Pacific Journal of Mathematics

CERTAIN SELF-HOMOTOPY EQUIVALENCES ON WEDGE PRODUCTS OF MOORE SPACES

HO WON CHOI AND KEE YOUNG LEE

Volume 272 No. 1 November 2014

PACIFIC JOURNAL OF MATHEMATICS
Vol. 272, No. 1, 2014
dx.doi.org/10.2140/pjm.2014.272.35

CERTAIN SELF-HOMOTOPY EQUIVALENCES ON WEDGE PRODUCTS OF MOORE SPACES

HO WON CHOI AND KEE YOUNG LEE

For a based 1-connected finite CW-complex X, let $\mathcal{E}(X)$ denote the group of homotopy classes of self-homotopy equivalences on X, and $\mathcal{E}_\sharp^{\dim+r}(X)$ the subgroup of $\mathcal{E}(X)$ of homotopy classes of self-homotopy equivalences on X that induce the identity homomorphism on the homotopy groups of X in dimensions $\leq \dim X + r$. For two given Moore spaces $M_1 = M(Z_q, n+1)$ and $M_2 = M(Z_p, n)$ with $n \geq 5$, we investigate the subsets of $[M_1, M_2]$ and $[M_2, M_1]$ consisting of homotopy classes of maps that induce the trivial homomorphism between the homotopy groups of M_1 and those of M_2 in dimensions $\leq \dim X + r$. Using the results of this investigation, we completely determine the subgroups $\mathcal{E}_\sharp^{\dim+r}\big(M(Z_q, n+1) \vee M(Z_p, n)\big)$, where p and q are positive integers, for $n \geq 5$ and $r = 0, 1$.

1. Introduction

If X and Y are based topological spaces, let $[X, Y]$ denote the set of homotopy classes of based maps from X to Y, let $\mathcal{E}(X)$ denote the subset of $[X, X]$ that consists of homotopy classes of self-homotopy equivalences of X and let $\mathcal{E}_\sharp^{\dim+r}(X)$ denote the set of homotopy classes of self-homotopy equivalences that induce the identity on the homotopy groups of X in dimensions at most $\dim X + r$. Then, $\mathcal{E}(X)$ is a group with a group operation given by the composition of homotopy classes, and $\mathcal{E}_\sharp^{\dim+r}(X)$ is a subgroup of $\mathcal{E}(X)$. The group $\mathcal{E}(X)$ and certain natural subgroups including $\mathcal{E}_\sharp^{\dim+r}(X)$ are fundamental objects in homotopy theory and have been studied extensively. For a survey of the known results and applications of $\mathcal{E}(X)$, see [Arkowitz 1990].

When G is an abelian group, we let $M(G, n)$ denote the Moore space, that is, the space with G as a single nonvanishing homology group at n-level. Also, in this case, $M(G, n)$ is a simply connected space. We note that if $n \geq 3$, then $M(G, n)$ is characterized by

$$\widetilde{H}_i(M(G, n)) \cong \begin{cases} G & \text{if } i = n, \\ 0 & \text{if } i \neq n. \end{cases}$$

This work was supported by a Korea University Grant.
MSC2010: primary 55P10, 55Q05, 55Q20; secondary 55Q40, 55Q52.
Keywords: self-homotopy equivalence, Moore space, homotopy group.

Let $C(G, n)$ denote the co-Moore space of type (G, n) defined by

$$\widetilde{H}^i(C(G,n)) \cong \begin{cases} G & \text{if } i = n, \\ 0 & \text{if } i \neq n. \end{cases}$$

If G is a finitely generated abelian group and $G = F \oplus T$, where F is a free abelian group of rank r and T is a finite group, then $M(G, n) = M(F, n) \vee M(T, n)$ and $C(G, n) = M(F, n) \vee M(T, n-1)$ for $n \geq 3$.

Arkowitz and Maruyama [1998] showed that $\mathcal{E}_\sharp^{\dim}(M(G, n)) \cong \oplus^{(r+s)s} Z_2$ and $\mathcal{E}_\sharp^{\dim+1}(M(G, n)) = 1$ for $n > 3$, where r is the rank of G and s is the number of 2-torsion summands in G. Moreover, they completely determined $\mathcal{E}_\sharp^{\dim}(C(G, n))$ for $n \geq 3$ by means of 2×2 matrices, where G is a finitely generated abelian group.

Jeong [2010] computed the groups $\mathcal{E}_\sharp^{\dim}(Y)$ for $Y = M(Z_p, n+1) \vee M(Z_p, n)$, $n \geq 5$ as follows:

$$\mathcal{E}_\sharp^{\dim}(Y) \cong \begin{cases} Z_p & \text{if } p \text{ is odd}, \\ Z_2 \oplus Z_2 & \text{if } p \equiv 2 \pmod 4, \\ Z_2 \oplus Z_2 \oplus Z_2 & \text{if } p \equiv 0 \pmod 4. \end{cases}$$

In this paper we study the self-homotopy equivalences on the wedge product $X = M(Z_q, n+1) \vee M(Z_p, n)$ for $n \geq 5$, where p and q are positive integers. For two given Moore spaces $M_1 = M(Z_q, n+1)$ and $M_2 = M(Z_p, n)$, we compute $[M_1, M_2]$ and $[M_2, M_1]$ and find their generators. Moreover, we investigate the subset of $[M_1, M_2]$ or $[M_2, M_1]$ that consists of elements whose induced homomorphisms are trivial between the homotopy groups of M_1 and those of M_2 in dimensions at most $\dim X + r$ with $r = 0, 1$. Using these results, we completely determine the groups $\mathcal{E}_\sharp^{\dim+r}(X)$ for $r = 0, 1$. As a result, we obtain Table 1 and the following:

$$\mathcal{E}_\sharp^{\dim+1}(X) \cong \begin{cases} 1 & \text{if } q \text{ is odd or } p \text{ is odd } (d = 1), \\ Z_d & \text{if } q \text{ is odd or } p \text{ is odd } (d \neq 1), \\ Z_{d/2} \oplus Z_2 & \text{if } p \equiv 0 \pmod 4 \text{ and } (p, 24) = 4 \text{ or } 12\ (d \neq 1), \\ Z_{d/2} & \text{if } p \equiv 0 \pmod 4 \text{ and } (p, 24) = 8 \text{ or } 24\ (d \neq 1), \\ Z_{d/2} & \text{if } q \equiv 2,\ p \equiv 2 \pmod 4, \\ Z_{d/2} \oplus Z_2 & \text{if } q \equiv 0,\ p \equiv 2 \pmod 4, \end{cases}$$

where d is the greatest common divisor of p and q.

The space X is neither a Moore space nor a co-Moore space but is characterized by finite homology groups and cohomology groups. That is,

$$\widetilde{H}_i(X) \cong \begin{cases} Z_p & \text{if } i = n, \\ Z_q & \text{if } i = n+1, \\ 0 & \text{otherwise}, \end{cases}$$

	q is odd $d=1$	q is odd $d\neq 1$	$q\equiv 2 \pmod 4$	$q\equiv 0 \pmod 4$
p is odd ($d=1$)	1	$\cdot$	Z_2	Z_2
p is odd ($d\neq 1$)	$\cdot$	Z_d	$Z_2\oplus Z_d$	$Z_2\oplus Z_d$
$p\equiv 2 \pmod 4$	1	Z_d	$Z_2\oplus Z_{d/2}\oplus Z_2$	$Z_2\oplus Z_{d/2}\oplus Z_4$
$p\equiv 0 \pmod 4$	1	Z_d	$Z_2\oplus Z_{d/2}\oplus Z_2\oplus Z_2$	$Z_2\oplus Z_{d/2}\oplus Z_2\oplus Z_2$

Table 1. Isomorphism class of the groups $\mathcal{E}_\sharp^{\dim}(X)$.

and

$$\widetilde{H}^i(X,\pi)\cong\begin{cases}\operatorname{Hom}(Z_p,\pi) & \text{if } i=n,\\ \operatorname{Ext}(Z_p,\pi)\oplus\operatorname{Hom}(Z_q,\pi) & \text{if } i=n+1,\\ \operatorname{Ext}(Z_q,\pi) & \text{if } i=n+2,\\ 0 & \text{otherwise.}\end{cases}$$

From this perspective, X is an interesting space for studying self-homotopy equivalences.

Throughout this paper, all topological spaces are based and have the based homotopy type of a finite 1-connected CW-complex. All maps and homotopies will preserve base points. For the spaces X and Y, we denote by $[X, Y]$ the set of homotopy classes of maps from X to Y. We do not distinguish between the notation of a map $X\to Y$ and that of its homotopy class in $[X, Y]$. If a group G is generated by a set $\{a_1,\dots,a_n\}$, then we denote the group by $G\{a_1,\dots,a_2\}$ or $G=\langle a_1,\dots,a_n\rangle$.

2. Preliminaries

Let X be a space. Then, we denote by SX the suspension of X and by S^nX the iterated suspension defined by $S^nX=S(S^{n-1}X)$. Let $f:A\to B$ be a map and let $C_f=B\cup_f CA$ be the mapping cone of f. Then, we have a Puppe sequence [1958] for f,

$$A\xrightarrow{f}B\xrightarrow{i}C_f\xrightarrow{\pi}SA\xrightarrow{Sf}SB\xrightarrow{Si}SC_f\xrightarrow{S\pi}S^2A\xrightarrow{S^2f}S^2B\longrightarrow\cdots,$$

such that the following sequence is exact for any space X:

$$\cdots\longrightarrow[SC_f,X]\xrightarrow{S\pi^*}[SB,X]\xrightarrow{Sf^*}[SA,X]\xrightarrow{\pi^*}[C_f,X]\xrightarrow{i^*}[B,X]\xrightarrow{f^*}[A,X],$$

where S^nf is a suspension map induced by f.

If A is m-connected and B is n-connected, then we have the following exact sequence for any CW-complex Y with dimension at most $m+n$ as a dual sequence of the above sequence [Blakers and Massey 1952]:

$$[Y, A] \xrightarrow{f_*} [Y, B] \xrightarrow{i_*} [Y, C_f] \xrightarrow{\pi_*} [Y, SA] \xrightarrow{Sf_*} [Y, SB] \longrightarrow \cdots .$$

Both sequences will be called *the exact sequences associated with the cofibration*

$$B \to C_f \to SA.$$

Proposition 2.1 [Arkowitz and Maruyama 1998]. *If X is $(k-1)$-connected, Y is $(l-1)$-connected, $k, l \geq 2$ and* $\dim P \leq k+l-1$, *then the projections $X \vee Y \to X$ and $X \vee Y \to Y$ induce a bijection*

$$[P, X \vee Y] \to [P, X] \oplus [P, Y].$$

Proposition 2.1 is a consequence of [Spanier 1966, p. 405] since the inclusion $X \vee Y \to X \times Y$ is a $(k+l-1)$-equivalence.

Next, we consider abelian groups G_1 and G_2 and Moore spaces $M_1 = M(G_1, n_1)$ and $M_2 = M(G_2, n_2)$. Let $X = M_1 \vee M_2$. We denote by $i_j : M_j \to X$ the inclusion and by $p_j : X \to M_j$ the projection, where $j = 1, 2$. If $f : X \to X$, then we define $f_{jk} : M_k \to M_j$ by $f_{jk} = p_j f i_k$ for $j, k = 1, 2$.

If $f : X \to Y$ is a map, then $f_{\sharp n} : \pi_n(X) \to \pi_n(Y)$ denotes the induced homomorphism in dimension n.

Proposition 2.2 [Arkowitz and Maruyama 1998]. *The function θ that assigns to each $f \in [X, X]$ the 2×2 matrix*

$$\theta(f) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

where $f_{jk} \in [M_k, M_j]$, is a bijection. In addition:

(1) $\theta(f+g) = \theta(f) + \theta(g)$, *so θ is an isomorphism* $[X, X] \to \bigoplus_{j,k=1,2} [M_k, M_j]$.

