*"] & {[S^{n+1}, S^{n+1}]}
\end{tikzcd}
$$

Lemma 3.4. *Let* $(p, q) \neq 1$. *Then, if either* $p \equiv 0 \pmod 4$ *and* $q \equiv 2 \pmod 4$ *or* $p \equiv 2 \pmod 4$ *and* $q \equiv 0 \pmod 4$, *we have* $[M_1, M_2] \cong Z_4 \oplus Z_2$.

Proof. Suppose that $p \equiv 0 \pmod 4$ and $q \equiv 2 \pmod 4$. With the results in [Araki and Toda 1965], we obtain the following diagram from the above diagram:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & Z_4 \arrow[d, "i_{2*}"] & & \\
0 \arrow[r] & Z_2 \oplus Z_2 \arrow[r, "\pi_1^*"] & {[M_1, M_2]} \arrow[r, "i_1^*"] \arrow[d, "\pi_{2*}"] & Z_2 \arrow[r] & 0 \\
 & & Z_2 \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}
$$

Thus, $[M_1, M_2]$ is isomorphic to one of three groups: Z_8, $Z_4 \oplus Z_2$ or $Z_2 \oplus Z_2 \oplus Z_2$. Since i_{2*} is injective, $[M_1, M_2]$ has an element of order 4. However, $Z_2 \oplus Z_2 \oplus Z_2$ does not have an element of order 4. Since π_1^* is injective, $[M_1, M_2]$ has a subgroup which is not cyclic. It follows that $[M_1, M_2] \neq Z_8$. Therefore, $[M_1, M_2] \cong Z_4 \oplus Z_2$.

Now, let $p \equiv 2 \pmod 4$ and $q \equiv 0 \pmod 4$. With the results in [Araki and Toda 1965], we obtain the following diagram from the above commutative diagram:

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ & & Z_2 \oplus Z_2 \arrow[d, "i_{2*}"] & & \\ 0 \arrow[r] & Z_4 \arrow[r, "\pi_1^*"] & {[M_1, M_2]} \arrow[r, "i_1^*"] \arrow[d, "\pi_{2*}"] & Z_2 \arrow[r] & 0 \\ & & Z_2 \arrow[d] & & \\ & & 0 & & \end{tikzcd}$$

Thus, $[M_1, M_2]$ is isomorphic to one of the three groups: Z_8, $Z_4 \oplus Z_2$ or $Z_2 \oplus Z_2 \oplus Z_2$. Since π_1^* is injective, $[M_1, M_2]$ has an element of order 4. However, $Z_2 \oplus Z_2 \oplus Z_2$ does not have an element of order 4. Since i_{2*} is injective, $[M_1, M_2]$ has a subgroup which is not cyclic. It follows that $[M_1, M_2] \neq Z_8$. Thus, $[M_1, M_2] \cong Z_4 \oplus Z_2$. □

By Lemma 3.4, $[M_1, M_2] \cong Z_4 \oplus Z_2$. However, $[M_1, M_2]$ has different generators under different conditions. Here we determine the generators.

If $p \equiv 0 \pmod 4$ and $q \equiv 2 \pmod 4$, then $[M_1, M_2] \cong Z_4 \oplus Z_2\{\alpha, \pi_1^*(\eta_2)\}$, where $\pi_1^*(\eta_1) = 2\alpha$ and $i_1^*(\alpha) = i_{2\sharp}(\eta)$.

If $p \equiv 2 \pmod 4$ and $q \equiv 0 \pmod 4$, then $[M_1, M_2] \cong Z_4 \oplus Z_2\{\pi_1^*(\bar{\eta}), \beta\}$, where $i_1^*(\beta) = i_{2\sharp}(\eta)$.

For a given homomorphism $h : G_1 \to G_2$, we have from Proposition 2.6 the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ext}(G_2, \pi_{n+1}(X)) \arrow[r] \arrow[d, "\bar{h}^\sharp"] & \pi_n(G_2; X) \arrow[r] \arrow[d, "h^*"] & \operatorname{Hom}(G_2, \pi_n(X)) \arrow[r] \arrow[d, "h^\sharp"] & 0 \\ 0 \arrow[r] & \operatorname{Ext}(G_1, \pi_{n+1}(X)) \arrow[r] & \pi_n(G_1; X) \arrow[r] & \operatorname{Hom}(G_1, \pi_n(X)) \arrow[r] & 0 \end{tikzcd}$$

where $\bar{h}^\sharp$ and $h^\sharp$ are induced by h and h^* is associated with h. This shows that the nonuniqueness of h^* is substantially limited. The measure of choice is bounded by the group

$$\operatorname{Hom}\big(\operatorname{Hom}(G_2, \pi_n(X)), \operatorname{Ext}(G_1, \pi_{n+1}(X))\big).$$

Lemma 3.5. *If $(p, q) = d \neq 1$, we have*

$$[M_1, M_2] \cong \begin{cases} Z_2 \oplus Z_2 & \text{if } p \equiv 2 \text{ and } q \equiv 2 \pmod 4, \\ Z_2 \oplus Z_2 \oplus Z_2 & \text{if } p \equiv 0 \text{ and } q \equiv 0 \pmod 4. \end{cases}$$

Proof. Suppose that $p \equiv 2 \pmod 4$ and $q \equiv 2 \pmod 4$. By the universal coefficient theorem for homotopy groups with coefficients, we have the short exact sequence

$$0 \to \mathrm{Ext}(Z_q, Z_4) \to [M_1, M_2] \to \mathrm{Hom}(Z_q, Z_2) \to 0.$$

Since $\mathrm{Ext}(Z_q, Z_4) \cong Z_{(q,4)} \cong Z_2$ and $\mathrm{Hom}(Z_q, Z_2) = Z_{(q,2)} = Z_2$, this sequence becomes

$$0 \to Z_2 \to [M_1, M_2] \to Z_2 \to 0.$$

Let $M_3 = M(Z_p, n+1)$. By the universal coefficient theorem for homotopy with coefficients, we have the sequence

$$0 \to \mathrm{Ext}(Z_p, Z_4) \to [M_3, M_2] \to \mathrm{Hom}(Z_p, Z_2) \to 0.$$

Similarly, this sequence becomes

$$0 \to Z_2 \to [M_3, M_2] \to Z_2 \to 0.$$

We may assume that $q \geq p$. Let $q = kd$ and $p = ld$, where $(k, l) = 1$. Then both k and l are odd. We define $h : Z_q \to Z_p$ by $h(\bar{1}) = \bar{l}$ with $\bar{s} = s + rZ \in Z_r$. Then, $\mathrm{im}(h)$ is congruent to Z_d in Z_p and h is a nontrivial homomorphism since $(q, p) = d \neq 1$. Thus, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Z_2 \arrow[r] \arrow[d, "\bar{h}^\sharp"] & {[M_3, M_2]} \arrow[r] \arrow[d, "h^*"] & Z_2 \arrow[r] \arrow[d, "h^\sharp"] & 0 \\
0 \arrow[r] & Z_2 \arrow[r] & {[M_1, M_2]} \arrow[r] & Z_2 \arrow[r] & 0
\end{tikzcd}$$

where $\bar{h}^\sharp : \mathrm{Ext}(Z_p, Z_4) \to \mathrm{Ext}(Z_q, Z_4)$ and $h^\sharp : \mathrm{Hom}(Z_p, Z_2) \to \mathrm{Hom}(Z_q, Z_2)$ are induced by h.

To show that $h^\sharp : \mathrm{Hom}(Z_p, Z_2) \to \mathrm{Hom}(Z_q, Z_2)$ is an isomorphism, it is sufficient to show that $h^\sharp$ is nontrivial. Let α be an nonzero element in $\mathrm{Hom}(Z_p, Z_2)$ such that $\alpha(\bar{1}) = \bar{1}$. Since $h^\sharp(\alpha) = \alpha \circ h \in \mathrm{Hom}(Z_q, Z_2)$ and $\alpha \circ h(\bar{1}) = \alpha(\bar{l}) = \bar{l} = \bar{1}$, where l is odd, it follows that $h^\sharp(\alpha)$ is a nontrivial homomorphism.