(2) $\theta(fg) = \theta(f)\theta(g)$, *where fg denotes composition in* $[X, X]$ *and* $\theta(f)\theta(g)$ *denotes matrix multiplication.*

(3) *If $\alpha_r : \pi_r(M_1) \oplus \pi_r(M_2) \to \pi_r(M_1 \vee M_2)$ is the homomorphism induced by the inclusions and $\beta_r : \pi_r(M_1 \vee M_2) \to \pi_r(M_1) \oplus \pi_r(M_2)$ the homomorphism induced by the projections respectively, then*

$$\beta_r f_{\sharp r} \alpha_r(x, y) = \big(f_{11\,\sharp r}(x) + f_{12\,\sharp r}(y),\ f_{21\,\sharp r}(x) + f_{22\,\sharp r}(y)\big)$$

for $x \in \pi_r(M_1)$ and $y \in \pi_r(M_2)$.

Proposition 2.3 [Araki and Toda 1965]. (1) $\pi_n(M(Z_q, n)) \cong Z_q$ *for all q.*

(2) $\pi_{n+1}(M(Z_q, n)) \cong \begin{cases} 0 & \text{if } q \text{ is odd}, \\ Z_2 & \text{if } q \text{ is even}. \end{cases}$

(3) $\pi_{n+2}(M(Z_q,n)) \cong \begin{cases} 0 & \textit{if } q \textit{ is odd}, \\ Z_4 & \textit{if } q \equiv 2 \pmod 4, \\ Z_2 \oplus Z_2 & \textit{if } q \equiv 0 \pmod 4). \end{cases}$

(4) $\pi_{n+3}(M(Z_q,n)) \cong \begin{cases} Z_{(q,24)} & \textit{if } q \textit{ is odd}, \\ Z_{(q,24)} \oplus Z_2 & \textit{if } q \textit{ is even}. \end{cases}$

The generators of $[S^{n+i}, S^n]$ can be summarized thus [Toda 1962]:

	$i<0$	$i=0$	$i=1$	$i=2$	$i=3$	$i=4,5$
$[S^{n+i}, S^n]$	0	Z	Z_2	Z_2	Z_{24}	0
Generator		ι	η	η^2	ν	0

Proposition 2.4 [Araki and Toda 1965].

(1) $[M(Z_q,n), S^n] \cong \begin{cases} 0 & \textit{if } q \textit{ is odd}, \\ Z_2 & \textit{if } q \textit{ is even}. \end{cases}$

(2) $[M(Z_q,n+1), S^n] \cong \begin{cases} 0 & \textit{if } q \textit{ is odd}, \\ Z_4 & \textit{if } q \equiv 2 \pmod 4), \\ Z_2 \oplus Z_2 & \textit{if } q \equiv 0 \pmod 4). \end{cases}$

Proposition 2.5 [Arkowitz and Maruyama 1998]. *For the Moore space* $X = M(G,n)$:

(1) $\mathcal{E}_\sharp^{\dim}(X) \cong \oplus^{(r+s)s} Z_2$, *where r is the rank of G and s is the number of* 2-*torsion summands in G.*

(2) $\mathcal{E}_\sharp^{\dim+1}(X) \cong 1$ *if* $n > 3$.

Proposition 2.6 (universal coefficient theorem for homotopy groups with coefficients [Hilton 1965]). *There is an exact sequence*

$$0 \to \operatorname{Ext}(G, \pi_{n+1}(X)) \to \pi_n(G;X) \to \operatorname{Hom}(G, \pi_n(X)) \to 0,$$

where $\pi_n(G;X)$, *the n-th homotopy group of X with coefficients in G, is given by* $\pi_n(G;X) = [M(G,n), X]$, *where* $M(G,n)$ *is a Moore space.*

3. Generators of the sets of homotopy classes on Moore spaces

In this section, we find generators of homotopy groups of Moore spaces and the sets of homotopy classes between two Moore spaces. Let

$$M_1 = M(Z_q, n+1) = S^{n+1} \cup_q e^{n+2} \quad \text{and} \quad M_2 = M(Z_p, n) = S^n \cup_p e^{n+1},$$

with $p, q \geq 1$. Then, there are two mapping cone sequences

$$S^{n+1} \xrightarrow{q\iota_1} S^{n+1} \xrightarrow{i_1} S^{n+1} \cup_q e^{n+2} \xrightarrow{\pi_1} S^{n+2} \xrightarrow{q\iota_1} S^{n+2}$$

and

$$S^n \xrightarrow{p_{\iota_2}} S^n \xrightarrow{i_2} S^n \cup_p e^{n+1} \xrightarrow{\pi_2} S^{n+1} \xrightarrow{p_{\iota_2}} S^{n+1},$$

where p_{ι_2} and q_{ι_1} are maps with degree p and q respectively.

Remark 3.1. We find generators of $\pi_m(M(Z_r, n))$, for $n \le m \le n+2$.

Recall that $\pi_n(M(Z_r, n)) \cong Z_r$. From the mapping cone sequence

$$S^n \xrightarrow{r_\iota} S^n \xrightarrow{i} M(Z_r, n) \xrightarrow{\pi} S^{n+1} \xrightarrow{r_\iota} S^{n+1},$$

we obtain the long exact sequence

$$\pi_n(S^n) \xrightarrow{r_{\iota\sharp}} \pi_n(S^n) \xrightarrow{i_\sharp} \pi_n(M(Z_r, n)) \xrightarrow{\pi_\sharp} \pi_n(S^{n+1}) \xrightarrow{r_{\iota\sharp}} \pi_n(S^{n+1}).$$

By the results in [Toda 1962], we have the sequence

$$Z\{\iota\} \xrightarrow{r_{\iota\sharp}} Z\{\iota\} \xrightarrow{i_\sharp} \pi_n(M(Z_r, n)) \longrightarrow 0,$$

so $i_\sharp$ is surjective. Thus, $\pi_n(M(Z_r, n)) \cong Z\{\iota\}/\operatorname{Im}(r_{\iota\sharp})$. Let $i_\sharp(\iota) = i$. Then, we can take i as a generator of $\pi_n(M(Z_r, n))$.

Next, we find a generator of $\pi_{n+1}(M(Z_r, n))$. There are two cases according to the parity of the positive integer r. If r is odd, then $\pi_{n+1}(M(Z_r, n))$ is trivial. If r is even, then we can take $i_\sharp(\eta)$ as a generator of $\pi_{n+1}(M(Z_r, n))$, where η is the generator of $\pi_{n+1}(S^n)$.

Finally, we find a generator of $\pi_{n+2}(M(Z_r, n))$. Consider the exact sequence

$$\pi_{n+2}(S^n) \xrightarrow{r_{\iota\sharp}} \pi_{n+2}(S^n) \xrightarrow{i_\sharp} \pi_{n+2}(M(Z_r, n)) \xrightarrow{\pi_\sharp} \pi_{n+2}(S^{n+1}) \xrightarrow{r_{\iota\sharp}} \pi_{n+2}(S^{n+1}).$$

Then by the results in [Toda 1962], we have the exact sequence

$$Z_2\{\eta^2\} \xrightarrow{r_{\iota\sharp}} Z_2\{\eta^2\} \xrightarrow{i_\sharp} \pi_{n+2}(M(Z_r, n)) \xrightarrow{\pi_\sharp} Z_2\{\eta\} \xrightarrow{q_{\iota\sharp}} Z_2\{\eta\}.$$

Since r is an even number, we obtain the exact sequence

$$0 \longrightarrow Z_2\{\eta^2\} \xrightarrow{i_\sharp} \pi_{n+2}(M(Z_r, n)) \xrightarrow{\pi_\sharp} Z_2\{\eta\} \longrightarrow 0.$$

If $r \equiv 2 \pmod 4$, then $\pi_{n+2}(M(Z_r, n)) \cong Z_4\{\bar{\eta}\}$ such that $i_\sharp(\eta^2) = 2\bar{\eta}$ and $\pi_\sharp(\bar{\eta}) = \eta$. On the other hand, if $r \equiv 0 \pmod 4$, then $\pi_{n+2}(M(Z_r, n)) \cong Z_2 \oplus Z_2\{\eta_1, \eta_2\}$ such that $i_\sharp(\eta^2) = \eta_1$ and $\pi_\sharp(\eta_2) = \eta$.

By Remark 3.1, it follows that

$$\pi_{n+1}(M_1) \cong Z_q\{i_1\}, \qquad \pi_n(M_2) \cong Z_p\{i_2\},$$
$$\pi_{n+2}(M_1) \cong Z_2\{i_{1\sharp}(\eta)\}, \qquad \pi_{n+1}(M_2) \cong Z_2\{i_{2\sharp}(\eta)\}.$$

Moreover, $\pi_{n+2}(M_2) \cong Z_4\{\bar{\eta}\}$ or $\pi_{n+2}(M_2) \cong Z_2 \oplus Z_2\{\eta_1, \eta_2\}$.

Lemma 3.2. *Let p and q be positive integers and (p, q) be the greatest common divisor of p and q. Consequently, if $(p, q) = d \neq 1$, then $[M_2, M_1] \cong Z_d\{\pi_2^*(i_1)\}$ and if $(p, q) = 1$, then $[M_2, M_1] \cong 0$.*

Proof. Consider the mapping cone sequence of M_2,

$$S^n \xrightarrow{p\iota_2} S^n \xrightarrow{i_2} S^n \cup_p e^{n+1} \xrightarrow{\pi_2} S^{n+1} \xrightarrow{p\iota_2} S^{n+1}.$$

This sequence induces the following exact sequence:

$$\pi_{n+1}(M_1) \xrightarrow{p\iota_2 *} \pi_{n+1}(M_1) \xrightarrow{\pi_2 *} [M_2, M_1] \xrightarrow{i_2 *} \pi_n(M_1) \xrightarrow{p\iota_2 *} \pi_n(M_1).$$

Since $\pi_{n+1}(M_1) \cong Z_q\{i_1\}$ and $\pi_n(M_1) \cong 0$, the exact sequence above becomes

$$Z_q\{i_1\} \xrightarrow{p^*_{\iota_2}} Z_q\{i_1\} \xrightarrow{\pi_2^*} [M_2, M_1] \longrightarrow 0.$$

If $(p, q) = 1$, the first $p^*_{\iota_2}$ is an isomorphism, so $[M_2, M_1] \cong 0$. Let $(p, q) = d \neq 1$. Then, since π_2^* is surjective and $p^*_{\iota_2}(i_1) = pi_1$, we have

$$[M_2, M_1] = \operatorname{im} \pi_2^* \cong Z_q\{i_1\}/\operatorname{im} p^*_{\iota_2} \cong Z_d\{\pi_2^*(i_1)\}.$$ □

Lemma 3.3. *If p or q is odd, then $[M_1, M_2] \cong 0$.*

Proof. Consider the mapping cone sequence of M_1,

$$S^{n+1} \xrightarrow{q\iota_1} S^{n+1} \xrightarrow{i_1} S^{n+1} \cup_q e^{n+2} \xrightarrow{\pi_1} S^{n+2} \xrightarrow{q\iota_1} S^{n+2}.$$

Then, we have the exact sequence

$$\pi_{n+2}(M_2) \xrightarrow{q^*_{\iota_1}} \pi_{n+2}(M_2) \xrightarrow{\pi_1^*} [M_1, M_2] \xrightarrow{i_1^*} \pi_{n+1}(M_2) \xrightarrow{q^*_{\iota_1}} \pi_{n+1}(M_2).$$

Let $p \equiv 2 \pmod 4$ and let q be odd. Then, since $\pi_{n+1}(M_2) \cong Z_2$ and $\pi_{n+2}(M_2) \cong Z_4$, we have the sequence

$$Z_4 \xrightarrow{q^*_{\iota_1}} Z_4 \xrightarrow{\pi_1^*} [M_1, M_2] \xrightarrow{i_1^*} Z_2 \xrightarrow{q^*_{\iota_1}} Z_2.$$

Furthermore, since $(q, 4) = 1$ and $(q, 2) = 1$, each $q^*_{\iota_1}$ is an isomorphism. Thus we have the exact sequence

$$0 \to [M_1, M_2] \to 0.$$

Therefore, $[M_1, M_2] \cong 0$.