Next, we show that $\bar{h}^\sharp : \mathrm{Ext}(Z_p, Z_4) \to \mathrm{Ext}(Z_q, Z_4)$ is an isomorphism. Consider the resolutions of Z_q and Z_p. Then we have following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r, "q"] \arrow[d, "h_1"] & Z \arrow[r, "\beta"] \arrow[d, "h_2"] & Z_q \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & Z \arrow[r, "p"] & Z \arrow[r, "\beta'"] & Z_p \arrow[r] & 0
\end{tikzcd}$$

See [Gray 1975, Lemma 25.3]. Now, we give precise definitions of the maps h_1, h_2 and $h^\sharp$. Since $\bar{l} = h(\bar{1}) = h \circ \beta(1) = \beta'(h_2(1))$, we have h_2 given by $h_2(1) = l$. Moreover, we can obtain h_1 using h_2. Since $p \circ h_1 = h_2 \circ q$, we have

$ph_1(1) = h_2(q) = qh_2(1) = dkl = pk$. Thus, h_1 is given by $h_1(1) = k$. If we consider the three homomorphisms $h^\sharp$, $h_1^\sharp$ and $h_2^\sharp$ induced by h, h_1 and h_2 respectively, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(Z_q, Z_4) \arrow[r, "\beta^*"] & \mathrm{Hom}(Z, Z_4) \arrow[r, "q^*"] & \mathrm{Hom}(Z, Z_4) \\
0 \arrow[r] & \mathrm{Hom}(Z_p, Z_4) \arrow[r, "\beta'^*"] \arrow[u, "h^\sharp"] & \mathrm{Hom}(Z, Z_4) \arrow[r, "p^*"] \arrow[u, "h_2^\sharp"] & \mathrm{Hom}(Z, Z_4) \arrow[u, "h_1^\sharp", "\cong"']
\end{tikzcd}$$

Next, we show that $h_1^\sharp$ is an isomorphism. We choose a generator α of $\mathrm{Hom}(Z, Z_4)$ such that $\alpha(1) = \bar{1}$. Then $h_1^\sharp(\alpha)(1) = (\alpha \circ h_1)(1) = \alpha(k) \neq 0 \pmod 2$ since k is odd. Therefore, $h_1^\sharp(\alpha)$ is a generator of $\mathrm{Hom}(Z, Z_4)$. Thus, $h_1^\sharp$ is an isomorphism.

By using $h_1^\sharp$, we determine the homomorphism $\bar{h}^\sharp : \mathrm{Ext}(Z_p, Z_4) \to \mathrm{Ext}(Z_q, Z_4)$. Since $q \equiv p \equiv 2 \pmod 4$ and

$$\mathrm{Ext}(Z_p, Z_4) = \mathrm{Hom}(Z, Z_4)/\operatorname{im}(p^*) \quad \text{and} \quad \mathrm{Ext}(Z_q, Z_4) = \mathrm{Hom}(Z, Z_4)/\operatorname{im}(q^*),$$

we have

$$\mathrm{Ext}(Z_p, Z_4) = \langle \alpha + \{2\alpha\} \rangle \quad \text{and} \quad \mathrm{Ext}(Z_q, Z_4) = \langle \alpha + \{2\alpha\} \rangle.$$

By well-known facts of homological algebra, $\bar{h}^\sharp : \mathrm{Ext}(Z_p, Z_4) \to \mathrm{Ext}(Z_q, Z_4)$ is given by $\bar{h}^\sharp(\alpha + \{2\alpha\}) = \alpha \circ h_1 + \{2\alpha\} \neq 0$. Therefore, $\bar{h}^\sharp$ is nontrivial. Thus, $\bar{h}^\sharp$ is an isomorphism.

By the five lemma, $h^* : [M_1, M_2] \to [M_3, M_2]$ is an isomorphism. From [Araki and Toda 1965], we have $[M_3, M_2] \cong Z_2 \oplus Z_2$. Therefore, $[M_1, M_2] \cong Z_2 \oplus Z_2$.

Next, we suppose that $q \equiv 0$ and $p \equiv 0 \pmod 4$.

From [Araki and Toda 1965] and the commutative diagram above Lemma 3.4, we obtain the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Z_2 \arrow[r, "\pi_1^*"] \arrow[d, "i_{2*}"] & Z_2 \oplus Z_2 \arrow[r, shift left, "i_1^*"] \arrow[d, "i_{2*}"] & Z_2 \arrow[r] \arrow[l, shift left, "r"] \arrow[d, shift left, "i_{2*}"] & 0 \\
0 \arrow[r] & Z_2 \oplus Z_2 \arrow[r, "\pi_1^*"] & {[M_1, M_2]} \arrow[r, "i_1^*"] \arrow[d, "\pi_{2*}"] & Z_2 \arrow[r] \arrow[u, shift left, "\theta"] & 0 \\
 & & Z_2 \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

Since the second row is a split exact sequence, there exists $r : [S^{n+1}, S^n] \to [M_1, S^n]$ such that $i_1^* \circ r = \mathrm{id}_{[S^{n+1}, S^n]}$. Moreover, since the third i_{2*} is an isomorphism, there exists $\theta : [S^{n+1}, M_2] \to [S^{n+1}, S^n]$ such that $\theta \circ i_{2*} = \mathrm{id}_{[S^{n+1}, S^n]}$ and $i_{2*} \circ \theta = \mathrm{id}_{[S^{n+1}, M_2]}$.

We define the map $k : [S^{n+1}, M_2] \to [M_1, M_2]$ by $k = i_{2*} \circ r \circ \theta$. Then, we have

$$\begin{aligned} i_1^* \circ k &= i_1^* \circ i_{2*} \circ r \circ \theta \\ &= i_{2*} \circ i_1^* \circ r \circ \theta \\ &= i_{2*} \circ \mathrm{id}_{[S^{n+1}, S^n]} \circ \theta \\ &= i_{2*} \circ \theta = \mathrm{id}_{[S^{n+1}, M_2]}. \end{aligned}$$

Therefore, the third row is a split exact sequence. Hence,

$$[M_1, M_2] \cong Z_2 \oplus Z_2 \oplus Z_2. \qquad \square$$

Now, we determine the generators of $[M_1, M_2]$ when either $p \equiv 2 \pmod 4$ and $q \equiv 2 \pmod 4$ or $p \equiv 0 \pmod 4$ and $q \equiv 0 \pmod 4$.

Let $p \equiv 2 \pmod 4$ and $q \equiv 2 \pmod 4$. By using the Puppe exact sequence, we have the following exact sequence:

$$\pi_{n+2}(M_2) \xrightarrow{q_{\iota_1}^*} \pi_{n+2}(M_2) \xrightarrow{\pi_1^*} [M_1, M_2] \xrightarrow{i_1^*} \pi_{n+1}(M_2) \xrightarrow{p_{\iota_1}^*} \pi_{n+1}(M_2).$$

By exactness, we obtain the exact sequence

$$0 \longrightarrow Z_2 \xrightarrow{\pi_1^*} [M_1, M_2] \xrightarrow{i_1^*} Z_2 \longrightarrow 0.$$

Thus, $[M_1, M_2] \cong Z_2 \oplus Z_2\{\pi_1^*(\bar{\eta}), \beta\}$, where $i_1^*(\beta) = i_{2\sharp}(\eta)$.

Next, we let $p \equiv 0 \pmod 4$ and $q \equiv 0 \pmod 4$. By a similar method we obtain $[M_1, M_2] \cong Z_2 \oplus Z_2 \oplus Z_2 \{\pi_1^*(\eta_1), \pi_1^*(\eta_2), \alpha\}$, where $i_1^*(\alpha) = i_{2\sharp}(\eta)$.

Remark 3.6. Here we determine the generators of $\pi_{n+3}(M(Z_q, n))$. By using the mapping cone sequence of the Moore space

$$S^n \xrightarrow{q_\iota} S^n \xrightarrow{i} M(Z_q, n) \xrightarrow{\pi} S^{n+1} \xrightarrow{q_\iota} S^{n+1},$$

we obtain a long exact sequence

$$\pi_{n+3}(S^n) \xrightarrow{q_{\iota\sharp}} \pi_{n+3}(S^n) \xrightarrow{i_\sharp} \pi_{n+3}(M(Z_q, n)) \xrightarrow{\pi_\sharp} \pi_{n+3}(S^{n+1}) \xrightarrow{q_{\iota\sharp}} \pi_{n+3}(S^{n+1}).$$

From the work by Toda [1962], we have

$$Z_{24}\{\nu\} \xrightarrow{q_{\iota\sharp}} Z_{24}\{\nu\} \xrightarrow{i_\sharp} \pi_{n+3}(M(Z_q, n)) \xrightarrow{\pi_\sharp} Z_2\{\eta^2\} \xrightarrow{q_{\iota\sharp}} Z_2\{\eta^2\}.$$

Thus, if q is odd, then $\pi_{n+3}(M(Z_q, n)) \cong Z_{(q,24)}\{i_\sharp(\nu)\}$, and if q is even, then $\pi_{n+3}(M(Z_q, n)) \cong Z_{(q,24)} \oplus Z_2\{i_\sharp(\nu), \overline{\eta^2}\}$ where $\pi_\sharp(\overline{\eta^2}) = \eta^2$.

Based on Remarks 3.1 and 3.6, we obtain for M_1 the table

	q odd	$q \equiv 2 \pmod 4$	$q \equiv 0 \pmod 4$
$\pi_{n+3}(M_1)$	0	Z_4	$Z_2 \oplus Z_2$
Generator		$\hat{\eta}$	η_3, η_4
Relation		$i_{1\sharp}(\eta^2) = 2\hat{\eta}, \pi_{1\sharp}(\hat{\eta}) = \eta$	$i_{1\sharp}(\eta^2) = \eta_3, \pi_{1\sharp}(\eta_4) = \eta$

while for M_2 we obtain

	p odd	$p \equiv 2 \pmod 4$	$p \equiv 0 \pmod 4$
$\pi_{n+3}(M_2)$	$Z_{(p,24)}$	$Z_{(p,24)} \oplus Z_2$	$Z_{(p,24)} \oplus Z_2$
Generator	$i_{2\sharp}(\nu)$	$i_{2\sharp}(\nu), \overline{\eta^2}$	$i_{2\sharp}(\nu), \overline{\eta^2}$
Relation		$\pi_{2\sharp}(\overline{\eta^2}) = \eta^2$	$\pi_{2\sharp}(\overline{\eta^2}) = \eta^2$

By Lemmas 3.4 and 3.5, we have the following table, where $\pi_1^*(\eta_1) = 2\alpha$, $i_1^*(\alpha) = i_{2\sharp}(\eta)$ and $i_1^*(\beta) = i_{2\sharp}(\eta)$:

	$[M_1, M_2]$	Generator
either q odd or p odd	0	
$q \equiv 2, p \equiv 0 \pmod 4$	$Z_4 \oplus Z_2$	$\alpha, \pi_1^*(\eta_2)$
$q \equiv 0, p \equiv 2 \pmod 4$	$Z_4 \oplus Z_2$	$\pi_1^*(\bar{\eta}), \beta$
$q \equiv p \equiv 2 \pmod 4$	$Z_2 \oplus Z_2$	$\pi_1^*(\bar{\eta}), \beta$
$q \equiv p \equiv 0 \pmod 4$	$Z_2 \oplus Z_2 \oplus Z_2$	$\pi_1^*(\eta_1), \pi_1^*(\eta_2), \alpha$

4. Computation of $\mathcal{E}_\sharp^{\dim+r}(M(Z_q, n+1) \vee M(Z_p, n))$ for $r = 0, 1$

In this section, we compute $\mathcal{E}_\sharp^{\dim+r}(M_1 \vee M_2)$, where $M_1 = M(Z_q, n+1) = S^{n+1} \cup_q e^{n+2}$ and $M_2 = M(Z_p, n) = S^n \cup_p e^{n+1}$ with $p, q \geq 1$. In [Jeong 2010], these groups were computed in the case of $p = q$. However, we compute those groups in the general case, that is, $p \neq q$ and $r = 0, 1$. Throughout this section we assume that $X = M_1 \vee M_2$. Note that $\pi_{n+k}(M_1 \vee M_2) \cong \pi_{n+k}(M_1) \oplus \pi_{n+k}(M_2)$ for $k \leq n$ by Proposition 2.1. Moreover, from Proposition 2.2, we can identify $f \in [X, X]$ with the 2×2 matrix

$$\theta(f) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

where $f_{11} \in [M_1, M_1]$, $f_{12} \in [M_2, M_1]$, $f_{21} \in [M_1, M_2]$, and $f_{22} \in [M_1, M_1]$.

Lemma 4.1. *Let $f \in [X, X]$ be given by*

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}.$$

Then $f \in \mathcal{E}(X)$ *if and only if* $f_{11} \in \mathcal{E}(M_1)$ *and* $f_{22} \in \mathcal{E}(M_2)$. *Additionally, if* $f \in \mathcal{E}_\sharp^{\dim}(X)$, *then* $f_{22} = 1$.

Proof. Let us denote by $h_{*n} : H_n(U) \to H_n(V)$ the induced homomorphism on the homology group from $h : U \to V$. Then, $f \in \mathcal{E}(X)$ if and only if f_* is an isomorphism if and only if f_{11*n+1} and f_{22*n} are isomorphisms if and only if $f_{11} \in \mathcal{E}(M_1)$ and $f_{22} \in \mathcal{E}(M_2)$. For the proof of the second statement, see [Jeong 2010, Lemma 3.3]. □

Let us denote by $g_{\sharp s} : \pi_s(U) \to \pi_s(V)$ the homomorphism induced by $g : U \to V$. It is clear from Lemma 4.1 that if $f \in \mathcal{E}(X)$, then $f_{\sharp(n+k)} : \pi_{n+k}(X) \to \pi_{n+k}(X)$ is given by

$$f_{\sharp(n+k)} = \begin{pmatrix} f_{11\sharp(n+k)} & f_{12\sharp(n+k)} \\ f_{21\sharp(n+k)} & f_{22\sharp(n+k)} \end{pmatrix},$$

where $f_{11\sharp(n+k)}$ and $f_{22\sharp(n+k)}$ are isomorphisms and $k \le n$.

Lemma 4.2. *If* $f \in \mathcal{E}(X)$ *and either* q *is odd or* p *is odd, then* $f_{12\sharp k} = 0$ *for* $k = 1, 2, \dots, n+2$.

Proof. Since M_1 is n-connected, we have $\pi_k(M_1) = 0$ for $k = 1, 2, \dots, n$. Thus it is sufficient to show that $f_{12\sharp k} = 0$ for $k = n+1, n+2$.

If p is odd, then $\pi_{n+1}(M_2)$ and $\pi_{n+2}(M_2)$ are trivial groups. Thus, $f_{12\sharp(n+1)} = f_{12\sharp(n+2)} = 0$.

Suppose that q is odd, p is even and $(p, q) = d \neq 1$. Then, $\pi_{n+1}(M_2) \cong Z_2\{i_{2\sharp}(\eta)\}$. Since $[M_2, M_1] \cong Z_d\{\pi_2^*(i_1)\}$, we have $f_{12\sharp n+1} = t\pi_2^*(i_1)_\sharp$ for some integer t such that $1 \le t \le d$. Thus, we have

$$f_{12\sharp(n+1)}(i_{2\sharp}(\eta)) = t\pi_2^*(i_1)(i_{2\sharp}(\eta)) = t(i_1 \circ \pi_2 \circ i_2 \circ \eta) = 0$$

because $\pi_2 \circ i_2$ is homotopic to a constant map. Hence, $f_{12\sharp(n+1)} = 0$. If $d = 1$, $[M_2, M_1] = 0$ and it is trivial.

For $k = n+2$, we are done since $\pi_{n+2}(M_1) = 0$. □

Here we introduce certain generators and elements of $[M_1, M_1]$ and $\mathcal{E}_\sharp^{\dim+r}(M_1)$ for $r = -1, 0, 1$ as described in [Jeong 2010].