In the case where $p \equiv 0 \pmod 4$ and q is odd, we can give a similar proof.

Next, let p be odd. Since $\pi_{n+1}(M_2)$ and $\pi_{n+2}(M_2)$ are trivial groups, so is $[M_1, M_2]$ by exactness. □

Let p and q be even. From the exact sequences associated with the cofibrations $S^{n+1} \to M_1 \to S^{n+2}$ and $S^n \to M_2 \to S^{n+1}$, we have the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
[S^{n+2}, S^n] & \xrightarrow{q^*_{\iota_1}} & [S^{n+2}, S^n] & \xrightarrow{\pi_1^*} & [M_1, S^n] & \xrightarrow{i_1^*} & [S^{n+1}, S^n] & \xrightarrow{q^*_{\iota_1}} & [S^{n+1}, S^n] \\
\downarrow p_{\iota_2 *} & & \downarrow p_{\iota_2 *} & & \downarrow p_{\iota_2 *} & & \downarrow p_{\iota_2 *} & & \downarrow p_{\iota_2 *} \\
[S^{n+2}, S^n] & \xrightarrow{q^*_{\iota_1}} & [S^{n+2}, S^n] & \xrightarrow{\pi_1^*} & [M_1, S^n] & \xrightarrow{i_1^*} & [S^{n+1}, S^n] & \xrightarrow{q^*_{\iota_1}} & [S^{n+1}, S^n] \\
\downarrow i_{2*} & & \downarrow i_{2*} & & \downarrow i_{2*} & & \downarrow i_{2*} & & \downarrow i_{2*} \\
[S^{n+2}, M_2] & \xrightarrow{q^*_{\iota_1}} & [S^{n+2}, M_2] & \xrightarrow{\pi_1^*} & [M_1, M_2] & \xrightarrow{i_1^*} & [S^{n+1}, M_2] & \xrightarrow{q^*_{\iota_1}} & [S^{n+1}, M_2] \\
\downarrow \pi_{2*} & & \downarrow \pi_{2*} & & \downarrow \pi_{2*} & & \downarrow \pi_{2*} & & \downarrow \pi_{2*} \\
[S^{n+2}, S^{n+1}] & \xrightarrow{q^*_{\iota_1}} & [S^{n+2}, S^{n+1}] & \xrightarrow{\pi_1^*} & [M_1, S^{n+1}] & \xrightarrow{i_1^*} & [S^{n+1}, S^{n+1}] & \xrightarrow{q^*_{\iota_1}} & [S^{n+1}, S^{n+1}] \\
\downarrow p_{\iota_2 *} & & \downarrow p_{\iota_2 *} & & \downarrow p_{\iota_2 *} & & \downarrow p_{\iota_2 *} & & \downarrow p_{\iota_2 *} \\
[S^{n+2}, S^{n+1}] & \xrightarrow{q^*_{\iota_1}} & [S^{n+2}, S^{n+1}] & \xrightarrow{\pi_1^*} & [M_1, S^{n+1}] & \xrightarrow{i_1^*} & [S^{n+1}, S^{n+1}] & \xrightarrow{q^*_{\iota_1}} & [S^{n+1}, S^{n+1}]
\end{array}
$$

Lemma 3.4. *Let* $(p, q) \neq 1$. *Then, if either* $p \equiv 0 \pmod 4$ *and* $q \equiv 2 \pmod 4$ *or* $p \equiv 2 \pmod 4$ *and* $q \equiv 0 \pmod 4$, *we have* $[M_1, M_2] \cong Z_4 \oplus Z_2$.

Proof. Suppose that $p \equiv 0 \pmod 4$ and $q \equiv 2 \pmod 4$. With the results in [Araki and Toda 1965], we obtain the following diagram from the above diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & Z_4 & & & & \\
 & & & & \downarrow i_{2*} & & & & \\
0 & \longrightarrow & Z_2 \oplus Z_2 & \xrightarrow{\pi_1^*} & [M_1, M_2] & \xrightarrow{i_1^*} & Z_2 & \longrightarrow & 0 \\
 & & & & \downarrow \pi_{2*} & & & & \\
 & & & & Z_2 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

Thus, $[M_1, M_2]$ is isomorphic to one of three groups: Z_8, $Z_4 \oplus Z_2$ or $Z_2 \oplus Z_2 \oplus Z_2$. Since i_{2*} is injective, $[M_1, M_2]$ has an element of order 4. However, $Z_2 \oplus Z_2 \oplus Z_2$ does not have an element of order 4. Since π_1^* is injective, $[M_1, M_2]$ has a subgroup which is not cyclic. It follows that $[M_1, M_2] \neq Z_8$. Therefore, $[M_1, M_2] \cong Z_4 \oplus Z_2$.

Now, let $p \equiv 2 \pmod 4$ and $q \equiv 0 \pmod 4$. With the results in [Araki and Toda 1965], we obtain the following diagram from the above commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & Z_2 \oplus Z_2 & & & & \\
 & & & & \downarrow{\scriptstyle i_{2*}} & & & & \\
0 & \longrightarrow & Z_4 & \xrightarrow{\pi_1^*} & [M_1, M_2] & \xrightarrow{i_1^*} & Z_2 & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \pi_{2*}} & & & & \\
 & & & & Z_2 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Thus, $[M_1, M_2]$ is isomorphic to one of the three groups: Z_8, $Z_4 \oplus Z_2$ or $Z_2 \oplus Z_2 \oplus Z_2$. Since π_1^* is injective, $[M_1, M_2]$ has an element of order 4. However, $Z_2 \oplus Z_2 \oplus Z_2$ does not have an element of order 4. Since i_{2*} is injective, $[M_1, M_2]$ has a subgroup which is not cyclic. It follows that $[M_1, M_2] \neq Z_8$. Thus, $[M_1, M_2] \cong Z_4 \oplus Z_2$. $\square$

By Lemma 3.4, $[M_1, M_2] \cong Z_4 \oplus Z_2$. However, $[M_1, M_2]$ has different generators under different conditions. Here we determine the generators.

If $p \equiv 0 \pmod 4$ and $q \equiv 2 \pmod 4$, then $[M_1, M_2] \cong Z_4 \oplus Z_2\{\alpha, \pi_1^*(\eta_2)\}$, where $\pi_1^*(\eta_1) = 2\alpha$ and $i_1^*(\alpha) = i_{2\sharp}(\eta)$.

If $p \equiv 2 \pmod 4$ and $q \equiv 0 \pmod 4$, then $[M_1, M_2] \cong Z_4 \oplus Z_2\{\pi_1^*(\bar{\eta}), \beta\}$, where $i_1^*(\beta) = i_{2\sharp}(\eta)$.

For a given homomorphism $h : G_1 \to G_2$, we have from Proposition 2.6 the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ext}(G_2, \pi_{n+1}(X)) & \longrightarrow & \pi_n(G_2; X) & \longrightarrow & \operatorname{Hom}(G_2, \pi_n(X)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \bar{h}^\sharp} & & \downarrow{\scriptstyle h^*} & & \downarrow{\scriptstyle h^\sharp} & & \\
0 & \longrightarrow & \operatorname{Ext}(G_1, \pi_{n+1}(X)) & \longrightarrow & \pi_n(G_1; X) & \longrightarrow & \operatorname{Hom}(G_1, \pi_n(X)) & \longrightarrow & 0
\end{array}$$

where $\bar{h}^\sharp$ and $h^\sharp$ are induced by h and h^* is associated with h. This shows that the nonuniqueness of h^* is substantially limited. The measure of choice is bounded by the group

$$\operatorname{Hom}\big(\operatorname{Hom}(G_2, \pi_n(X)), \operatorname{Ext}(G_1, \pi_{n+1}(X))\big).$$

Lemma 3.5. *If $(p, q) = d \neq 1$, we have*

$$[M_1, M_2] \cong \begin{cases} Z_2 \oplus Z_2 & \text{if } p \equiv 2 \text{ and } q \equiv 2 \pmod 4, \\ Z_2 \oplus Z_2 \oplus Z_2 & \text{if } p \equiv 0 \text{ and } q \equiv 0 \pmod 4. \end{cases}$$

Proof. Suppose that $p \equiv 2 \pmod 4$ and $q \equiv 2 \pmod 4$. By the universal coefficient theorem for homotopy groups with coefficients, we have the short exact sequence

$$0 \to \mathrm{Ext}(Z_q, Z_4) \to [M_1, M_2] \to \mathrm{Hom}(Z_q, Z_2) \to 0.$$

Since $\mathrm{Ext}(Z_q, Z_4) \cong Z_{(q,4)} \cong Z_2$ and $\mathrm{Hom}(Z_q, Z_2) = Z_{(q,2)} = Z_2$, this sequence becomes

$$0 \to Z_2 \to [M_1, M_2] \to Z_2 \to 0.$$

Let $M_3 = M(Z_p, n+1)$. By the universal coefficient theorem for homotopy with coefficients, we have the sequence

$$0 \to \mathrm{Ext}(Z_p, Z_4) \to [M_3, M_2] \to \mathrm{Hom}(Z_p, Z_2) \to 0.$$

Similarly, this sequence becomes

$$0 \to Z_2 \to [M_3, M_2] \to Z_2 \to 0.$$

We may assume that $q \geq p$. Let $q = kd$ and $p = ld$, where $(k, l) = 1$. Then both k and l are odd. We define $h : Z_q \to Z_p$ by $h(\bar{1}) = \bar{l}$ with $\bar{s} = s + rZ \in Z_r$. Then, $\mathrm{im}(h)$ is congruent to Z_d in Z_p and h is a nontrivial homomorphism since $(q, p) = d \neq 1$. Thus, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_2 & \longrightarrow & [M_3, M_2] & \longrightarrow & Z_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \bar{h}^{\sharp}} & & \downarrow{\scriptstyle h^{*}} & & \downarrow{\scriptstyle h^{\sharp}} & & \\ 0 & \longrightarrow & Z_2 & \longrightarrow & [M_1, M_2] & \longrightarrow & Z_2 & \longrightarrow & 0 \end{array}$$

where $\bar{h}^{\sharp} : \mathrm{Ext}(Z_p, Z_4) \to \mathrm{Ext}(Z_q, Z_4)$ and $h^{\sharp} : \mathrm{Hom}(Z_p, Z_2) \to \mathrm{Hom}(Z_q, Z_2)$ are induced by h.

To show that $h^{\sharp} : \mathrm{Hom}(Z_p, Z_2) \to \mathrm{Hom}(Z_q, Z_2)$ is an isomorphism, it is sufficient to show that $h^{\sharp}$ is nontrivial. Let α be an nonzero element in $\mathrm{Hom}(Z_p, Z_2)$ such that $\alpha(\bar{1}) = \bar{1}$. Since $h^{\sharp}(\alpha) = \alpha \circ h \in \mathrm{Hom}(Z_q, Z_2)$ and $\alpha \circ h(\bar{1}) = \alpha(\bar{l}) = \bar{l} = \bar{1}$, where l is odd, it follows that $h^{\sharp}(\alpha)$ is a nontrivial homomorphism.