Remark 4.3. Let $M_1 = M(Z_q, n+1)$ be a Moore space with q is even. By Proposition 2.5, $\mathcal{E}_\sharp^{\dim}(M_1) \cong Z_2$ and $\mathcal{E}_\sharp^{\dim+1}(M_1) = 1$. In this remark, we describe the generator of $\mathcal{E}_\sharp^{\dim}(M_1)$ explicitly.

Consider the mapping cone sequence

$$S^{n+1} \xrightarrow{q_{\iota_1}} S^{n+1} \xrightarrow{i_1} S^{n+1} \cup_q e^{n+2} \xrightarrow{\pi_1} S^{n+2} \xrightarrow{q_{\iota_1}} S^{n+2}\ .$$

Then, we have the following exact sequence:

$$\pi_{n+2}(M_1) \xrightarrow{q_{\iota_1}^*} \pi_{n+2}(M_1) \xrightarrow{\pi_1^*} [M_1, M_1] \xrightarrow{i_1^*} \pi_{n+1}(M_1) \xrightarrow{q_{\iota_1}^*} \pi_{n+1}(M_1)\ .$$

Since $\pi_{n+2}(M_1) \cong Z_2\{i_1\eta\}$ and $\pi_{n+1}(M_1) \cong Z_q\{1\}$, we have the short exact sequence

$$0 \longrightarrow Z_2\{i_1\eta\} \xrightarrow{\pi_1^*} [M_1, M_1] \xrightarrow{i_1^*} Z_q\{1\} \longrightarrow 0 .$$

By [Araki and Toda 1965, Theorem 4.1],

$$[M_1, M_1] \cong \begin{cases} Z_{2q}\{1\} & \text{if } q \equiv 2 \pmod 4, \\ Z_q \oplus Z_2\{1, i_1 \circ \eta \circ \pi_1\} & \text{if } q \equiv 0 \pmod 4, \end{cases}$$

and

$$\pi_1^*(i_1 \circ \eta) = i_1 \circ \eta \circ \pi_1 \in [M_1, M_1].$$

Let $i_1 \circ \eta \circ \pi_1 = \epsilon$. Then, ϵ has order 2 and $1 + \epsilon \in [M_1, M_1]$. Since $n \geq 5$, we have that $1 + \epsilon$ is a suspension map. Thus,

$$(1+\epsilon) \circ (1+\epsilon) \simeq 1 \circ (1+\epsilon) + \epsilon \circ (1+\epsilon) = 1 + \epsilon + \epsilon + \epsilon \circ \epsilon = 1 + 2\epsilon + \epsilon^2.$$

If $q \equiv 2 \pmod 4$, then $i_1 \circ \eta \circ \pi_1 = q1$ and $\epsilon^2 = i_1 \circ \eta \circ \pi_1 \circ i_1 \circ \eta \circ \pi_1$. Since $\pi_1 \circ i_1 = 0$ and ϵ has order 2, we have $2\epsilon = 0$ and $\epsilon^2 = 0$. Thus, $(1+\epsilon) \circ (1+\epsilon) \simeq 1$ and $1 + \epsilon \in \mathcal{E}(M_1)$.

Since each $\alpha \in \pi_{n+r}(M_1)$ is a suspension map, for $r = 1, 2, 3$, we have

$$(1+\epsilon)_\sharp(\alpha) = \alpha + \epsilon \circ \alpha.$$

Since $\pi_{n+1}(M_1) \cong Z_q\{i_1\}$ and $\epsilon_\sharp(i_1) = i_1 \circ \eta \circ \pi_1 \circ i_1 = 0$, we have $1 + \epsilon \in \mathcal{E}_\sharp^{\dim -1}(M_1)$.

Since $\pi_{n+2}(M_1) \cong Z_2\{i_{1\sharp}(\eta)\}$ and $\epsilon_\sharp(i_{1\sharp}(\eta)) = i_1 \circ \eta \circ \pi_1 \circ i_1 \circ \eta = 0$, we have $1 + \epsilon \in \mathcal{E}_\sharp^{\dim}(M_1)$.

Since $\pi_{n+3}(M_1) \cong Z_4\{\hat{\eta}\}$ and

$$\epsilon_\sharp(\hat{\eta}) = i_1 \circ \eta \circ \pi_1 \circ \hat{\eta} = i_1 \circ \eta \circ \eta = i_1 \circ \eta^2 = 2\hat{\eta} \neq 0,$$

we have $1 + \epsilon \notin \mathcal{E}_\sharp^{\dim+1}(M_1)$.

We obtain similar results in the case of $q \equiv 0 \pmod 4$.

Theorem 4.4. *If $X = M_1 \vee M_2$ and $(p, q) = 1$, then*

$$\mathcal{E}_\sharp^{\dim}(X) \cong \begin{cases} 1 & \text{if } q \text{ is odd}, \\ Z_2 & \text{if } q \text{ is even and } p \text{ is odd}. \end{cases}$$

Proof. Let $(q, p) = 1$. Then, either q or p is odd. By Lemmas 3.2 and 3.3, we have $[M_2, M_1] = 0$ and $[M_1, M_2] = 0$.

If q is odd, then $\mathcal{E}_\sharp^{\dim}(M_1) = 1$ and $\mathcal{E}_\sharp^{\dim}(M_2) = 1$ by Proposition 2.5 and Lemma 4.1. Therefore $\mathcal{E}_\sharp^{\dim}(X) = 1$.

If p is odd and q is even, then $\mathcal{E}_\sharp^{\dim}(M_1) \cong Z_2\{1+\epsilon\}$ and $\mathcal{E}_\sharp^{\dim}(M_2) = 1$ by Proposition 2.5, Lemma 4.1, and Remark 4.3. Thus, we have

$$\mathcal{E}_\sharp^{\dim}(X) \cong \left\{ \begin{pmatrix} 1+\epsilon & 0 \\ 0 & 1 \end{pmatrix} \middle| \, \epsilon \in Z_2 \ \{i_1\eta\pi_1\} \right\},$$

where η is the generator of $\pi_{n+2}(S^{n+1})$. □

Theorem 4.5. *If $X = M_1 \vee M_2$ and $(p,q) = d \neq 1$, then*

$$\mathcal{E}_\sharp^{\dim}(X) \cong \begin{cases} Z_d & \text{if } q \text{ is odd,} \\ Z_2 \oplus Z_d & \text{if } q \text{ is even and } p \text{ is odd.} \end{cases}$$

Proof. By Lemmas 3.2 and 3.3, we have $[M_2, M_1] \cong Z_d\{\pi_2^*(i_1)\}$ and $[M_1, M_2] = 0$. Moreover, $f_{12\sharp k} = 0$ for $k = 1, 2, \ldots, n+2$ by Lemma 4.2.

Thus, if q is odd, then we have

$$\mathcal{E}_\sharp^{\dim}(X) \cong \left\{ \begin{pmatrix} 1 & f_{12} \\ 0 & 1 \end{pmatrix} \middle| \, f_{12} \in Z_d \ \{\pi_2^*(i_1)\} \right\},$$

but if q is even and p is odd, then we have

$$\mathcal{E}_\sharp^{\dim}(X) \cong \left\{ \begin{pmatrix} 1+\epsilon & f_{12} \\ 0 & 1 \end{pmatrix} \middle| \, f_{12} \in Z_d \ \{\pi_2^*(i_1)\}, \epsilon \in Z_2 \ \{i_1\eta\pi_1\} \right\}. \qquad \square$$

Let f_{12} be an element of $[M_2, M_1] \cong Z_d\{\pi_2^*(i_1)\}$, Then $f_{12} = s\pi_2^*(i_1)$ for $1 \le s \le d$.

Lemma 4.6. *For $f = \left(\begin{smallmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{smallmatrix}\right) \in \mathcal{E}(X)$, let p and q be even. Then, $f_{12\sharp k} = 0$ for $k = 1, 2, \ldots, n+1$.*

Proof. Since M_1 is n-connected, $\pi_k(M_1) = 0$ for $k = 1, 2, \ldots, n$. Thus, it is sufficient to show that $f_{12\sharp(n+1)} = 0$. Since $[M_2, M_1] \cong Z_d\{\pi_2^*(i_1)\}$ by Lemma 3.2 and f_{12} belongs to $[M_2, M_1]$, we have $f_{12} = s\pi_2^*(i_1)$ for some $1 \le s \le d$. Moreover, $\pi_{n+1}(M_2) \cong Z_2\{i_{2\sharp}(\eta)\}$ by Remark 3.1. Thus, we have

$$f_{12\sharp(n+1)}(i_{2\sharp}(\eta)) = s\pi_2^*(i_1)(i_{2\sharp}(\eta)) = s(i_1 \circ \pi_2 \circ i_2 \circ \eta) = 0$$

since $\pi_2 \circ i_2$ is homotopic to the constant map. □

Lemma 4.7. *Let p and q be even and $f_{12} = s\pi_2^*(i_1)$ be an element of $[M_2, M_1] \cong Z_d\{\pi_2^*(i_1)\}$ for $1 \le s \le d$. Then, $f_{12\sharp(n+2)} \neq 0$ if s is odd, and $f_{12\sharp(n+2)} = 0$ if s is even.*

Proof. First, we note that $\pi_{n+2}(M_1) \cong Z_2\{i_{\sharp 1}(\eta)\}$.

Suppose that $p \equiv 0 \pmod 4$. Since $\pi_{n+2}(M_2) \cong Z_2 \oplus Z_2\{\eta_1, \eta_2\}$, we have

$$\pi_2^*(i_1) = \pi_2^*(i_1)(\eta_1) = \pi_2^*(i_1)(i_{2\sharp}(\eta^2)) = i_1 \circ \pi_2 \circ i_2 \circ \eta^2 = 0$$

and

$$\pi_2^*(i_1)(\eta_2) = i_1 \circ \pi_2 \circ \eta_2 = i_1 \circ \eta \neq 0.$$

Thus, $f_{12\sharp(n+2)}(\eta_1) = 0$ for all f_{12}. Moreover, if $s = 2l$ for some $1 \le l \le d/2$, then

$$s\pi_2^*(i_1)(\eta_2) = si_1 \circ \pi_2 \circ \eta_2 = 2li_1 \circ \eta = 0.$$

Therefore, each element in $\langle 2\pi_2^*(i_1)\rangle \cong Z_{d/2}$ induces the trivial homomorphism on $\pi_{n+2}(M_2)$. However, if $s = 2l+1$ for some $0 \le l < d/2 - 1$, then

$$s\pi_2^*(i_1)(\eta_2) = si_1 \circ \pi_2 \circ \eta_2 = (2l+1)i_1 \circ \eta = i_1 \circ \eta \ne 0.$$

Thus, if f_{12} does not belong to $\langle 2\pi_2^*(i_1)\rangle \cong Z_{d/2}$, then $f_{12\sharp(n+2)} \ne 0$.

Suppose that $p \equiv 2 \pmod 4$. Since $\pi_{n+2}(M_2) \cong Z_4\{\overline{\eta}\}$, we have

$$\pi_2^*(i_1)_\sharp(\overline{\eta}) = i_1 \circ \pi_2 \circ \overline{\eta} = i_1 \circ \eta = i_{1\sharp}(\eta) \ne 0.$$

If $s = 2k$ for some $1 \le l \le d/2$, then

$$s\pi_2^*(i_1)_\sharp(\overline{\eta}) = si_1 \circ \pi_2 \circ \overline{\eta} = si_1 \circ \eta = 2li_{1\sharp}(\eta) = 0.$$

Thus, each element in $\langle 2\pi_2^*(i_1)\rangle \cong Z_{d/2}$ induces the trivial homomorphism on $n+2$.

However, if $s = 2l+1$ for some $0 \le l \le d/2 - 1$, then

$$s\pi_2^*(i_1)_\sharp(\overline{\eta}) = si_1 \circ \pi_2 \circ \overline{\eta} = si_1 \circ \eta = (2l+1)i_{1\sharp}(\eta) = i_{1\sharp}(\eta) \ne 0.$$

Thus if f_{12} does not belong to $\langle 2\pi_2^*(i_1)\rangle \cong Z_{d/2}$, then $f_{12\sharp(n+2)} \ne 0$. □

Theorem 4.8. *Let p and q be even and let $X = M_1 \vee M_2$. Then if $(p, q) = d \ne 1$, we have*

$$\mathcal{E}_\sharp^{\dim}(X) \cong \begin{cases} Z_2 \oplus Z_{d/2} \oplus Z_2 \oplus Z_2 & \text{if } q \equiv 2,\ p \equiv 0 \pmod 4, \\ Z_2 \oplus Z_{d/2} \oplus Z_4 & \text{if } q \equiv 0,\ p \equiv 2 \pmod 4, \\ Z_2 \oplus Z_{d/2} \oplus Z_2 & \text{if } q \equiv 2,\ p \equiv 2 \pmod 4, \\ Z_2 \oplus Z_{d/2} \oplus Z_2 \oplus Z_2 & \text{if } q \equiv 0,\ p \equiv 0 \pmod 4. \end{cases}$$

Proof. By Proposition 2.5, $\mathcal{E}_\sharp^{\dim}(M_1) \cong Z_2$ and $\mathcal{E}_\sharp^{\dim}(M_2) = 1$. By Lemma 4.6, for each $f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \in \mathcal{E}(X)$, we have $f_{12\sharp k} = 0$ for $k = 1, 2, \ldots, n+1$. By Lemma 4.7, each element in $\langle 2\pi_2^*(i_1)\rangle \cong Z_{d/2}$ induces the trivial homomorphism on $\pi_{n+2}(M_2)$. Furthermore, if f_{12} does not belong to $\langle 2\pi_2^*(i_1)\rangle \cong Z_{d/2}$, then $f_{12\sharp(n+2)} \ne 0$. Thus, it is sufficient to investigate $f_{21\sharp n}$, $f_{21\sharp(n+1)}$ and $f_{21\sharp(n+2)}$.

Case 1. Let $q \equiv 2 \pmod 4$ and $p \equiv 0 \pmod 4$. From Lemma 3.4, we obtain $[M_1, M_2] \cong Z_4 \oplus Z_2\{\alpha, \pi_1^*(\eta_2)\}$, where $\pi_1^*(\eta_1) = 2\alpha$ and $i_1^*(\alpha) = i_{2\sharp}(\eta)$.

Since M_1 is n-connected, $\pi_n(M_1) = 0$. Thus, $f_{21\sharp n} = 0$.

Since $\pi_{n+1}(M_1) \cong Z_q\{i_1\}$, we have $\pi_1^*(\eta_2)_\sharp(i_1) = \eta_2 \circ \pi_1 \circ i_1 = 0$.

Conversely, since $\pi_{n+1}(M_2) \cong Z_2\{i_{2\sharp}(\eta)\}$ and $\alpha_\sharp(i_1) = \alpha \circ i_1 = i_{2\sharp}(\eta) \ne 0$, we have $(2\alpha)_\sharp = 0$ and $(3\alpha)_\sharp \ne 0$. Moreover, since $\pi_{n+2}(M_1) \cong Z_2\{i_{1\sharp}(\eta)\}$, we have $\pi_1^*(\eta_2)_\sharp(i_{1\sharp}(\eta)) = \eta_2 \circ \pi_1 \circ i_1 \circ \eta = 0$. Hence, $\begin{pmatrix} 1 & 0 \\ f_{21} & 1 \end{pmatrix}$ belongs to $\mathcal{E}_\sharp^{\dim}(X)$ if $f_{21} \in Z_2 \oplus Z_2\{2\alpha, \pi_1^*(\eta_2)\}$.

Therefore,

$$\mathcal{E}_\sharp^{\dim}(X) \cong \left\{ \begin{pmatrix} 1+\epsilon & f_{12} \\ f_{21} & 1 \end{pmatrix} \middle| \, f_{12} \in \langle 2\pi_2^*(i_1)\rangle, \, f_{21} \in \langle 2\alpha\rangle \oplus \langle \pi_1^*(\eta_2)\rangle \right\},$$

where $\epsilon \in \langle i_1\eta\pi_1\rangle$.