Next, we show that $\bar{h}^{\sharp} : \mathrm{Ext}(Z_p, Z_4) \to \mathrm{Ext}(Z_q, Z_4)$ is an isomorphism. Consider the resolutions of Z_q and Z_p. Then we have following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \xrightarrow{q} & Z & \xrightarrow{\beta} & Z_q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & Z & \xrightarrow{p} & Z & \xrightarrow{\beta'} & Z_p & \longrightarrow & 0 \end{array}$$

See [Gray 1975, Lemma 25.3]. Now, we give precise definitions of the maps h_1, h_2 and $h^{\sharp}$. Since $\bar{l} = h(\bar{1}) = h \circ \beta(1) = \beta'(h_2(1))$, we have h_2 given by $h_2(1) = l$. Moreover, we can obtain h_1 using h_2. Since $p \circ h_1 = h_2 \circ q$, we have

$ph_1(1)=h_2(q)=qh_2(1)=dkl=pk$. Thus, h_1 is given by $h_1(1)=k$. If we consider the three homomorphisms $h^\sharp$, $h_1^\sharp$ and $h_2^\sharp$ induced by h, h_1 and h_2 respectively, we have the following commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}(Z_q, Z_4) & \xrightarrow{\beta^*} & \mathrm{Hom}(Z, Z_4) & \xrightarrow{q^*} & \mathrm{Hom}(Z, Z_4) \\
& {\scriptstyle h^\sharp}\Big\uparrow & & {\scriptstyle h_2^\sharp}\Big\uparrow & & {\scriptstyle h_1^\sharp}\Big\uparrow{\scriptstyle \cong} \\
0 \longrightarrow & \mathrm{Hom}(Z_p, Z_4) & \xrightarrow{\beta'^*} & \mathrm{Hom}(Z, Z_4) & \xrightarrow{p^*} & \mathrm{Hom}(Z, Z_4)
\end{array}$$

Next, we show that $h_1^\sharp$ is an isomorphism. We choose a generator α of $\mathrm{Hom}(Z, Z_4)$ such that $\alpha(1)=\bar{1}$. Then $h_1^\sharp(\alpha)(1)=(\alpha\circ h_1)(1)=\alpha(k)\neq 0 \pmod 2$ since k is odd. Therefore, $h_1^\sharp(\alpha)$ is a generator of $\mathrm{Hom}(Z, Z_4)$. Thus, $h_1^\sharp$ is an isomorphism.

By using $h_1^\sharp$, we determine the homomorphism $\bar{h}^\sharp : \mathrm{Ext}(Z_p, Z_4)\to\mathrm{Ext}(Z_q, Z_4)$. Since $q\equiv p\equiv 2 \pmod 4$ and

$$\mathrm{Ext}(Z_p, Z_4)=\mathrm{Hom}(Z, Z_4)/\operatorname{im}(p^*) \quad\text{and}\quad \mathrm{Ext}(Z_q, Z_4)=\mathrm{Hom}(Z, Z_4)/\operatorname{im}(q^*),$$

we have

$$\mathrm{Ext}(Z_p, Z_4)=\langle\alpha+\{2\alpha\}\rangle \quad\text{and}\quad \mathrm{Ext}(Z_q, Z_4)=\langle\alpha+\{2\alpha\}\rangle.$$

By well-known facts of homological algebra, $\bar{h}^\sharp : \mathrm{Ext}(Z_p, Z_4)\to\mathrm{Ext}(Z_q, Z_4)$ is given by $\bar{h}^\sharp(\alpha+\{2\alpha\})=\alpha\circ h_1+\{2\alpha\}\neq 0$. Therefore, $\bar{h}^\sharp$ is nontrivial. Thus, $\bar{h}^\sharp$ is an isomorphism.

By the five lemma, $h^* : [M_1, M_2]\to[M_3, M_2]$ is an isomorphism. From [Araki and Toda 1965], we have $[M_3, M_2]\cong Z_2\oplus Z_2$. Therefore, $[M_1, M_2]\cong Z_2\oplus Z_2$.

Next, we suppose that $q\equiv 0$ and $p\equiv 0 \pmod 4$.

From [Araki and Toda 1965] and the commutative diagram above Lemma 3.4, we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & Z_2 & \xrightarrow{\pi_1^*} & Z_2\oplus Z_2 & \underset{r}{\overset{i_1^*}{\longleftrightarrow}} & Z_2 & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle i_{2*}} & & \Big\downarrow{\scriptstyle i_{2*}} & & {\scriptstyle \theta}\Big\updownarrow{\scriptstyle i_{2*}} & & \\
0 & \longrightarrow & Z_2\oplus Z_2 & \xrightarrow{\pi_1^*} & [M_1, M_2] & \xrightarrow{i_1^*} & Z_2 & \longrightarrow & 0 \\
 & & & & \Big\downarrow{\scriptstyle \pi_{2*}} & & & & \\
 & & & & Z_2 & & & & \\
 & & & & \Big\downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Since the second row is a split exact sequence, there exists $r : [S^{n+1}, S^n] \to [M_1, S^n]$ such that $i_1^* \circ r = \mathrm{id}_{[S^{n+1}, S^n]}$. Moreover, since the third i_{2*} is an isomorphism, there exists $\theta : [S^{n+1}, M_2] \to [S^{n+1}, S^n]$ such that $\theta \circ i_{2*} = \mathrm{id}_{[S^{n+1}, S^n]}$ and $i_{2*} \circ \theta = \mathrm{id}_{[S^{n+1}, M_2]}$.

We define the map $k : [S^{n+1}, M_2] \to [M_1, M_2]$ by $k = i_{2*} \circ r \circ \theta$. Then, we have

$$\begin{aligned} i_1^* \circ k &= i_1^* \circ i_{2*} \circ r \circ \theta \\ &= i_{2*} \circ i_1^* \circ r \circ \theta \\ &= i_{2*} \circ \mathrm{id}_{[S^{n+1}, S^n]} \circ \theta \\ &= i_{2*} \circ \theta = \mathrm{id}_{[S^{n+1}, M_2]}. \end{aligned}$$

Therefore, the third row is a split exact sequence. Hence,

$$[M_1, M_2] \cong Z_2 \oplus Z_2 \oplus Z_2. \qquad \square$$

Now, we determine the generators of $[M_1, M_2]$ when either $p \equiv 2 \pmod 4$ and $q \equiv 2 \pmod 4$ or $p \equiv 0 \pmod 4$ and $q \equiv 0 \pmod 4$.

Let $p \equiv 2 \pmod 4$ and $q \equiv 2 \pmod 4$. By using the Puppe exact sequence, we have the following exact sequence:

$$\pi_{n+2}(M_2) \xrightarrow{q_{\iota_1}^*} \pi_{n+2}(M_2) \xrightarrow{\pi_1^*} [M_1, M_2] \xrightarrow{i_1^*} \pi_{n+1}(M_2) \xrightarrow{p_{\iota_1}^*} \pi_{n+1}(M_2).$$

By exactness, we obtain the exact sequence

$$0 \longrightarrow Z_2 \xrightarrow{\pi_1^*} [M_1, M_2] \xrightarrow{i_1^*} Z_2 \longrightarrow 0.$$

Thus, $[M_1, M_2] \cong Z_2 \oplus Z_2\{\pi_1^*(\overline{\eta}), \beta\}$, where $i_1^*(\beta) = i_{2\sharp}(\eta)$.

Next, we let $p \equiv 0 \pmod 4$ and $q \equiv 0 \pmod 4$. By a similar method we obtain $[M_1, M_2] \cong Z_2 \oplus Z_2 \oplus Z_2 \{\pi_1^*(\eta_1), \pi_1^*(\eta_2), \alpha\}$, where $i_1^*(\alpha) = i_{2\sharp}(\eta)$.

Remark 3.6. Here we determine the generators of $\pi_{n+3}(M(Z_q, n))$. By using the mapping cone sequence of the Moore space

$$S^n \xrightarrow{q_\iota} S^n \xrightarrow{i} M(Z_q, n) \xrightarrow{\pi} S^{n+1} \xrightarrow{q_\iota} S^{n+1},$$

we obtain a long exact sequence

$$\pi_{n+3}(S^n) \xrightarrow{q_{\iota\sharp}} \pi_{n+3}(S^n) \xrightarrow{i_\sharp} \pi_{n+3}(M(Z_q, n)) \xrightarrow{\pi_\sharp} \pi_{n+3}(S^{n+1}) \xrightarrow{q_{\iota\sharp}} \pi_{n+3}(S^{n+1}).$$

From the work by Toda [1962], we have

$$Z_{24}\{\nu\} \xrightarrow{q_{\iota\sharp}} Z_{24}\{\nu\} \xrightarrow{i_\sharp} \pi_{n+3}(M(Z_q, n)) \xrightarrow{\pi_\sharp} Z_2\{\eta^2\} \xrightarrow{q_{\iota\sharp}} Z_2\{\eta^2\}.$$

Thus, if q is odd, then $\pi_{n+3}(M(Z_q, n)) \cong Z_{(q,24)}\{i_\sharp(\nu)\}$, and if q is even, then $\pi_{n+3}(M(Z_q, n)) \cong Z_{(q,24)} \oplus Z_2\{i_\sharp(\nu), \overline{\eta^2}\}$ where $\pi_\sharp(\overline{\eta^2}) = \eta^2$.

Based on Remarks 3.1 and 3.6, we obtain for M_1 the table

	q odd	$q \equiv 2 \pmod 4$	$q \equiv 0 \pmod 4$
$\pi_{n+3}(M_1)$	0	Z_4	$Z_2 \oplus Z_2$
Generator		$\hat{\eta}$	η_3, η_4
Relation		$i_{1\sharp}(\eta^2) = 2\hat{\eta}, \pi_{1\sharp}(\hat{\eta}) = \eta$	$i_{1\sharp}(\eta^2) = \eta_3, \pi_{1\sharp}(\eta_4) = \eta$

while for M_2 we obtain

	p odd	$p \equiv 2 \pmod 4$	$p \equiv 0 \pmod 4$
$\pi_{n+3}(M_2)$	$Z_{(p,24)}$	$Z_{(p,24)} \oplus Z_2$	$Z_{(p,24)} \oplus Z_2$
Generator	$i_{2\sharp}(\nu)$	$i_{2\sharp}(\nu), \overline{\eta^2}$	$i_{2\sharp}(\nu), \overline{\eta^2}$
Relation		$\pi_{2\sharp}(\overline{\eta^2}) = \eta^2$	$\pi_{2\sharp}(\overline{\eta^2}) = \eta^2$

By Lemmas 3.4 and 3.5, we have the following table, where $\pi_1^*(\eta_1) = 2\alpha$, $i_1^*(\alpha) = i_{2\sharp}(\eta)$ and $i_1^*(\beta) = i_{2\sharp}(\eta)$:

	$[M_1, M_2]$	Generator
either q odd or p odd	0	
$q \equiv 2, p \equiv 0 \pmod 4$	$Z_4 \oplus Z_2$	$\alpha, \pi_1^*(\eta_2)$
$q \equiv 0, p \equiv 2 \pmod 4$	$Z_4 \oplus Z_2$	$\pi_1^*(\bar{\eta}), \beta$
$q \equiv p \equiv 2 \pmod 4$	$Z_2 \oplus Z_2$	$\pi_1^*(\bar{\eta}), \beta$
$q \equiv p \equiv 0 \pmod 4$	$Z_2 \oplus Z_2 \oplus Z_2$	$\pi_1^*(\eta_1), \pi_1^*(\eta_2), \alpha$

4. Computation of $\mathcal{E}_\sharp^{\dim+r}(M(Z_q, n+1) \vee M(Z_p, n))$ for $r = 0, 1$

In this section, we compute $\mathcal{E}_\sharp^{\dim+r}(M_1 \vee M_2)$, where $M_1 = M(Z_q, n+1) = S^{n+1} \cup_q e^{n+2}$ and $M_2 = M(Z_p, n) = S^n \cup_p e^{n+1}$ with $p, q \geq 1$. In [Jeong 2010], these groups were computed in the case of $p = q$. However, we compute those groups in the general case, that is, $p \neq q$ and $r = 0, 1$. Throughout this section we assume that $X = M_1 \vee M_2$. Note that $\pi_{n+k}(M_1 \vee M_2) \cong \pi_{n+k}(M_1) \oplus \pi_{n+k}(M_2)$ for $k \leq n$ by Proposition 2.1. Moreover, from Proposition 2.2, we can identify $f \in [X, X]$ with the 2×2 matrix

$$\theta(f) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

where $f_{11} \in [M_1, M_1]$, $f_{12} \in [M_2, M_1]$, $f_{21} \in [M_1, M_2]$, and $f_{22} \in [M_1, M_1]$.

Lemma 4.1. *Let $f \in [X, X]$ be given by*

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}.$$

Then $f \in \mathcal{E}(X)$ *if and only if* $f_{11} \in \mathcal{E}(M_1)$ *and* $f_{22} \in \mathcal{E}(M_2)$. *Additionally, if* $f \in \mathcal{E}_\sharp^{\dim}(X)$, *then* $f_{22} = 1$.

Proof. Let us denote by $h_{*n} : H_n(U) \to H_n(V)$ the induced homomorphism on the homology group from $h : U \to V$. Then, $f \in \mathcal{E}(X)$ if and only if f_* is an isomorphism if and only if f_{11*n+1} and f_{22*n} are isomorphisms if and only if $f_{11} \in \mathcal{E}(M_1)$ and $f_{22} \in \mathcal{E}(M_2)$. For the proof of the second statement, see [Jeong 2010, Lemma 3.3]. □

Let us denote by $g_{\sharp s} : \pi_s(U) \to \pi_s(V)$ the homomorphism induced by $g : U \to V$. It is clear from Lemma 4.1 that if $f \in \mathcal{E}(X)$, then $f_{\sharp(n+k)} : \pi_{n+k}(X) \to \pi_{n+k}(X)$ is given by

$$f_{\sharp(n+k)} = \begin{pmatrix} f_{11\sharp(n+k)} & f_{12\sharp(n+k)} \\ f_{21\sharp(n+k)} & f_{22\sharp(n+k)} \end{pmatrix},$$

where $f_{11\sharp(n+k)}$ and $f_{22\sharp(n+k)}$ are isomorphisms and $k \le n$.

Lemma 4.2. *If* $f \in \mathcal{E}(X)$ *and either* q *is odd or* p *is odd, then* $f_{12\sharp k} = 0$ *for* $k = 1, 2, \dots, n+2$.

Proof. Since M_1 is n-connected, we have $\pi_k(M_1) = 0$ for $k = 1, 2, \dots, n$. Thus it is sufficient to show that $f_{12\sharp k} = 0$ for $k = n+1, n+2$.

If p is odd, then $\pi_{n+1}(M_2)$ and $\pi_{n+2}(M_2)$ are trivial groups. Thus, $f_{12\sharp(n+1)} = f_{12\sharp(n+2)} = 0$.

Suppose that q is odd, p is even and $(p, q) = d \neq 1$. Then, $\pi_{n+1}(M_2) \cong Z_2\{i_{2\sharp}(\eta)\}$. Since $[M_2, M_1] \cong Z_d\{\pi_2^*(i_1)\}$, we have $f_{12\sharp n+1} = t\pi_2^*(i_1)_\sharp$ for some integer t such that $1 \le t \le d$. Thus, we have

$$f_{12\sharp(n+1)}(i_{2\sharp}(\eta)) = t\pi_2^*(i_1)(i_{2\sharp}(\eta)) = t(i_1 \circ \pi_2 \circ i_2 \circ \eta) = 0$$

because $\pi_2 \circ i_2$ is homotopic to a constant map. Hence, $f_{12\sharp(n+1)} = 0$. If $d = 1$, $[M_2, M_1] = 0$ and it is trivial.

For $k = n+2$, we are done since $\pi_{n+2}(M_1) = 0$. □

Here we introduce certain generators and elements of $[M_1, M_1]$ and $\mathcal{E}_\sharp^{\dim+r}(M_1)$ for $r = -1, 0, 1$ as described in [Jeong 2010].

Remark 4.3. Let $M_1 = M(Z_q, n+1)$ be a Moore space with q is even. By Proposition 2.5, $\mathcal{E}_\sharp^{\dim}(M_1) \cong Z_2$ and $\mathcal{E}_\sharp^{\dim+1}(M_1) = 1$. In this remark, we describe the generator of $\mathcal{E}_\sharp^{\dim}(M_1)$ explicitly.

Consider the mapping cone sequence

$$S^{n+1} \xrightarrow{q_{\iota_1}} S^{n+1} \xrightarrow{i_1} S^{n+1} \cup_q e^{n+2} \xrightarrow{\pi_1} S^{n+2} \xrightarrow{q_{\iota_1}} S^{n+2}.$$

Then, we have the following exact sequence:

$$\pi_{n+2}(M_1) \xrightarrow{q_{\iota_1}^*} \pi_{n+2}(M_1) \xrightarrow{\pi_1^*} [M_1, M_1] \xrightarrow{i_1^*} \pi_{n+1}(M_1) \xrightarrow{q_{\iota_1}^*} \pi_{n+1}(M_1).$$

Since $\pi_{n+2}(M_1) \cong Z_2\{i_1\eta\}$ and $\pi_{n+1}(M_1) \cong Z_q\{1\}$, we have the short exact sequence

$$0 \longrightarrow Z_2\{i_1\eta\} \xrightarrow{\pi_1^*} [M_1, M_1] \xrightarrow{i_1^*} Z_q\{1\} \longrightarrow 0\ .$$

By [Araki and Toda 1965, Theorem 4.1],

$$[M_1, M_1] \cong \begin{cases} Z_{2q}\{1\} & \text{if } q \equiv 2 \pmod 4, \\ Z_q \oplus Z_2\{1, i_1 \circ \eta \circ \pi_1\} & \text{if } q \equiv 0 \pmod 4, \end{cases}$$

and

$$\pi_1^*(i_1 \circ \eta) = i_1 \circ \eta \circ \pi_1 \in [M_1, M_1].$$

Let $i_1 \circ \eta \circ \pi_1 = \epsilon$. Then, ϵ has order 2 and $1 + \epsilon \in [M_1, M_1]$. Since $n \geq 5$, we have that $1 + \epsilon$ is a suspension map. Thus,

$$(1+\epsilon) \circ (1+\epsilon) \simeq 1 \circ (1+\epsilon) + \epsilon \circ (1+\epsilon) = 1 + \epsilon + \epsilon + \epsilon \circ \epsilon = 1 + 2\epsilon + \epsilon^2.$$

If $q \equiv 2 \pmod 4$, then $i_1 \circ \eta \circ \pi_1 = q1$ and $\epsilon^2 = i_1 \circ \eta \circ \pi_1 \circ i_1 \circ \eta \circ \pi_1$. Since $\pi_1 \circ i_1 = 0$ and ϵ has order 2, we have $2\epsilon = 0$ and $\epsilon^2 = 0$. Thus, $(1+\epsilon) \circ (1+\epsilon) \simeq 1$ and $1 + \epsilon \in \mathcal{E}(M_1)$.

Since each $\alpha \in \pi_{n+r}(M_1)$ is a suspension map, for $r = 1, 2, 3$, we have

$$(1+\epsilon)_\sharp(\alpha) = \alpha + \epsilon \circ \alpha.$$

Since $\pi_{n+1}(M_1) \cong Z_q\{i_1\}$ and $\epsilon_\sharp(i_1) = i_1 \circ \eta \circ \pi_1 \circ i_1 = 0$, we have $1 + \epsilon \in \mathcal{E}_\sharp^{\dim -1}(M_1)$.

Since $\pi_{n+2}(M_1) \cong Z_2\{i_{1\sharp}(\eta)\}$ and $\epsilon_\sharp(i_{1\sharp}(\eta)) = i_1 \circ \eta \circ \pi_1 \circ i_1 \circ \eta = 0$, we have $1 + \epsilon \in \mathcal{E}_\sharp^{\dim}(M_1)$.

Since $\pi_{n+3}(M_1) \cong Z_4\{\hat{\eta}\}$ and

$$\epsilon_\sharp(\hat{\eta}) = i_1 \circ \eta \circ \pi_1 \circ \hat{\eta} = i_1 \circ \eta \circ \eta = i_1 \circ \eta^2 = 2\hat{\eta} \neq 0,$$

we have $1 + \epsilon \notin \mathcal{E}_\sharp^{\dim+1}(M_1)$.

We obtain similar results in the case of $q \equiv 0 \pmod 4$.

Theorem 4.4. *If $X = M_1 \vee M_2$ and $(p, q) = 1$, then*

$$\mathcal{E}_\sharp^{\dim}(X) \cong \begin{cases} 1 & \text{if } q \text{ is odd}, \\ Z_2 & \text{if } q \text{ is even and } p \text{ is odd}. \end{cases}$$

Proof. Let $(q, p) = 1$. Then, either q or p is odd. By Lemmas 3.2 and 3.3, we have $[M_2, M_1] = 0$ and $[M_1, M_2] = 0$.

If q is odd, then $\mathcal{E}_\sharp^{\dim}(M_1) = 1$ and $\mathcal{E}_\sharp^{\dim}(M_2) = 1$ by Proposition 2.5 and Lemma 4.1. Therefore $\mathcal{E}_\sharp^{\dim}(X) = 1$.

If p is odd and q is even, then $\mathcal{E}_\sharp^{\dim}(M_1) \cong Z_2\{1+\epsilon\}$ and $\mathcal{E}_\sharp^{\dim}(M_2) = 1$ by Proposition 2.5, Lemma 4.1, and Remark 4.3. Thus, we have

$$\mathcal{E}_\sharp^{\dim}(X) \cong \left\{ \begin{pmatrix} 1+\epsilon & 0 \\ 0 & 1 \end{pmatrix} \,\middle|\, \epsilon \in Z_2 \ \{i_1\eta\pi_1\} \right\},$$

where η is the generator of $\pi_{n+2}(S^{n+1})$. □

Theorem 4.5. *If $X = M_1 \vee M_2$ and $(p,q) = d \neq 1$, then*

$$\mathcal{E}_\sharp^{\dim}(X) \cong \begin{cases} Z_d & \text{if } q \text{ is odd,} \\ Z_2 \oplus Z_d & \text{if } q \text{ is even and } p \text{ is odd.} \end{cases}$$

Proof. By Lemmas 3.2 and 3.3, we have $[M_2, M_1] \cong Z_d\{\pi_2^*(i_1)\}$ and $[M_1, M_2] = 0$. Moreover, $f_{12\sharp k} = 0$ for $k = 1, 2, \ldots, n+2$ by Lemma 4.2.

Thus, if q is odd, then we have

$$\mathcal{E}_\sharp^{\dim}(X) \cong \left\{ \begin{pmatrix} 1 & f_{12} \\ 0 & 1 \end{pmatrix} \,\middle|\, f_{12} \in Z_d \ \{\pi_2^*(i_1)\} \right\},$$

but if q is even and p is odd, then we have

$$\mathcal{E}_\sharp^{\dim}(X) \cong \left\{ \begin{pmatrix} 1+\epsilon & f_{12} \\ 0 & 1 \end{pmatrix} \,\middle|\, f_{12} \in Z_d \ \{\pi_2^*(i_1)\}, \epsilon \in Z_2 \ \{i_1\eta\pi_1\} \right\}.$$ □

Let f_{12} be an element of $[M_2, M_1] \cong Z_d\{\pi_2^*(i_1)\}$, Then $f_{12} = s\pi_2^*(i_1)$ for $1 \leq s \leq d$.

Lemma 4.6. *For $f = \left(\begin{smallmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{smallmatrix}\right) \in \mathcal{E}(X)$, let p and q be even. Then, $f_{12\sharp k} = 0$ for $k = 1, 2, \ldots, n+1$.*

Proof. Since M_1 is n-connected, $\pi_k(M_1) = 0$ for $k = 1, 2, \ldots, n$. Thus, it is sufficient to show that $f_{12\sharp(n+1)} = 0$. Since $[M_2, M_1] \cong Z_d\{\pi_2^*(i_1)\}$ by Lemma 3.2 and f_{12} belongs to $[M_2, M_1]$, we have $f_{12} = s\pi_2^*(i_1)$ for some $1 \leq s \leq d$. Moreover, $\pi_{n+1}(M_2) \cong Z_2\{i_{2\sharp}(\eta)\}$ by Remark 3.1. Thus, we have

$$f_{12\sharp(n+1)}(i_{2\sharp}(\eta)) = s\pi_2^*(i_1)(i_{2\sharp}(\eta)) = s(i_1 \circ \pi_2 \circ i_2 \circ \eta) = 0$$

since $\pi_2 \circ i_2$ is homotopic to the constant map. □

Lemma 4.7. *Let p and q be even and $f_{12} = s\pi_2^*(i_1)$ be an element of $[M_2, M_1] \cong Z_d\{\pi_2^*(i_1)\}$ for $1 \leq s \leq d$. Then, $f_{12\sharp(n+2)} \neq 0$ if s is odd, and $f_{12\sharp(n+2)} = 0$ if s is even.*

Proof. First, we note that $\pi_{n+2}(M_1) \cong Z_2\{i_{\sharp 1}(\eta)\}$.