Case 2. Let $q \equiv 0 \pmod 4$ and $p \equiv 2 \pmod 4$. From Lemma 3.4, we obtain $[M_1, M_2] \cong Z_4 \oplus Z_2\{\pi_1^*(\bar{\eta}), \beta\}$, where $i_1^*(\beta) = i_{2\sharp}(\eta)$.

Since $\pi_n(M_1) = 0$, we have $f_{21\sharp n} = 0$. However, since $\pi_{n+1}(M_1) \cong Z_q\{i_1\}$ and $\pi_{n+1}(M_2) \cong Z_2\{i_{2\sharp}(\eta)\}$, we have $\pi_1^*(\bar{\eta})_\sharp(i_1) = \bar{\eta}\circ\pi_1\circ i_1 = 0$, but $\beta_\sharp(i_1) = \beta\circ i_1 = i_{2\sharp}(\eta) \neq 0$.

For the generator $\pi_1^*(\bar{\eta})$ of $[M_1, M_2] \cong Z_4 \oplus Z_2\{\pi_1^*(\bar{\eta}), \beta\}$ and the generator $i_{1\sharp}(\eta)$ of $\pi_{n+2}(M_1) \cong Z_2\{i_{1\sharp}(\eta)\}$, we have $\pi_1^*(\bar{\eta})_\sharp(i_{1\sharp}(\eta)) = \bar{\eta}\circ\pi_1\circ i_{1\sharp}(\eta) = 0$.

Hence, $\left(\begin{smallmatrix} 1 & 0 \\ f_{21} & 1 \end{smallmatrix}\right)$ belongs to $\mathcal{E}_\sharp^{\dim}(X)$ if $f_{21} \in \langle \pi_1^*(\bar{\eta})\rangle$.

Therefore,

$$\mathcal{E}_\sharp^{\dim}(X) \cong \left\{ \begin{pmatrix} 1+\epsilon & f_{12} \\ f_{21} & 1 \end{pmatrix} \middle| \, f_{12} \in \langle 2\pi_2^*(i_1)\rangle, \, f_{21} \in \langle \pi_1^*(\bar{\eta})\rangle \right\},$$

where $\epsilon \in \langle i_1\eta\pi_1\rangle$.

Case 3. Let $q \equiv 2 \pmod 4$ and $p \equiv 2 \pmod 4$. From Lemma 3.5, we obtain $[M_1, M_2] \cong Z_2 \oplus Z_2\{\pi_1^*(\bar{\eta}), \beta\}$, where $i_1^*(\beta) = i_{2\sharp}(\eta)$.

First, we recall that $f_{21\sharp n} = 0$ since $\pi_n(M_1) = 0$.

Since $\pi_{n+1}(M_1) \cong Z_q\{i_1\}$ and $\pi_{n+1}(M_2) \cong Z_2\{i_{2\sharp}(\eta)\}$, we have $\pi_1^*(\bar{\eta})_\sharp(i_1) = \bar{\eta}\circ\pi_1\circ i_1 = 0$, but $\beta_\sharp(i_1) = \beta\circ i_1 = i_{2\sharp}(\eta) \neq 0$. Moreover, since $\pi_{n+2}(M_1) \cong Z_2\{i_{1\sharp}(\eta)\}$, we have $\pi_1^*(\bar{\eta})_\sharp(i_{1\sharp}(\eta)) = \bar{\eta}\circ\pi_1\circ i_1\circ\eta = 0$.

Hence, if $f_{21} \in \langle \pi_1^*(\bar{\eta})\rangle$, then $\left(\begin{smallmatrix} 1 & 0 \\ f_{21} & 1 \end{smallmatrix}\right)$ belongs to $\mathcal{E}_\sharp^{\dim}(X)$. However, if $f_{21} \in \langle\beta\rangle$, this cannot be the case. Therefore,

$$\mathcal{E}_\sharp^{\dim}(X) \cong \left\{ \begin{pmatrix} 1+\epsilon & f_{12} \\ f_{21} & 1 \end{pmatrix} \middle| \, f_{12} \in \langle 2\pi_2^*(i_1)\rangle, \, f_{21} \in \langle \pi_1^*(\bar{\eta})\rangle \right\},$$

where $\epsilon \in \langle i_1\eta\pi_1\rangle$.

Case 4. Let $q \equiv 0 \pmod 4$ and $p \equiv 0 \pmod 4$. From Lemma 3.5, we obtain $[M_1, M_2] \cong Z_2 \oplus Z_2 \oplus Z_2\{\pi_1^*(\eta_1), \pi_1^*(\eta_2), \alpha\}$, where $i_1^*(\alpha) = i_{2\sharp}(\eta)$. First, we note that $f_{21\sharp n} = 0$ since $\pi_n(M_1) = 0$.

Since $\pi_{n+1}(M_1) \cong Z_q\{i_1\}$ and $\pi_{n+1}(M_2) \cong Z_2\{i_{2\sharp}(\eta)\}$, we have $\pi_1^*(\eta_1)_\sharp(i_1) = \eta_1\circ\pi_1\circ i_1 = 0$ and $\pi_1^*(\eta_2)_\sharp(i_1) = \eta_2\circ\pi_1\circ i_1 = 0$, but $\alpha_\sharp(i_1) = \alpha\circ i_1 = i_{2\sharp}(\eta) \neq 0$. Also, since $\pi_{n+2}(M_1) \cong Z_2\{i_{1\sharp}(\eta)\}$, we have $\pi_1^*(\eta_1)_\sharp(i_{1\sharp}(\eta)) = \eta_1\circ\pi_1\circ i_{1\sharp}(\eta) = 0$ and $\pi_1^*(\eta_2)_\sharp(i_{1\sharp}(\eta)) = \eta_2\circ\pi_1\circ i_{1\sharp}(\eta) = 0$.

Hence, if $f_{21} \in \langle \pi_1^*(\eta_1)\rangle \oplus \langle \pi_1^*(\eta_2)\rangle$, then $\left(\begin{smallmatrix} 1 & 0 \\ f_{21} & 1 \end{smallmatrix}\right)$ belongs to $\mathcal{E}_\sharp^{\dim}(X)$. However, if $f_{21} \in \langle\alpha\rangle$, this cannot be the case. Therefore,

$$\mathcal{E}_{\sharp}^{\dim}(X) \cong \left\{ \begin{pmatrix} 1+\epsilon & f_{12} \\ f_{21} & 1 \end{pmatrix} \middle| \; f_{12} \in \langle 2\pi_2^*(i_1)\rangle, \, f_{21} \in \pi_1^*(\langle \eta_1\rangle \oplus \langle \eta_2\rangle) \right\},$$

where $\epsilon \in \langle i_1 \eta \pi_1\rangle$. □

From Theorems 4.4–4.8, we obtain Table 1 (see page 37).

Theorem 4.9. *Let $X = M_1 \vee M_2$, $n \geq 5$ and $(q, p) = d$. Then we have*

$$\mathcal{E}_{\sharp}^{\dim+1}(X) \cong \begin{cases} 1 & \text{if } q \text{ is odd or } p \text{ is odd } (d=1), \\ Z_d & \text{if } q \text{ is odd or } p \text{ is odd } (d \neq 1), \\ Z_{d/2} \oplus Z_2 & \text{if } p \equiv 0 \pmod 4 \text{ and } (p, 24) = 4 \text{ or } 12 (d \neq 1), \\ Z_{d/2} & \text{if } p \equiv 0 \pmod 4 \text{ and } (p, 24) = 8 \text{ or } 24 (d \neq 1), \\ Z_{d/2} & \text{if } q \equiv 2,\ p \equiv 2 \pmod 4, \\ Z_{d/2} \oplus Z_2 & \text{if } q \equiv 0,\ p \equiv 2 \pmod 4. \end{cases}$$

Proof. By virtue of Remark 4.3, Theorem 4.4 and the fact that $\mathcal{E}_{\sharp}^{\dim+1}(X) \subseteq \mathcal{E}_{\sharp}^{\dim}(X)$, we have $\mathcal{E}_{\sharp}^{\dim+1}(X) = 1$ if $(p, q) = 1$.