Suppose that $p \equiv 0 \pmod 4$. Since $\pi_{n+2}(M_2) \cong Z_2 \oplus Z_2\{\eta_1, \eta_2\}$, we have

$$\pi_2^*(i_1) = \pi_2^*(i_1)(\eta_1) = \pi_2^*(i_1)(i_{2\sharp}(\eta^2)) = i_1 \circ \pi_2 \circ i_2 \circ \eta^2 = 0$$

and

$$\pi_2^*(i_1)(\eta_2) = i_1 \circ \pi_2 \circ \eta_2 = i_1 \circ \eta \neq 0.$$

Thus, $f_{12\sharp(n+2)}(\eta_1)=0$ for all f_{12}. Moreover, if $s=2l$ for some $1\le l\le d/2$, then

$$s\pi_2^*(i_1)(\eta_2)=si_1\circ\pi_2\circ\eta_2=2li_1\circ\eta=0.$$

Therefore, each element in $\langle 2\pi_2^*(i_1)\rangle\cong Z_{d/2}$ induces the trivial homomorphism on $\pi_{n+2}(M_2)$. However, if $s=2l+1$ for some $0\le l<d/2-1$, then

$$s\pi_2^*(i_1)(\eta_2)=si_1\circ\pi_2\circ\eta_2=(2l+1)i_1\circ\eta=i_1\circ\eta\neq 0.$$

Thus, if f_{12} does not belong to $\langle 2\pi_2^*(i_1)\rangle\cong Z_{d/2}$, then $f_{12\sharp(n+2)}\neq 0$.

Suppose that $p\equiv 2 \pmod 4$. Since $\pi_{n+2}(M_2)\cong Z_4\{\overline{\eta}\}$, we have

$$\pi_2^*(i_1)_\sharp(\overline{\eta})=i_1\circ\pi_2\circ\overline{\eta}=i_1\circ\eta=i_{1\sharp}(\eta)\neq 0.$$

If $s=2k$ for some $1\le l\le d/2$, then

$$s\pi_2^*(i_1)_\sharp(\overline{\eta})=si_1\circ\pi_2\circ\overline{\eta}=si_1\circ\eta=2li_{1\sharp}(\eta)=0.$$

Thus, each element in $\langle 2\pi_2^*(i_1)\rangle\cong Z_{d/2}$ induces the trivial homomorphism on $n+2$.

However, if $s=2l+1$ for some $0\le l\le d/2-1$, then

$$s\pi_2^*(i_1)_\sharp(\overline{\eta})=si_1\circ\pi_2\circ\overline{\eta}=si_1\circ\eta=(2l+1)i_{1\sharp}(\eta)=i_{1\sharp}(\eta)\neq 0.$$

Thus if f_{12} does not belong to $\langle 2\pi_2^*(i_1)\rangle\cong Z_{d/2}$, then $f_{12\sharp(n+2)}\neq 0$. □

Theorem 4.8. *Let p and q be even and let $X=M_1\vee M_2$. Then if $(p,q)=d\neq 1$, we have*

$$\mathcal{E}_\sharp^{\dim}(X)\cong\begin{cases} Z_2\oplus Z_{d/2}\oplus Z_2\oplus Z_2 & \text{if } q\equiv 2,\ p\equiv 0 \pmod 4,\\ Z_2\oplus Z_{d/2}\oplus Z_4 & \text{if } q\equiv 0,\ p\equiv 2 \pmod 4,\\ Z_2\oplus Z_{d/2}\oplus Z_2 & \text{if } q\equiv 2,\ p\equiv 2 \pmod 4,\\ Z_2\oplus Z_{d/2}\oplus Z_2\oplus Z_2 & \text{if } q\equiv 0,\ p\equiv 0 \pmod 4).\end{cases}$$

Proof. By Proposition 2.5, $\mathcal{E}_\sharp^{\dim}(M_1)\cong Z_2$ and $\mathcal{E}_\sharp^{\dim}(M_2)=1$. By Lemma 4.6, for each $f=\left(\begin{smallmatrix} f_{11} & f_{12}\\ f_{21} & f_{22}\end{smallmatrix}\right)\in\mathcal{E}(X)$, we have $f_{12\sharp k}=0$ for $k=1,2,\ldots,n+1$. By Lemma 4.7, each element in $\langle 2\pi_2^*(i_1)\rangle\cong Z_{d/2}$ induces the trivial homomorphism on $\pi_{n+2}(M_2)$. Furthermore, if f_{12} does not belong to $\langle 2\pi_2^*(i_1)\rangle\cong Z_{d/2}$, then $f_{12\sharp(n+2)}\neq 0$. Thus, it is sufficient to investigate $f_{21\sharp n}$, $f_{21\sharp(n+1)}$ and $f_{21\sharp(n+2)}$.

Case 1. Let $q\equiv 2 \pmod 4$ and $p\equiv 0 \pmod 4$. From Lemma 3.4, we obtain $[M_1,M_2]\cong Z_4\oplus Z_2\{\alpha,\pi_1^*(\eta_2)\}$, where $\pi_1^*(\eta_1)=2\alpha$ and $i_1^*(\alpha)=i_{2\sharp}(\eta)$.

Since M_1 is n-connected, $\pi_n(M_1)=0$. Thus, $f_{21\sharp n}=0$.

Since $\pi_{n+1}(M_1)\cong Z_q\{i_1\}$, we have $\pi_1^*(\eta_2)_\sharp(i_1)=\eta_2\circ\pi_1\circ i_1=0$.

Conversely, since $\pi_{n+1}(M_2)\cong Z_2\{i_{2\sharp}(\eta)\}$ and $\alpha_\sharp(i_1)=\alpha\circ i_1=i_{2\sharp}(\eta)\neq 0$, we have $(2\alpha)_\sharp=0$ and $(3\alpha)_\sharp\neq 0$. Moreover, since $\pi_{n+2}(M_1)\cong Z_2\{i_{1\sharp}(\eta)\}$, we have $\pi_1^*(\eta_2)_\sharp(i_{1\sharp}(\eta))=\eta_2\circ\pi_1\circ i_1\circ\eta=0$. Hence, $\left(\begin{smallmatrix} 1 & 0\\ f_{21} & 1\end{smallmatrix}\right)$ belongs to $\mathcal{E}_\sharp^{\dim}(X)$ if $f_{21}\in Z_2\oplus Z_2\{2\alpha,\pi_1^*(\eta_2)\}$.

Therefore,

$$\mathcal{E}_\sharp^{\dim}(X) \cong \left\{ \begin{pmatrix} 1+\epsilon & f_{12} \\ f_{21} & 1 \end{pmatrix} \,\middle|\, f_{12} \in \langle 2\pi_2^*(i_1) \rangle,\ f_{21} \in \langle 2\alpha \rangle \oplus \langle \pi_1^*(\eta_2) \rangle \right\},$$

where $\epsilon \in \langle i_1 \eta \pi_1 \rangle$.

Case 2. Let $q \equiv 0 \pmod 4$ and $p \equiv 2 \pmod 4$. From Lemma 3.4, we obtain $[M_1, M_2] \cong Z_4 \oplus Z_2\{\pi_1^*(\bar\eta), \beta\}$, where $i_1^*(\beta) = i_{2\sharp}(\eta)$.

Since $\pi_n(M_1) = 0$, we have $f_{21\sharp n} = 0$. However, since $\pi_{n+1}(M_1) \cong Z_q\{i_1\}$ and $\pi_{n+1}(M_2) \cong Z_2\{i_{2\sharp}(\eta)\}$, we have $\pi_1^*(\bar\eta)_\sharp(i_1) = \bar\eta \circ \pi_1 \circ i_1 = 0$, but $\beta_\sharp(i_1) = \beta \circ i_1 = i_{2\sharp}(\eta) \neq 0$.

For the generator $\pi_1^*(\bar\eta)$ of $[M_1, M_2] \cong Z_4 \oplus Z_2\{\pi_1^*(\bar\eta), \beta\}$ and the generator $i_{1\sharp}(\eta)$ of $\pi_{n+2}(M_1) \cong Z_2\{i_{1\sharp}(\eta)\}$, we have $\pi_1^*(\bar\eta)_\sharp(i_{1\sharp}(\eta)) = \bar\eta \circ \pi_1 \circ i_{1\sharp}(\eta) = 0$.

Hence, $\left(\begin{smallmatrix} 1 & 0 \\ f_{21} & 1 \end{smallmatrix}\right)$ belongs to $\mathcal{E}_\sharp^{\dim}(X)$ if $f_{21} \in \langle \pi_1^*(\bar\eta) \rangle$.

Therefore,

$$\mathcal{E}_\sharp^{\dim}(X) \cong \left\{ \begin{pmatrix} 1+\epsilon & f_{12} \\ f_{21} & 1 \end{pmatrix} \,\middle|\, f_{12} \in \langle 2\pi_2^*(i_1) \rangle,\ f_{21} \in \langle \pi_1^*(\bar\eta) \rangle \right\},$$

where $\epsilon \in \langle i_1 \eta \pi_1 \rangle$.

Case 3. Let $q \equiv 2 \pmod 4$ and $p \equiv 2 \pmod 4$. From Lemma 3.5, we obtain $[M_1, M_2] \cong Z_2 \oplus Z_2\{\pi_1^*(\bar\eta), \beta\}$, where $i_1^*(\beta) = i_{2\sharp}(\eta)$.

First, we recall that $f_{21\sharp n} = 0$ since $\pi_n(M_1) = 0$.

Since $\pi_{n+1}(M_1) \cong Z_q\{i_1\}$ and $\pi_{n+1}(M_2) \cong Z_2\{i_{2\sharp}(\eta)\}$, we have $\pi_1^*(\bar\eta)_\sharp(i_1) = \bar\eta \circ \pi_1 \circ i_1 = 0$, but $\beta_\sharp(i_1) = \beta \circ i_1 = i_{2\sharp}(\eta) \neq 0$. Moreover, since $\pi_{n+2}(M_1) \cong Z_2\{i_{1\sharp}(\eta)\}$, we have $\pi_1^*(\bar\eta)_\sharp(i_{1\sharp}(\eta)) = \bar\eta \circ \pi_1 \circ i_1 \circ \eta = 0$.

Hence, if $f_{21} \in \langle \pi_1^*(\bar\eta) \rangle$, then $\left(\begin{smallmatrix} 1 & 0 \\ f_{21} & 1 \end{smallmatrix}\right)$ belongs to $\mathcal{E}_\sharp^{\dim}(X)$. However, if $f_{21} \in \langle \beta \rangle$, this cannot be the case. Therefore,

$$\mathcal{E}_\sharp^{\dim}(X) \cong \left\{ \begin{pmatrix} 1+\epsilon & f_{12} \\ f_{21} & 1 \end{pmatrix} \,\middle|\, f_{12} \in \langle 2\pi_2^*(i_1) \rangle,\ f_{21} \in \langle \pi_1^*(\bar\eta) \rangle \right\},$$

where $\epsilon \in \langle i_1 \eta \pi_1 \rangle$.

Case 4. Let $q \equiv 0 \pmod 4$ and $p \equiv 0 \pmod 4$. From Lemma 3.5, we obtain $[M_1, M_2] \cong Z_2 \oplus Z_2 \oplus Z_2\{\pi_1^*(\eta_1), \pi_1^*(\eta_2), \alpha\}$, where $i_1^*(\alpha) = i_{2\sharp}(\eta)$. First, we note that $f_{21\sharp n} = 0$ since $\pi_n(M_1) = 0$.

Since $\pi_{n+1}(M_1) \cong Z_q\{i_1\}$ and $\pi_{n+1}(M_2) \cong Z_2\{i_{2\sharp}(\eta)\}$, we have $\pi_1^*(\eta_1)_\sharp(i_1) = \eta_1 \circ \pi_1 \circ i_1 = 0$ and $\pi_1^*(\eta_2)_\sharp(i_1) = \eta_2 \circ \pi_1 \circ i_1 = 0$, but $\alpha_\sharp(i_1) = \alpha \circ i_1 = i_{2\sharp}(\eta) \neq 0$. Also, since $\pi_{n+2}(M_1) \cong Z_2\{i_{1\sharp}(\eta)\}$, we have $\pi_1^*(\eta_1)_\sharp(i_{1\sharp}(\eta)) = \eta_1 \circ \pi_1 \circ i_{1\sharp}(\eta) = 0$ and $\pi_1^*(\eta_2)_\sharp(i_{1\sharp}(\eta)) = \eta_2 \circ \pi_1 \circ i_{1\sharp}(\eta) = 0$.

Hence, if $f_{21} \in \langle \pi_1^*(\eta_1) \rangle \oplus \langle \pi_1^*(\eta_2) \rangle$, then $\left(\begin{smallmatrix} 1 & 0 \\ f_{21} & 1 \end{smallmatrix}\right)$ belongs to $\mathcal{E}_\sharp^{\dim}(X)$. However, if $f_{21} \in \langle \alpha \rangle$, this cannot be the case. Therefore,

$$\mathcal{E}_\sharp^{\dim}(X) \cong \left\{ \begin{pmatrix} 1+\epsilon & f_{12} \\ f_{21} & 1 \end{pmatrix} \,\middle|\, f_{12} \in \langle 2\pi_2^*(i_1)\rangle,\ f_{21} \in \pi_1^*(\langle \eta_1\rangle \oplus \langle \eta_2\rangle) \right\},$$

where $\epsilon \in \langle i_1 \eta \pi_1 \rangle$. □

From Theorems 4.4–4.8, we obtain Table 1 (see page 37).

Theorem 4.9. *Let $X = M_1 \vee M_2$, $n \geq 5$ and $(q, p) = d$. Then we have*

$$\mathcal{E}_\sharp^{\dim+1}(X) \cong \begin{cases} 1 & \text{if } q \text{ is odd or } p \text{ is odd } (d=1), \\ Z_d & \text{if } q \text{ is odd or } p \text{ is odd } (d \neq 1), \\ Z_{d/2} \oplus Z_2 & \text{if } p \equiv 0 \pmod 4 \text{ and } (p,24)=4 \text{ or } 12 (d\neq 1), \\ Z_{d/2} & \text{if } p \equiv 0 \pmod 4 \text{ and } (p,24)=8 \text{ or } 24 (d\neq 1), \\ Z_{d/2} & \text{if } q \equiv 2,\ p \equiv 2 \pmod 4, \\ Z_{d/2} \oplus Z_2 & \text{if } q \equiv 0,\ p \equiv 2 \pmod 4. \end{cases}$$

Proof. By virtue of Remark 4.3, Theorem 4.4 and the fact that $\mathcal{E}_\sharp^{\dim+1}(X) \subseteq \mathcal{E}_\sharp^{\dim}(X)$, we have $\mathcal{E}_\sharp^{\dim+1}(X) = 1$ if $(p, q) = 1$.

By Proposition 2.5, we have $\mathcal{E}_\sharp^{\dim+1}(M_1) = 1$. Thus, it is sufficient to identify $f_{12\sharp(n+3)}$ and $f_{21\sharp(n+3)}$. First, we note that $[M_2, M_1] \cong Z_d\{\pi_2^*(i_1)\}$ by Lemma 3.2.

Case 1. Suppose that q is odd or p is odd and $(p, q) = d \neq 1$. Since $[M_1, M_2] = 0$ by Lemma 3.3, we only investigate $f_{12\sharp(n+3)}$.

If q is odd, $f_{12\sharp(n+3)} = 0$ since $\pi_{n+3}(M_1) = 0$. If q is even and p is odd, $\pi_{n+3}(M_2) \cong Z_{(p,24)}\{i_{2\sharp}(\nu)\}$. Since

$$\pi_2^*(i_1)_\sharp(i_{2\sharp}(\nu)) = i_1 \circ \pi_2 \circ i_2 \circ \nu = 0,$$

we have $f_{12\sharp(n+3)} = 0$ for each $f_{12} \in [M_2, M_1]$. Therefore,

$$\mathcal{E}_\sharp^{\dim+1}(X) \cong \left\{ \begin{pmatrix} 1 & f_{12} \\ 0 & 1 \end{pmatrix} \,\middle|\, f_{12} \in \langle \pi_2^*(i_1)\rangle \right\}.$$

Case 2. Suppose that $q \equiv 2 \pmod 4$ and $p \equiv 0 \pmod 4$. First, we note that

$$\pi_{n+3}(M_2) \cong Z_{(p,24)} \oplus Z_2\ \{i_{2\sharp}(\nu), \overline{\eta^2}\}$$

and that $\pi_{n+3}(M_1) \cong Z_4\{\hat{\eta}\}$ by Proposition 2.3. Let $f_{12} = s\pi_2^*(i_1)$. If $s = 2l$ for some $1 \leq l \leq d/2$, then

$$s\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = 2l\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = 4l\hat{\eta} = 0$$

since

$$\pi_2^*(i_1)_\sharp(i_{2\sharp}(\nu)) = i_1 \circ \pi_2 \circ i_2 \circ \nu = 0$$

and

$$\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = i_1 \circ \pi_2 \circ \overline{\eta^2} = i_{1\sharp}(\eta^2) = 2\hat{\eta} \neq 0 \in \pi_{n+3}(M_1) \cong Z_4.$$

Further, if $s = 2l + 1$ for some $0 \le l \le d/2 - 1$, then

$$s\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = (2l+1)\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = 4l\hat{\eta} + 2\hat{\eta} = 2\hat{\eta} \neq 0.$$

Thus, each $f_{12} \in \langle 2\pi_2^*(i_1)\rangle \cong Z_{d/2}$ induces the trivial homomorphism on $\pi_{n+3}(M_2)$. However, if f_{12} does not belong to $\langle 2\pi_2^*(i_1)\rangle$, then $f_{12\sharp(n+3)} \neq 0$.

Let us investigate $f_{21\sharp(n+3)}$. Note that $[M_1, M_2] \cong Z_4 \oplus Z_2\{\alpha, \pi_1^*(\eta_2)\}$ and $\pi_{n+3}(M_1) \cong Z_4\{\hat{\eta}\}$ with $\pi_1^*(\eta_1) = 2\alpha$, $i_1^*(\alpha) = i_{2\sharp}(\eta)$, $i_{1\sharp}(\eta^2) = 2\hat{\eta}$ and $\pi_{1\sharp}(\hat{\eta}) = \eta$. Since $\pi_{2\sharp}(\eta_2 \circ \eta) = \eta^2$, we have

$$\pi_1^*(\eta_2)_\sharp(\hat{\eta}) = \eta_2 \circ \pi_1 \circ \hat{\eta} = \eta_2 \circ \eta \neq 0.$$

Moreover, since $\eta^3 = 4\nu$ [Toda 1962, (5.5)], we have

$$2\alpha_\sharp(\hat{\eta}) = 2\alpha \circ \hat{\eta} = \eta_1 \circ \pi_1 \circ \hat{\eta} = \eta_1 \circ \eta = i_{1\sharp}(\eta^2) \circ \eta = i_2 \circ \eta^3 = 4i_{2\sharp}(\nu).$$

Therefore, $\alpha_\sharp(\hat{\eta}) = 2i_{2\sharp}(\nu)$. Since $(p, 24)$ is a multiple of 4, we have $\alpha_\sharp(\hat{\eta}) = 2i_{2\sharp}(\nu) \neq 0$ and $3\alpha_\sharp(\hat{\eta}) = 6i_{2\sharp}(\nu) \neq 0$.

Since ν is 2-primary, if $(p, 24) = 4$ or $(p, 24) = 12$, then $2\alpha_\sharp(\hat{\eta}) = 0$, and if $(p, 24) = 8$ or $(p, 24) = 24$, then $2\alpha_\sharp(\hat{\eta}) \neq 0$. Thus, each $f_{21} \in \langle 2\alpha\rangle$ induces the trivial homomorphism on $\pi_{n+3}(M_1)$ provided that $(p, 24) = 4$ or $(p, 24) = 12$.

Therefore, if $(p, 24) = 4$ or $(p, 24) = 12$, we have

$$\mathcal{E}_\sharp^{\dim+1}(X) \cong \left\{ \begin{pmatrix} 1 & f_{12} \\ f_{21} & 1 \end{pmatrix} \,\middle|\, f_{12} \in \langle 2\pi_2^*(i_1)\rangle,\ f_{21} \in \langle 2\alpha\rangle \right\},$$

and if $(p, 24) = 8$ or 24, we have

$$\mathcal{E}_\sharp^{\dim+1}(X) \cong \left\{ \begin{pmatrix} 1 & f_{12} \\ 1 & 1 \end{pmatrix} \,\middle|\, f_{12} \in \langle 2\pi_2^*(i_1)\rangle \right\}.$$

Case 3. Suppose that $q \equiv 0 \pmod 4$ and $p \equiv 2 \pmod 4$. We note that

$$\pi_{n+3}(M_2) \cong Z_{(p,24)} \oplus Z_2\{i_{2\sharp}(\nu), \overline{\eta^2}\},$$
$$\pi_{n+3}(M_1) \cong Z_2 \oplus Z_2\{\eta_3, \eta_4\}$$

and $[M_1, M_2] \cong Z_4 \oplus Z_2\{\pi_1^*(\overline{\eta}), \beta\}$. First, we investigate $f_{12\sharp(n+3)}$. Let $f_{12} = s\pi_2^*(i_1) \in [M_2, M_1] \cong Z_d\{\pi_2^*(i_1)\}$. Then, we have

$$\pi_2^*(i_1)_\sharp(i_{2\sharp}(\nu)) = i_1 \circ \pi_2 \circ i_2 \circ \nu = 0$$

and

$$\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = i_1 \circ \pi_2 \circ \overline{\eta^2} = i_1 \circ \eta^2 \neq 0.$$

If $s = 2l$ for some $1 \le l \le d/2$, then $2l\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = 2li_1 \circ \eta^2 = 0$, because $i_1 \circ \eta^2 = \eta_3 \in \pi_{n+3}(M_1)$. However, if $s = 2l + 1$ for some $0 \le l \le d/2 - 1$, then $(2l+1)\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = (2k+1)i_1 \circ \eta^2 = i_1 \circ \eta^2 \neq 0$.

Thus, any $f_{12} \in \langle 2\pi_2^*(i_1)\rangle \cong Z_{d/2}$ induces the trivial homomorphism on $\pi_{n+3}(M_2)$. However, for $f_{12} \notin \langle 2\pi_2^*(i_1)\rangle$, we have $f_{12\sharp n+3} \neq 0$.

Next, we investigate $f_{21\sharp(n+3)}$. Because $[M_1, M_2] \cong Z_4 \oplus Z_2\{\pi_1^*(\bar{\eta}), \beta\}$ and $\beta_{\sharp(n+2)} \neq 0$, we check only the generators $\pi_1^*(\bar{\eta})$. For η_3, we have

$$\pi_1^*(\bar{\eta})_\sharp(\eta_3) = \bar{\eta} \circ \pi_1 \circ \eta_3 = \bar{\eta} \circ \pi_1 \circ i_{1\sharp}(\eta^2) = 0.$$

For η_4, we have

$$\pi_1^*(\bar{\eta})_\sharp(\eta_4) = \bar{\eta} \circ \pi_1 \circ \eta_4 = \bar{\eta} \circ \eta \neq 0$$

since $\pi_{2\sharp}(\bar{\eta} \circ \eta) = \eta^2 \neq 0$.

However, $2\pi_1^*(\bar{\eta})_\sharp(\eta_4) = \bar{\eta} \circ \pi_1 \circ 2\eta_4 = 0$.

Thus, every $f_{21} \in \langle 2\pi_1^*(\bar{\eta})\rangle$ induces the trivial homomorphism on $n+3$.