By Proposition 2.5, we have $\mathcal{E}_{\sharp}^{\dim+1}(M_1) = 1$. Thus, it is sufficient to identify $f_{12\sharp(n+3)}$ and $f_{21\sharp(n+3)}$. First, we note that $[M_2, M_1] \cong Z_d\{\pi_2^*(i_1)\}$ by Lemma 3.2.

Case 1. Suppose that q is odd or p is odd and $(p, q) = d \neq 1$. Since $[M_1, M_2] = 0$ by Lemma 3.3, we only investigate $f_{12\sharp(n+3)}$.

If q is odd, $f_{12\sharp(n+3)} = 0$ since $\pi_{n+3}(M_1) = 0$. If q is even and p is odd, $\pi_{n+3}(M_2) \cong Z_{(p,24)}\{i_{2\sharp}(\nu)\}$. Since

$$\pi_2^*(i_1)_\sharp(i_{2\sharp}(\nu)) = i_1 \circ \pi_2 \circ i_2 \circ \nu = 0,$$

we have $f_{12\sharp(n+3)} = 0$ for each $f_{12} \in [M_2, M_1]$. Therefore,

$$\mathcal{E}_{\sharp}^{\dim+1}(X) \cong \left\{ \begin{pmatrix} 1 & f_{12} \\ 0 & 1 \end{pmatrix} \middle| \; f_{12} \in \langle \pi_2^*(i_1)\rangle \right\}.$$

Case 2. Suppose that $q \equiv 2 \pmod 4$ and $p \equiv 0 \pmod 4$. First, we note that

$$\pi_{n+3}(M_2) \cong Z_{(p,24)} \oplus Z_2 \ \{i_{2\sharp}(\nu), \overline{\eta^2}\}$$

and that $\pi_{n+3}(M_1) \cong Z_4\{\hat{\eta}\}$ by Proposition 2.3. Let $f_{12} = s\pi_2^*(i_1)$. If $s = 2l$ for some $1 \leq l \leq d/2$, then

$$s\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = 2l\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = 4l\hat{\eta} = 0$$

since

$$\pi_2^*(i_1)_\sharp(i_{2\sharp}(\nu)) = i_1 \circ \pi_2 \circ i_2 \circ \nu = 0$$

and

$$\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = i_1 \circ \pi_2 \circ \overline{\eta^2} = i_{1\sharp}(\eta^2) = 2\hat{\eta} \neq 0 \in \pi_{n+3}(M_1) \cong Z_4.$$

Further, if $s = 2l + 1$ for some $0 \le l \le d/2 - 1$, then

$$s\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = (2l+1)\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = 4l\hat{\eta} + 2\hat{\eta} = 2\hat{\eta} \neq 0.$$

Thus, each $f_{12} \in \langle 2\pi_2^*(i_1)\rangle \cong Z_{d/2}$ induces the trivial homomorphism on $\pi_{n+3}(M_2)$. However, if f_{12} does not belong to $\langle 2\pi_2^*(i_1)\rangle$, then $f_{12\sharp(n+3)} \neq 0$.

Let us investigate $f_{21\sharp(n+3)}$. Note that $[M_1, M_2] \cong Z_4 \oplus Z_2\{\alpha, \pi_1^*(\eta_2)\}$ and $\pi_{n+3}(M_1) \cong Z_4\{\hat{\eta}\}$ with $\pi_1^*(\eta_1) = 2\alpha$, $i_1^*(\alpha) = i_{2\sharp}(\eta)$, $i_{1\sharp}(\eta^2) = 2\hat{\eta}$ and $\pi_{1\sharp}(\hat{\eta}) = \eta$. Since $\pi_{2\sharp}(\eta_2 \circ \eta) = \eta^2$, we have

$$\pi_1^*(\eta_2)_\sharp(\hat{\eta}) = \eta_2 \circ \pi_1 \circ \hat{\eta} = \eta_2 \circ \eta \neq 0.$$

Moreover, since $\eta^3 = 4\nu$ [Toda 1962, (5.5)], we have

$$2\alpha_\sharp(\hat{\eta}) = 2\alpha \circ \hat{\eta} = \eta_1 \circ \pi_1 \circ \hat{\eta} = \eta_1 \circ \eta = i_{1\sharp}(\eta^2) \circ \eta = i_2 \circ \eta^3 = 4i_{2\sharp}(\nu).$$

Therefore, $\alpha_\sharp(\hat{\eta}) = 2i_{2\sharp}(\nu)$. Since $(p, 24)$ is a multiple of 4, we have $\alpha_\sharp(\hat{\eta}) = 2i_{2\sharp}(\nu) \neq 0$ and $3\alpha_\sharp(\hat{\eta}) = 6i_{2\sharp}(\nu) \neq 0$.

Since ν is 2-primary, if $(p, 24) = 4$ or $(p, 24) = 12$, then $2\alpha_\sharp(\hat{\eta}) = 0$, and if $(p, 24) = 8$ or $(p, 24) = 24$, then $2\alpha_\sharp(\hat{\eta}) \neq 0$. Thus, each $f_{21} \in \langle 2\alpha\rangle$ induces the trivial homomorphism on $\pi_{n+3}(M_1)$ provided that $(p, 24) = 4$ or $(p, 24) = 12$.

Therefore, if $(p, 24) = 4$ or $(p, 24) = 12$, we have

$$\mathcal{E}_\sharp^{\dim+1}(X) \cong \left\{ \begin{pmatrix} 1 & f_{12} \\ f_{21} & 1 \end{pmatrix} \,\middle|\, f_{12} \in \langle 2\pi_2^*(i_1)\rangle,\ f_{21} \in \langle 2\alpha\rangle \right\},$$

and if $(p, 24) = 8$ or 24, we have

$$\mathcal{E}_\sharp^{\dim+1}(X) \cong \left\{ \begin{pmatrix} 1 & f_{12} \\ 1 & 1 \end{pmatrix} \,\middle|\, f_{12} \in \langle 2\pi_2^*(i_1)\rangle \right\}.$$

<u>Case 3</u>. Suppose that $q \equiv 0 \pmod 4$ and $p \equiv 2 \pmod 4$. We note that

$$\pi_{n+3}(M_2) \cong Z_{(p,24)} \oplus Z_2\{i_{2\sharp}(\nu), \overline{\eta^2}\},$$
$$\pi_{n+3}(M_1) \cong Z_2 \oplus Z_2\{\eta_3, \eta_4\}$$

and $[M_1, M_2] \cong Z_4 \oplus Z_2\{\pi_1^*(\overline{\eta}), \beta\}$. First, we investigate $f_{12\sharp(n+3)}$. Let $f_{12} = s\pi_2^*(i_1) \in [M_2, M_1] \cong Z_d\{\pi_2^*(i_1)\}$. Then, we have

$$\pi_2^*(i_1)_\sharp(i_{2\sharp}(\nu)) = i_1 \circ \pi_2 \circ i_2 \circ \nu = 0$$

and

$$\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = i_1 \circ \pi_2 \circ \overline{\eta^2} = i_1 \circ \eta^2 \neq 0.$$

If $s = 2l$ for some $1 \le l \le d/2$, then $2l\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = 2li_1 \circ \eta^2 = 0$, because $i_1 \circ \eta^2 = \eta_3 \in \pi_{n+3}(M_1)$. However, if $s = 2l + 1$ for some $0 \le l \le d/2 - 1$, then $(2l+1)\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = (2k+1)i_1 \circ \eta^2 = i_1 \circ \eta^2 \neq 0$.

Thus, any $f_{12} \in \langle 2\pi_2^*(i_1)\rangle \cong Z_{d/2}$ induces the trivial homomorphism on $\pi_{n+3}(M_2)$. However, for $f_{12} \notin \langle 2\pi_2^*(i_1)\rangle$, we have $f_{12\sharp n+3} \neq 0$.

Next, we investigate $f_{21\sharp(n+3)}$. Because $[M_1, M_2] \cong Z_4 \oplus Z_2\{\pi_1^*(\overline{\eta}), \beta\}$ and $\beta_{\sharp(n+2)} \neq 0$, we check only the generators $\pi_1^*(\overline{\eta})$. For η_3, we have

$$\pi_1^*(\overline{\eta})_\sharp(\eta_3) = \overline{\eta} \circ \pi_1 \circ \eta_3 = \overline{\eta} \circ \pi_1 \circ i_{1\sharp}(\eta^2) = 0.$$

For η_4, we have

$$\pi_1^*(\overline{\eta})_\sharp(\eta_4) = \overline{\eta} \circ \pi_1 \circ \eta_4 = \overline{\eta} \circ \eta \neq 0$$

since $\pi_{2\sharp}(\overline{\eta} \circ \eta) = \eta^2 \neq 0$.

However, $2\pi_1^*(\overline{\eta})_\sharp(\eta_4) = \overline{\eta} \circ \pi_1 \circ 2\eta_4 = 0$.

Thus, every $f_{21} \in \langle 2\pi_1^*(\overline{\eta})\rangle$ induces the trivial homomorphism on $n+3$.

Therefore, we have

$$\mathcal{E}_\sharp^{\dim+1}(X) \cong \left\{ \begin{pmatrix} 1 & f_{12} \\ f_{21} & 1 \end{pmatrix} \middle| \; f_{12} \in \langle 2\pi_2^*(i_1)\rangle,\ f_{21} \in \langle 2\pi_1^*(\overline{\eta})\rangle \right\}.$$

Case 4. Suppose that $q \equiv 2 \pmod 4$ and $p \equiv 2 \pmod 4$. Note that $\pi_{n+3}(M_2) \cong Z_{(p,24)} \oplus Z_2\{i_{2\sharp}(\nu), \overline{\eta^2}\}$ and $\pi_{n+3}(M_1) \cong Z_4\{\hat{\eta}\}$. First, we investigate $f_{12\sharp(n+3)}$. For the generator $\pi_2^*(i_1)$ of $[M_2, M_1]$, we have

$$\pi_2^*(i_1)_\sharp(i_{2\sharp}(\nu)) = i_1 \circ \pi_2 \circ i_2 \circ \nu = 0$$

and

$$\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = i_1 \circ \pi_2 \circ \overline{\eta^2} = i_1 \circ \eta^2 = 2\hat{\eta} \neq 0.$$

Let $f_{12} = s\pi_2^*(i_1)$. If $s = 2l$ for $1 \leq l \leq d/2$, then $s\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = 4l\hat{\eta} = 0$, and if $s = 2l+1$ for $0 \leq l \leq d/2 - 1$, then $s\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = (4l+2)\hat{\eta} = 2\hat{\eta} \neq 0$.

Thus, each $f_{12} \in \langle 2\pi_2^*(i_1)\rangle \cong Z_{d/2}$ induces the trivial homomorphism on $n+3$. However, for $f_{12} \notin \langle 2\pi_2^*(i_1)\rangle$, we have $f_{12\sharp(n+3)} \neq 0$.

Next, we investigate $f_{21\sharp(n+3)}$. Note that $[M_1, M_2] \cong Z_2 \oplus Z_2\{\pi_1^*(\overline{\eta}), \beta\}$. Since $\beta_{\sharp n+2} \neq 0$, we consider only the generator $\pi_1^*(\overline{\eta})$.

Since $\pi_{2\sharp}(\overline{\eta} \circ \eta) = \pi_2 \circ \overline{\eta} \circ \eta = \eta^2 \neq 0$, we have $\pi_1^*(\overline{\eta})_\sharp(\hat{\eta}) = \overline{\eta} \circ \pi_1 \circ \hat{\eta} = \overline{\eta} \circ \eta \neq 0$. Therefore, no f_{21} induces a trivial homomorphism.

Thus, we have

$$\mathcal{E}_\sharp^{\dim+1}(X) \cong \left\{ \begin{pmatrix} 1 & f_{12} \\ 1 & 1 \end{pmatrix} \middle| \; f_{12} \in \langle 2\pi_2^*(i_1)\rangle \right\}.$$

Case 5. Suppose that $q \equiv 0 \pmod 4$ and $p \equiv 0 \pmod 4$. Note that $\pi_{n+3}(M_2) \cong Z_{(p,24)} \oplus Z_2\{i_{2\sharp}(\nu), \overline{\eta^2}\}$ and $\pi_{n+3}(M_1) \cong Z_2 \oplus Z_2\{\eta_3, \eta_4\}$.

First, we investigate $f_{12\sharp(n+3)}$. For the generator $\pi_2^*(i_1)$ of $[M_2, M_1]$, we have

$$\pi_2^*(i_1)_\sharp(i_{2\sharp}(\nu)) = i_1 \circ \pi_2 \circ i_2 \circ \nu = 0$$

and

$$\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = i_1 \circ \pi_2 \circ \overline{\eta^2} = i_1 \circ \eta^2 \neq 0.$$

Let $f_{12} = s\pi_2^*(i_1)$. If $s = 2l$ for $1 \le l \le d/2$, then

$$s\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = 2li_1 \circ \eta^2 = l2\eta_3 = 0.$$

However, if $s = 2l+1$ for $0 \le l \le d/2-1$, then

$$s\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = (2l+1)i_1 \circ \eta^2 = \eta_3 \neq 0.$$

Thus, each $f_{12} \in \langle 2\pi_2^*(i_1)\rangle \cong Z_{d/2}$ induces the trivial homomorphism on $n+3$. However, for $f_{12} \notin \langle 2\pi_2^*(i_1)\rangle$, we have $f_{12\sharp(n+3)} \neq 0$.

Next, we consider $f_{21\sharp(n+3)}$. Note that

$$[M_1, M_2] \cong Z_2 \oplus Z_2 \oplus Z_2 \ \{\ \pi_1^*(\eta_1), \pi_1^*(\eta_2), \alpha\}.$$

Since $\alpha_{\sharp(n+2)} = 0$, we consider only the generators $\pi_1^*(\eta_1)$ and $\pi_1^*(\eta_2)$. For $\pi_1^*(\eta_1)$, we have

$$\pi_1^*(\eta_1)_\sharp(\eta_3) = \eta_1 \circ \pi_1 \circ \eta_3 = \eta_1 \circ \pi_1 \circ i_1\eta^2 = 0$$

and

$$\pi_1^*(\eta_1)_\sharp(\eta_4) = \eta_1 \circ \pi_1 \circ \eta_4 = \eta_1 \circ \eta = i_{2\sharp}(\eta^2) \circ \eta = 4i_{2\sharp}(\nu).$$

Thus, if $(p, 24) = 4$ or $(p, 24) = 12$, then $\pi_1^*(\eta_1)_\sharp(\eta_4) = 4i_{1\sharp}(\nu) = 0$, and if $(p, 24) = 8$ or $(p, 24) = 24$, then $\pi_1^*(\eta_1)_\sharp(\eta_4) = 4i_{1\sharp}(\nu) \neq 0$.

Since $\pi_{2\sharp}(\eta_2 \circ \eta) = \eta^2$, we have $\pi_1^*(\eta_2)_\sharp(\eta_4) = \eta_2 \circ \pi_1 \circ \eta_4 = \eta_2 \circ \eta \neq 0$.

Therefore, if $(p, 24) = 4$ or $(p, 24) = 12$, we have

$$\mathcal{E}_\sharp^{\dim+1}(X) \cong \left\{ \begin{pmatrix} 1 & f_{12} \\ f_{21} & 1 \end{pmatrix} \,\middle|\, f_{12} \in \langle 2\pi_2^*(i_1)\rangle,\ f_{21} \in \langle \pi_2^*(\eta_1)\rangle \right\},$$

and if $(p, 24) = 8$ or $(p, 24) = 24$, we have

$$\mathcal{E}_\sharp^{\dim+1}(X) \cong \left\{ \begin{pmatrix} 1 & f_{12} \\ 1 & 1 \end{pmatrix} \,\middle|\, f_{12} \in \langle 2\pi_2^*(i_1)\rangle \right\}. \qquad \square$$

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