Therefore, we have

$$\mathcal{E}_\sharp^{\dim+1}(X) \cong \left\{ \begin{pmatrix} 1 & f_{12} \\ f_{21} & 1 \end{pmatrix} \middle| \ f_{12} \in \langle 2\pi_2^*(i_1)\rangle,\ f_{21} \in \langle 2\pi_1^*(\bar{\eta})\rangle \right\}.$$

Case 4. Suppose that $q \equiv 2 \pmod 4$ and $p \equiv 2 \pmod 4$. Note that $\pi_{n+3}(M_2) \cong Z_{(p,24)} \oplus Z_2\{i_{2\sharp}(\nu), \overline{\eta^2}\}$ and $\pi_{n+3}(M_1) \cong Z_4\{\hat{\eta}\}$. First, we investigate $f_{12\sharp(n+3)}$. For the generator $\pi_2^*(i_1)$ of $[M_2, M_1]$, we have

$$\pi_2^*(i_1)_\sharp(i_{2\sharp}(\nu)) = i_1 \circ \pi_2 \circ i_2 \circ \nu = 0$$

and

$$\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = i_1 \circ \pi_2 \circ \overline{\eta^2} = i_1 \circ \eta^2 = 2\hat{\eta} \neq 0.$$

Let $f_{12} = s\pi_2^*(i_1)$. If $s = 2l$ for $1 \leq l \leq d/2$, then $s\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = 4l\hat{\eta} = 0$, and if $s = 2l+1$ for $0 \leq l \leq d/2 - 1$, then $s\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = (4l+2)\hat{\eta} = 2\hat{\eta} \neq 0$.

Thus, each $f_{12} \in \langle 2\pi_2^*(i_1)\rangle \cong Z_{d/2}$ induces the trivial homomorphism on $n+3$. However, for $f_{12} \notin \langle 2\pi_2^*(i_1)\rangle$, we have $f_{12\sharp(n+3)} \neq 0$.

Next, we investigate $f_{21\sharp(n+3)}$. Note that $[M_1, M_2] \cong Z_2 \oplus Z_2\{\pi_1^*(\bar{\eta}), \beta\}$. Since $\beta_{\sharp n+2} \neq 0$, we consider only the generator $\pi_1^*(\bar{\eta})$.

Since $\pi_{2\sharp}(\bar{\eta} \circ \eta) = \pi_2 \circ \bar{\eta} \circ \eta = \eta^2 \neq 0$, we have $\pi_1^*(\bar{\eta})_\sharp(\hat{\eta}) = \bar{\eta} \circ \pi_1 \circ \hat{\eta} = \bar{\eta} \circ \eta \neq 0$. Therefore, no f_{21} induces a trivial homomorphism.

Thus, we have

$$\mathcal{E}_\sharp^{\dim+1}(X) \cong \left\{ \begin{pmatrix} 1 & f_{12} \\ 1 & 1 \end{pmatrix} \middle| \ f_{12} \in \langle 2\pi_2^*(i_1)\rangle \right\}.$$

Case 5. Suppose that $q \equiv 0 \pmod 4$ and $p \equiv 0 \pmod 4$. Note that $\pi_{n+3}(M_2) \cong Z_{(p,24)} \oplus Z_2\{i_{2\sharp}(\nu), \overline{\eta^2}\}$ and $\pi_{n+3}(M_1) \cong Z_2 \oplus Z_2\{\eta_3, \eta_4\}$.

First, we investigate $f_{12\sharp(n+3)}$. For the generator $\pi_2^*(i_1)$ of $[M_2, M_1]$, we have

$$\pi_2^*(i_1)_\sharp(i_{2\sharp}(\nu)) = i_1 \circ \pi_2 \circ i_2 \circ \nu = 0$$

and

$$\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = i_1 \circ \pi_2 \circ \overline{\eta^2} = i_1 \circ \eta^2 \neq 0.$$

Let $f_{12} = s\pi_2^*(i_1)$. If $s = 2l$ for $1 \le l \le d/2$, then

$$s\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = 2li_1 \circ \eta^2 = l2\eta_3 = 0.$$

However, if $s = 2l + 1$ for $0 \le l \le d/2 - 1$, then

$$s\pi_2^*(i_1)_\sharp(\overline{\eta^2}) = (2l+1)i_1 \circ \eta^2 = \eta_3 \ne 0.$$

Thus, each $f_{12} \in \langle 2\pi_2^*(i_1)\rangle \cong Z_{d/2}$ induces the trivial homomorphism on $n+3$. However, for $f_{12} \notin \langle 2\pi_2^*(i_1)\rangle$, we have $f_{12\sharp(n+3)} \ne 0$.

Next, we consider $f_{21\sharp(n+3)}$. Note that

$$[M_1, M_2] \cong Z_2 \oplus Z_2 \oplus Z_2 \ \{ \pi_1^*(\eta_1), \pi_1^*(\eta_2), \alpha\}.$$

Since $\alpha_{\sharp(n+2)} = 0$, we consider only the generators $\pi_1^*(\eta_1)$ and $\pi_1^*(\eta_2)$. For $\pi_1^*(\eta_1)$, we have

$$\pi_1^*(\eta_1)_\sharp(\eta_3) = \eta_1 \circ \pi_1 \circ \eta_3 = \eta_1 \circ \pi_1 \circ i_1\eta^2 = 0$$

and

$$\pi_1^*(\eta_1)_\sharp(\eta_4) = \eta_1 \circ \pi_1 \circ \eta_4 = \eta_1 \circ \eta = i_{2\sharp}(\eta^2) \circ \eta = 4i_{2\sharp}(\nu).$$

Thus, if $(p, 24) = 4$ or $(p, 24) = 12$, then $\pi_1^*(\eta_1)_\sharp(\eta_4) = 4i_{1\sharp}(\nu) = 0$, and if $(p, 24) = 8$ or $(p, 24) = 24$, then $\pi_1^*(\eta_1)_\sharp(\eta_4) = 4i_{1\sharp}(\nu) \ne 0$.

Since $\pi_{2\sharp}(\eta_2 \circ \eta) = \eta^2$, we have $\pi_1^*(\eta_2)_\sharp(\eta_4) = \eta_2 \circ \pi_1 \circ \eta_4 = \eta_2 \circ \eta \ne 0$.

Therefore, if $(p, 24) = 4$ or $(p, 24) = 12$, we have

$$\mathcal{E}_\sharp^{\dim+1}(X) \cong \left\{ \begin{pmatrix} 1 & f_{12} \\ f_{21} & 1 \end{pmatrix} \,\middle|\, f_{12} \in \langle 2\pi_2^*(i_1)\rangle,\ f_{21} \in \langle \pi_2^*(\eta_1)\rangle \right\},$$

and if $(p, 24) = 8$ or $(p, 24) = 24$, we have

$$\mathcal{E}_\sharp^{\dim+1}(X) \cong \left\{ \begin{pmatrix} 1 & f_{12} \\ 1 & 1 \end{pmatrix} \,\middle|\, f_{12} \in \langle 2\pi_2^*(i_1)\rangle \right\}. \qquad \square$$

References

[Araki and Toda 1965] S. Araki and H. Toda, "Multiplicative structures in mod q cohomology theories, I", *Osaka J. Math.* **2** (1965), 71–115. MR 32 #449 Zbl 0129.15201

[Arkowitz 1990] M. Arkowitz, "The group of self-homotopy equivalences—a survey", pp. 170–203 in *Groups of self-equivalences and related topics* (Montreal, 1988), edited by R. A. Piccinini, Lecture Notes in Math. **1425**, Springer, Berlin, 1990. MR 91i:55001 Zbl 0713.55004

[Arkowitz and Maruyama 1998] M. Arkowitz and K.-i. Maruyama, "Self-homotopy equivalences which induce the identity on homology, cohomology or homotopy groups", *Topology Appl.* **87**:2 (1998), 133–154. MR 99i:55009 Zbl 0929.55008

[Blakers and Massey 1952] A. L. Blakers and W. S. Massey, "The homotopy groups of a triad, II", *Ann. of Math.* (2) **55** (1952), 192–201. MR 13,485f Zbl 0046.40604

[Gray 1975] B. Gray, *Homotopy theory: An introduction to algebraic topology*, Pure and Applied Mathematics **64**, Academic Press, New York, 1975. MR 53 #6528 Zbl 0322.55001

[Hilton 1965] P. Hilton, *Homotopy theory and duality*, Gordon and Breach, New York, 1965. MR 33 #6624 Zbl 0155.50801

[Jeong 2010] M. H. Jeong, "Certain subgroups of self-homotopy equivalences of the wedge of two Moore spaces", *Commun. Korean Math. Soc.* **25**:1 (2010), 111–117. MR 2011a:55008 Zbl 1213.55005

[Puppe 1958] D. Puppe, "Homotopiemengen und ihre induzierten Abbildungen, I", *Math. Z.* **69** (1958), 299–344. MR 20 #6698 Zbl 0092.39803

[Spanier 1966] E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966. MR 35 #1007 Zbl 0145.43303

[Toda 1962] H. Toda, *Composition methods in homotopy groups of spheres*, Annals of Mathematics Studies **49**, Princeton University Press, 1962. MR 26 #777 Zbl 0101.40703

Received August 2, 2013. Revised October 17, 2013.

HO WON CHOI
DEPARTMENT OF MATHEMATICS
KOREA UNIVERSITY
SEOUL 702-701
SOUTH KOREA
howon@korea.ac.kr

KEE YOUNG LEE
DEPARTMENT OF MATHEMATICS
KOREA UNIVERSITY
SEJONG 425-791
SOUTH KOREA
keyolee@korea.ac.kr

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

EDITORS

Don Blasius (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI
CALIFORNIA INST. OF TECHNOLOGY
INST. DE MATEMÁTICA PURA E APLICADA
KEIO UNIVERSITY
MATH. SCIENCES RESEARCH INSTITUTE
NEW MEXICO STATE UNIV.
OREGON STATE UNIV.
STANFORD UNIVERSITY
UNIV. OF BRITISH COLUMBIA
UNIV. OF CALIFORNIA, BERKELEY
UNIV. OF CALIFORNIA, DAVIS
UNIV. OF CALIFORNIA, LOS ANGELES
UNIV. OF CALIFORNIA, RIVERSIDE
UNIV. OF CALIFORNIA, SAN DIEGO
UNIV. OF CALIF., SANTA BARBARA
UNIV. OF CALIF., SANTA CRUZ
UNIV. OF MONTANA
UNIV. OF OREGON
UNIV. OF SOUTHERN CALIFORNIA
UNIV. OF UTAH
UNIV. OF WASHINGTON
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2014 is US $410/year for the electronic version, and $535/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

© 2014 Mathematical Sciences Publishers

PACIFIC JOURNAL OF MATHEMATICS

Volume 272 No. 1 November 2014

Nonconcordant links with homology cobordant zero-framed surgery manifolds 1
JAE CHOON CHA and MARK POWELL

Certain self-homotopy equivalences on wedge products of Moore spaces 35
HO WON CHOI and KEE YOUNG LEE

Modular transformations involving the Mordell integral in Ramanujan's lost notebook 59
YOUN-SEO CHOI

The D-topology for diffeological spaces 87
J. DANIEL CHRISTENSEN, GORDON SINNAMON and ENXIN WU

On the Atkin polynomials 111
AHMAD EL-GUINDY and MOURAD E. H. ISMAIL

Evolving convex curves to constant-width ones by a perimeter-preserving flow 131
LAIYUAN GAO and SHENGLIANG PAN

Hilbert series of certain jet schemes of determinantal varieties 147
SUDHIR R. GHORPADE, BOYAN JONOV and B. A. SETHURAMAN

On a Liu–Yau type inequality for surfaces 177
OUSSAMA HIJAZI, SEBASTIÁN MONTIEL and SIMON RAULOT

Nonlinear Euler sums 201
ISTVÁN MEZŐ

Boundary limits for fractional Poisson a-extensions of L^p boundary functions in a cone 227
LEI QIAO and TAO ZHAO

Jacobi–Trudi determinants and characters of minimal affinizations 237
STEVEN V SAM

Normal families of holomorphic mappings into complex projective space concerning shared hyperplanes 245
LIU YANG, CAIYUN FANG and XUECHENG PANG